Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) — 9758/01

Paper 1

September 2019

3 hours

Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name, index number and class on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For teachers' use:

Qn	Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Q9	Q10	Q11	Q12	Total
Score													
Max Score	4	5	6	7	7	7	8	9	10	13	12	12	100

This question paper consists of **26** printed pages and **2** blank pages.

1 A confectionary bakes 20 banana cakes, 50 chocolate cakes and 30 durian cakes every day. The total price of 1 banana cake, 1 chocolate cake and 1 durian cake is \$29.50. On a particular day, at 7 pm, the confectionary has collected \$730 from the sales of the cakes, and there were half the banana cakes, one-tenth of the chocolate cakes and one third of the durian cakes left. In order to sell as many cakes as possible, all cakes were discounted by 40% from their respective selling price from 7 pm onwards. By closing time, all the cakes were sold and the total revenue for the entire day was \$880. Determine the selling price of each type of cake before discount. [4]

2 **(a)** Without using a calculator, solve $\dfrac{30-11x}{x^2-9} \leq -2.$ [3]

(b) Solve $(a-3bx^2)\mathrm{e}^{ax-bx^3} < 0,$ where a and b are positive constants. [2]

3 The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ with respect to origin O, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel.

(i) Given that B lies on the line segment AC such that $\overrightarrow{BC} = 5\mathbf{b} - \mu\mathbf{a}$, find the value of μ. Hence find $\overrightarrow{OC}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) The point N is the midpoint of OC. The line segment AN meets OB at point E. Find the position vector of E. [4]

4 The function f is defined as follows:

$$\mathrm{f}(x) = x + \frac{1}{x-a}, \quad a < x \leq b$$

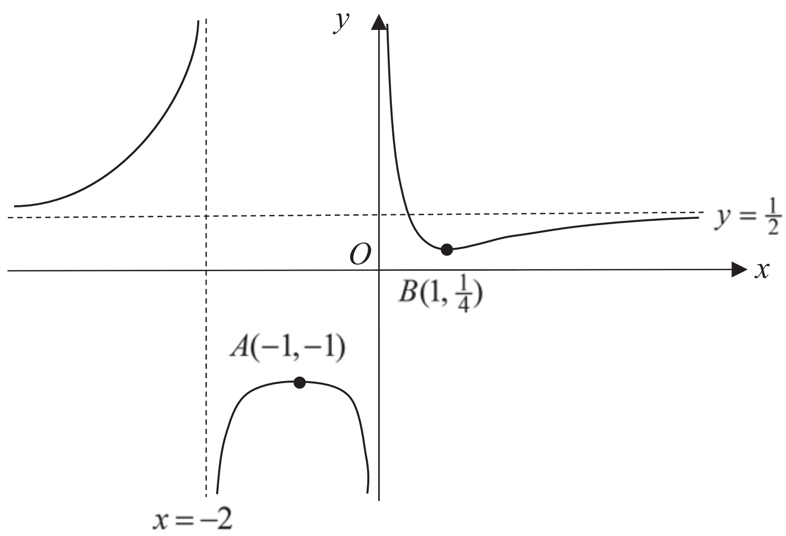
where a is a positive constant.

(i) Given that f^{-1} exist, show that $b \leq a+1$. [2]

(ii) Given that $a = 1$ and $b = 2$, find $\mathrm{f}^{-1}(x)$ and the domain of f^{-1}. [5]

5 **(a)** Describe a sequence of two transformations that maps the graph of $y=\ln\left(\dfrac{x^2}{x+1}\right)$ onto the graph of $y=\ln\left(\dfrac{2x+1}{4x^2}\right)$. [2]

(b) The diagram below shows the graph of $y=\mathrm{f}(x)$. It has a maximum point at $A(-1,-1)$ and a minimum point at $B(1,\frac{1}{4})$. The graph has asymptotes $y=\frac{1}{2}$, $x=0$ and $x=-2$.



Sketch, on separate diagrams, the graphs of

(i) $y = \mathrm{f}(2-x)$, [2]

(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

stating clearly the equations of any asymptotes, coordinates of any points of intersection with both axes and the points corresponding to A and B.

6 The sequence of complex numbers $\{w_n\}$ are defined as follows

$$w_n = \frac{\left[1+(n-1)\mathrm{i}\right]\left[1+(n+1)\mathrm{i}\right]}{\left(1+n\mathrm{i}\right)^2} \text{ for } n \in \mathbb{Z}^+.$$

(i) Show that $\arg(w_n) = p\left[\arg(1+(n-1)\mathrm{i})\right] + q\left[\arg(1+n\mathrm{i})\right] + r\left[\arg(1+(n+1)\mathrm{i}\right]$, where p, q and r are constants to be determined. [1]

Consider a related sequence $\{z_n\}$ where $z_n = w_1 w_2 \ldots w_n$, the product of the first n terms of the above sequence.

(ii) Use the method of differences to show that $\arg z_n = -\frac{1}{4}\pi - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]$. [4]

(iii) Deduce the limit of $\arg(z_n)$ as $n \to \infty$. Hence write down a linear relationship between $\text{Re}(z_n)$ and $\text{Im}(z_n)$ as $n \to \infty$. [2]

7 A curve C has parametric equations $x = 4\sin 2\theta - 2,\ y = 3 - 4\cos 2\theta$ for $0 \le \theta < \pi$.

(i) Find a cartesian equation of C. Give the geometrical interpretation of C. [3]

(ii) P is a point on C where $\theta = \frac{3}{8}\pi$. The tangent at P meets the y-axis at the point T and the normal at P meets the y-axis at the point N. Find the exact area of triangle NPT. [5]

8 The equations of two planes P_1 and P_2 are $x - 2y + 3z = 4$ and $3x + 2y - z = 4$ respectively.

(i) The planes P_1 and P_2 intersect in a line L. Find a vector equation of L. [2]

The equation of a third plane P_3 is $5x - ky + 6z = 1$, where k is a constant.

(ii) Given that the three planes have no point in common, find the value of k. [2]

Use the value of k found in part **(ii)** for the rest of the question.

(iii) Given Q is a point on L meeting the x-y plane, find the shortest distance from Q to P_3. [3]

(iv) By considering the plane containing Q and parallel to P_3 or otherwise, determine whether the origin O and Q are on the same or opposite side of P_3. [2]

9 It is given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\mathrm{e}^{-y} - 1$ and that $y = 0$ when $x = 0$.

(i) (a) Show that $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -\left(a + b\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$, where a and b are constants to be determined. [2]

(b) Hence, find the first three non-zero terms in the Maclaurin series expansion for y. [2]

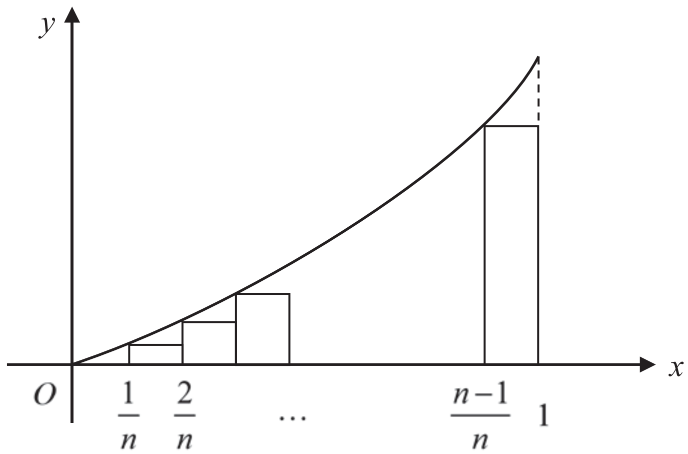
(ii) Find the particular solution of the differential equation, giving your answer in the form $y = \mathrm{f}(x)$. [4]

(iii) Denoting the answer in **(i)(b)** as $\mathrm{g}(x)$, for $x \geq 0$, find the set of values of x for which the value of $\mathrm{g}(x)$ is within ± 0.05 of the value of $\mathrm{f}(x)$. [2]

10 **(a)** Find $\int \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x$. [3]

(b) Using the substitution $t = \tan x$, find $\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,\mathrm{d}x$. [4]

(c)



The diagram above shows part of the graph of $y = x^2 + 3x$, with rectangles approximating the area under the curve from $x = 0$ to $x = 1$. The area under the curve may be approximated by the total area, A, of $(n-1)$ rectangles each of width $\frac{1}{n}$. Given that $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, show that $A = \frac{(n-1)(11n-1)}{6n^2}$.

Explain briefly how the value of $\int_0^1 x^2 + 3x \, dx$ can be deduced from this expression, and hence find this value exactly without integration. [6]

11 The department of statistics of a country has developed two mathematical functions to analyse the foreign worker policy. The first function f models the amount of strain to the country's infrastructure (housing, transportation, utilities and access to medical care etc.) based on the number of foreign workers allowed into the country and is defined as follows:

$$\mathrm{f}(x) = (x-5)^3 + 200, \quad 0 \le x \le 15$$

where x denotes the number of foreign workers, in ten thousands, allowed into the country and $\mathrm{f}(x)$ denotes the amount strain to the country's infrastructure.

The second function g models the happiness index, from 0 (least happy) to 1 (most happy), of the country's local population based on the amount of strain to the country's infrastructure and is defined as follows:

$$\mathrm{g}(w) = \ln\left(\mathrm{e} - \frac{w}{1000}\right), \quad 0 \le w \le 1000(\mathrm{e}-1)$$

where w denotes the amount of strain to the country's infrastructure and $\mathrm{g}(w)$ denotes the happiness index of the country's local population.

(i) The composite function gf models the happiness index based on the number of foreign workers, show that this function exists. [2]

(ii) Find range of values for the happiness index of the country's local population if its government plans to allow 70,000 to 110,000 foreign workers into the country. [3]

(iii) Determine whether the happiness index increases or decreases as x increases. [3]

A third function h models the gross domestic product (GDP) of the country based on the number of foreign workers (in ten thousands), x, allowed into the country and is defined as follows:

$$\mathrm{h}(x) = 400 - (x-10)^2, \quad 0 \le x \le 15$$

where $\mathrm{h}(x)$ denotes the GDP in billions of dollars.

(iv) Find the range of values for the GDP if the government plans to have a happiness index from 0.7 to 0.9 in order to secure an electoral win for the coming elections. [4]

12 In a particular chemical reaction, every 2 grams of U and 1 gram of V are combined and converted to form 3 grams of W. Let u, v and w denote the mass (in grams) of U, V and W respectively present at time t (in minutes). According to the law of mass action, the rate of change of w with respect to t is proportional to the product of u and v. Initially, $u = 40$, $v = 50$ and $w = 0$.

(i) Show that $\dfrac{\mathrm{d}w}{\mathrm{d}t} = k(w-60)(w-150)$, where k is a positive constant. [3]

It is observed that when $t = 5,\ w = 10$.

(ii) Find w when $t = 20,$ giving your answer to two decimal places. [7]

(iii) What happens to w for large values of t? [2]

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Name:		Index Number:		Class:	

DUNMAN HIGH SCHOOL
Preliminary Examination
Year 6

MATHEMATICS (Higher 2) **9758/02**

Paper 2

September 2019

3 hours

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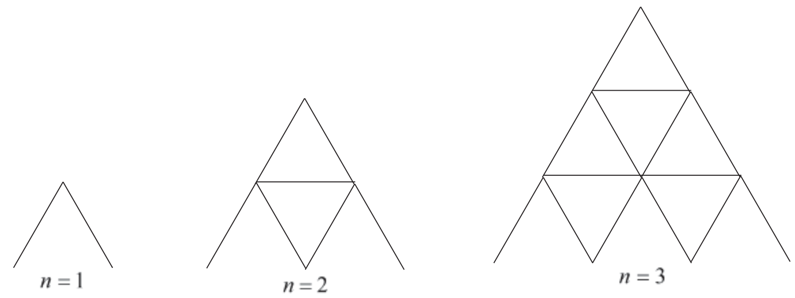
Score												
Max Score	**7**	**8**	**8**	**8**	**9**	**4**	**7**	**9**	**12**	**12**	**16**	**100**

 This question paper consists of **23** printed pages and **1** blank page.

Section A: Pure Mathematics [40 marks]

Commented [jo1]: APY

1 Ryan has playing cards which he stacks into pyramids. He will begin by stacking up cards to form a pyramid with 1 level, followed by another pyramid with 2 levels and so forth. The pyramids with n levels for different values of n are shown below:



Let S_n denotes the number of cards in a pyramid with n levels. It is given that $S_n = an^2 + bn + c$ for some constants a, b and c.

(i) Give an expression of the number of additional cards needed to form a pyramid of nth level from $(n-1)$th level. Leave your expression in terms of a, b and n. [2]

(ii) Find the values of a, b and c. [2]

(iii) Hence prove that S_n is the sum of an arithmetic progression and state the common difference. [2]

(iv) One pyramid of each level from 1 to 23 is formed. Find the total number of cards required to form these 23 pyramids. [1]

2 A curve C has equation $3x^2 - 2xy + 5y^2 = 14$.

Commented [jo2]: CHC – new qn

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x - y}{x - 5y}$. [2]

(ii) Find the exact x-coordinates of the points on the curve C at which the tangent is parallel to the y-axis. [3]

(iii) A point $P(x, y)$ moves along the curve C in such a way that y decreases at a constant rate of 7 units per second. Given that x increases at the instant when $y = 1$, find the corresponding rate of change in x. [3]

3 The complex number z is such that $az^2 + bz + a = 0$ where a and b are real constants. It is given that $z = z_0$ is a solution to this equation where $\operatorname{Im}(z_0) \neq 0$.

Commented [OMFJ3]: HG - ok

(i) Verify that $z = \dfrac{1}{z_0}$ is the other solution. Hence show that $|z_0| = 1$. [4]

Take $\text{Im}(z_0) = \frac{1}{2}$ for the rest of the question.

(ii) Find the possible complex numbers for z_0. [2]

(iii) If $\text{Re}(z_0) > 0,$ find b in terms a. [2]

4 The complex number w has modulus $\sqrt{2}$ and argument $\frac{1}{4}\pi$ and the complex number z has modulus $\sqrt{2}$ and argument $\frac{5}{6}\pi$.

Commented [jo4]: APY

(i) By first expressing w and z in the form $x+\mathrm{i}y$, find the exact real and imaginary parts of $w+z$. [3]

(ii) On the same Argand diagram, sketch the points P, Q, R representing the complex numbers z, w and $z+w$ respectively. State the geometrical shape of the quadrilateral $OPRQ$. [3]

(iii) Referring to the Argand diagram in part **(ii)**, find $\arg(w+z)$ and show that $\tan\left(\frac{11}{24}\pi\right) = \dfrac{a+\sqrt{2}}{\sqrt{6}+b}$ where a and b are constants to be determined. [2]

5 The curves C_1 and C_2 have equations $y=\dfrac{x-b}{x-a}$ and $y=\dfrac{x-b}{b}$ respectively, where a and b are constants with $1<a<b$.

Commented [OMFJ5]: JO

(i) Show that the x-coordinates of the points of intersection of C_1 and C_2 are b and $a+b$. Hence sketch C_1 and C_2 on a single diagram, labelling any points of intersection with the axes and the equations of any asymptotes. [4]

(ii) Using the diagram, solve $\dfrac{x-b}{x-a} \geq \dfrac{x-b}{b}$. [2]

(iii) Let $a = 2$ and $b = 3$. The region bounded by C_1 and C_2 is rotated through 4 right angles about the y-axis to form a solid of revolution of volume V. Find the numerical value of V, giving your answer correct to 3 decimal places. [3]

Commented [jo6]: new qn to add in volume

Section B: Probability and Statistics [60 marks]

6 Nine gifts, three of which are identical and the rest are distinct, are distributed among five people without restrictions on the number of gifts a person can have. By first considering the number of ways to distribute the distinct gifts or otherwise, find the number of way that the nine gifts can be distributed. [4]

Commented [jo7]: check answer

7 In a school survey, a group of 80 students are asked about how much time per week (to nearest hour) they spend on their co-curricular activities (CCA). The readings are shown below:

Commented [jo8]: EC - ok

	CCA (hours)		
	3 or less	**4 to 6**	**7 or more**
Boy	17	20	10
Girl	18	$15-k$	k

A student is selected random from the group. Defining the events as follows:

G : The student is a girl.
L : The student spends 6 hours or less weekly.
M : The student spends 4 hours or more weekly.

Find the following probabilities in terms of k.

(i) $\mathrm{P}(L' \cup M')$ [2]

(ii) $\mathrm{P}(G \mid L')$ [1]

(iii) Given that $\mathrm{P}(L \cap M) = \dfrac{2}{5}$, find the value of k. Hence determine if L and M are independent, justifying your answer. [3]

(iv) If the events G and $(L \cap M)$ are mutually exclusive, find the value of k. [1]

8 Sharron who is an amateur swimmer has been attending swimming lessons. She records her time taken to swim 50 metres each month. Her best timing, t seconds, recorded each month x, for the first 7 months is as follows.

Month x	1	2	3	4	5	6	7
Time taken, t	115	87	75	67	62	61	55

(i) Draw a scatter diagram showing these timings. [1]

(ii) It is desired to predict Sharron's timings on future swims. Explain why, in this context, neither a linear nor a quadratic model is likely to be appropriate. [2]

Commented [jo9]: Nat – ok

It is decided to fit a model of the form $t = a + \frac{b}{x}$ where a and b are constants.

(iii) State with a reason whether each of a and b is positive or negative. [2]

(iv) Find the product moment correlation coefficient and the constants a and b. [2]

At the 8th month, Sharron recorded her best timing and calculated the regression line using all the data from the first 8 months to be $t = 48.28 + \frac{69.45}{x}$.

(v) Find her best timing, to the nearest second, at the 8th month. [2]

9 The time taken, T (in minutes), for a 17-year-old student to complete a 5-km run is a random variable with mean 30. After a new training programme is introduced for these students, a random sample of n students is taken. The mean time and standard deviation for the sample are found to be 28.9 minutes and 4.0 minutes respectively.

Commented [OMFJ10]: YCH – shared H1 (use as contextual)

(a) Find the unbiased estimate of the population variance in terms of n. [1]

(b) Using $n = 40$,

(i) carry out a test at the 10% significance level to determine if the mean time taken has changed. State appropriate hypotheses for the test and define any symbols you use. [4]

(ii) State what it means by the p-value in this context. [1]

(iii) Give a reason why no assumptions about the population are needed in order for the test to be valid. [1]

(c) The trainer claims instead that the new training programme is able to improve the mean of T, 30 minutes, by at least 5%. The school wants to test his claim.

(i) Write down the null and alternative hypothesis. [1]

(ii) Using the existing sample, the school carried out a test at 1% significance level and found that there was sufficient evidence to reject the trainer's claim. Find the set of values that n can take, stating any necessary assumption(s) needed to carry out the test. [4]

Commented [jo11]: FM marker in case of t-test

Commented [jo12]: check if want to mark down

10 The speeds of an e-scooter (X km/h) and a pedestrian (Y km/h) measured on a particular stretch of footpath are normally distributed with mean and variance as follows:

Commented [jo13]: EC – OK (use as contextual)

	mean	variance
X	12.3	9.9
Y	μ	σ^2

It is known that $\mathrm{P}(Y < 5.2) = \mathrm{P}(Y \geq 7.0) = 0.379$.

(i) State the value of μ and find the value of σ. [2]

(ii) Given that the speeds of half of the e-scooters measured are found to be within a km/h of the mean, find a. [2]

(iii) A LTA officer stationed himself at the footpath and measured the speeds of 50 e-scooters at random. Find the probability that the 50th e-scooter is the 35th to exceed LTA's legal speed limit of 10 km/h. [3]

(iv) On another day, the LTA officer randomly measured the speeds of 6 e-scooters and 15 pedestrians. Find the probability that the mean speed of the e-scooters is more than twice the mean speed of the pedestrians captured. [3]

(v) Find the probability that the mean speed of n randomly chosen e-scooters is more than 10 km/h, if n is large. [2]

11 **(a)** At a funfair, Alice pays \$3 to play a game by tossing a fair dice until she gets a '6'. Let X be the number of times that the player tosses a fair dice until he gets a '6'. The prize, S (in dollars), that the player may win is given by the following function:

Commented [jo14]: CHC – ok (to update alternative solution)

$$S = \begin{cases} 8, & \text{if } X = 1, \\ 4, & \text{if } 2 \le X \le k, \\ 0, & \text{otherwise.} \end{cases}$$

where k is a positive integer.

(i) Show that $\mathrm{P}(2 \le X \le k) = (\frac{5}{6}) - (\frac{5}{6})^k$. Hence draw up a table showing the probability distributions of S. [4]

(ii) Find the least value of k such that Alice is expected to earn a profit. [3]

(b) Alice uses a computer program to simulate 80 tosses of a biased coin. Let Y be the random variable denoting the number of heads obtained and p be the probability of obtaining a head. It is given that $80+\mathrm{E}(Y)=6\mathrm{Var}(Y)$.

(i) Find the exact value of p. [3]

(ii) Find the probability of obtaining at least 30 heads, given that the first 5 tosses are heads. [3]

(iii) Alice executes the program 50 times. Find the probability that the mean number of heads, $\overline{Y}$, is less than 25. [3]

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2019 Year 6 H2 Math Prelim P1 Mark Scheme

Qn	Suggested Solution	
1		

	Number sold before 7 pm	Number left after 7 pm
Banana	10	10
Chocolate	45	5
Durian	20	10

Let the selling price of banana cake, chocolate cake, durian cake before discount be \$$b$, \$$c$, \$$d$ respectively.

$$a+b+c=29.50 \quad \cdots(1)$$

$$10b+45c+20d=730$$
$$2b+9c+4d=146 \quad \cdots(2)$$

$$0.6(10b+5c+10d)=880-730$$
$$10b+5c+10d=250$$
$$2b+c+2d=50 \quad \cdots(3)$$

Solving (1), (2), (3) using GC, $a=8.50$, $b=9$, $c=12$
The selling price of banana cake, chocolate cake and durian cake is \$8.50, \$9 and \$12 respectively.

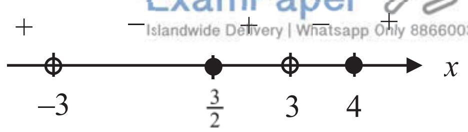
Qn	Suggested Solution	
2(a)		

$$\frac{30-11x}{x^2-9}\le -2$$

$$\frac{30-11x+2(x^2-9)}{x^2-9}\le 0$$

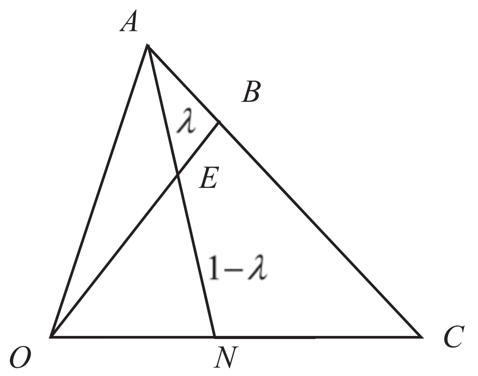
$$\frac{2x^2-11x+12}{x^2-9}\le 0$$

$$\frac{(2x-3)(x-4)}{(x-3)(x+3)}\le 0$$

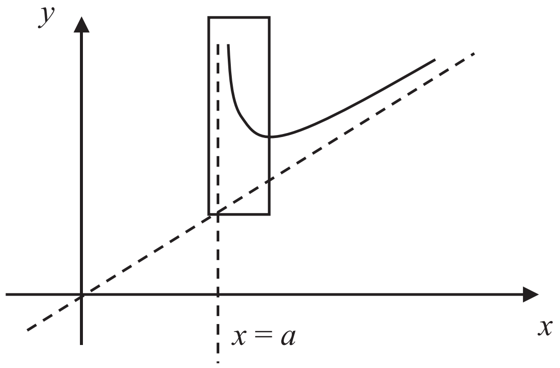


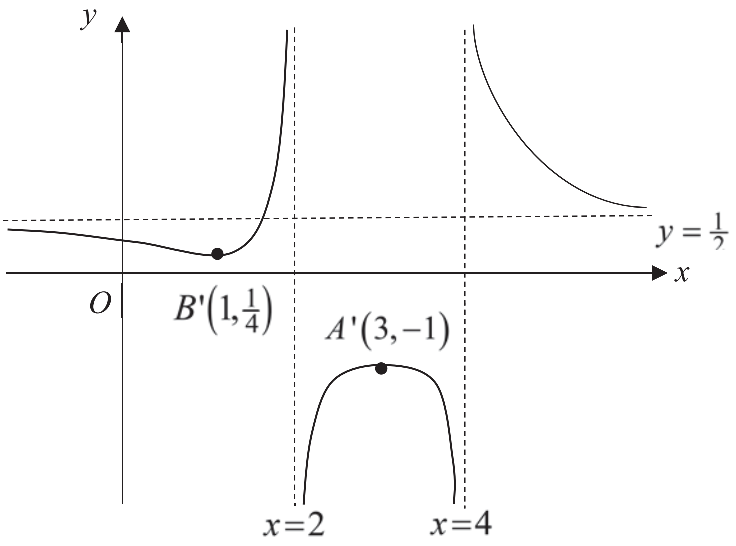
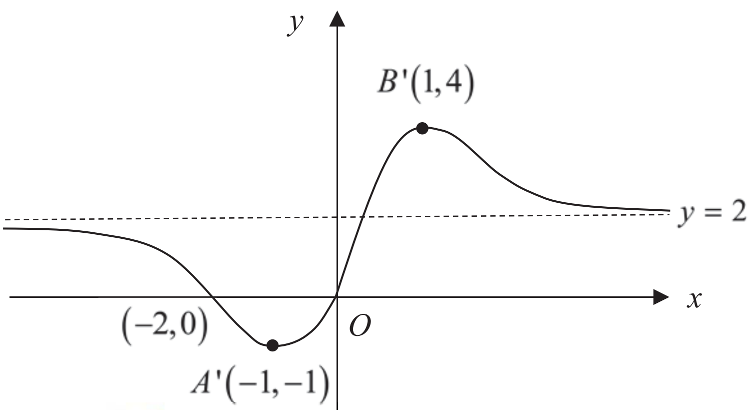
$\therefore -3<x\le \frac{3}{2}$ or $3<x\le 4$

(b)	$(a-3bx^2)e^{ax-bx^3}<0$ $a-3bx^2<0 \quad$ since $e^{ax-bx^3}>0$ for all x $x^2>\dfrac{a}{3b}$ $x>\sqrt{\dfrac{a}{3b}}$ or $x<-\sqrt{\dfrac{a}{3b}}$	

Qn	Suggested Solution	
3(i)	Since A, B and C are collinear and $\overrightarrow{AB}=\mathbf{b}-\mathbf{a}$ $\therefore \mu=5$ $\overrightarrow{OC}=\overrightarrow{OB}+\overrightarrow{BC}$ $=\mathbf{b}+(5\mathbf{b}-5\mathbf{a})$ $=6\mathbf{b}-5\mathbf{a}$	
(ii)	 $\overrightarrow{OE}=k\mathbf{b}$ $\overrightarrow{OE}=\lambda\overrightarrow{ON}+(1-\lambda)\overrightarrow{OA}$ $=\dfrac{\lambda}{2}(6\mathbf{b}-5\mathbf{a})+(1-\lambda)\mathbf{a}$ $=3\lambda\mathbf{b}+\left(1-\dfrac{7}{2}\lambda\right)\mathbf{a}$ $1-\dfrac{7}{2}\lambda=0\Rightarrow\lambda=\dfrac{2}{7}\Rightarrow k=\dfrac{6}{7}$ $\therefore\overrightarrow{OE}=\mu\mathbf{b}=\dfrac{6}{7}\mathbf{b}$	

Qn	Suggested Solution	

4 (i)	Since the shape of the curve is  For f to be 1-1, the largest b can take is the x-coordinate of the turning point. $\mathrm{f}'(x)=1-\dfrac{1}{(x-a)^2}$ $1-\dfrac{1}{(x-a)^2}=0$ $x=a\pm1$ x-coordinate of turning point is $a+1$, since $b>a$ For graph to be 1-1, $b\le a+1$,	
(ii)	Let $y=\mathrm{f}(x)$ $y=x+\dfrac{1}{x-1}$ $(x-1)y=x(x-1)+1$ $xy-y=x^2-x+1$ $x^2-(1+y)x+1+y=0$ $\left(x-\dfrac{(1+y)}{2}\right)^2-\dfrac{(1+y)^2}{4}+1+y=0$ $\left(x-\dfrac{(1+y)}{2}\right)^2=\dfrac{(y-1)^2}{4}-1$ $x=\dfrac{(1+y)}{2}\pm\sqrt{\dfrac{(y-1)^2}{4}-1}$ Since $\left(\dfrac{3}{2},\dfrac{7}{2}\right)$ is a point on the curve of $y=\mathrm{f}(x)$, $x=\dfrac{(1+y)}{2}-\sqrt{\dfrac{(y-1)^2}{4}-1}$ $\mathrm{f}^{-1}(x)=\dfrac{(1+x)}{2}-\sqrt{\dfrac{(x-1)^2}{4}-1}$ The domain of f^{-1} is the range of $\mathrm{f}=[3,\infty)$.	

Qn	Suggested Solution	
5(a)	Series of transformations: $y=\ln\dfrac{x^2}{x+1}$ $\downarrow$ **1. Reflect in the x-axis** (replace y with $-y$) $y=-\ln\dfrac{x^2}{x+1}=\ln\dfrac{x+1}{x^2}$ $\downarrow$ **2. Scale by factor $\frac{1}{2}$ parallel to the x-axis** (replace x with $2x$) $y=\ln\dfrac{2x+1}{(2x)^2}=\ln\dfrac{2x+1}{4x^2}$	
(b)(i)		
(ii)		

Qn	Suggested Solution	
6	$\arg(w_n)=\arg[1+(n-1)\mathrm{i}]-2\arg(1+n\mathrm{i})+\arg[1+(n+1)\mathrm{i}]$	

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<tr><td>(i)</td><td></td><td></td></tr>
<tr><td>(ii)</td><td>

$$\begin{aligned}
&\arg z_n\\
&= \arg(w_1 w_2 \ldots w_n)\\
&= \arg(w_1) + \arg(w_2) + \ldots \arg(w_n)\\
&= \sum_{k=1}^{n} \arg w_k\\
&= \sum_{k=1}^{n} \arg\left(\frac{[1+(k-1)\mathrm{i}][1+(k+1)\mathrm{i}]}{(1+k\mathrm{i})^2}\right)\\
&= \sum_{k=1}^{n}\left[\arg[1+(k-1)\mathrm{i}] - 2\arg(1+k\mathrm{i}) + \arg[1+(k+1)\mathrm{i}]\right]
\end{aligned}$$

$$= \left\{\begin{array}{llllll}
 & [\arg(1) & - & 2\arg(1+\mathrm{i}) & + & \arg(1+2\mathrm{i})]\\
+ & [\arg(1+\mathrm{i}) & - & 2\arg(1+2\mathrm{i}) & + & \arg(1+3\mathrm{i})]\\
+ & [\arg(1+2\mathrm{i}) & - & 2\arg(1+3\mathrm{i}) & + & \arg(1+4\mathrm{i})\\
 & & & \vdots & & \\
+ & [\arg[1+(n-2)\mathrm{i}] & - & 2\arg[1+(n-1)\mathrm{i}] & + & \arg(1+n\mathrm{i})]\\
+ & [\arg[1+(n-1)\mathrm{i}] & - & 2\arg(1+n\mathrm{i}) & + & \arg[1+(n+1)\mathrm{i}]]
\end{array}\right.$$

$$\begin{aligned}
&= \arg(1) - \arg(1+\mathrm{i}) - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]\\
&= -\tfrac{1}{4}\pi - \arg(1+n\mathrm{i}) + \arg[1+(n+1)\mathrm{i}]
\end{aligned}$$

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<tr><td>(iii)</td><td>

As $n \to \infty$, $\arg(1+n\mathrm{i}) \to \frac{1}{2}\pi$ and $\arg[1+(n+1)\mathrm{i}] \to \frac{1}{2}\pi$

Hence $\arg z_n \to -\frac{1}{4}\pi$.

(argand diagram with y = -x line to show argument)

Thus $\mathrm{Re}(z_n) = -\mathrm{Im}(z_n)$

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<tr><td>7
(i)</td><td>

$4\sin 2\theta = x+2$

$16\sin^2 2\theta = (x+2)^2$ ---- (1)

$4\cos 2\theta = 3-y$

$16\cos^2 2\theta = (3-y)^2$ ---- (2)

(1) + (2) gives

$(x+2)^2 + (y-3)^2 = 16$

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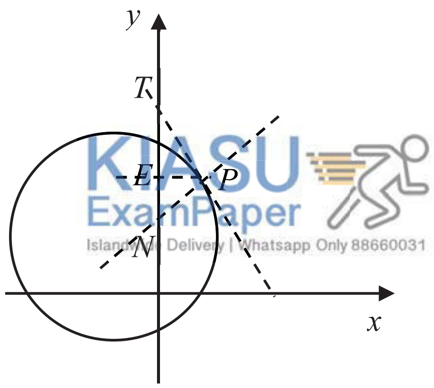
	Hence C is a circle with centre $(-2,3)$ and radius 4 units.	
(ii)	$\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=8\cos 2\theta$ and $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=8\sin 2\theta$ gives $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\tan 2\theta$ For $\theta=\dfrac{3}{8}\pi$, $x=2\sqrt{2}-2$ $y=3+2\sqrt{2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$ Equation of tangent: $y-3-2\sqrt{2}=-1(x-2\sqrt{2}+2)$ Equation of normal: $y-3-2\sqrt{2}=x-2\sqrt{2}+2$ So $\mathrm{T}(0,1+4\sqrt{2})$ and $\mathrm{N}(0,5)$ Hence the area of triangle NPT $=\dfrac{1}{2}(4\sqrt{2}-4)(2\sqrt{2}-2)$ $=(2\sqrt{2}-2)(2\sqrt{2}-2)$ $=12-8\sqrt{2}$ units^2 Alternatively, Let E be the point closest to P along the y-axis. Since $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$ at P, the triangle TPE is such that $ET=EP$ and $\measuredangle TEP=90^o$. 	

	The normal at P i.e. $\frac{dy}{dx}=1$. the triangle NPE is such that $EN = EP$ and $\measuredangle NEP = 90^o$. Therefore the two triangles are congruent, and the area of triangle NPT $=2\left[\frac{1}{2}\left(2\sqrt{2}-2\right)\left(2\sqrt{2}-2\right)\right]$ $=\left(2\sqrt{2}-2\right)^2$ $=12-8\sqrt{2}$	

Qn	Suggested Solution	
8 (i)	$x-2y+3z=4$ ---- (1) $3x+2y-z=4$ ---- (2) Solving (1) and (2) using GC gives $x=2-0.5z$ $y=-1+1.25z$ $z=z$ Hence L : $\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}-2\\5\\4\end{pmatrix}$, $\lambda\in\mathbb{R}$	
(ii)	P_3 : $\mathbf{r}\bullet\begin{pmatrix}5\\-k\\6\end{pmatrix}=1$ If the three planes have no point in common, $\begin{pmatrix}-2\\5\\4\end{pmatrix}\bullet\begin{pmatrix}5\\-k\\6\end{pmatrix}=0$ $\Rightarrow -10-5k+24=0$ $\therefore\ k=2.8$	

(iii)	$\overrightarrow{OQ}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ Distance required $=\dfrac{\left\lvert 1-\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\rvert}$ $=\dfrac{\lvert 1-12.8\rvert}{\sqrt{68.84}}=1.42$ units (3 s.f.) **Alternative** $\overrightarrow{OQ}=\begin{pmatrix}2\\-1\\0\end{pmatrix}$ and let $\overrightarrow{OY}=\begin{pmatrix}0\\0\\1/6\end{pmatrix}$ where Y is a point on P_3 Shortest distance from Q to P_3 $=\dfrac{\left\lvert\overrightarrow{YZ}\bullet\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\rvert}{\sqrt{5^2+(-2.8)^2+6^2}}=\dfrac{\left\lvert\begin{pmatrix}2\\-1\\-1/6\end{pmatrix}\bullet\begin{pmatrix}5\\-2.8\\6\end{pmatrix}\right\rvert}{\sqrt{68.84}}=1.42$ units	
(iv)	Plane containing Q and parallel to P_3: $5x-2.8y+6z=d$ where $d=\begin{pmatrix}2\\-1\\0\end{pmatrix}\cdot\begin{pmatrix}5\\-2.8\\6\end{pmatrix}=5(2)-2.8(-1)+6(0)=12.8$ $\therefore 5x-2.8y+6z=12.8$ Since $12.8>1>0$, P_3 is in between the above plane and the origin. Thus O and Q are on the opposite sides of P_3.	

Qn	Suggested Solution	
9(i) **(a)**	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\mathrm{e}^{-y}-1$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\left(-\mathrm{e}^{-y}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $=-\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}y}{\mathrm{d}x}$ $\frac{\mathrm{d}^3y}{\mathrm{d}x^2}=-\left[\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right]=-\left(1+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$	
(b)	$\frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-\left[\left(1+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^3y}{\mathrm{d}x^3}+2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]$ When $x=0,\ y=0$ (given) $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{2},\quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{4},\quad \frac{\mathrm{d}^3y}{\mathrm{d}x^3}=0,\quad \frac{\mathrm{d}^4y}{\mathrm{d}x^4}=-\frac{1}{8}$ $y=-\frac{1}{2}x+\frac{\frac{1}{4}}{2!}x^2+0-\frac{\frac{1}{8}}{4!}x^3+\ldots$ $=-\frac{1}{2}x+\frac{1}{8}x^2-\frac{1}{192}x^4+\ldots$	
(ii)	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\mathrm{e}^{-y}-1$ $\frac{1}{\frac{1}{2}\mathrm{e}^{-y}-1}\frac{\mathrm{d}y}{\mathrm{d}x}=1$ $\int\frac{1}{\frac{1}{2}\mathrm{e}^{-y}-1}\,\mathrm{d}y=\int 1\,\mathrm{d}x$ $\int\frac{\mathrm{e}^{y}}{\frac{1}{2}-\mathrm{e}^{y}}\,\mathrm{d}y=x+C$ $-\ln\lvert\frac{1}{2}-\mathrm{e}^{y}\rvert=x+C$ $\frac{1}{2}-\mathrm{e}^{y}=\pm\mathrm{e}^{-x+C}=A\mathrm{e}^{-x}$ $y=\ln(\frac{1}{2}-A\mathrm{e}^{-x})$ When $x=0,\ y=0$ $0=\ln(\frac{1}{2}-A\mathrm{e}^{0})$ $A=-\frac{1}{2}$ $\therefore y=\ln(\frac{1}{2}+\frac{1}{2}\mathrm{e}^{-x})$	

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Alternative (for integration)

$$\int \frac{1}{\frac{1}{2}\mathrm{e}^{-y}-1}\,\mathrm{d}y = x+C$$

$$\int \frac{1-\frac{1}{2}\mathrm{e}^{-y}+\frac{1}{2}\mathrm{e}^{-y}}{\frac{1}{2}\mathrm{e}^{-y}-1}\,\mathrm{d}y = x+C$$

$$\int -1-\frac{(-\frac{1}{2}\mathrm{e}^{-y})}{\frac{1}{2}\mathrm{e}^{-y}-1}\,\mathrm{d}y = x+C$$

$$-y-\ln\left|\tfrac{1}{2}\mathrm{e}^{-y}-1\right| = x+C$$

$$\ln \mathrm{e}^{-y}-\ln\left|\tfrac{1}{2}\mathrm{e}^{-y}-1\right| = x+C$$

$$\ln\left|\frac{\mathrm{e}^{-y}}{\frac{1}{2}\mathrm{e}^{-y}-1}\right| = x+C$$

$$\ln\left|\frac{1}{\frac{1}{2}-\mathrm{e}^{y}}\right| = x+C$$

$$-\ln\left|\tfrac{1}{2}-\mathrm{e}^{y}\right| = x+C$$

$$\vdots$$

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(iii)

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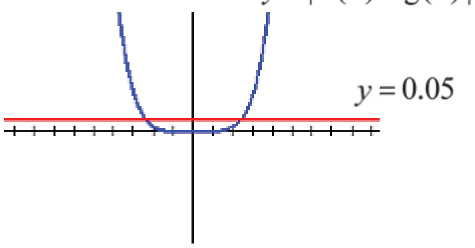
$\lvert \mathrm{f}(x)-\mathrm{g}(x)\rvert < 0.05$



From GC, $\{x\in\mathbb{R} : 0\le x<2.43\}$

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10(i)

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$$\int \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x = \left[x\sqrt{2x-1}\right]-\int\sqrt{2x-1}\,\mathrm{d}x$$

$$= x\sqrt{2x-1}-\frac{1}{3}\left((2x-1)^{\frac{3}{2}}\right)+C$$

$$= \sqrt{2x-1}\left(x-\frac{1}{3}(2x-1)\right)+C$$

$$= \frac{1}{3}\sqrt{2x-1}\,(x+1)+C$$

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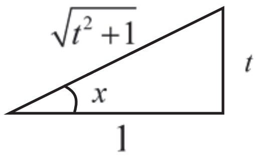
$$\int \frac{x}{\sqrt{2x-1}}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x-1+1}{\sqrt{2x-1}}\,\mathrm{d}x$$
$$= \frac{1}{2}\int \sqrt{2x-1}\,\mathrm{d}x + \frac{1}{2}\int \frac{1}{\sqrt{2x-1}}\,\mathrm{d}x$$
$$= \frac{1}{2}\frac{(2x-1)^{\frac{3}{2}}}{\frac{3}{2}(2)} + \frac{1}{2}\frac{(2x-1)^{\frac{1}{2}}}{\frac{1}{2}(2)} + C$$
$$= \frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}} + C$$

(ii)

$x = \tan^{-1} t, \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{1+t^2}, \qquad \sin x = \frac{t}{\sqrt{t^2+1}}$



$$\int \frac{1}{4\cos^2 x + 9\sin^2 x}\,\mathrm{d}x$$
$$= \int \frac{1}{4+5\sin^2 x}\,\mathrm{d}x$$
$$= \int \frac{1}{4+5\frac{t^2}{t^2+1}} \cdot \frac{1}{1+t^2}\,\mathrm{d}t$$
$$= \int \frac{1}{4+9t^2}\,\mathrm{d}t$$
$$= \frac{1}{9}\int \frac{1}{(\frac{2}{3})^2+t^2}\,\mathrm{d}t$$
$$= \frac{1}{6}\tan^{-1}\frac{3t}{2} + C = \frac{1}{6}\tan^{-1}\frac{3\tan x}{2} + C$$

(iii)

$$A = \frac{1}{n}\left(\mathrm{f}\left(\frac{1}{n}\right) + \mathrm{f}\left(\frac{2}{n}\right) + \ldots + \mathrm{f}\left(\frac{n-1}{n}\right)\right)$$
$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \ldots + \left(\frac{n-1}{n}\right)^2 + 3\left(\frac{1}{n}\right) + 3\left(\frac{2}{n}\right) + \ldots + 3\left(\frac{n-1}{n}\right)\right]$$
$$= \frac{1}{n}\left[\frac{1}{n^2}(1^2+2^2+\ldots+(n-1)^2) + \frac{3}{n}(1+2+\ldots+(n-1))\right]$$
$$= \frac{1}{n^3}\left(\frac{1}{6}(n-1)(n)(2n-1)\right) + \frac{3}{n^2}\left(\frac{n-1}{2}(n)\right)$$
$$= \frac{(n-1)(2n-1+9n)}{6n^2} = \frac{(n-1)(11n-1)}{6n^2}$$

$A \to \int_0^1 x^2 + 3x\,\mathrm{d}x$ as $n \to \infty$

in particular,

	$\dfrac{(n-1)(11n-1)}{6n^2}=\dfrac{11n^2-12n+1}{6n^2}=\dfrac{11-\frac{12}{n}+\frac{1}{n^2}}{6}\to\dfrac{11}{6}$	

Qn	Suggested Solution	
11(i)	$R_{\mathrm{f}}=\left[75,1200\right],\ D_{\mathrm{g}}=\left[0,1000(\mathrm{e}-1)\right]$ Since $R_{\mathrm{f}}\subset D_{\mathrm{g}}$, the composite function gf exist.	
(ii)	$[7,\ 11]\xrightarrow{\mathrm{f}}[208,\ 416]\xrightarrow{\mathrm{g}}[0.834,\ 0.920]$ The range of values for the happiness index is [0.834, 0.920]	
(iii)	Since f is an increasing function and g is a decreasing function, the composite function gf will be a decreasing function. e.g. for $b>a$ f is an increasing function $\Rightarrow$ $\mathrm{f}(b)>\mathrm{f}(a)$ g is a decreasing function $\Rightarrow$ $\mathrm{gf}(b)<\mathrm{gf}(a)$ **Alternative** Differentiate and deduce negative gradient	
(iv)	$[8.8859,\ 12.961]\xleftarrow{\mathrm{f}^{-1}}[258.68,\ 704.53]\xleftarrow{\mathrm{g}^{-1}}[0.7,\ 0.9]$ The number of foreign workers allowed in the country can be from 88859 to 129610. $[8.8859,\ 12.961]\xrightarrow{\mathrm{h}}[391,\ 400]$ Take note that h(x) is a quadratic expression, thus the range of GDP will be 391 billion to 400 billion dollars.	

Qn	Suggested Solution	
12(i)	Amount of U in time t $= 40 - \frac{2}{2+1}w = 40 - \frac{2}{3}w$ Amount of V in time t $= 50 - \frac{1}{3}w$ $\frac{\mathrm{d}w}{\mathrm{d}t} = k_1\left(40 - \frac{2}{3}w\right)\left(50 - \frac{1}{3}w\right), \; k_1 \in \mathbb{R}^+$ as amt. of $w\uparrow$ $= k_1\left(-\frac{2}{3}\right)(w-60)\left(-\frac{1}{3}\right)(w-150)$ $= k(w-60)(w-150), \quad k = \frac{2}{9}k_1$	
(ii)	$\frac{\mathrm{d}w}{\mathrm{d}t} = k(w-60)(w-150)$ $\frac{1}{(w-60)(w-150)}\frac{\mathrm{d}w}{\mathrm{d}t} = k$ $\frac{1}{w^2-210w+9000}\frac{\mathrm{d}w}{\mathrm{d}t} = k$ $\frac{1}{(w-105)^2-45^2}\frac{\mathrm{d}w}{\mathrm{d}t} = k$ Integrating w.r.t. t: $\frac{1}{2(45)}\ln\left\|\frac{(w-105)-45}{(w-105)+45}\right\| = kt + C, \; k$ an arbitrary constant $\left\|\frac{w-150}{w-60}\right\| = \mathrm{e}^{90C}\mathrm{e}^{90kt}$ $\frac{w-150}{w-60} = A\mathrm{e}^{90kt}$, where $A = \pm\mathrm{e}^{90C}$ When $t=0, \; w=0$: $\frac{-150}{-60} = A$ $\therefore A = \frac{5}{2}$ When $t=5, \; w=10$:	

	$\dfrac{10-150}{10-60}=\dfrac{5}{2}e^{90k(5)}$ $k=\dfrac{1}{450}\ln\dfrac{28}{25}$ $\therefore \dfrac{w-150}{w-60}=\dfrac{5}{2}e^{\left(\frac{1}{5}\ln\frac{28}{25}\right)t}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{t}{5}}$ When $t=20$, $\dfrac{w-150}{w-60}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{20}{5}}=3.93379$ $w(3.93379-1)=60(3.93379)-150$ $w=29.3229=29.32$ (2 d.p.)	
(iii)	$\dfrac{w-150}{w-60}=\dfrac{5}{2}\left(\dfrac{28}{25}\right)^{\frac{t}{5}}$ As $t\to\infty$, RHS $\to\infty$ i.e. $w-60\to 0$ $\therefore w\to 60$ Method 2: (remove from solution) Use graph of dw/dt vs w and deduce equilibrium (or equivalent deductions)	

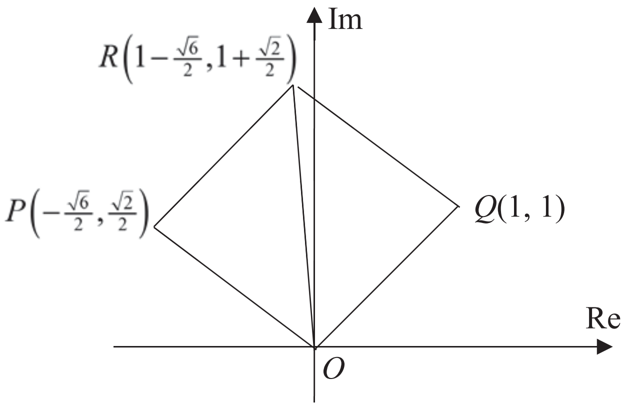
2019 Year 6 H2 Math Prelim P2 Mark Scheme

Qn	Suggested Solution	
1(i)	$S_n - S_{n-1}$ $= an^2 + bn + c - (a(n-1)^2 + b(n-1) + c)$ $= 2an - a + b$ Total number of additional cards need is $2an - a + b$	
(ii)	Additional cards to form 2nd level from 1st level $= 5$ $4a - a + b = 5 \Rightarrow 3a + b = 5$ --- (1) Additional cards to form 3rd level from 2nd level $= 8$ $6a - a + b = 8 \Rightarrow 5a + b = 8$ ---(2) Solving both (1) and (2), $a = \frac{3}{2}, b = \frac{1}{2}$. Using $S_1 = 2 \Rightarrow \frac{3}{2}(1)^2 + \frac{1}{2}(1) + c = 2 \Rightarrow c = 0$. **Alternative** Substituting different values of n, $n = 1$: $a + b + c = 2$ $n = 2$: $4a + 2b + c = 7$ $n = 3$: $9a + 3b + c = 15$ From GC, $a = 1.5$, $b = 0.5$ and $c = 0$ **Alternative** n = 1, number of cards = 2 n = 2, number of cards = 2 + 5 n = 3, number of cards = 2 + 5 + 8 $S_n = \frac{n}{2}[2(2) + (n-1)(3)] = \frac{n}{2}(3n+1) = 1.5n^2 + 0.5n$ $\therefore$ $a = 1.5$, $b = 0.5$ and $c = 0$	
(ii)	$u_n = 3n - 1$ $u_n - u_{n-1} = (3n-1) - (3(n-1)-1) = 3$ (constant) Thus Sn is a sum of AP with common difference 3.	
(iii)	$\sum_{n=1}^{23} S_n = \sum_{n=1}^{23} (1.5n^2 + 0.5n) = 6624$	

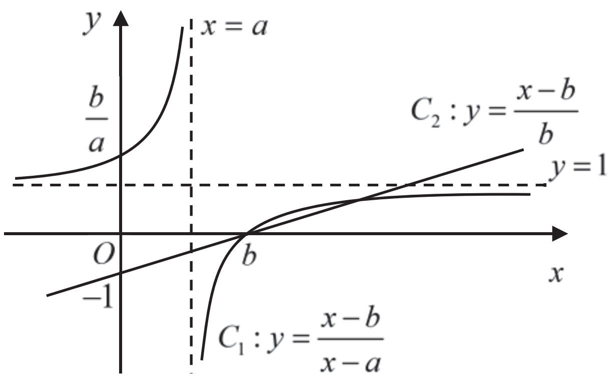
Qn	Suggested Solution	
2 (i)	$3x^2 - 2xy + 5y^2 = 14$ ---- (1) Differentiate (1) implicitly wrt x: $6x - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 10y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $(2x - 10y)\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 2y$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x - y}{x - 5y}$ (shown)	
(ii)	$x - 5y = 0 \Rightarrow y = 0.2x$ Sub $y = 0.2x$ into (1): $3x^2 - 2x(0.2x) + 5(0.2x)^2 = 14$ $2.8x^2 = 14$ $x = \pm\sqrt{5}$	
(iii)	When $y = 1$, $3x^2 - 2x - 9 = 0$ Therefore, $x = -1.4305$ or $x = 2.0972$ $\frac{\mathrm{d}y}{\mathrm{d}t} = (\frac{\mathrm{d}y}{\mathrm{d}x})(\frac{\mathrm{d}x}{\mathrm{d}t})$ $-7 = (\frac{3x-1}{x-5})(\frac{\mathrm{d}x}{\mathrm{d}t})$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{7(5-x)}{3x-1}$ When $x = 2.0972$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 3.84$ units per second (3 s.f.)	

Qn	Suggested Solution	
3(i)	LHS $= a\left(\frac{1}{z_0}\right)^2 + b\left(\frac{1}{z_0}\right) + a$ $= \left(\frac{1}{z_0}\right)^2 (a + bz_0 + az_0^{\ 2})$ $= 0$ $\because a + bz_0 + az_0^{\ 2} = 0$ Thus $z = \frac{1}{z_0}$ is a solution. Since a and b are real constants,	

	$\frac{1}{z_0} = z_0^*$ $z_0 z_0^* = 1$ $\lvert z_0 \rvert^2 = 1$ Since $\lvert z_0 \rvert > 0$, $\lvert z_0 \rvert = 1$ **Alternative for first part:** Let second root be z_1 product of roots $z_0 z_1 = \frac{a}{a} = 1$ $\therefore z_1 = \frac{1}{z_0}$	
(ii)	Let $z_0 = x_0 + \mathrm{i}y_0$ Since $\mathrm{Im}(z_0) = \frac{1}{2}$, $y_0 = \frac{1}{2}$. From part (i), $\lvert z_0 \rvert = 1$ $\sqrt{x_0^2 + y_0^2} = 1$ $\sqrt{x_0^2 + \left(\frac{1}{2}\right)^2} = 1$ $x_0 = \pm\frac{\sqrt{3}}{2}$ $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$ or $-\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$	
(iii)	Since $\mathrm{Re}(z_0) > 0$, $z_0 = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}$. Subst into $az_0^2 + bz_0 + a = 0$, $a\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right)^2 + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $a\left(\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) + b\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) + a = 0$ $\left(\frac{3}{2}a + \frac{\sqrt{3}}{2}b\right) + \mathrm{i}\left(\frac{1}{2}b + \frac{\sqrt{3}}{2}a\right) = 0$ $\therefore b = -\sqrt{3}a$	

Qn	Suggested Solution	
4(i)	$w=\sqrt{2}\left(\cos\frac{1}{4}\pi+\mathrm{i}\sin\frac{1}{4}\pi\right)$ $=1+\mathrm{i}$ $z=\sqrt{2}\left(\cos\frac{5}{6}\pi+\mathrm{i}\sin\frac{5}{6}\pi\right)$ $=-\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}\mathrm{i}$ $w+z=\left(1-\frac{\sqrt{6}}{2}\right)+\left(1+\frac{\sqrt{2}}{2}\right)\mathrm{i}$	
(ii)	 *OPRQ* is a rhombus	
(iii)	Note that *OR* bisects the angle *POQ* since *OPRQ* is a rhombus. Thus $\arg(w+z)=\frac{1}{2}\left(\frac{1}{4}\pi+\frac{5}{6}\pi\right)=\frac{13}{24}\pi$. $\tan\left(\frac{11}{24}\pi\right)=\frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{6}}{2}-1}$ $=\frac{2+\sqrt{2}}{\sqrt{6}-2}$ $\therefore a=2, b=-2$	

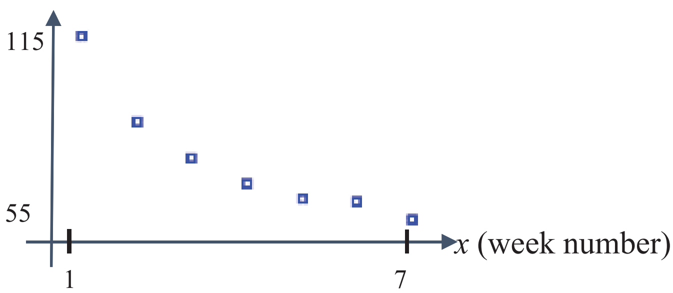
Qn	Suggested Solution	
5(i)	Graphs intersect at:	

	$\frac{x-b}{x-a}=\frac{x-b}{b}$ $b(x-b)=(x-b)(x-a)$ $(x-b)(x-a-b)=0$ $x=b$ or $x=a+b$  	
(ii)	$\therefore x<a$ or $b \le x \le a+b$	
(iii)	From GC, point of intersection at $(5, \frac{2}{3})$ $V=\pi\int_0^{\frac{2}{3}} \underbrace{x_2^{\ 2}}_{C_2} - \underbrace{x_1^{\ 2}}_{C_1} \, dy$ $=\pi\int_0^{\frac{2}{3}} (3y+3)^2-\left(\frac{2y-3}{y-1}\right)^2 \, dy$ $=5.742$ (3 d.p.)	

Qn	Suggested Solution	
6	For distinct gifts, 5^6 ways Now considering the distinct gifts, Case 1: 3 person get 1 gift No of ways = ${}^5C_3 \times 5^6 = 156250$ Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2(2)\times 5^6 = 312500$ Case 3: 1 person get 3 gifts No of ways = ${}^5C_1 \times 5^6 = 78125$ Total number of ways $=156250+312500+78125=546875$ **Alternative**	

	<u>Stage 1: Distribute 6 distinct gifts among 5 people</u> No of ways = 5^6 <u>Stage 2: Distribute 3 identical gifts among 5 people</u> Case 1: 3 person get 1 gift No of ways = ${}^5C_3 = 10$ Case 2: 1 person get 1 gift, another person gets 2 gifts No of ways = ${}^5C_2(2) = 20$ Case 3: 1 person get 3 gifts No of ways = ${}^5C_1 = 5$ Total number of ways = $(10+20+5)5^6 = 546875$	

Qn	Suggested Solution (updated 26 Sep)	
7(i)	$\mathrm{P}(L'\cup M') = \dfrac{80-n(L\cap M)}{80}$ 4 to 6 hours $= \dfrac{80-(35-k)}{80} = \dfrac{45+k}{80}$ **ALT** $\mathrm{P}(L'\cup M') = \mathrm{P}(L)+P(M')-\mathrm{P}(L'\cap M')$ $= \dfrac{10+k}{80}+\dfrac{35}{80}-0$ $= \dfrac{45+k}{80}$	
(ii)	$\mathrm{P}(G \mid L') = \dfrac{P(G\cap L')}{P(L')} = \dfrac{k}{k+10}$	
(iii)	Given $\mathrm{P}(L\cap M) = \dfrac{2}{5}$ From table: $\mathrm{P}(L\cap M) = \dfrac{20+(15-k)}{80} = \dfrac{35-k}{80}$ Solving: $k=3$ $P(L)P(M) = \dfrac{67}{80}\times\dfrac{45}{80} = \dfrac{603}{1280} \neq \dfrac{2}{5}$ Since $P(L\cap M) \neq P(L)P(M)$, L and M are **NOT** independent **ALT** $P(L) = \dfrac{70-k}{80} = \dfrac{67}{80}$ $\mathrm{P}(L \mid M) = \dfrac{35-k}{45} = \dfrac{32}{45} \neq \dfrac{67}{80}$ Since $P(L) \neq \mathrm{P}(L \mid M)$, L and M are **NOT** independent	
(iv)	Since $\mathrm{P}(G\cap(L\cap M)) = 0$ $\Rightarrow 15-k=0$ $\therefore k=15$	

Qn	Suggested Solution	
8 (i)		
(ii)	A linear model would predict her timing to decrease at a constant rate and eventually negative, which is not possible as there is a limit to how fast a person can swim. A quadratic model would predict that her timings would have a minimum and then increase at an increasing rate, which is also not appropriate.	
(iii)	Based on the scatter diagram and the model, as x increases t decreases at a decreasing rate, therefore b is positive. a has to be positive as it represents the best possible timing that Sharron can swim in the long run.	
(iv)	From GC, $r = 0.991$ $b = 67.69$ $a = 49.50$	
(v)	Let m be the best timing Sharron has at the 8$^{\text{th}}$ month. $\overline{\left(\frac{1}{x}\right)} = 0.33973$ We know that $\left(\overline{\frac{1}{x}}, \bar{t}\right)$ is on the regression line $t = 48.28 + 69.45\left(\frac{1}{x}\right)$. $\bar{t} = 48.28 + 69.45(0.33973) = 71.874$ $\frac{522+m}{8} = 71.874$ $m = 52.992$ Sharron best timing is 53 seconds at the 8th month	

Qn	Suggested Solution	
9 (a)	An unbiased estimate for the population variance : $s^2 = \frac{n}{n-1}\left(4^2\right) = \frac{16n}{n-1}$ minutes^2	
(b) (i)	Let μ be the population mean time taken for a 17-year-old student to complete a 5 km run. To test at 10 % significance level, $H_0 : \mu = 30.0$ min $H_1 : \mu \neq 30.0$ min For $n = 40$, $s^2 = \frac{16(40)}{39} = \frac{640}{39}$ Test Statistic: Under H_0, $\overline{T} \sim \mathrm{N}\left(30.0, \frac{640/39}{40}\right)$ approximately by Central Limit Theorem since n is large p-value $= 2\mathrm{P}\left(\overline{T} \leq 28.9\right) = 0.0859 \leq 0.10$, we reject H_0 and conclude that there is sufficient evidence at the 10 % significance level that the population mean time taken has changed.	
(ii)	The p-value is the probability of obtaining a sample mean at least as extreme as the given sample, assuming that the population mean time taken has not changed from 30.0 min. **OR** The p-value is the smallest significance level to conclude that the population mean time has changed from 30.0 min.	
(iii)	Since the sample size of 40 is large, by Central Limit Theorem, $\overline{T}$ follows a normal distribution approximately. Thus no assumptions are needed.	
(c) (i)	New population mean timing = $0.95 \times 30 = 28.5$ min To test at 5 % significance level, $H_0 : \mu = 28.5$ min $H_1 : \mu > 28.5$ min	
(ii)	Assumption: n is large for Central Limit Theorem to apply. Test Statistic: Under H_0, $\overline{T} \sim \mathrm{N}\left(28.5, \frac{4.0^2}{n-1}\right)$ approximately by Central Limit Theorem	

	For H_0 to be rejected, we need $\mathrm{P}\left(\overline{T} \geq 28.9\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \dfrac{28.9-28.5}{\frac{4}{\sqrt{n-1}}}\right) \leq 0.01$ $\mathrm{P}\left(Z \geq \dfrac{\sqrt{n-1}}{10}\right) \leq 0.01$ $\dfrac{\sqrt{n-1}}{10} \geq 2.3263$ $n \geq 542.2$ Thus required set = $\left\{n \in \mathbb{Z} : n \geq 543\right\}$	

Qn	Suggested Solution	
10 (i)	By symmetry, $\mu = \dfrac{5.2+7.0}{2} = 6.1$ $\text{P}(Y < 5.2) = \text{P}(Y \geq 7.0) = 0.379$ $\text{P}\left(Z < \dfrac{5.2-6.1}{\sigma}\right) = 0.379 \Rightarrow \dfrac{-0.9}{\sigma} = -0.308108$ $\sigma = 2.92105 = 2.92$ (3sf)	
(ii)	$X \sim N(12.3, 9.9)$ $\text{P}(\lvert X-12.3 \rvert < a) = 0.5$ $\text{P}(12.3 - a < X < 12.3 + a) = 0.5$ From GC, $12.3 - a = 10.1777$ $a = 2.1223 = 2.12$ (3sf)	
	Alternative $\text{P}(\lvert X-12.3 \rvert < a) = 0.5$ $\text{P}(\lvert Z \rvert < \dfrac{a}{\sqrt{9.9}}) = 0.5$ $\text{P}(Z < -\dfrac{a}{\sqrt{9.9}}) = 0.25 \Rightarrow \quad -\dfrac{a}{\sqrt{9.9}} = -0.674489$ $a = 2.12$ (3sf)	
(iii)	$\text{P}(X > 10 = 0.76761$ Let W = number of e-scooters that exceed speed limit, out of 49 $W \sim \text{B}(49, \text{P}(X > 10))$ i.e. $W \sim \text{B}(49, 0.76761)$ Probability required $= \text{P}(W = 34) \times 0.76761$ $= 0.61022 \times 0.76761$ $= 0.046840 = 0.0468$ (3sf)	
(iv)	Want: $\text{P}\left(\dfrac{X_1 + \ldots + X_6}{6} > 2\left(\dfrac{Y_1 + \ldots + Y_{15}}{15}\right)\right)$ $= \text{P}(\overline{X} - 2\overline{Y} > 0)$ $\overline{X} - 2\overline{Y} \sim \text{N}\left(12.3 - 2(6.1), \dfrac{9.9}{6} + \dfrac{4}{15}(2.92105^2)\right)$ i.e. $\overline{X} - 2\overline{Y} \sim \text{N}(0.1, 3.92533)$ $\therefore \text{P}(\overline{X} - 2\overline{Y} > 0) = 0.520$ (3sf)	

(v)	Let T = Total speed of n e-scooters $\overline{T} \sim \mathrm{N}(12.3, \frac{9.9}{n})$ $\mathrm{P}(\overline{T} > 10) = \mathrm{P}(Z > \frac{10-12.3}{\sqrt{\frac{9.9}{n}}})$ $= \mathrm{P}(Z > -0.73098\sqrt{n}) = 1$ (since n is large) **<u>Alternative</u>** As n gets larger, $\overline{x} \to \mu = 12.3 > 10$ Thus mean speed of these n e-scooters >10 with probability 1	

<table>
<tr><th>Qn</th><th>Suggested Solution</th><th></th></tr>
<tr><td>11 (a)(i)</td><td>

Method 1: direct computation

$\mathrm{P}(2 \le X \le k)$

$= \mathrm{P}(X=2)+\mathrm{P}(X=3)+\mathrm{P}(X=4)+\ldots+\mathrm{P}(X=k)$

$= (\tfrac{5}{6})(\tfrac{1}{6})+(\tfrac{5}{6})^2(\tfrac{1}{6})+(\tfrac{5}{6})^3(\tfrac{1}{6})+(\tfrac{5}{6})^4(\tfrac{1}{6})+\ldots+(\tfrac{5}{6})^{k-1}(\tfrac{1}{6})$

$= (\tfrac{1}{6})\left[(\tfrac{5}{6})+(\tfrac{5}{6})^2+(\tfrac{5}{6})^3+(\tfrac{5}{6})^4+\ldots+(\tfrac{5}{6})^{k-1}\right]$

$= (\tfrac{1}{6})\left[\dfrac{(\tfrac{5}{6})(1-(\tfrac{5}{6})^{k-1})}{1-(\tfrac{5}{6})}\right]$

$= (\tfrac{5}{6})-(\tfrac{5}{6})^k$

Method 2: complement method

$\mathrm{P}(2 \le X \le k)$

$= 1-\mathrm{P}(X=1)-\underbrace{\mathrm{P}(X>k)}_{\text{first } k \text{ are not } 6's}$

$= 1-\tfrac{1}{6}-(\tfrac{5}{6})^k$

$= \tfrac{5}{6}-(\tfrac{5}{6})^k$

s	8	4	0
$\mathrm{P}(S=s)$	$\tfrac{1}{6}$	$(\tfrac{5}{6})-(\tfrac{5}{6})^k$	$(\tfrac{5}{6})^k$

</td><td></td></tr>
<tr><td>(ii)</td><td>

From GC,

$\mathrm{E}(S)=\dfrac{8}{6}+4\left(\dfrac{5}{6}-\left(\dfrac{5}{6}\right)^k\right)=\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k$

$\mathrm{E}(\text{Profit})=\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k-3>0$

$\dfrac{14}{3}-4\left(\dfrac{5}{6}\right)^k-3>0$

$\left(\dfrac{5}{6}\right)^k<\dfrac{5}{12}$

$k>4.802$

Least value of k is 5.

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<tr><td>(b)(i)</td><td>

$Y \sim \mathrm{B}(80, p)$

$80+80p=480p(1-p)$

$1+p=6p-6p^2$

$6p^2-5p+1=0$

$p=\dfrac{1}{3}$ or $p=\dfrac{1}{2}$ (rejected as coin is not fair)

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(ii)	Let W be the number of heads obtained in the last 75 tosses $W \sim \mathrm{B}(75, \frac{1}{3})$ Required probability $= \mathrm{P}(W \geq 25)$ $= 1 - \mathrm{P}(W \leq 24)$ $= 0.543$ **Alternative** Use conditional probability	
(iii)	$\overline{Y} \sim \mathrm{N}(\frac{80}{3}, \frac{16}{45})$ approximately by central limit theorem since the sample size of 50 is large $\mathrm{P}(\overline{Y} < 25) = 0.00259 \quad$ (3 s.f.)	