1 Let a and b be positive integers.

(a) **(i)** Prove that $\gcd(a,b)=\gcd(a,b-a)$ [3]

(ii) Using the result from part **(i)**, evaluate $\gcd(72,120)$. [2]

(b) Prove that for any integers a, b and c, $\gcd(a,\gcd(b,c))=\gcd(\gcd(a,b),c)$ [5]

2 This question is about the third series of coins in Singapore. This series consists of coins of five different denominations: 5 cents, 10 cents, 20 cents, 50 cents and 1 dollar.

(a) Thirteen coins are to be selected. In how many ways can this be done

(i) if there are no restrictions? [2]

(ii) if at least one coin of each denomination and at most three 5 cent coins are selected? [4]

(b) A sequence of thirteen coins is to be formed. In how many ways can this be done

(i) if no two adjacent coins have the same denomination, [1]

(ii) if at least one coin of each denomination must be used? [3]

3 **(i)** Prove that for any integers x and y and for any prime p,

$$(x+y)^p \equiv x^p + y^p \pmod{p}.$$ [2]

(ii) Using induction, show that for any prime p, $a^p \equiv a \pmod{p}$ for all positive integers a. [4]

(iii) Show that if n is not a multiple of 4, then $\sum_{i=1}^{4} i^n \equiv 0 \pmod{5}$. [4]

4 **(a)** For $a,b,c,p,q,r \in \mathbb{R}^+$, prove that $\sqrt{ap}+\sqrt{bq}+\sqrt{cr} \le \sqrt{(a+b+c)(p+q+r)}$. [3]

(b) For $x,y,z \in \mathbb{R}^+$, prove that $\dfrac{xyz}{(x+y)(y+z)(x+z)} \le \dfrac{1}{8}$. [5]

(c) For $x,y,z \in \mathbb{R}^+$, and using the results of parts **(a)** and **(b)**, prove that

$$\sqrt{\frac{2x}{x+y}}+\sqrt{\frac{2y}{y+z}}+\sqrt{\frac{2z}{z+x}} \le 3.$$ [7]

Hint: $\dfrac{2x}{x+y}=\dfrac{2x(y+z)(z+x)}{(x+y)(y+z)(z+x)}$

5 Eugene runs once every day. Each of his runs is either a threshold run, a tempo run or a recovery run. Over a period of *n* consecutive days, he does not do a threshold run on two consecutive days and he does not do a recovery run for more than two consecutive days.

Let a_n, b_n, c_n be the number of possible ways Eugene can run over a period of *n* consecutive days where the first day is a threshold run, a tempo run or a recovery run respectively.

(i) For $n \in \mathbb{Z}^+$, explain why

$$a_{n+1} = b_n + c_n,$$
$$b_{n+1} = a_n + b_n + c_n,$$
$$c_{n+2} = a_{n+1} + b_{n+1} + a_n + b_n.$$

[4]

(ii) Hence express a_{n+4} in terms of $a_{n+3}, a_{n+2}, a_{n+1}$ and a_n for $n \in \mathbb{Z}^+$. [3]

(iii) Show that $a_5 = 40$. [2]

(iv) Find the number of ways Eugene can run over a period of 5 consecutive days. [1]

6 **(a)** Use suitable substitution(s), prove that, for $n \geq 1$

(i) $\int_0^1 \left(1-x^2\right)^n \mathrm{d}x = I_{2n+1}$; [2]

(ii) $\int_0^1 \left(1+x^2\right)^{-n} \mathrm{d}x < I_{2n-2}$ [2]

where $I_k = \int_0^{\frac{\pi}{2}} \left(\cos\theta\right)^k \mathrm{d}\theta, k \geq 0$.

(b) Using standard series from the List of Formulae (MF26), show that $1 - x^2 \leq \mathrm{e}^{-x^2} \leq \left(1+x^2\right)^{-1}$ for $x \geq 0$. Hence prove that $\sqrt{n} I_{2n+1} \leq \int_0^{\sqrt{n}} \mathrm{e}^{-y^2}\, \mathrm{d}y < \sqrt{n} I_{2n-2}$ for $n \geq 1$. [5]

(c) Given that $\sqrt{k} I_k \to \sqrt{\frac{\pi}{2}}$ when $k \to \infty$, show that $\int_0^{\infty} \mathrm{e}^{-x^2}\, \mathrm{d}x = \frac{\sqrt{\pi}}{2}$. [2]

(d) Let $U_n = \int_0^{\infty} x^{2n} \mathrm{e}^{-x^2} \mathrm{d}x,\ n = 0, 1, 2, \ldots$.

Given that $x^{2n-1}\mathrm{e}^{-x^2} \to 0$ as $x \to \infty$, prove that $U_n = \frac{2n-1}{2} U_{n-1}$ for $n \geq 1$.

Hence prove that $\int_0^{\infty} x^{2n} \mathrm{e}^{-x^2} \mathrm{d}x = \frac{(2n)!\sqrt{\pi}}{2^{2n+1} n!}$. [4]

7 **(i)** Given that $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are collinear points, show that

$$(x_3 - x_2)y_1 + (x_1 - x_3)y_2 + (x_2 - x_1)y_3 = 0$$ [2]

The functions f and g are defined on the real numbers and satisfy the equation

$$\mathrm{g}(\mathrm{f}(x+y)) = \mathrm{f}(x) + (2x+y)\mathrm{g}(y)$$

(ii) Show that $\mathrm{g}(\mathrm{f}(-x)) = \mathrm{f}(x)$, and hence prove that $\mathrm{f}(-x-y) = \mathrm{f}(x) + (2x+y)\mathrm{g}(y)$. [3]

(iii) For real numbers a, b and c, show that

$$\mathrm{f}(-a) = \mathrm{f}(-b) + (a-b)\mathrm{g}(a+b)$$
$$\mathrm{f}(-b) = \mathrm{f}(-c) + (b-c)\mathrm{g}(b+c)$$
$$\mathrm{f}(-c) = \mathrm{f}(-a) + (c-a)\mathrm{g}(c+a)$$ [2]

(iv) By showing that $\mathrm{g}(x)$ is a linear function, prove that

$$\mathrm{f}(x) = \mathrm{g}(x) = 0 \text{ or } \mathrm{f}(x) = x^2 + C \text{ and } \mathrm{g}(x) = x$$

where C is a constant. [8]

8 **(a)** A sequence $\{x_n\}$ is defined by $x_n = \sum_{m=1}^{n} \frac{1}{m^m}$. Given that $m^m \geq 2^m$ for all $m \geq 2$,

Show that

(i) the sequence is bounded by $\frac{3}{2}$, [2]

(ii) the sequence converges. [2]

(b) The Fibonacci numbers are defined by $F_0 = 0, F_1 = 1$ and, for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$.

(i) Prove that $F_r \leq 2^{r-1}F_1$ for all $r \geq 1$. [2]

(ii) Let $S_n = \sum_{r=1}^{n} \frac{F_r}{9^r}$. Show that

$$\sum_{r=1}^{n} \frac{F_{r+1}}{9^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{9^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{9^{r-1}} = 71S_n - 9F_1 - F_0 + \frac{F_n}{9^n} + \frac{F_{n+1}}{9^{n-1}}.$$ [4]

(iii) Show that $\sum_{r=1}^{\infty} \frac{F_r}{9^r} = \frac{9}{71}$. [3]

(iv) Given that $\sum_{r=7}^{\infty} \frac{F_r}{9^r} < 2\times10^{-6}$, and using the result in **(iii)**, find the value of $\frac{1}{71}$ up to the first six decimal places. [2]

End of Paper