Chapter 7 (Pure Mathematics) : Integration Techniques

Objectives:

At the end of the chapter, you should be able to

(a)	understand that integration is the reverse of differentiation
(b)	find the integral of x^n, for any rational *n*, and, e^x, together with constant multiples, sums and differences
(c)	find the integral of $(ax+b)^n$, for any rational *n*, and $e^{(ax+b)}$

Content

References

1. New Additional Mathematics, Ho Soo Thong (Msc, Dip Ed), Khor Nyak Hiong (Bsc, Dip Ed)

2. New Syllabus Additional Mathematics (7^{th} Edition), Shinglee Publishers Ptd Ltd

7.1 Introduction

If you have the gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 5$ and some related information, can you find the equation of the curve y? The answer is yes. The process of getting y from $\frac{\mathrm{d}y}{\mathrm{d}x}$ is called Integration, also sometimes known as anti-differentiation.

Example 1

Given the gradient function $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, find the <u>general equation</u> of y.

Solution:

We know that if we differentiate x^2, we will get $2x$. Hence $y = x^2$.

But this is not the complete picture, because if you differentiate $x^2 + 1$, you will also get the same answer, $2x$. The same is true for $y = x^2 + 2$.

In fact if you differentiate x^2 plus any constant, you get the same answer.

Therefore $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \quad \Rightarrow \quad y = x^2 + C$, where C is an arbitrary constant.

This reversal of differentiation, known as integration, is represented as follows:

$$\int 2x \ \mathrm{d}x = x^2 + C$$

C is the "Constant of Integration". It is there because of **functions whose derivative are $2x$ is not unique.** For example, $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 - 4) = 2x$; $\frac{\mathrm{d}}{\mathrm{d}x}(x^2 + 20) = 2x$, etc…

7.2 Definition

Integration is the reverse process of Differentiation.

If $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{F}(x) = \mathrm{f}(x)$, then

$$\int \mathrm{f}(x) \ \mathrm{d}x = \mathrm{F}(x) + C .$$

Integral symbol

Function we want to integrate

Variable to integrate

where $\frac{\mathrm{dF}(x)}{\mathrm{d}x} = \mathrm{f}(x)$ and C is an arbitrary constant.

7.2.1 Basic Properties of Indefinite Integral

Let f and g be two functions. Then,

1. $\int k \, \mathrm{d}x = kx + C$, where C is an arbitrary constant
2. Multiply function by constant: $\int k\mathrm{f}(x) \, \mathrm{d}x = k\int \mathrm{f}(x) \, \mathrm{d}x$
3. Sum and Difference of functions: $\int \left[\mathrm{f}(x) \pm \mathrm{g}(x)\right] \mathrm{d}x = \int \mathrm{f}(x) \, \mathrm{d}x \pm \int \mathrm{g}(x) \, \mathrm{d}x$

7.3 Integrate Basic Functions

7.3.1 Integrate Algebraic Functions

Recall when we differentiate x^n, we multiply by the number in the power and reduce the power by 1. i.e. $\frac{\mathrm{d}(x^n)}{\mathrm{d}x} = nx^{n-1}$

In integration, we raise the power by 1 and divide by $n+1$. See (a)

Note: For (a) to (d) below, C represents an arbitrary constant.

	Common functions	Function	Integral
(a)	Power where $n \neq -1$	$\int x^n \, \mathrm{d}x$	$= \frac{x^{n+1}}{n+1} + C \, , n \neq -1$
		$\int (ax+b)^n \, \mathrm{d}x$	$= \frac{(ax+b)^{n+1}}{a(n+1)} + C \, , n \neq -1$
(b)	Constant	$\int k \, \mathrm{d}x = \int kx^0 \, \mathrm{d}x$	$= kx + C$

Special Case (when power $n = -1$)

(c)	Power where $n = -1$	$\int x^{-1} \, \mathrm{d}x = \int \frac{1}{x} \mathrm{d}x$	$= \ln\lvert x\rvert + C$
(d)		$\int (ax+b)^{-1} \, \mathrm{d}x = \int \frac{1}{ax+b} \mathrm{d}x$	$= \frac{\ln\lvert ax+b\rvert}{a} + C$

Example 2 (Using results (a) to (b))

Find the following indefinite integrals.

Solution:

(i)	$\int x^3 \, dx = \frac{x^4}{4} + C$, where C is an arbitrary constant	**(vi)**	$\int (x+2) \, dx = \frac{x^2}{2} + 2x + C$ OR $\int (x+2) \, dx$ $= \frac{(x+2)^2}{2(1)} + C = \frac{(x+2)^2}{2} + C$, where C is an arbitrary constant
(ii)	$\int 2 \, dx = \int 2x^0 dx = 2x + C$, where C is an arbitrary constant	**(vii)**	$\int (3x-5)^6 dx = \frac{(3x-5)^7}{7(3)} + C$ $= \frac{(3x-5)^7}{21} + C,$ where C is an arbitrary constant
(iii)	$\int \sqrt{x} \, dx = \int x^{\frac{1}{2}} dx$ $= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + C$ $= \frac{2}{3} x^{\frac{3}{2}} + C,$ where C is an arbitrary constant	**(viii)**	$\int \frac{2}{(3x+4)^3} \, dx = \int 2(3x+4)^{-3} \, dx$ $= 2\frac{(3x+4)^{-2}}{-2(3)} + C$ $= -\frac{1}{3(3x+4)^2} + C,$ where C is an arbitrary constant
(iv)	$\int \frac{1}{\sqrt{x}} \, dx = \int x^{-\frac{1}{2}} \, dx$ $= \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$ $= 2x^{\frac{1}{2}} + C,$ where C is an arbitrary constant	**(ix)**	$\int \frac{2}{\sqrt{3x+1}} \, dx = \int 2(3x+1)^{-\frac{1}{2}} \, dx$ $= 2\frac{(3x+1)^{\frac{1}{2}}}{\frac{1}{2}(3)} + C$ $= \frac{4}{3}\sqrt{3x+1} + C,$ where C is an arbitrary constant
(v)	$\int \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C,$ where C is an arbitrary constant	**(x)**	$\int x(x-2) \, dx = \int (x^2 - 2x) \, dx$ $= \frac{x^3}{3} - 2.\frac{x^2}{2} + C$

			$=\frac{x^3}{3}-x^2+C,$ where C is an arbitrary constant

Example 3 (using results (c) and (d))

Find the following indefinite integrals.

Solution:

(i)	$\int \frac{3}{2x}\,\mathrm{d}x=\frac{3}{2}\int \frac{1}{x}\,\mathrm{d}x$ $=\frac{3}{2}\ln\lvert x\rvert+C,$ where C is an arbitrary constant	**(iii)**	$\int \frac{3}{1-2x}\,\mathrm{d}x=3\int \frac{1}{1-2x}\,\mathrm{d}x$ $=3\frac{\ln\lvert 1-2x\rvert}{-2}+C$ $=-\frac{3}{2}\ln\lvert 1-2x\rvert+C,$ where C is an arbitrary constant
(ii)	$\int \frac{1}{2x+1}\,\mathrm{d}x=\frac{\ln\lvert 2x+1\rvert}{2}+C,$ where C is an arbitrary constant	**(iv)**	$\int\left(\frac{1}{(1-3x)^2}+\frac{1}{4x+3}\right)\mathrm{d}x$ $=\int(1-3x)^{-2}\,\mathrm{d}x+\int(4x+3)^{-1}\,\mathrm{d}x$ $=\frac{(1-3x)^{-1}}{-1(-3)}+\frac{\ln\lvert 4x+3\rvert}{4}+C$ $=\frac{1}{3(1-3x)}+\frac{\ln\lvert 4x+3\rvert}{4}+C,$ where C is an arbitrary constant

7.3.2 Integrate Exponential Function

Function	**Integral**
$\int e^x \mathrm{d}x$	$= e^x + C$, where C is an arbitrary constant
$\int e^{ax+b} \mathrm{d}x$	$= \dfrac{1}{a} e^{ax+b} + C$, where C is an arbitrary constant

Example 4
Find the following indefinite integrals.

Solution:

(i)	$\int 2e^x \, \mathrm{d}x = 2\int e^x \, \mathrm{d}x$ $= 2e^x + C,$ where C is an arbitrary constant	**(iv)**	$\int e^{7x-4} \, \mathrm{d}x = \dfrac{e^{7x-4}}{7} + C,$ where C is an arbitrary constant
(ii)	$\int \dfrac{1}{e^{2x}} \, \mathrm{d}x = \int e^{-2x} \mathrm{d}x$ $= \dfrac{e^{-2x}}{-2} + C$ $= -\dfrac{1}{2e^{2x}} + C,$ where C is an arbitrary constant	**(v)**	$\int \dfrac{e^{2x} - 3}{e^x} \, \mathrm{d}x = \int \left(\dfrac{e^{2x}}{e^x} - \dfrac{3}{e^x} \right) \mathrm{d}x$ $= \int \left(e^x - 3e^{-x} \right) \mathrm{d}x$ $= e^x - 3\dfrac{e^{-x}}{-1} + C$ $= e^x + \dfrac{3}{e^x} + C,$ where C is an arbitrary constant
(iii)	$\int \sqrt{e^x} \, \mathrm{d}x = \int \left(e^x\right)^{\frac{1}{2}} \mathrm{d}x$ $= \int e^{\frac{x}{2}} \, \mathrm{d}x$ $= \dfrac{e^{\frac{x}{2}}}{\frac{1}{2}} + C$ $= 2e^{\frac{x}{2}} + C,$ where C is an arbitrary constant	**(vi)**	$\int (e^x - e^{3x})^2 \mathrm{d}x = \int (e^x - e^{3x})^2 \mathrm{d}x$ $= \int (e^{2x} - 2e^{4x} + e^{6x}) \, \mathrm{d}x$ $= \dfrac{e^{2x}}{2} - \dfrac{2e^{4x}}{4} + \dfrac{e^{6x}}{6} + C$ $= \dfrac{e^{2x}}{2} - \dfrac{e^{4x}}{2} + \dfrac{e^{6x}}{6} + C,$ where C is an arbitrary constant

Example 5

Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\sqrt{2x^2+1}\right] = \frac{2x}{2x^2+1}$. Hence find $\int \frac{3x}{2x^2+1}\,\mathrm{d}x$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\sqrt{2x^2+1}\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\ln(2x^2+1)\right]\\ &= \frac{1}{2}\left(\frac{4x}{2x^2+1}\right)\\ &= \frac{2x}{2x^2+1} \quad \text{(Shown)}\end{aligned}$$

$$\begin{aligned}\int \frac{3x}{2x^2+1}\,\mathrm{d}x &= \frac{3}{2}\int \frac{2x}{2x^2+1}\,\mathrm{d}x\\ &= \frac{3}{2}\left[\ln\sqrt{(2x^2+1)}\right] + C \text{ where } C \text{ is an arbitrary constant}\end{aligned}$$

Exercise 1

1. Find the following indefinite integrals.

(i) $\int\left(3x^5+2x-4\right)\,\mathrm{d}x$ (ii) $\int\frac{1-x}{\sqrt{x}}\,\mathrm{d}x$ (iii) $\int(t-2)(t+3)\,\mathrm{d}t$

(iv) $\int\sqrt{(5x+7)^3}\,\mathrm{d}x$ (v) $\int 2e^{3x}\,\mathrm{d}x$ (vi) $\int e^{3x-5}\,\mathrm{d}x$

Solution:

(i) $\int\left(3x^5+2x-4\right)\mathrm{d}x = \frac{3x^6}{6}+\frac{2x^2}{2}-4x+C$ $= \frac{x^6}{2}+x^2-4x+C,$ where C is an arbitrary constant	(ii) $\int\frac{1-x}{\sqrt{x}}\,\mathrm{d}x = \int\left(x^{-\frac{1}{2}}-x^{\frac{1}{2}}\right)\mathrm{d}x$ $= \frac{x^{\frac{1}{2}}}{\frac{1}{2}}-\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+C$ $= 2\sqrt{x}-\frac{2}{3}x^{\frac{3}{2}}+C,$ where C is an arbitrary constant
(iii) $\int(t-2)(t-3)\mathrm{d}t = \int(t^2+t-6)\,\mathrm{d}t$ $= \frac{t^3}{3}+\frac{t^2}{2}-6t+C,$ where C is an arbitrary constant	(iv)

	$\int\sqrt{(5x+7)^3}\mathrm{d}x=\int(5x+7)^{\frac{3}{2}}\mathrm{d}x$ $=\frac{(5x+7)^{\frac{5}{2}}}{\frac{5}{2}(5)}+C$ $=\frac{2}{25}(5x+7)^{\frac{5}{2}}+C,$ where C is an arbitrary constant
(v) $\int 2e^{3x}\mathrm{d}x=\frac{2e^{3x}}{3}+C$, where C is an arbitrary constant	(vi) $\int e^{3x-5}\mathrm{d}x=\frac{e^{3x-5}}{3}+C,$ where C is an arbitrary constant

2. **DHS Prelim 8865/2018/Q2**

(i) Differentiate $\sqrt{(e^{2x}+1)^3}$ with respect to x.

(ii) Hence find $\int e^x\sqrt{(e^{4x}+e^{2x})}\,\mathrm{d}x$.

Solution:

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{(e^{2x}+1)^3}=\frac{\mathrm{d}}{\mathrm{d}x}(e^{2x}+1)^{\frac{3}{2}}=\frac{3}{2}(e^{2x}+1)^{\frac{1}{2}}\left(2e^{2x}\right)=3e^{2x}\sqrt{(e^{2x}+1)}$

(ii) $\int e^x\sqrt{(e^{4x}+e^{2x})}\,\mathrm{d}x$

$=\int e^x\sqrt{[e^{2x}(e^{2x}+1)]}\,\mathrm{d}x$

$=\frac{1}{3}\int 3e^{2x}\sqrt{(e^{2x}+1)}\,\mathrm{d}x$

$=\frac{1}{3}\left[\sqrt{(e^{2x}+1)^3}\right]+C$

3. **NJC Prelim 8865/2018/Q3**

(i) Differentiate $\ln\left(x^2+9\right)$.

(ii) Express $\frac{2x^2-x+9}{(1-x)\left(x^2+9\right)}$ in the form $\frac{A}{1-x}+\frac{Bx}{x^2+9}$ where A and B where are integers to be determined.

(iii) Hence find $\int\frac{2x^2-x+9}{(1-x)\left(x^2+9\right)}\,\mathrm{d}x$.

Solution:

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\ln(x^2+9)\right]=\dfrac{2x}{x^2+9}$

(ii) $$\frac{A}{1-x}+\frac{Bx}{x^2+9}=\frac{A(x^2+9)+(1-x)(Bx)}{(1-x)(x^2+9)}$$
$$=\frac{Ax^2+9A+Bx-Bx^2}{(1-x)(x^2+9)}$$
$$=\frac{\left(A-B\right)x^2+Bx+9A}{(1-x)(x^2+9)}$$

Comparing coefficients with $\dfrac{2x^2-x+9}{(1-x)(x^2+9)}$:

$x^2: A-B=2$

$x: B=-1$

constant $: 9A=9 \Rightarrow A=1$

Therefore, $A=1, B=-1$

Hence $\dfrac{2x^2-x+9}{(1-x)(x^2+9)}=\dfrac{1}{1-x}-\dfrac{x}{x^2+9}$ (shown)

(iii) $$\int\frac{2x^2-x+9}{(1-x)(x^2+9)}\,\mathrm{d}x=\int\frac{1}{1-x}-\frac{x}{x^2+9}\,\mathrm{d}x$$
$$=\int\frac{1}{1-x}\mathrm{d}x-\frac{1}{2}\int\frac{2x}{x^2+9}\,\mathrm{d}x$$
$$=-\ln\left|1-x\right|-\frac{1}{2}\ln(x^2+9)+C$$

Answer:

1(i) $\dfrac{x^6}{2}+x^2-4x+C$	1(ii) $2\sqrt{x}-\dfrac{2}{3}x^{\frac{3}{2}}+C$
1(iii) $\dfrac{t^3}{3}+\dfrac{t^2}{2}-6t+C$	1(iv) $\dfrac{2}{25}\left(5x+7\right)^{\frac{5}{2}}+C$
1(v) $\dfrac{2e^{3x}}{3}+C$	1(vi) $\dfrac{e^{3x-5}}{3}+C$
For (i) to (iii) above, C represents an arbitrary constant.	
2. $3\mathrm{e}^{2x}\sqrt{(\mathrm{e}^{2x}+1)}$; $\dfrac{1}{3}\left[\sqrt{(\mathrm{e}^{2x}+1)^3}\right]+C$	
3. (i) $\dfrac{2x}{x^2+9}$; (ii) $A=1, B=-1$; (iii) $-\ln\left|1-x\right|-\dfrac{1}{2}\ln(x^2+9)+C$	

7.4 Miscellaneous Examples

Example 6

Given that the gradient of a curve is $2x^2+7x$ and that the curve passes through the origin, determine the equation of the curve.

Solution:

Given $\frac{\mathrm{d}y}{\mathrm{d}x}=2x^2+7x$,

$y=\int\left(2x^2+7x\right)\mathrm{d}x$

$=2\left(\frac{x^3}{3}\right)+7\left(\frac{x^2}{2}\right)+C,$

where C is an arbitrary constant

$=\frac{2}{3}x^3+\frac{7}{2}x^2+C\cdots(1)$

Since curve passes through the origin,

$0=0+C$

$C=0$

Therefore the equation of the curve is $y=\frac{2}{3}x^3+\frac{7}{2}x^2$.

Note: (0, 0) satisfies the equation $y=\frac{2}{3}x^3+\frac{7}{2}x^2+C$

Substitute $x = 0$, $y = 0$ into (1)

Example 7

Given that the rate of change of s with respect to t is given by $\frac{\mathrm{d}s}{\mathrm{d}t}=3t^2-7$ and that $s = 6$ when $t = 0$, find s in terms of t.

Solution:

Given $\frac{\mathrm{d}s}{\mathrm{d}t}=3t^2-7$,

$s=\int\left(3t^2-7\right)\mathrm{d}t$

$=\frac{3t^3}{3}-7t+C,$ where C is an arbitrary constant $\cdots(1)$

When $s = 6$ and $t = 0$

$6=0-0+C$

$C=6$

$\therefore\ s=t^3-7t+6$

Note:
Substitute $s = 6$, $t = 0$ into (1)

Exercise 2

1. Find the following indefinite integrals.

(i) $\int\left(t^{\frac{5}{2}}+\frac{1}{t^2}\right)\mathrm{d}t$ (ii) $\int(2x+9)^5\ \mathrm{d}x$ (iii) $\int\frac{3}{3-2x}\ \mathrm{d}x$

(iv) $\int\frac{1}{5x+3}\ \mathrm{d}x$ (v) $\int\frac{7}{\sqrt{2-3x}}\ \mathrm{d}x$ (vi) $\int\frac{e^{1-x}+3}{e^{x+1}}\ \mathrm{d}x$

Solution:

For (i) to (vi) below, C represents an arbitrary constant.

(i) $\int\left(t^{\frac{5}{2}}+\frac{1}{t^2}\right)dt=\int\left(t^{\frac{5}{2}}+t^{-2}\right)dt$ $=\frac{t^{\frac{7}{2}}}{\frac{7}{2}}+\frac{t^{-1}}{-1}+C$ $=\frac{2}{7}t^{\frac{7}{2}}-\frac{1}{t}+C$	(ii) $\int(2x+9)^5\,dx=\frac{(2x+9)^6}{6(2)}+C$ $=\frac{(2x+9)^6}{12}+C$
(iii) $\int\frac{3}{3-2x}dx=\frac{3\ln\lvert 3-2x\rvert}{-2}+C$ $=-\frac{3}{2}\ln\lvert 3-2x\rvert+C$	(iv) $\int\frac{1}{5x+3}dx=\frac{\ln\lvert 5x+3\rvert}{5}+C$
(v) $\int\frac{7}{\sqrt{2-3x}}dx=\int 7(2-3x)^{-\frac{1}{2}}dx$ $=7\frac{(2-3x)^{\frac{1}{2}}}{\frac{1}{2}(-3)}+C$ $=\frac{-14}{3}\sqrt{2-3x}+C$	(vi) $\int\frac{e^{1-x}+3}{e^{x+1}}dx=\int e^{1-x-x-1}+3e^{-x-1}dx$ $=\int e^{-2x}+3e^{-x-1}dx$ $=\frac{e^{-2x}}{-2}+\frac{3e^{-x-1}}{-1}+C$ $=-\frac{1}{2e^{2x}}-\frac{3}{e^{x+1}}+C$

2. Given that the gradient of a curve is $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{x}}-2x$ and that the curve passes through the point $(4,-2)$, find the equation of the curve.

Solution:

$\frac{dy}{dx}=\frac{1}{\sqrt{x}}-2x$ $y=\int \frac{1}{\sqrt{x}}-2x\ dx$ $=\int x^{-\frac{1}{2}}-2x\ dx$ $=\frac{x^{\frac{1}{2}}}{\frac{1}{2}}-2\left(\frac{x^2}{2}\right)+C=2\sqrt{x}-x^2+C$ --(1), where C is an arbitrary constant	Substitute $(4,-2)$ into (1), $-2=2\sqrt{4}-4^2+C$ $C=-2-4+16$ $=10$ Equation of the curve is $y=2\sqrt{x}-x^2+10$

3. The gradient of a curve at any point is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=4-2x$. Explain why the curve has a maximum value when $x=2$. If the maximum value of the curve is 1, find the equation of the curve.

Solution:

$\frac{\mathrm{d}y}{\mathrm{d}x}=4-2x$ When $\frac{dy}{dx}=0$, $4-2x=0$ $2x=4$ $x=2$ $\frac{d^2y}{dx^2}=-2<0$ Therefore, the curve has a maximum value when $x=2$.	$y=\int(4-2x\ )\mathrm{d}x$ $y=4x-\frac{2x^2}{2}+C$ $=4x-x^2+C\cdots(1)$ where C is an arbitrary constant Substitute $(2,1)$ into (1), $1=4(2)-2^2+C$ $C=1-8+4$ $=-3$ Equation of the curve is $y=4x-x^2-3$.

4. The curve for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + \frac{k}{x^2}$, where k is a constant, passes through the points $(1,1)$ and $(2,12)$. Find the equation of the curve and the stationary point.

Solution:

$\frac{dy}{dx} = 2x + \frac{k}{x^2}$ $y = \int 2x + \frac{k}{x^2}\,dx$ $= \int 2x + kx^{-2}\,dx$ $= 2\left(\frac{x^2}{2}\right) + k\,\frac{x^{-1}}{-1} + C$ $= x^2 - \frac{k}{x} + C \cdots (1)$ where C is an arbitrary constant Substitute $(1,1)$ into (1), $1 = 1 - k + C$ $-k + C = 0 \cdots (2)$ Substitute $(2,12)$ into (1), $12 = 2^2 - \frac{k}{2} + C$ $-\frac{k}{2} + C = 8 \cdots (3)$	Solving (2) and (3) using GC, $k = C = 16$ Equation of the curve is $y = x^2 - \frac{16}{x} + 16$ $\frac{dy}{dx} = 2x + \frac{16}{x^2}$ When $\frac{dy}{dx} = 0$, $2x + \frac{16}{x^2} = 0$ $2x = -\frac{16}{x^2}$ $x^3 = -8$ $x = -2$ When $x = -2$, $y = (-2)^2 - \frac{16}{-2} + 16$ $= 4 + 8 + 16$ $= 28$ $(-2, 28)$ is the stationary point.

Answers:

1(i) $\frac{2}{7}t^{\frac{7}{2}} - \frac{1}{t} + C$	(ii) $\frac{(2x+9)^6}{12} + C$
(iii) $-\frac{3}{2}\ln\lvert 3-2x\rvert + C$	1(iv) $\frac{\ln\lvert 5x+3\rvert}{5} + C$
1(v) $\frac{-14}{3}\sqrt{2-3x} + C$	1(vi) $-\frac{1}{2e^{2x}} - \frac{3}{e^{x+1}} + C$
2. $y = 2\sqrt{x} - x^2 + 10$	3. $y = 4x - x^2 - 3$
4. $y = x^2 - \frac{16}{x} + 16$, $(-2, 28)$	

For 1(i) to 1(vi) above, C represents an arbitrary constant.

Practice Questions

1. [NYJC/2018/BT2/Q3a]

Find $\displaystyle\int \frac{\left(x^2+3\right)^2}{x^3}\,\mathrm{d}x$.

[Ans: $\dfrac{x^2}{2}+6\ln|x|-\dfrac{9}{2x^2}+C$]

Solution:

$$\int \frac{\left(x^2+3\right)^2}{x^3}\,\mathrm{d}x=\int \frac{x^4+6x^2+9}{x^3}\,\mathrm{d}x$$
$$=\int x+\frac{6}{x}+\frac{9}{x^3}\,\mathrm{d}x$$
$$=\frac{x^2}{2}+6\ln|x|-\frac{9}{2x^2}+C$$

where C is an arbitrary constant

2. [DHS/2018/BT1/Q2b]

Find $\displaystyle\int\left(3\mathrm{e}^{1-2t}+\frac{1}{\sqrt{(2t+1)}}\right)\mathrm{d}t$.

[Ans: $-\dfrac{3}{2}\mathrm{e}^{1-2t}+\sqrt{(2t+1)}+C$, where C is an arbitrary constant]

Solution:

$$\int\left(3\mathrm{e}^{1-2t}+\frac{1}{\sqrt{(2t+1)}}\right)\mathrm{d}t$$
$$=\int\left(3\mathrm{e}^{1-2t}+(2t+1)^{-\frac{1}{2}}\right)\mathrm{d}t$$
$$=\frac{3\mathrm{e}^{1-2t}}{-2}+\frac{(2t+1)^{\frac{1}{2}}}{\frac{1}{2}(2)}+C$$
$$=-\frac{3}{2}\mathrm{e}^{1-2t}+\sqrt{(2t+1)}+C,$$ where C is an arbitrary constant

3. [AJC Promo 8865/2018/Q3b]

(i) Find $\displaystyle\int\left(\mathrm{e}^{1-2x}+\frac{1}{2x}+\frac{1}{x\sqrt{x}}\right)\mathrm{d}x$, simplifying your answers.

(ii) Find $\displaystyle\int \frac{1}{2\left(1+3x\right)^2}\,\mathrm{d}x$.

[Ans: (i) $= -\frac{e^{1-2x}}{2} + \frac{1}{2}\ln|x| - \frac{2}{\sqrt{x}} + C$ (ii) $-\frac{1}{6(1+3x)} + C$]

Solution:

(i) $\int \left(e^{1-2x} + \frac{1}{2x} + \frac{1}{x\sqrt{x}} \right) dx$

$= -\frac{e^{1-2x}}{2} + \frac{1}{2}\ln|x| - 2x^{-\frac{1}{2}} + C = -\frac{e^{1-2x}}{2} + \frac{1}{2}\ln|x| - \frac{2}{\sqrt{x}} + C$, C is an arbitrary constant

(ii)

$\int \frac{1}{2(1+3x)^2}\, dx = \frac{1}{2}\int (1+3x)^{-2} dx$

$= \frac{1}{2}\left[\frac{(1+3x)^{-1}}{-1(3)} \right] + C$

$= -\frac{1}{6(1+3x)} + C$, where C is an arbitrary constant

4. [NJC/2018/BT2/Q6]

(i) Verify that $x + 4 + \frac{25}{x-4} = \frac{x^2+9}{x-4}$.

(ii) Given that $y = \ln\left(e^{x^2} - 1\right)$, find $\frac{dy}{dx}$.

(iii) Hence, find $\int \frac{x^2+9}{x-4} + \frac{2xe^{x^2}}{e^{x^2}-1} dx$.

[Ans: (ii) $\frac{dy}{dx} = \frac{2xe^{x^2}}{e^{x^2}-1}$ (iii) $\frac{x^2}{2} + 4x + 25\ln(x-4) + \ln\left(e^{x^2}-1\right) + C$]

Solution:

(i) $\text{LHS} = x + 4 + \frac{25}{x-4} = \frac{(x+4)(x-4)+25}{x-4} = \frac{x^2-16+25}{x-4}$

$= \frac{x^2-9}{x-4} = \text{RHS}$

(ii) $y = \ln\left[\left(e^{x^2} - 1\right)\right]$

$\frac{dy}{dx} = \frac{2xe^{x^2}}{e^{x^2}-1}$

(iii)

$$\int \frac{x^2+9}{x-4}+\frac{2xe^{x^2}}{e^{x^2}-1}\,dx$$
$$=\int x+4+\frac{25}{x-4}+\frac{2xe^{x^2}}{e^{x^2}-1}\,dx$$
$$=\frac{x^2}{2}+4x+25\ln(x-4)+\ln\left(e^{x^2}-1\right)+C, \text{where C is an arbitrary constant}$$

5. [DHS Promo 8865/2018/Q3]

(a) (i) For $0<x<\frac{3}{2}$, show that $\frac{d}{dx}\ln(x^3(3-2x))=\frac{a+bx}{x(3-2x)}$, where a and b are constants to be determined.

(ii) Hence solve $\int \frac{27-24x}{x(3-2x)}\,dx$.

(b) Find $\int\left[me^{m^2-2x}+3(mx)^2\right]dx$, where m is a constant.

[Ans: (a)(i) $\frac{9-8x}{x(3-2x)}$ (a)(ii) $3\ln(x^3(3-2x))+C$ (b) $-\frac{1}{2}me^{m^2-2x}+m^2x^3+c$]

Solution:

(a)(i)
$$\frac{d}{dx}\ln(x^3(3-2x))$$
$$=\frac{d}{dx}\left(3\ln x+\ln(3-2x)\right)$$
$$=\frac{3}{x}-\frac{2}{3-2x}$$
$$=\frac{3(3-2x)-2x}{x(3-2x)}$$
$$=\frac{9-8x}{x(3-2x)}, \text{ where } a=9, b=-8 \text{ (shown)}$$

Alternative Method
$$\frac{d}{dx}\ln(x^3(3-2x))$$
$$=\frac{d}{dx}\left(\ln\left(3x^3-2x^4\right)\right)$$
$$=\frac{9x^2-8x^3}{3x^3-2x^4}$$
$$=\frac{9-8x}{3x-2x^2}$$
$$=\frac{9-8x}{x(3-2x)}, \text{ where } a=9, b=-8 \text{ (shown)}$$

(a)(ii)
$$\int \frac{27-24x}{x(3-2x)}\,dx=3\int \frac{9-8x}{x(3-2x)}\,dx$$
$$=3\ln(x^3(3-2x))+c, \text{ where c is an arbitrary constant}$$

(b)
$$\int\left[me^{m^2-2x}+3(mx)^2\right]dx$$
$$=m\int e^{m^2-2x}dx+3m^2\int x^2dx.$$
$$=-\frac{1}{2}me^{m^2-2x}+m^2x^3+c, \text{ where c is an arbitrary constant}$$

Summary

In the table below, '*k*', '*a*' and '*b*' are constants, and *C* is an arbitrary constant.

	Common functions		**Function**	**Integral**
(a)	Constant		$\int k \, \mathrm{d}x$	$= kx + C$
(b)	Variable		$\int x \, \mathrm{d}x$	$= \frac{x^2}{2} + C$
(c)	Square		$\int x^2 \, \mathrm{d}x$	$= \frac{x^3}{3} + C$
(d)	Power where $n \neq -1$		$\int x^n \, \mathrm{d}x$	$= \frac{x^{n+1}}{n+1} + C \, , n \neq -1$
(e)			$\int (ax+b)^n \, \mathrm{d}x$	$= \frac{(ax+b)^{n+1}}{a(n+1)} + C \, , n \neq -1$
(f)	Power where $n = -1$		$\int x^{-1} \, \mathrm{d}x = \int \frac{1}{x} \mathrm{d}x$	$= \ln\lvert x \rvert + C$
(g)			$\int (ax+b)^{-1} \, \mathrm{d}x = \int \frac{1}{ax+b} \mathrm{d}x$	$= \frac{\ln\lvert ax+b \rvert}{a} + C$
(h)	Exponential		$\int e^x \mathrm{d}x$	$= e^x + C$
(i)			$\int e^{ax+b} \mathrm{d}x$	$= \frac{1}{a} e^{ax+b} + C$