**1** **(a)** Given a differential equation of the form $\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{g}\left(\frac{x}{y}\right)$, use the substitution $x = vy$ to show that

$$\int \frac{1}{\mathrm{g}(v) - v}\,\mathrm{d}v = \int \frac{1}{y}\,\mathrm{d}y .$$

[2]

**(b)** Show that the differential equation

$$xy\mathrm{e}^{\left(\frac{x}{y}\right)^2}\frac{\mathrm{d}x}{\mathrm{d}y} = y^2 + \left(y^2 + x^2\right)\mathrm{e}^{\left(\frac{x}{y}\right)^2}, \quad x, y \neq 0$$

can be written in the form $\frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{g}\left(\frac{x}{y}\right)$. [2]

**(c)** It is given that the curve of the differential equation **(b)** passes through the point $\left(\sqrt{\ln 9}, -\sqrt{2}\right)$.

Solve the differential equation given in **(b)**, leaving your answer in the form $y = \mathrm{h}\left(\frac{x}{y}\right)$. [7]

**2** **(a)** Show that $\dfrac{\binom{n-1}{k-1}}{\binom{n}{k}} = \dfrac{k}{n}$ for any positive integers $n$, $k$ where $n \geq k$. [2]

It is given that $a_1, a_2, \ldots, a_n \in \mathbb{R}^+$ and $s$ is a positive integer where $s \leq n$.

Let $S$ be $\sum a_{i_1} a_{i_2} \ldots a_{i_s}$, where the sum is taken over all $i_1, \ldots, i_s \in \{1, 2, \ldots, n\}$ such that $1 \leq i_1 < i_2 < \ldots < i_s \leq n$.

**(b)** Find, in terms of $n$ and $s$, the number of terms in $S$. [1]

**(c)** Let $m$ be a fixed integer where $1 \leq m \leq n$.
Find, in terms of $n$ and $s$, the number of terms in $S$ such that $m \in \{i_1, i_2, \ldots, i_s\}$. [1]

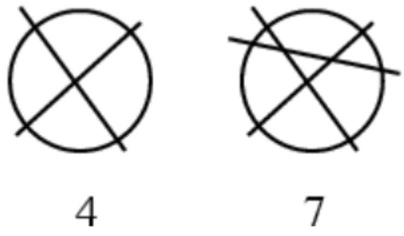
**(d)** Let $g = \left(a_1 a_2 \cdots a_n\right)^{\frac{1}{n}}$. Prove that

$$\left(1 + a_1\right)\left(1 + a_2\right)\cdots\left(1 + a_n\right) \geq \left(1 + g\right)^n .$$

[6]

**3** In this question, we will examine different ways of dividing circles.

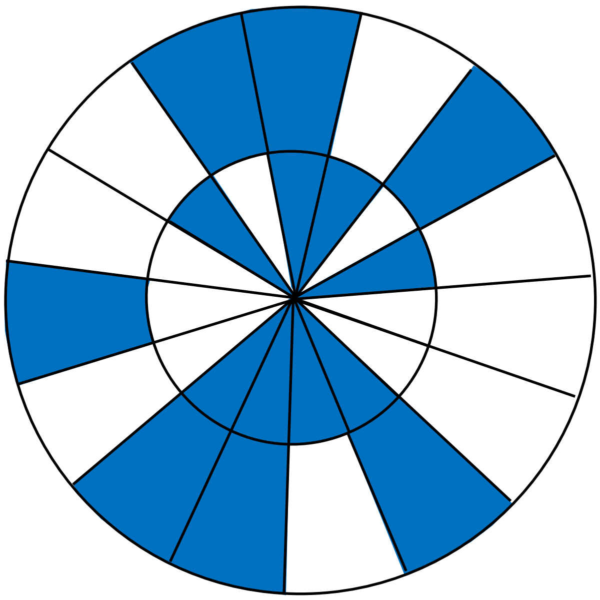
**(a)** Consider the number of regions formed when a circle is cut $n$ times. The cases where the maximum number $m_n$ is achieved for $n = 2$ and 3 is illustrated below with the values of $m_n$ stated.




Illustrate the case where the maximum number $m_n$ is achieved for $n = 4$. Hence deduce, with justification, an expression for $m_n - m_{n-1}$ in terms of $n$ and solve for $m_n$. Explain briefly why this represents the **maximum** number of regions. [7]

**(b)** Suppose a circle is divided into $2n+1$ congruent sectors, with $n$ of them randomly coloured black and the other $n+1$ randomly coloured white. A smaller concentric circle is placed on the larger circle and also divided into $2n+1$ congruent sectors, with $n+1$ of them randomly coloured black and the other $n$ randomly coloured white.

A possible case for $n = 7$ is illustrated below.



Prove that, for all $n$, we can always find $n+1$ sectors with matched colours by suitably rotating the smaller circle if necessary. [5]

**4** The Fibonacci series is defined by

$$F_1 = F_2 = 1, \qquad F_{n+2} = F_{n+1} + F_n \text{ for } n \geq 1.$$

**(a)** Show that $\sum_{i=1}^{n} F_{2i} = F_{2n+1} - F_1$, and state a similar result for $\sum_{i=1}^{n} F_{2i+1}$. [3]

Let *S* be the set of all integers that can be written in the form $F_{n_1} + F_{n_2} + ... + F_{n_t}$, where $n_i$ is a sequence of positive integers such that $n_1 \geq 2$ and $n_{i+1} > n_i + 1$ for all $1 \leq i \leq t-1$.

For example,

5 is in *S* because, $5 = F_5$.

54 is in *S* because, $54 = 2+5+13+34 = F_3 + F_5 + F_7 + F_9$.

190 is in *S* because, $190 = 1+3+8+34+144 = F_2 + F_4 + F_6 + F_9 + F_{12}$.

**(b)** Show that both 55 and 191 are in *S*. [2]

**(c)** **(i)** Suppose *k* is in *S*, where $k = F_2 + F_{u_2} + F_{u_3} + ... + F_{u_s}$.

Show that $k+1$ is also in *S*. [3]

**(ii)** Hence, use mathematical induction to prove that all positive integers are in *S*. [6]

**5** An ice-cream shop sells single scoop cones with *k* different flavours available. You may assume that the shop does not allow mixing of flavours in any single scoop.

**(a)** An order of *n* single scoop cones is made where $n \geq k$. Given that all *k* flavours are bought and the order includes an odd number of cones for each flavour, state a condition between *n* and *k* and find the number of such possible orders. [3]

Choosing from the *k* possible flavours, a group of *n* kids orders one single scoop cone each.

**(b)** By considering all possible orders made by the *n* kids**,** explain why

$$k^n = \sum_{r=0}^{k} \mathrm{S}(n,r)\, {}^kP_r$$

where S(*n*, *r*) denotes the number of ways to partition *n* distinct objects into *r* disjoint, non-empty subsets and ${}^kP_r = k(k-1)(k-2)\ldots(k-r+1)$. [4]

**(c)** Apply the principle of inclusion and exclusion to enumerate the number of possible orders which include all *k* flavours and show that

$$\mathrm{S}(n,k) = \frac{1}{k!}\sum_{r=0}^{k} c_r (k-r)^n,$$

where $c_r$ are expressions, in terms of *r* and *k*, to be determined. [5]

**6** The functions f and g are defined as follows.

$$\mathrm{f}(x) = 2x$$
$$\mathrm{g}(x) = x + 1$$

An arrangement of functions f and g is a composition of functions f and g, where each function can be composed any number of times, but at least once each. For example, $\mathrm{fgfgfgfgf}$ and $\mathrm{gf}^3\mathrm{g}$ are both arrangements of f and g.

**(a)** Given that $\mathrm{h}(x) = ax + b$ describes a function that is equal to an arrangement of f and g, find the set of possible values of *a* and *b*. [2]

**(b)** **(i)** Show that $\mathrm{g}^2\mathrm{f}(x) = \mathrm{fg}(x)$. [1]

**(ii)** List all the arrangements of f and g that are equal to the function $4x + 4$. [2]

**(c)** **(i)** Find an expression for the arrangement $\mathrm{g}^i\mathrm{fg}^j\mathrm{fg}^k(x)$ in terms of *x*, where *i*, *j* and *k* are non-negative integers. [1]

**(ii)** Hence, or otherwise, show that for all positive integers *m*, the number of arrangements of f and g that are equal to the function $4x + 4m$ is $(m+1)^2$. [5]

**(iii)** By using a suitable bijection, show that the number of arrangements of f and g that are equal to the function $4x + 4m$ is equal to the number of arrangements of f and g that are equal to the function $4x + 4m + 1$, where *m* is a positive integer. [3]

**7** In this question, all variables represent positive integers.

The greatest common divisor of *x*, *y* and *z*, written $\gcd(x, y, z)$, is the largest positive integer that divides each of *x*, *y* and *z*.

We say that $(x, y, z)$ is a Pythagorean triple if $x^2 + y^2 = z^2$. If, in addition, $\gcd(x, y, z) = 1$, we say that $(x, y, z)$ is a **primitive** Pythagorean triple. Examples of primitive Pythagorean triples are $(3,4,5)$, $(5,12,13)$, $(7,24,25)$ & $(9,40,41)$.

**(a)** **(i)** Find consecutive integers *a* and *b* such that $(11, a, b)$ is a Pythagorean triple. [1]

**(ii)** By an appropriate generalisation, show that there exist infinitely many primitive Pythagorean triples. [2]

**(b)** **(i)** Find integers *x*, *y* and *z* satisfying $\gcd(x, y, z) = 1$, and $\gcd(xy, yz, zx) > 1$. [1]

**(ii)** Show that if $(x, y, z)$ is a primitive Pythagorean triple, then $\gcd(xy, yz, zx) = 1$. [5]

**(c)** Let $(x, y, z)$ be a Pythagorean triple. Show that $(xy)^4 + (yz)^4 + (zx)^4 = \left(z^4 - x^2y^2\right)^2$. [2]

**(d)** Deduce carefully that the equation $u^4 + v^4 + w^4 = t^2$ has infinitely many integer solutions such that $\gcd(u, v, w) = 1$. [2]

**8** A Gaussian integer is a complex number where the real and imaginary parts are both integers.

**(a)** Given any complex number $z$, show that there is a Gaussian integer $w$ such that $|z-w| \le \frac{1}{\sqrt{2}}$. [3]

**(b)** Suppose that $s$, $t$ are Gaussian integers with $t \ne 0$. By considering the complex number $\frac{s}{t}$, deduce that there are Gaussian integers $q$, $r$ such that $|r| < |t|$ and $s = qt + r$. [2]

**(c)** Let $s$ and $t$ be the Gaussian integers $5+4\mathrm{i}$ and $1+2\mathrm{i}$ respectively. By considering Gaussian integers near $\frac{5+4\mathrm{i}}{1+2\mathrm{i}}$, show that there are exactly 3 pairs of Gaussian integers $(q, r)$ such that $|r| < |t|$ and $s = qt + r$, and find these pairs. [4]

**(d)** Let $s$, $t$ be Gaussian integers such that $\frac{s}{t}$ is **not** a Gaussian integer, $t \ne 0$. Prove that there are always at least two pairs of Gaussian integers $(q, r)$ such that $|r| < |t|$ and $s = qt + r$. [5]

**End of Paper**