2021 Further Mathematics Paper 1 (9649/1)

1 Consider the differential equation

$$\frac{dy}{dx} = \frac{1+e^{x+y}}{xy},$$

where $y = 1$ when $x = 1$.

(a) Use one step of both the Euler method and the improved Euler method to calculate approximations to the value of y when $x = 1.5$, giving each answer correct to 3 decimal places. [4]

(b) Explain the large discrepancy between these two answers. [1]

[Solution]

(a) Given $x_0 = 1, y_0 = 1, h = 0.5$ and $f(x, y) = \dfrac{1+e^{x+y}}{xy}$.

By Euler method,

$y_1 = y_0 + 0.5f(x_0, y_0) = 5.194528 \approx 5.195$ (to 3 dp)

By Improved Euler method,

$u_1 = y_0 + 0.5f(x_0, y_0) = 5.194528 \approx 5.195$

$y_1 = y_0 + \dfrac{0.5}{2}\left[f(x_0, y_0) + f(x_1, u_1)\right] \approx 29.053$ (to 3 dp)

(b) The Euler method uses the slope at the point (x_0, y_0) to obtain the estimate of y_1, while the Improved Euler method uses the average of the slope based on the initial point and the point (x_1, u_1). Since the values of f(x_0, y_0) and f(x_1, u_1) differ greatly, there is a large discrepancy between the two answers.

2 **(a)** Showing full working, determine the eigenvalues and corresponding eigenvectors of the matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}.$$ [5]

(b) Describe the geometrical significance of these eigenvectors with respect to the plane transformation given by the matrix **M**. [1]

[Solution]

(a) Let λ be the eigenvalue of the matrix **M**

Then $\begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$(1 - \lambda)(-2 - \lambda) - 4 = 0$

$-2 - \lambda + 2\lambda + \lambda^2 - 4 = 0 \Rightarrow \lambda^2 + \lambda - 6 = 0$

$\Rightarrow (\lambda + 3)(\lambda - 2) = 0$

$\lambda = 2$ or -3

If $\lambda = 2$, $\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow -x + 2y = 0$ and $2x - 4y = 0$ $\Rightarrow x = 2y$

A corresponding eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

If $\lambda = -3$, $\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow 4x + 2y = 0$ and $2x + y = 0$ $\Rightarrow y = -2x$

A corresponding eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

(b) Points on the line $x = 2y$ and $y = -2x$ under **M** will be map to another point on the same line respectively – that means the lines are $x = 2y$ and $y = -2x$ are invariant under the transformation represented by **M.**

3 For real numbers x, and any positive integer n, the function F_n is defined by

$$F_n(x) = \frac{x(x+1)(x+2)\ldots(x+n-1)}{n!}.$$

Prove by induction that, for all positive integers n, $F_n\left(\frac{1}{2}\right) < \frac{1}{\sqrt{2n+1}}$. [7]

[Solution]

Let P(n) be the statement

"$F_n(\tfrac{1}{2}) < \frac{1}{\sqrt{2n+1}}$ for $n = 1, 2, 3, \ldots$. where $F_n(\tfrac{1}{2}) = \frac{\frac{1}{2}(\frac{1}{2}+1)\ldots(\frac{1}{2}+n-1)}{n!}$ "

n = 1, $F_1(\tfrac{1}{2}) = \frac{\frac{1}{2}}{1!} = \frac{1}{2} < \frac{1}{\sqrt{3}}$ as $2 > \sqrt{3}$. Thus P(1) is true.

Assume P(k) is true for some positive integer k, that is $F_k(\tfrac{1}{2}) < \frac{1}{\sqrt{2k+1}}$.

When $n = k + 1$, to prove $F_{k+1}(\tfrac{1}{2}) < \frac{1}{\sqrt{2k+3}}$

LHS = $F_{k+1}(\tfrac{1}{2}) = \frac{\frac{1}{2}(\frac{1}{2}+1)\ldots(\frac{1}{2}+k-1)}{k!}\,\frac{(\frac{1}{2}+k)}{k+1}$

$$< \frac{1}{\sqrt{2k+1}}\,\frac{(\frac{1}{2}+k)}{k+1} = \frac{1}{\sqrt{2k+1}}\,\frac{2k+1}{2k+2}$$

$$= \frac{\sqrt{2k+1}}{2k+2} = \frac{\sqrt{(2k+1)(2k+3)}}{(2k+2)\sqrt{2k+3}}$$

$$= \frac{\sqrt{4k^2+8k+3}}{(2k+2)\sqrt{2k+3}} < \frac{\sqrt{4k^2+8k+4}}{(2k+2)\sqrt{2k+3}} = \frac{\sqrt{(2k+2)^2}}{(2k+2)\sqrt{2k+3}}$$

$$< \frac{1}{\sqrt{2k+3}}$$

So P(k) is true $\Rightarrow$ P(k + 1) is true, by Mathematical Induction, $\mathrm{F}_n(\tfrac{1}{2}) < \dfrac{1}{\sqrt{2n+1}}$ for all positive integers n .

4 Let C denote the curve $y = \dfrac{x}{1+ax}$ between the origin and the point $\left(b, \dfrac{b}{1+ab}\right)$, where a and b are positive constants.

When C is rotated about the x-axis, the volume generated is V. When C is rotated about the y-axis, the volume generated is W.

(a) Determine V in terms of a and b. [5]

(b) Show that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x^2}.$$

Hence or otherwise, show that $W = V$. [4]

[Solution]

(a) By disc method,

$$V = \pi\int_0^b \left(\frac{x}{1+ax}\right)^2 \mathrm{d}x$$

$$= \pi\int_0^b \frac{x^2}{a^2x^2+2ax+1}\,\mathrm{d}x$$

$$= \pi\int_0^b \frac{\frac{1}{a^2}\left(a^2x^2+2ax+1\right)-\frac{2}{a}x-\frac{1}{a^2}}{a^2x^2+2ax+1}\,\mathrm{d}x \text{ (or use long division)}$$

$$= \pi\int_0^b \frac{1}{a^2}-\frac{\frac{2}{a}x+\frac{1}{a^2}}{a^2x^2+2ax+1}\,\mathrm{d}x$$

$$= \pi\int_0^b \frac{1}{a^2}\,\mathrm{d}x - \pi\int_0^b \frac{\frac{1}{a^3}\left(2a^2x+2a\right)-\frac{1}{a^2}}{a^2x^2+2ax+1}\,\mathrm{d}x$$

$$= \pi\left[\tfrac{1}{a^2}x\right]_0^b - \pi\left[\frac{1}{a^3}\ln\left(1+ax\right)^2 - \frac{1}{a^2}\frac{\left(1+ax\right)^{-1}}{-a}\right]_0^b$$

$$= \pi\left(\frac{b}{a^2}\right) - \pi\left(\frac{2}{a^3}\ln\left(1+ab\right)+\frac{1}{a^3\left(1+ab\right)}-\frac{1}{a^3}\right)$$

$$= \frac{\pi}{a^3}\left(ab-2\ln\left(1+ab\right)-\frac{1}{1+ab}+1\right)$$

(b) $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left(1+ax\right)-x\left(a\right)}{\left(1+ax\right)^2} = \frac{1}{\left(1+ax\right)^2} = \left(\frac{x^2}{\left(1+ax\right)^2}\right)\frac{1}{x^2} = \frac{y^2}{x^2} \text{ (shown)}$$

By disc method, $W = \pi \int_{y=0}^{y=\frac{b}{1+ab}} x^2 \, dy$

$$= \pi \int_{x=0}^{x=b} x^2 \left(\frac{dy}{dx}\right) dx = \pi \int_{x=0}^{x=b} y^2 dx$$

Since $\frac{dy}{dx} = \frac{y^2}{x^2}$

$= V$ (shown)

5 Consider the equation $f(x) = 0$, where $f(x) = x e^{x-1} + x - 2 - \delta$ and δ is very close to zero.

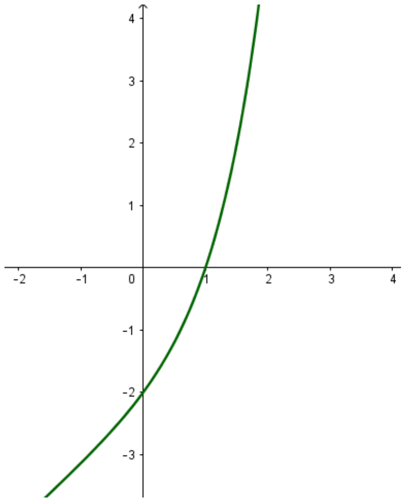
(a) Sketch the curve $y = x e^{x-1} + x - 2$. Hence show that $f(x) = 0$ has a single root, α, and that α is very close to 1. [4]

(b) Use two iterations of the Newton-Raphson method, with initial approximation $x_0 = 1$, to show that $\alpha \approx 1 + \frac{1}{3}\delta - \frac{1}{18}\delta^2$, where terms in δ^3 and higher powers of δ have been ignored. [6]

[Solution]

(a) Graph of $y = xe^{x-1} + x - 2$

We note that when $x = 1$, $y = 1 + 1 - 2 = 0$ and there is only one zero.



$f(1) = -\delta$

$f(1.1) = 1.1e^{0.1} + 1.1 - 2 - \delta = 0.316 - \delta$

If δ is very small and positive, $f(1) < 0$ and $f(1.1) > 0$

$\Rightarrow 1 < \alpha < 1.1$

$f(0.9) = 0.9e^{-0.1} + 0.9 - 2 - \delta = -0.286 - \delta$

If δ is very small and negative, $f(1) > 0$ and $f(0.9) < 0$

$\Rightarrow 0.9 < \alpha < 1$

In both cases, α is close to 1.

(b) $f'(x) = xe^{x-1} + e^{x-1} + 1 = (x+1)e^{x-1} + 1$

Using the Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n e^{x_n-1} + x_n - 2 - \delta}{(x_n+1)e^{x_n-1} + 1}$

$$= \frac{x_n^2 e^{x_n-1} + x_n e^{x_n-1} + x_n - [x_n e^{x_n-1} + x_n - 2 - \delta]}{(x_n+1)e^{x_n-1} + 1}$$

$$= \frac{x_n^2 e^{x_n-1} + 2 + \delta}{(x_n+1)e^{x_n-1} + 1}$$

$x_0 = 1$

$x_1 = \frac{3+\delta}{3} = 1 + \frac{\delta}{3}$

$$x_2 = \frac{(1+\frac{\delta}{3})^2 e^{\frac{\delta}{3}} + 2 + \delta}{(2+\frac{\delta}{3})e^{\frac{\delta}{3}} + 1} = \frac{(1+\frac{2\delta}{3}+\frac{\delta^2}{9})(1+\frac{2\delta}{3}+\frac{\delta^2}{18}) + 2 + \delta}{(2+\frac{\delta}{3})(1+\frac{2\delta}{3}+\frac{\delta^2}{18}) + 1}$$

$$= \frac{1+\frac{\delta}{3}+\frac{\delta^2}{18}+\frac{2\delta}{3}+\frac{2\delta^2}{9}+\frac{\delta^2}{18}+2+\delta}{2+\frac{2\delta}{3}+\frac{2\delta^2}{18}+\frac{\delta}{3}+\frac{\delta^2}{9}+1} = \frac{3+2\delta+\frac{\delta^2}{3}}{3+\delta+\frac{\delta^2}{6}}$$

$$= (3+2\delta+\tfrac{\delta^2}{3})3^{-1}[1+(\tfrac{\delta}{3}+\tfrac{\delta^2}{2})]^{-1}$$

$$= \frac{1}{3}(3+2\delta+\tfrac{\delta^2}{3})[1-(\tfrac{\delta}{3}+\tfrac{\delta^2}{18})+(\tfrac{\delta}{3}+\tfrac{\delta^2}{18})^2+\ldots]$$

$$= \frac{1}{3}(3+2\delta+\tfrac{\delta^2}{3})[1-\tfrac{\delta}{3}-\tfrac{\delta^2}{18}+\tfrac{\delta^2}{9}+\ldots]$$

$$= \frac{1}{3}(3+2\delta+\tfrac{\delta^2}{3})[1-\tfrac{\delta}{3}+\tfrac{\delta^2}{18}+\ldots]$$

$$= \frac{1}{3}(3+2\delta+\tfrac{\delta^2}{3}-\delta-\tfrac{2\delta^2}{3}+\tfrac{3\delta^2}{18}+\ldots)$$

$$= \frac{1}{3}(3+\delta-\tfrac{3\delta^2}{18}+\ldots) = 1+\frac{\delta}{3}-\frac{\delta^2}{18}+\ldots$$

6 The curve E has cartesian equation $\left(x^2+y^2\right)^2 = 5x^3+xy^2$.

(a) Determine the polar equation of E in the form $r = \mathrm{f}(\theta)$, where f is a function of $\cos\theta$ only. [2]

(b) Use integration by parts to show that, for integers $n \geqslant 2$,

$$\int_0^{\frac{1}{2}\pi} \cos^n\theta \, \mathrm{d}\theta = \left(\frac{n-1}{n}\right)\int_0^{\frac{1}{2}\pi} \cos^{n-2}\theta \, \mathrm{d}\theta. \quad [4]$$

(c) Find, in an exact form, the total area enclosed by E. Full working must be shown. [5]

[Solution]

(a) $\left(x^2+y^2\right)^2 = 5x^3+xy^2$

$$r^4 = 5r^3\cos^3\theta + r^3\cos\theta\sin^2\theta$$

$$r = \cos\theta\left(5\cos^2\theta+\sin^2\theta\right)$$

$$= \cos\theta\left(1+4\cos^2\theta\right)$$

$$= 4\cos^3\theta+\cos\theta$$

(b) $\int_0^{\frac{1}{2}\pi} \cos^n\theta \, \mathrm{d}\theta = \int_0^{\frac{1}{2}\pi} \cos^{n-1}\theta\cos\theta \, \mathrm{d}\theta$

$$= \left[\sin\theta\cos^{n-1}\theta\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{1}{2}\pi}(n-1)\sin^2\theta\cos^{n-2}\theta \, \mathrm{d}\theta$$

$$= (n-1)\int_0^{\frac{1}{2}\pi}\left(1-\cos^2\theta\right)\cos^{n-2}\theta \, \mathrm{d}\theta$$

$$= (n-1)\left[\int_0^{\frac{1}{2}\pi}\cos^{n-2}\theta \, \mathrm{d}\theta - \int_0^{\frac{1}{2}\pi}\cos^n\theta \, \mathrm{d}\theta\right]$$

$$\int_0^{\frac{1}{2}\pi}\cos^n\theta \, \mathrm{d}\theta + (n-1)\int_0^{\frac{1}{2}\pi}\cos^n\theta \, \mathrm{d}\theta = (n-1)\int_0^{\frac{1}{2}\pi}\cos^{n-2}\theta \, \mathrm{d}\theta$$

$$n\int_0^{\frac{1}{2}\pi}\cos^n\theta \, \mathrm{d}\theta = (n-1)\int_0^{\frac{1}{2}\pi}\cos^{n-2}\theta \, \mathrm{d}\theta$$

$$\int_0^{\frac{1}{2}\pi} \cos^n \theta \, d\theta = \left(\frac{n-1}{n}\right) \int_0^{\frac{1}{2}\pi} \cos^{n-2} \theta \, d\theta \text{ (shown)}$$

(c) Area enclosed $= 2 \times \frac{1}{2} \int_0^{\frac{\pi}{2}} \left(4\cos^3\theta + \cos\theta\right)^2 d\theta$

$$= \int_0^{\frac{\pi}{2}} \left(16\cos^6\theta + 8\cos^4\theta + \cos^2\theta\right) d\theta$$

Using the result in (b),

$$\int_0^{\frac{1}{2}\pi} \cos^2\theta \, d\theta = \left(\frac{1}{2}\right) \int_0^{\frac{1}{2}\pi} 1 \, d\theta = \frac{1}{2}[\theta]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$$

$$\int_0^{\frac{1}{2}\pi} \cos^4\theta \, d\theta = \frac{3}{4} \int_0^{\frac{1}{2}\pi} \cos^2\theta \, d\theta = \frac{3}{4}\left(\frac{\pi}{4}\right) = \frac{3\pi}{16}$$

$$\int_0^{\frac{1}{2}\pi} \cos^6\theta \, d\theta = \frac{5}{6} \int_0^{\frac{1}{2}\pi} \cos^4\theta \, d\theta = \frac{5}{6}\left(\frac{3\pi}{16}\right) = \frac{15\pi}{96}$$

$$\therefore \text{ Area enclosed} = 16\left(\frac{15\pi}{96}\right) + 8\left(\frac{3\pi}{16}\right) + \left(\frac{\pi}{4}\right) = \frac{17\pi}{4}$$

7 The general equation of the family of all quadratic curves, which includes the conic sections, is

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0. \quad (*)$$

(a) In the first instance, it is given that $A = 1$ and $C = 0$.

(i) In order for (∗) to describe a circle, state a necessary condition on the value of B. [1]

(ii) Assume this necessary condition on B holds. Find the necessary condition on D, E and F for (∗) to describe a circle. Explain what happens when this condition on D, E and F doesn't hold. [5]

(b) In a different case, it is given that $C = 1$. Find, in terms of the coefficients of (∗), the value of the constant k for which the transformation with matrix $\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}$ reduces (∗) to a form with zero xy term. [5]

[Solution]

(a) A = 1 and C = 0

(i) For (*) to be a circle, B = A = 1

(ii) A = B = 1 and C = 0

$x^2 + y^2 +$ + Dx + Ey + F = 0

$(x + \frac{D}{2})^2 + (y + \frac{E}{2})^2 - \frac{D^2}{4} - \frac{E^2}{4} + F = 0$

$(x + \frac{D}{2})^2 + (y + \frac{E}{2})^2 = \frac{D^2}{4} + \frac{E^2}{4} - F$

For (*) to be a circle, $\frac{D^2}{4} + \frac{E^2}{4} - F > 0 \Rightarrow D^2 + E^2 - 4F > 0$

If $D^2 + E^2 - 4F = 0$ then $x = -\frac{D}{2}$ and $y = -\frac{E}{2}$ thus (*) is the point $(-\frac{D}{2}, -\frac{E}{2})$

If $D^2 + E^2 - 4F < 0$ then no solution

(b) Let $\begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} k & 1 \\ -1 & k \end{pmatrix}^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{k^2+1}\begin{pmatrix} k & -1 \\ 1 & k \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$

Substitute $x = \frac{1}{k^2+1}(kx'-y')$ and $y = \frac{1}{k^2+1}(x'+ky')$ into the original equation with $C = 1$

We have:

$$A(kx'-y')^2 + B(x'+ky')^2 + C(kx'-y')(x'+ky') + D(kx'-y') + E(x'+ky') + F(k^2+1) = 0$$

Making the coefficient of the $x'y'$ term = 0

That is: $k^2 + 2(B-A)k - 1 = 0$

$$\text{So } k = \frac{-2(B-A) \pm \sqrt{4(B-A)^2+4}}{2} = A - B \pm \sqrt{1+(B-A)^2}$$

Note: if we do by actual rotation matrix $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ for clockwise rotation of angle θ, we need to "normalise" the given matrix and work out the angle – a bit more troublesome.

8 It has been discovered that an aggressive and destructive type of hornet was accidentally introduced into a forest. It is estimated that the initial population of hornets, at the start of 2019, was 3000 and that the population of hornets at the start of 2020 was 4500.

From previous studies, it is known that the increase in hornet population from year n to year $(n+1)$ is four times the increase from year $(n-1)$ to year n.

(a) **(i)** Let H_n denote the population of hornets n years after the start of 2019. Find an expression for H_n in terms of n. [5]

(ii) Deduce the predicted size of the hornet population at the start of 2030. Give your answer correct to 3 significant figures. [1]

An ecological group takes steps to eliminate the hornets from the forest by the start of 2030. In order to achieve this, they implement a "harvesting" method of pest-control which, at the end of each year, removes a proportion h of the population which was alive at the end of the previous year.

(b) Given that this pest-control process began in 2020, determine the appropriate value of h, giving your answer to 3 significant figures. [8]

[Solution]

(a)(i) H_n denote the population of hornets *n* years after the start of 2019

$H_0 = 3000$|

$H_1 = 4500$ (1 year after the start of 2019 that is at the end of 1st year or start of 2020)

$H_{n+1} - H_n = 4(H_n - H_{n-1})$

$H_{n+1} - 5H_n + 4H_{n-1} = 0$

Auxiliary equation: $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 4$ or 1

General solution: $H_n = A\left(4^n\right) + B$ where $n = 0, 1, 2, \ldots$.

When $n = 0$, $3000 = A + B$ ---- (1)

When $n = 1$, $4500 = 4A + B$ --- (2)

$(2)-(1)$: $1500 = 3A \Rightarrow A = 500$

$B = 2500$

Thus $H_n = 500(4^n) + 2500$ for $n = 0, 1, 2, \ldots$.

(ii) At the start of 2030, $n = 11$

$H_{11} = 500(4^{11}) + 2500 = 2097154500 \approx 210 \times 10^7$ (Correct to 3 sf)

(b) With "harvesting"

$H_{n+1} = 5H_n - 4H_{n-1} - hH_n$ for $n \geq 2$

$H_{n+1} + (h-5)H_n + 4H_{n-1} = 0$

$m^2 + (h-5)m + 4 = 0$

$$m = \frac{(5-h) \pm \sqrt{(h-5)^2 - 16}}{2}$$

Since $0 < h < 1$, m has two distinct solutions.

Hence $H_n = A\left(\frac{(5-h)+\sqrt{(h-5)^2-16}}{2}\right)^n + B\left(\frac{(5-h)-\sqrt{(h-5)^2-16}}{2}\right)^n$ for $A, B \in \mathbb{R}$

$H_0 = 3000 \Rightarrow A + B = 3000$

$$H_1 = 4500 \Rightarrow A\left(\frac{(5-h)+\sqrt{(h-5)^2-16}}{2}\right) + B\left(\frac{(5-h)-\sqrt{(h-5)^2-16}}{2}\right)$$

$$= (3000-B)\left(\frac{(5-h)+\sqrt{(h-5)^2-16}}{2}\right) + B\left(\frac{(5-h)-\sqrt{(h-5)^2-16}}{2}\right)$$

Hence $4500 - 7500 + 1500h - 1500\sqrt{(h-5)^2-16} = -B\sqrt{(h-5)^2-16}$

$$\Rightarrow B = 1500\left(1 + \frac{2-h}{\sqrt{(h-5)^2-16}}\right)$$

and $A = 3000 - B = 1500\left(1 - \frac{2-h}{\sqrt{(h-5)^2-16}}\right)$

Therefore $H_{11} = 0 \Rightarrow$

$$\left[\left(1-\frac{2-h}{\sqrt{(h-5)^2-16}}\right)\left(\frac{(5-h)+\sqrt{(h-5)^2-16}}{2}\right)^{11}+\left(1+\frac{2-h}{\sqrt{(h-5)^2-16}}\right)\left(\frac{(5-h)-\sqrt{(h-5)^2-16}}{2}\right)^{11}\right]=0$$

From GC: $h = 0.83499 = 0.835$ (3 sf)

9 Let $\omega = \cos\left(\frac{2}{11}\pi\right) + \mathrm{i}\sin\left(\frac{2}{11}\pi\right)$.

(a) **(i)** Write down, in terms of ω, the eleven roots of $z^{11} - 1 = 0$. [1]

(ii) Show that $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} = -1$. [1]

The complex numbers α and β are given by

$$\alpha = \omega + \omega^3 + \omega^4 + \omega^5 + \omega^9 \text{ and } \beta = \omega^{-1} + \omega^{-3} + \omega^{-4} + \omega^{-5} + \omega^{-9}.$$

(b) **(i)** Find, in its simplest form, the quadratic equation whose roots are α and β. [5]

(ii) Show that $|\alpha| = \sqrt{3}$. [2]

(iii) **Do not use a calculator in answering this part.**

By considering $C = \mathrm{Re}(\alpha)$ and $S = \mathrm{Im}(\alpha)$, prove that

$$\sin\left(\tfrac{2}{11}\pi\right) + \sin\left(\tfrac{6}{11}\pi\right) + \sin\left(\tfrac{8}{11}\pi\right) + \sin\left(\tfrac{10}{11}\pi\right) + \sin\left(\tfrac{18}{11}\pi\right) = \tfrac{1}{2}\sqrt{11}.$$

Full working must be shown. [3]

[Solution]

Given $\omega = \cos\frac{2\pi}{11} + i\sin\frac{2\pi}{11}$

(a)

$z^{11} - 1 = 0 \Rightarrow z^{11} = \mathrm{e}^{\mathrm{i}(2k\pi)}$ for $k = 0, 1, 2, 3, \ldots, 10$

(we choose these k values for the requirement of the subsequent parts)

$\Rightarrow z = e^{i\frac{2k\pi}{11}}$

(i) So in terms of ω, the roots are $1, \omega^n$ where $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$

(ii) Coefficient of z^{10} is 0 in the polynomial equation $z^{11} - 1 = 0$

Sum of roots = 0

Thus $1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} = 0$

$\Rightarrow \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10} = -1$

OR $\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10}$

$= \dfrac{\omega\left(1-\omega^{10}\right)}{1-\omega}$ (sum of GP)

$= \dfrac{\omega - \omega^{11}}{1-\omega}$

$= \frac{\omega - 1}{1-\omega} = -1$ (since $\omega^{11} = 1$)

(b) Given $\alpha = \omega + \omega^3 + \omega^4 + \omega^5 + \omega^9$ and $\beta = \omega^{-1} + \omega^{-3} + \omega^{-4} + \omega^{-5} + \omega^{-9}$

Since $\omega^{11} = 1$ so $\beta = \omega^{11}(\omega^{-1} + \omega^{-3} + \omega^{-4} + \omega^{-5} + \omega^{-9})$

$= \omega^{10} + \omega^8 + \omega^7 + \omega^6 + \omega^2$

Sum of roots: $\alpha + \beta = \omega + \omega^3 + \omega^4 + \omega^5 + \omega^9 + \omega^{10} + \omega^8 + \omega^7 + \omega^6 + \omega^2 = -1$

Product of roots: $\alpha\beta = (\omega + \omega^3 + \omega^4 + \omega^5 + \omega^9)(\omega^{10} + \omega^8 + \omega^7 + \omega^6 + \omega^2)$

$= 1 + \omega^9 + \omega^8 + \omega^7 + \omega^3 +$

$\omega^2 + 1 + \omega^{10} + \omega^9 + \omega^5 +$

$\omega^3 + \omega + 1 + \omega^{10} + \omega^6 +$

$\omega^4 + \omega^2 + \omega + 1 + \omega^7 +$

$\omega^8 + \omega^6 + \omega^5 + \omega^4 + 1$

$= 5 + 2(\omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 + \omega^7 + \omega^8 + \omega^9 + \omega^{10})$

$= 5 - 2 = 3$

Using $\omega^{11} = 1$

The quadratic equation is: $z^2 + z + 3 = 0$

(ii) $z = \frac{-1 \pm \sqrt{1-12}}{2} = \frac{1}{2}(-1 \pm \sqrt{11}i)$. So either $\alpha = \frac{1}{2}(-1+\sqrt{11}i)$ or $\frac{1}{2}(-1-\sqrt{11}i)$

$\Rightarrow |\alpha| = \frac{1}{2}\sqrt{12} = \sqrt{3}$

(iii) $\text{Im}(\alpha) = \text{Im}(\omega + \omega^3 + \omega^4 + \omega^5 + \omega^9)$

$= \sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} + \sin\frac{18\pi}{11}$

$\Rightarrow \sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} + \sin\frac{18\pi}{11} = \frac{\sqrt{11}}{2}$ or $-\frac{\sqrt{11}}{2}$

$\sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} + \sin\frac{18\pi}{11}$

$= \sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} - \sin\frac{4\pi}{11}$

If $\sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} - \sin\frac{4\pi}{11} = -\frac{\sqrt{11}}{2}$ then

$\sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} = \sin\frac{4\pi}{11} - \frac{\sqrt{11}}{2} < 1 - \frac{\sqrt{11}}{2} < 0$

This is not possible as each sin of each angle on the LHS is positive.

Thus $\sin\frac{2\pi}{11} + \sin\frac{6\pi}{11} + \sin\frac{8\pi}{11} + \sin\frac{10\pi}{11} - \sin\frac{4\pi}{11} = \frac{\sqrt{11}}{2}$

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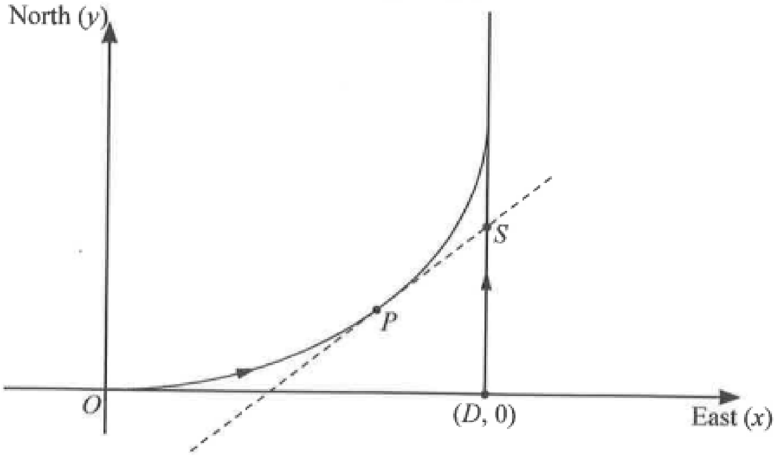

At noon, ship P sets sail from port O at constant speed $V\,\text{m s}^{-1}$ in pursuit of ship S which is travelling due north at constant speed $U\,\text{m s}^{-1}$, where $U < V$. At noon, S is D m due east of O.

P is always steered directly towards S, so that its instantaneous direction is tangential to its *pursuit-curve*, as shown in the diagram above.

At time t seconds after noon, P is at the point which is x m east and y m north of O.

(a) **(i)** Write down

- the coordinates of S at time t seconds after noon,
- an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ at time t. [2]

(ii) Let $m = \frac{\mathrm{d}y}{\mathrm{d}x}$. Explain why, with reference to the distance travelled by P, $\frac{\mathrm{d}}{\mathrm{d}x}(Vt) = \sqrt{1+m^2}$. [1]

(iii) Using all of these results, show that $(D-x)\frac{\mathrm{d}m}{\mathrm{d}x} = \lambda\sqrt{1+m^2}$, where $\lambda = \frac{U}{V}$. [3]

(b) Use the substitution $m = \tan\theta$ to deduce the result $m + \sqrt{1+m^2} = \left(1 - \frac{x}{D}\right)^{-\lambda}$. [5]

(c) Determine, in the form $y = \mathrm{f}(x)$, the equation of the *pursuit-curve*. [4]

[Solution]:

(a)(i) Coordinates of S = (D, Ut)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \text{Gradient } PS \text{ (at time } t) = \frac{Ut - y}{D - x}$$

(ii) Distance travelled by P = arc length OP

$$= \int_0^x \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x = \int_0^x \sqrt{1+m^2}\,\mathrm{d}x$$

Since P travelled at a constant speed $V\text{ms}^{-1}$, the distance travelled by P is also given by Vt.

Hence $Vt = \int_0^x \sqrt{1+m^2}\,\mathrm{d}x$

Diffentiate the above w.r.t x: $\frac{d}{dx}(Vt)=\sqrt{1+m^2}$

(iii) From (a)(i), $\frac{dy}{dx}=\frac{Ut-y}{D-x}$

$$\Rightarrow (D-x)\frac{dy}{dx}=Ut-y$$

$$\Rightarrow (D-x)m=Ut-y$$

Differentiate w.r.t x: $(D-x)\frac{dm}{dx}-m=U\frac{dt}{dx}-\frac{dy}{dx}$

$$(D-x)\frac{dm}{dx}-m=U\frac{\sqrt{1+m^2}}{V}-m$$

$$(D-x)\frac{dm}{dx}=\lambda\sqrt{1+m^2}\text{, where }\lambda=\frac{U}{V}$$

(from (a) (ii)

$$\frac{d}{dx}(Vt)=\sqrt{1+m^2}$$

$$\Rightarrow V\frac{dt}{dx}=\sqrt{1+m^2}$$

$$\Rightarrow \frac{dt}{dx}=\frac{\sqrt{1+m^2}}{V}$$

(b) $m=\tan\theta$ **---- (1)**

$\frac{dm}{dx}=\sec^2\theta\frac{d\theta}{dx}$ **---- (2)**

Sub (1) and (2) into DE in part (iii):

$$(D-x)\sec^2\theta\frac{d\theta}{dx}=\lambda\sqrt{1+\tan^2\theta}$$

$$\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\frac{d\theta}{dx}=\frac{\lambda}{D-x}$$

$$\sec\theta\frac{d\theta}{dx}=\frac{\lambda}{D-x}$$

$$\int\sec\theta\,d\theta=\int\frac{\lambda}{D-x}\,dx$$

$$\ln|\sec\theta+\tan\theta|=-\lambda\ln|D-x|+\ln c$$

$$\ln|\sec\theta+\tan\theta|=\ln c|D-x|^{-\lambda}$$

$$\sec\theta+\tan\theta=\pm c(D-x)^{-\lambda}$$

$$\sec\theta+\tan\theta=A(D-x)^{-\lambda}\text{, where }A=\pm c$$

$$\sqrt{1+\tan^2\theta}+\tan\theta=A(D-x)^{-\lambda}$$

$$\sqrt{1+m^2}+m=A(D-x)^{-\lambda}$$

At noon, $t=0$, $x=0$, $y=0$ $\Rightarrow m=0$

$1=A(D)^{-\lambda}\Rightarrow A=D^{\lambda}$

$$\therefore\sqrt{1+m^2}+m=D^{\lambda}(D-x)^{-\lambda}=D^{\lambda}D^{-\lambda}\left(1-\frac{x}{D}\right)^{-\lambda}$$

$$\therefore\sqrt{1+m^2}+m=\left(1-\frac{x}{D}\right)^{-\lambda}\text{ (deducd)}$$

(c) $\sqrt{1+m^2}+m=\left(1-\frac{x}{D}\right)^{-\lambda}$

$$\sqrt{1+m^2} = \left(1-\frac{x}{D}\right)^{-\lambda} - m$$

$$1+m^2 = \left(1-\frac{x}{D}\right)^{-2\lambda} - 2\left(1-\frac{x}{D}\right)^{-\lambda} m + m^2$$

$$1 = \left(1-\frac{x}{D}\right)^{-2\lambda} - 2\left(1-\frac{x}{D}\right)^{-\lambda} m$$

$$2\left(1-\frac{x}{D}\right)^{-\lambda} m = \left(1-\frac{x}{D}\right)^{-2\lambda} - 1$$

$$m = \frac{1}{2}\left[\left(1-\frac{x}{D}\right)^{-\lambda} - \left(1-\frac{x}{D}\right)^{\lambda}\right]$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left[\left(1-\frac{x}{D}\right)^{-\lambda} - \left(1-\frac{x}{D}\right)^{\lambda}\right]$$

Integrate w.r.t x: $y = \frac{1}{2}\int\left[\left(1-\frac{x}{D}\right)^{-\lambda} - \left(1-\frac{x}{D}\right)^{\lambda}\right]\mathrm{d}x$

$$y = \frac{1}{2}\left[\frac{D\left(1-\frac{x}{D}\right)^{-\lambda+1}}{-(-\lambda+1)} - \frac{D\left(1-\frac{x}{D}\right)^{\lambda+1}}{-(\lambda+1)}\right] + C$$

$$= \frac{1}{2}\left[\frac{D\left(1-\frac{x}{D}\right)^{\lambda+1}}{1+\lambda} - \frac{D\left(1-\frac{x}{D}\right)^{1-\lambda}}{1-\lambda}\right] + C$$

When $x = 0$, $y = 0$, $0 = \frac{1}{2}\left[\frac{D}{1+\lambda} - \frac{D}{1-\lambda}\right] + C \Rightarrow C = \frac{D\lambda}{1-\lambda^2}$

$$\therefore y = \frac{D}{2(1+\lambda)}\left(1-\frac{x}{D}\right)^{\lambda+1} - \frac{D}{2(1-\lambda)}\left(1-\frac{x}{D}\right)^{1-\lambda} + \frac{D\lambda}{1-\lambda^2}$$