ST ANDREW'S JUNIOR COLLEGE

PRELIMINARY EXAMINATION

MATHEMATICS
Higher 2 **9740/1**

Monday **31 AUG 2015** **3 hours**

READ THESE INSTRUCTIONS FIRST

Write your name, civics group and index number on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.

Answer **all** the questions. Total marks is 100.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically state otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematic steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together.

This document consists of 6 printed pages including this page.

1 Without using a calculator, solve the inequality

$$\frac{5x^2-x-14}{2x^2+x-3} \leq 3.$$ **[4]**

2 Prove by mathematical induction that,

$$\sin x+\sin 11x+\sin 21x+\ldots+\sin(10n+1)x=\frac{\cos 4x-\cos(10n+6)x}{2\sin 5x}$$

where $\sin 5x \neq 0$, for non-negative integers n. **[5]**

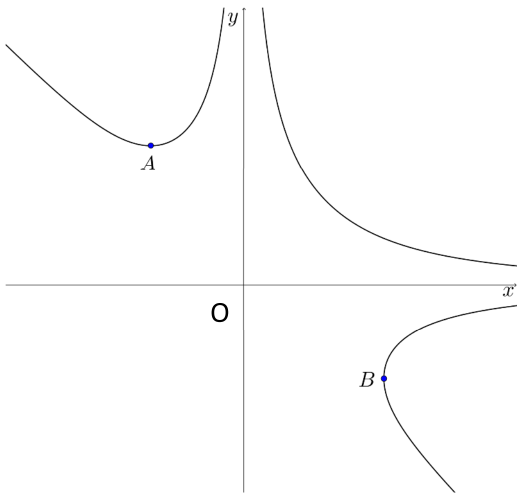
3 The function f is defined by

$$\mathrm{f}(x)=\frac{3x-5}{x-2} \quad \text{for } x\in \mathbb{R}, x \neq 2.$$

(i) Show by differentiation that f is a decreasing function on any interval in the domain. **[2]**

(ii) Find $\mathrm{f}^{-1}(x)$, stating its domain. **[3]**

4 The diagram shows the curve with equation $x^3y^2+x^2y^3=1$. The curve is symmetrical about the line $y=x$ and has a stationary point A.



(i) Find the exact coordinates of A. **[5]**

(ii) The point B is the point of the curve at which the tangent is parallel to the y-axis. Hence, or otherwise, state the exact coordinates of B. **[1]**

5 The ellipse C_1 with the equation $\frac{x^2}{6^2}+\frac{y^2}{3^2}=1$ is transformed by a stretch with scale factor k parallel to the x-axis, followed by a translation of m units in the positive x-direction, followed by a translation of n units in the negative y-direction, to form C_2.

(i) Find the equation of C_2 in terms of *k, m*, and *n*. **[3]**

(ii) Given that C_2 is a circle centred at $(4,-7)$, find the values of *k*, *m* and *n*. **[2]**

6 The variables *u* and *t* are related by $\left(t^2+1\right)\frac{\mathrm{d}u}{\mathrm{d}t}=5t$.

(i) Find the particular solution of the differential equation for which $u=3$ when $t=0$.

[4]

(ii) What can be said about the gradient of every solution curve as $t \to \pm\infty$? **[1]**

(iii) On a single diagram, sketch the curve represented by the result in part (i), together with another member of the family of solution curves. **[2]**

7 A curve *C* has parametric equations

$$x=3-\cos^3\theta, \quad y=4+\sin\theta, \text{ where } 0\le\theta\le\pi.$$

(i) Sketch *C*, showing clearly the coordinates of the points on the curve where $\theta=0$ and π. **[2]**

(ii) Without using a calculator, find the exact area of the region bounded by *C* and the line $y=4$. **[6]**

8 **(i)** Show that $\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}\equiv\frac{r+4}{r(r+1)(r+2)}$. **[2]**

(ii) Hence find $S_n=\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}$, giving your answer in the form $k-\mathrm{f}(n)$, where k is a constant. **[4]**

(iii) Use your answer to part (**ii**) to find $\sum_{r=2}^{n}\frac{r^2+3r-4}{r\left(r^2-1\right)(r+2)}$ **[3]**

9 The function g, with domain $\{x \in \mathbb{R} : 1 \le x \le 6\}$, has six of its function values given in the table below.

x	1	2	3	4	5	6
$\mathrm{g}(x)$	4	5	1	3	2	5

(i) Use the table to explain why g does not have an inverse function. **[2]**

(ii) Find $\mathrm{g}^3(3)$. Hence find the set of all positive integers n for which $\mathrm{g}^n(3) = 4$. **[3]**

It is also known that for $k = 1, 2, 3, 4, 5$

$$\mathrm{g}(x) = \mathrm{g}(k) + \left(\mathrm{g}(k+1) - \mathrm{g}(k)\right)(x-k) \qquad \text{for } k < x < k+1.$$

(iii) Evaluate $\mathrm{g}(1.5)$ and $\mathrm{g}(2.7)$. **[2]**

(iv) Sketch the graph of $y = \mathrm{g}(x)$. **[2]**

(v) Find the range of values of k for which the equation $\mathrm{g}(x) = k$ has four real distinct roots. **[1]**

10 Relative to the origin O, the position vectors of A and B are $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. It is also know that $\angle OAB$ has a size of $\frac{\pi}{6}$ radians.

(i) State an unit vector that is parallel to $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **[1]**

The point M is on the line AB such that $\angle MOA$ is equal to $\angle OAB$.

(ii) Using sine rule for the triangle OAM, or otherwise, find the length of AM in terms of $\mathbf{a}$. Hence find $\overrightarrow{AM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. **[3]**

It is given further that $\mathbf{a} \bullet \mathbf{b} = 0$.

(iii) Hence, by considering $\overrightarrow{AM}$ and another suitable vector, prove that the shortest distance from M to the line OA can be expressed as $\dfrac{|\mathbf{a}||\mathbf{b}|}{\sqrt{3}|\mathbf{b}-\mathbf{a}|}$ units. **[4]**

11 It is given that $\ln(1+y)=\tan^{-1}x$.

(i) Prove that $\left(1+x^2\right)\dfrac{dy}{dx}=1+y$. **[2]**

(ii) Find the first three non-zero terms in the Maclaurin's series for y. **[3]**

(iii) **(a)** Use your answer to part **(ii)** to give an approximation for $\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)\,dx$.

(b) Use your calculator to find an accurate value for $\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)dx$. How can the approximation in **(iii)(a)** be made better? **[4]**

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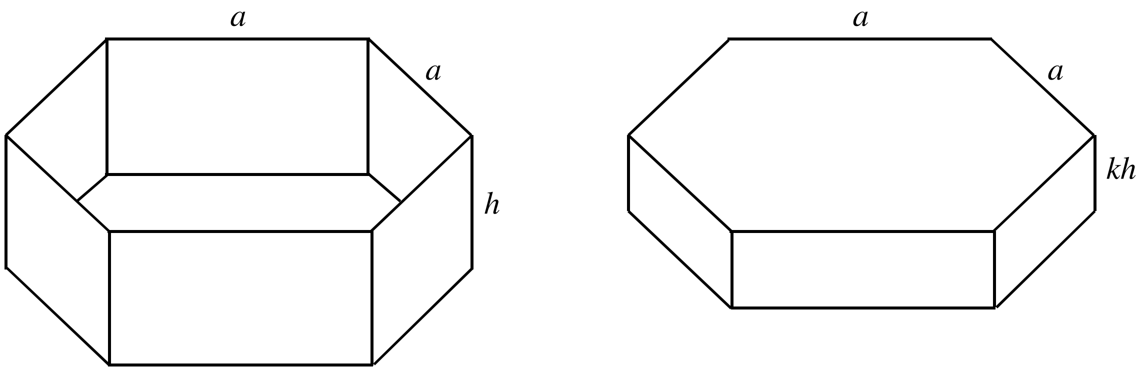

Box　　　　Lid

A jewel company makes a box of volume 100 cm^3 and of negligible thickness in the shape of a prism with the cross-sectional area of a regular hexagon. Each side of the hexagon is a cm and the height of the box is h cm. The lid of the box has depth kh cm where $0<k\le 1$ (see diagram).

(i) Show that the area of the hexagonal cross-section of the box can be expressed as $\dfrac{3\sqrt{3}}{2}a^2$. **[3]**

(ii) Given that the volume of the box can be expressed as $V=\dfrac{3\sqrt{3}}{2}a^2h$, use differentiation to find the value of a that gives the minimum total external surface area of the box and lid in terms of k. **[5]**

(iii) Find the ratio $\dfrac{h}{a}$ in this case, simplifying your answer. **[1]**

(iv) Find the values for which $\dfrac{h}{a}$ must lie. **[2]**

13 **(a)** A complex number $z = x+iy$, where $x, y \in \mathbb{R}$, is represented by the point P in an Argand diagram. The complex number $w = \dfrac{z-2i}{z+4}$, where $z \neq -4$, has its real part zero. By using $z = x+iy$, or otherwise, show that the locus of P in the Argand diagram is a circle. Hence find the equation of the circle, stating clearly its centre and radius. **[4]**

(b) The complex number z satisfies the relations $|z+2-i| \leq \sqrt{5}$ and

$\arg(z-1+2i) = \dfrac{3\pi}{4}$.

(i) Illustrate both of these relations on a single Argand diagram. **[4]**

(ii) Find the exact minimum and maximum value of $|z-1+2i|$. **[2]**

(iii) Find the minimum and maximum values of $\arg(w-1+2i)$, where w satisfies $|w+2-i| = \sqrt{5}$. **[3]**

End of Paper