H2 Topic 10 – Oscillations

Swings are one of a child's earliest interactions with free oscillations and resonance. Find out why the frequency of oscillations does not differ with the mass of the person on it, and how our parents, siblings and friends naturally supply extra bursts of energy into the swing system under resonant conditions.

Content
- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations, resonance

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations

(b) investigate the motion of an oscillator using experimental and graphical methods

(c) show an understanding of and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency

(d) recall and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion

(e) recognise and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$

(f) recognise and use the equations $v = v_0 \cos \omega t$ and $v = \pm\omega\sqrt{x_0^2 - x^2}$

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion

(h) describe the interchange between kinetic and potential energy during simple harmonic motion

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and to the importance of critical damping in applications such as a car suspension system

(j) describe practical examples of forced oscillations and resonance

(k) describe graphically how the amplitude of a forced oscillation changes with driving frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance

(l) show an appreciation that there are some circumstances in which resonance is useful, and other circumstances in which resonance should be avoided

10.0 Introduction

We have studied the kinematics of bodies in (i) straight line, (ii) projectile and (iii) circular motions.

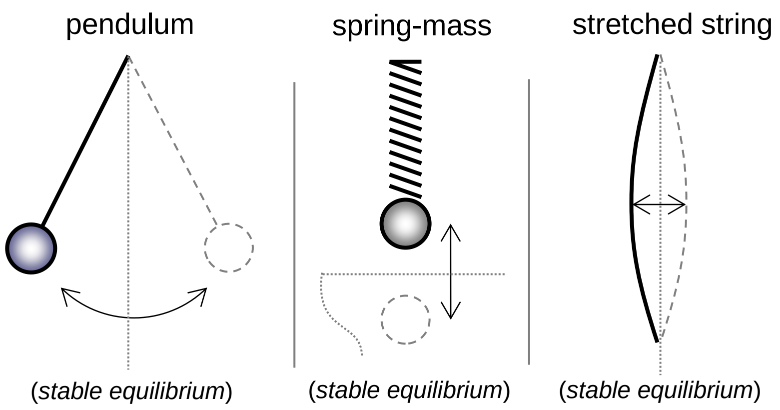
Oscillatory motions are another type of motion that can be often seen in everyday life: the swinging pendulum of a grandfather clock, the vibrations of a bus, and the waves on the strings of a guitar. Other non-visible examples include the electro-magnetic oscillations emitted by wireless equipment, gaseous atoms and molecules that propagate sound waves, and the lattice vibrations of atoms.

To better understand how these systems work, it is important we learn the mechanics of oscillations.

10.1 Oscillations

An **oscillation** is

a to-and-fro motion between two limits



If an oscillation repeats over time, it is a *periodic* motion.

An oscillation begins when a system is perturbed from a condition of stable equilibrium.

Force(s) tend to arise as a result of the initial disturbance, to bring the system *back* to the state of stable equilibrium – these are generally known as *restoring forces*.

The initial perturbation introduces an amount of *energy*. Within the system, a continuous and periodic interchange of kinetic and potential energy takes place – there is usually maximal amounts of potential energy where/when the kinetic energy is at a minimal, and vice versa.

10.1.1 Free Oscillations

Free oscillations are oscillations with

constant amplitude and
without energy loss or gain
as there is no driving or resistive forces acting on it.

Free oscillations occur when a system is displaced from its state of stable equilibrium and is then allowed to move or respond without restraint.

There is no external force applied or resisting its motion.

Therefore there is no gain or loss of total energy of the oscillating system.

The natural frequency of a simple pendulum is dependent only on its length – so swings (look at it from the side) typically exhibit the same frequency regardless of the weight of the child.

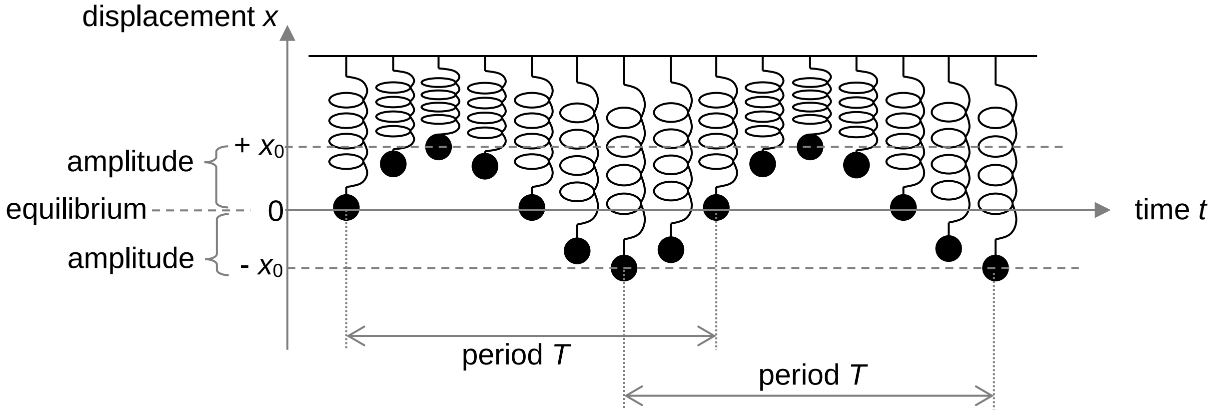
When a system oscillates freely, it does so at its *natural frequency*, which depends on some physical factors such as its dimensions, mass or elasticity.

Natural frequency is the

frequency at which a body will vibrate when there is no driving or resistive forces acting on it.

10.1.2 Quantities of Oscillation

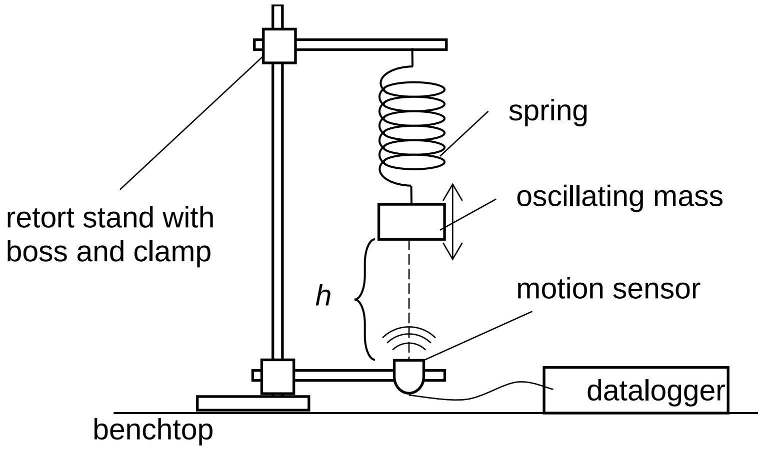
We refer to a vertical spring-mass system attached to a ceiling to help familiarize ourselves with the various quantities used to describe an oscillation.



equilibrium position	position where no net force acts on the oscillating mass
displacement x	distance in a specified direction from equilibrium position of oscillating mass
amplitude x_0	maximum displacement from equilibrium position
period T	time taken for one complete oscillation of the oscillating mass
frequency f	number of complete oscillations per unit time. (unit: Hertz, Hz) $\boxed{f = \frac{1}{T}}$
phase ϕ	an angular measure (in either degrees or radians) of the fraction of a cycle completed by the oscillating mass i.e. completion of one complete cycle corresponds to a phase $\phi = 360°$ or 2π rad
phase difference $\Delta\phi$	measure of how much an oscillation is out of step with another oscillation if two oscillations are in-phase: $\Delta\phi = 0$ out-of-phase: $\Delta\phi \neq 0$ anti-phase: $\Delta\phi = \pi$ OR $180°$
angular frequency ω	defined as the product of 2π and frequency. unit: radian s^{-1} $\boxed{\omega = 2\pi f}$ (i) is effectively rate of change of phase of oscillating mass $\omega = \frac{d\phi}{dt}$ (ii) shares same symbol ω as circular motion's angular velocity (rate of change of angular displacement $\omega = \frac{d\theta}{dt}$) but are different quantities!

10.1.3 Experimental Investigation of a Vertical Oscillator

The oscillation of a vertical spring-mass system can be investigated using a motion sensor connected to a data logger.



A motion sensor can measure position, velocity and acceleration of moving objects by emitting ultrasonic pulses and determining the time lag of pulses reflected back to the sensor.

Because it needs a surface to reflect the ultrasound, it is not suitable if (i) the moving object is small (like a metal ball bearing) or (ii) soft (high absorbance of sound waves).

- Set up the experiment as shown above.
- Displace the mass vertically downwards and release it to set into vertical oscillations.
- Measure the variation with time of height *h* using a motion sensor connected to a data-logger
- Start the data-logger when the oscillations are steady.

equilibrium position	h_{eqm} =	10.0 cm
amplitude	h_0 =	2.0 cm
period	T =	3.2 s
frequency	f =	0.313 Hz
angular frequency	ω =	1.96 rad s^{-1}
displacement when t = 2.9 s	$h(2.9)$ =	1.0 cm
equation for position *h* in cm*	$h(t)$ =	$10-2\cos\left(\frac{2\pi}{3.2}t-\frac{18\pi}{16}\right)$

**Start from* $h=\cos\theta=\cos(\omega t)$. *At* t = 1.8 s, *the function is a negative cosine of amplitude 2 spanning till t = 5 s, so* $h=-2\cos\left(\left(\frac{2\pi}{T}\right)(t-1.8)\right)$. *Translate vertically up by 10 to yield equation.*

10.2 Simple Harmonic Motion (SHM)

Simple harmonic motion is a type of oscillatory motion where the

acceleration is

[magnitude] directly proportional to displacement from the equilibrium position

and

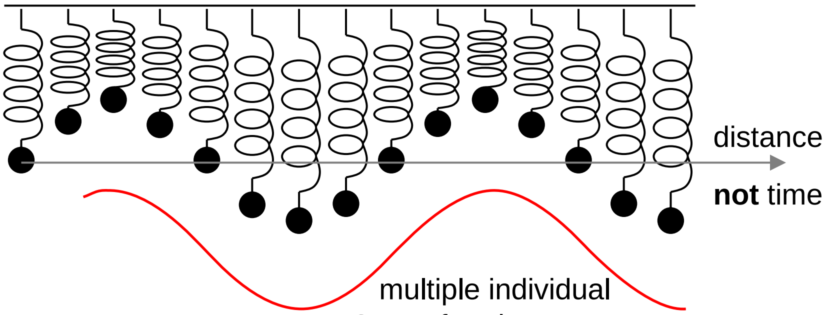
[direction] directed opposite to displacement

$$a = -\omega^2 x$$

a : acceleration (m s^{-1})
ω : angular frequency (rad s^{-1})
x : displacement from equilibrium position (m)

Note:
- ω^2 is a *constant of proportionality* between a and x
- negative sign indicates that a and x are in the *opposite directions*

SHM is mathematically the simplest case of free oscillations. Yet by itself, it allows us to gain a deep understanding of periodic motion and provides a basis to describe more complicated oscillations.

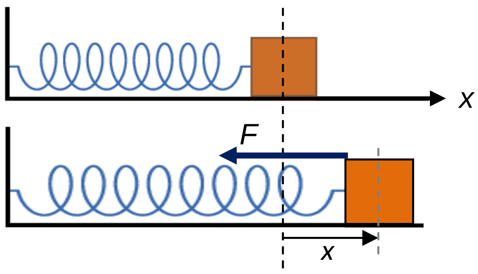


SHM typically describes the motion of a single body. When multiple bodies (adjacent to each other in space) move individually with their own SHMs, we can witness *progressive* or *stationary* waves when we "zoom out" of the individual view and regard the many-bodies as a bulk system.

10.2.1 Horizontal Spring-Mass Systems

A block of mass m rests on a smooth surface and is attached to the end of a light spring of force constant k.

From x = 0 the equilibrium position (i.e. when spring is neither compressed nor stretched), the block is displaced by a distance x to the right.



By Newton's 2nd law of motion:

$$F_{\text{net}} = ma$$

$$k|x| = ma$$

$$a = \left(\frac{k}{m}\right)|x| = -\left(\frac{k}{m}\right)x$$

since a and x are opposite in directions

Compare this equation with $a = -\omega^2 x$

to get $\omega = \sqrt{\dfrac{k}{m}}$

Since the relationship between a and x is in the form of the defining equation for SHM, the oscillations of a horizontal spring-mass system is simple harmonic with

- angular frequency $\omega = \sqrt{\dfrac{k}{m}}$
- natural frequency $f_{\text{natural}} = \dfrac{1}{2\pi}\omega = \dfrac{1}{2\pi}\sqrt{\dfrac{k}{m}}$
- period $T = 2\pi\sqrt{\dfrac{m}{k}}$

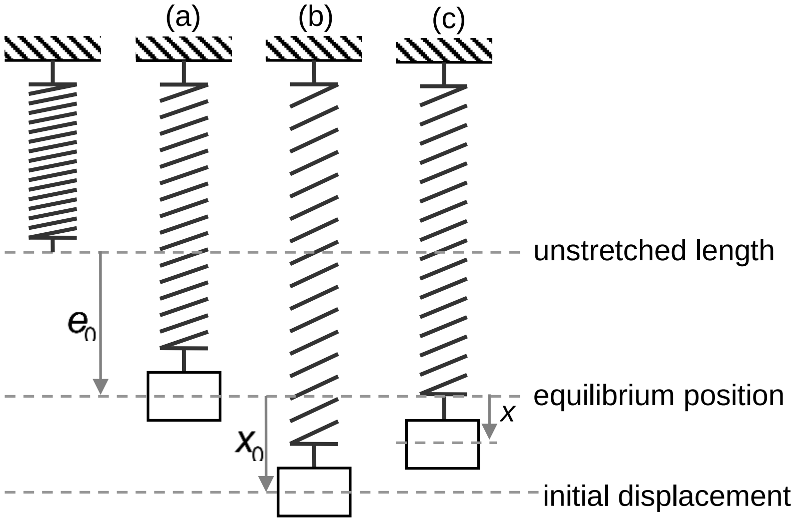
10.2.2 Vertical Spring-Mass Systems

In (a), spring is stretched by e_0 by the weight of the mass. Since it is at equilibrium, there is no resultant force:

$$F_{\text{spring}} = W$$
$$ke_0 = mg$$

In (b), the mass is pulled down and released to set into oscillation about equilibrium with amplitude of x_0.

In (c) which is at a displacement of x below the equilibrium



By Newton's 2nd law of motion:

$$F_{\text{net}} = ma$$
$$F_{\text{spring}} - mg = ma$$
$$k\left(e_0 + |x|\right) - mg = ma$$
$$mg + k|x| - mg = ma$$
$$a = \left(\frac{k}{m}\right)|x|$$

$a = -\left(\frac{k}{m}\right)x$ since a and x are opposite in directions

Compare with $a = -\omega^2 x$

to get $\omega = \sqrt{\frac{k}{m}}$

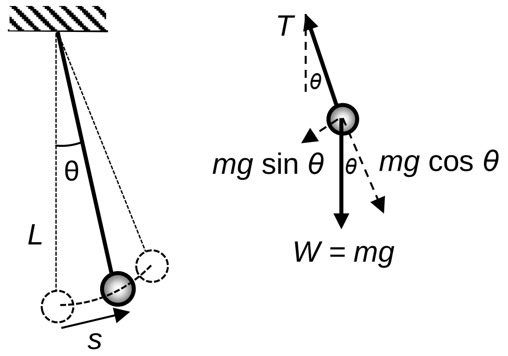
Since the relationship between a and x is in the form of the defining equation for SHM, the oscillations of a horizontal spring-mass system is simple harmonic with

- angular frequency $\omega = \sqrt{\frac{k}{m}}$
- natural frequency $f_{\text{natural}} = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$
- period $T = 2\pi\sqrt{\frac{m}{k}}$

Quick Check: Does a bouncing ball exhibit SHM?

No. Its acceleration is constant whenever it is displaced from its equilibrium position (ground). Its acceleration is not directly proportional to its displacement from its equilibrium position.

10.2.3 Simple Pendulum



A pendulum bob of mass *m* is hung from a ceiling by a light inextensible string of length *L*.

The bob is displaced initially by an angle *θ* from the vertical.

The component of the weight parallel to the string is balanced by tension $T = mg\cos\theta$.

$mg\sin\theta$ remains as the net force acting on the pendulum bob. The restoring force is the component of *W* tangential to pendulum's motion.

by Newton's 2nd law of motion:

$$F_{\text{net}} = ma$$
$$mg\sin\theta = ma$$
$$a = g\sin\theta$$

for small angles $\sin\theta \approx \theta = \dfrac{s}{L}$

$$a = \left(\frac{g}{L}\right)s$$

$a = -\left(\dfrac{g}{L}\right)s$ since a and s are opposite in directions

Compare with $a = -\omega^2 x$

to get $\omega = \sqrt{\dfrac{g}{L}}$

Since the relationship between *a* and *s* is in the form of the defining equation for SHM, the oscillations of a horizontal spring-mass system is simple harmonic with

- angular frequency $\omega = \sqrt{\dfrac{g}{L}}$
- natural frequency $f_{\text{natural}} = \dfrac{1}{2\pi}\omega = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{L}}$
- period $T = 2\pi\sqrt{\dfrac{L}{g}}$

(i) The period or frequency of a simple pendulum near surface of Earth depends only on the length of the pendulum.

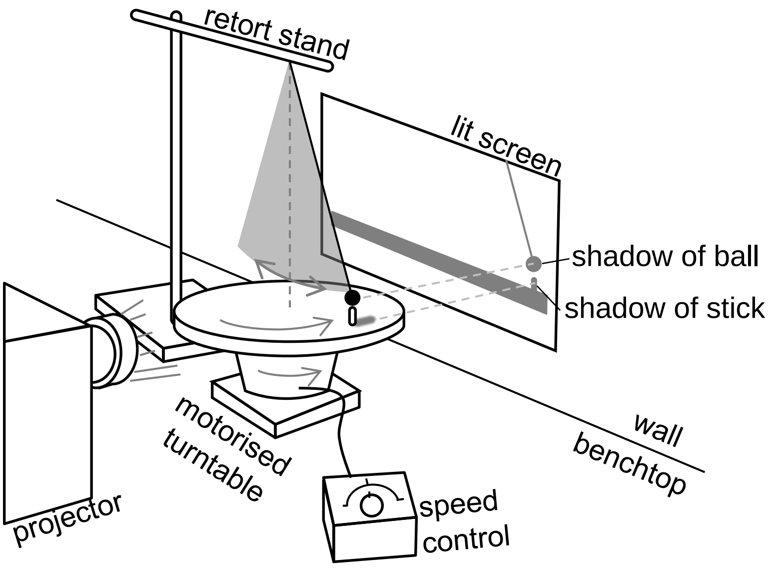
(ii) Note the small angle approximation used; this is why pendulum practicals insist on small oscillations.

10.2.4 Comparing Simple Harmonic Motion with Circular Motion

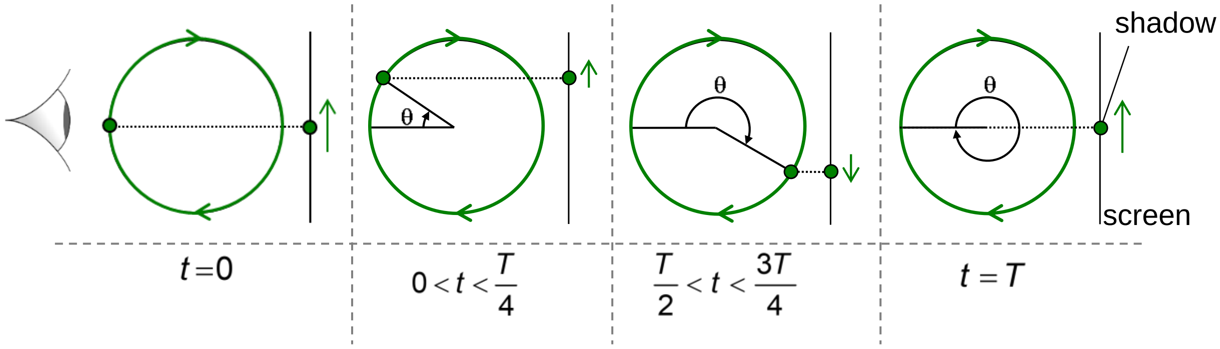
We can visualise SHM as a 2D projection of uniform circular motion.

In the setup (right), a motorized circular turntable with a stick attached to its rim rotates with an adjustable angular *speed* ω of circular motion. A simple pendulum oscillates directly above the turntable.

A projector provides a collimated light beam and casts the shadows of the ball and the pendulum on a screen. The angular speed of the turntable is adjusted such that the two shadows move in sync, one above the other.



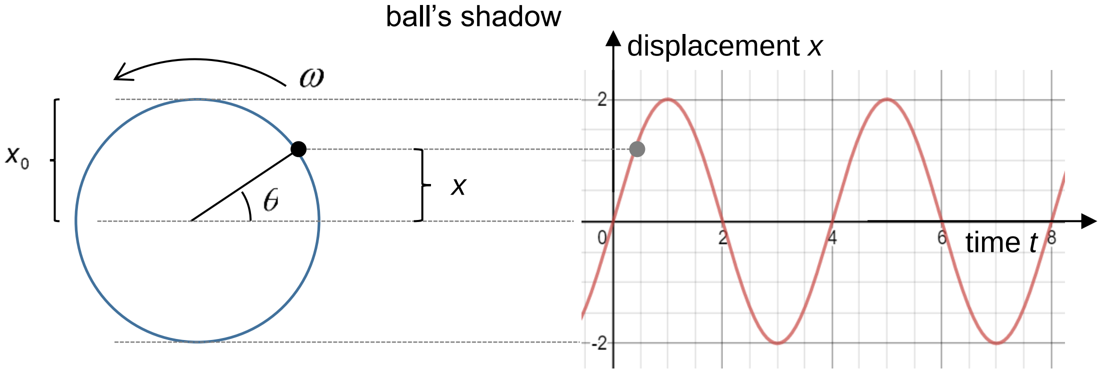
While the ball performs uniform circular motion, its shadow exhibits SHM.



10.3 Kinematics of Simple Harmonic Motion

The variation with time of displacement of a body under SHM is sinusoidal (sine or cosine function). In this scenario,

- at time t = 0, the body is at the equilibrium point (x = 0).
- some time t later, the displacement of the body is x and the phase angle is $\theta = \omega t$.



displacement *x*	velocity *v*	acceleration *a*
$x = x_0 \sin\theta$ $= x_0 \sin(\omega t)$	$v = \dfrac{dx}{dt}$ $= (\omega x_0)\cos(\omega t)$	$a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$ $= (-\omega^2 x_0)\sin(\omega t)$ $= -\omega^2 x$

We say that $x = x_0 \sin(\omega t)$ is a solution to the defining equation of SHM. Mathematically, it means that $x = x_0 \sin(\omega t)$ can always fulfil the condition of $\dfrac{d^2x}{dt^2} = -kx$ where the proportionality constant is $k = \omega^2$.[If time t = 0 when $x = x_0$, then $x = x_0 \cos(\omega t)$ would have been the defining solution]

There will be times when it would be useful to eliminate the variable t from the equations. Using the trigonometric identity

$$\sin^2 wt + \cos^2 wt = 1$$

$$\left(\frac{x}{x_0}\right)^2 + \left(\frac{v}{\omega x_0}\right)^2 = 1$$

$$\omega^2 x^2 + v^2 = \omega^2 x_0^2$$

Hence the velocity of a body under SHM is given by

$$\boxed{v = \pm\omega\sqrt{x_o^2 - x^2}}$$

Thus, the kinematic relationships of a S.H.M. can be summarized as follows:

	variation with x	variation with t	
		starting at $\boldsymbol{x = 0}$ when $t = 0$	starting at $\boldsymbol{x = x_0}$ when $t = 0$
x		$x = x_0 \sin(\omega t)$	$x = x_0 \cos(\omega t)$
v	$v = \pm\omega\sqrt{x_0^2 - x^2}$	$v = \omega x_0 \cos(\omega t)$	$v = -\omega x_0 \sin(\omega t)$
a	$a = -\omega^2 x$	$a = -\omega^2 x_0 \sin(\omega t)$	$a = -\omega^2 x_0 \cos(\omega t)$

Quick Check: How would the variation of x, v and a with respect to t differ when oscillator starts moving in the negative direction?

Example 1 (a)

An oscillation has amplitude 5.0 mm and a period T of 0.180 s. At time t = 0, its displacement is at the equilibrium position and travelling downwards.

Determine for the oscillation, its

(i) angular frequency,

(ii) maximum velocity and

(iii) maximum acceleration.

(i)
$$\omega = 2\pi f = \frac{2\pi}{T} = \frac{2\pi}{0.18} = 34.9 \text{ rad s}^{-1}$$

(ii)
$$v = \pm\omega\sqrt{x_0^2 - x^2}$$
$$|v_0| = \omega\sqrt{x_0^2 - 0} = |\omega x_0| = \left(\frac{2\pi}{T}\right)x_0 = \left(\frac{2\pi}{0.18}\right)\left(5\times10^{-3}\right) = 0.175 \text{ m s}^{-1}$$

(iii)
$$a = -\omega^2 x$$
$$a_0 = \left|\omega^2 x_0\right| = \left(\frac{2\pi}{T}\right)^2 x_0 = \left(\frac{2\pi}{0.18}\right)^2\left(5\times10^{-3}\right) = 6.09 \text{ m s}^{-2}$$

Example 1 (b),(c)

Write an expression for the following relationships, taking upwards as positive.

(i) displacement x with time t,

(ii) velocity v with time t, and

(iii) acceleration a with time t.

(i) x

$$= -x_0 \sin(\omega t)$$

$$= -(0.0050)\sin\left(\frac{2\pi}{0.180}t\right)$$

(ii) v

$$= -\omega x_0 \cos(\omega t)$$

$$= (-0.175)\cos\left(\frac{2\pi}{0.180}t\right)$$

(iii) a

$$= \omega^2 x_0 \sin(\omega t)$$

$$= (6.09)\sin\left(\frac{2\pi}{0.180}t\right)$$

(c) Determine the displacement, velocity and acceleration when

(i) t = 0.030 s, and

(ii) t = 0.210 s

(i)

$$x = -0.0050\sin\left(\frac{2\pi}{0.180}(0.030)\right) = -0.00433 \text{ m}$$

$$v = -0.175\cos\left(\frac{2\pi}{0.180}(0.030)\right) = -0.0875 \text{ m s}^{-1}$$

$$a = 6.09\sin\left(\frac{2\pi}{0.180}(0.030)\right) = 5.27 \text{ m s}^{-2}$$

(ii)

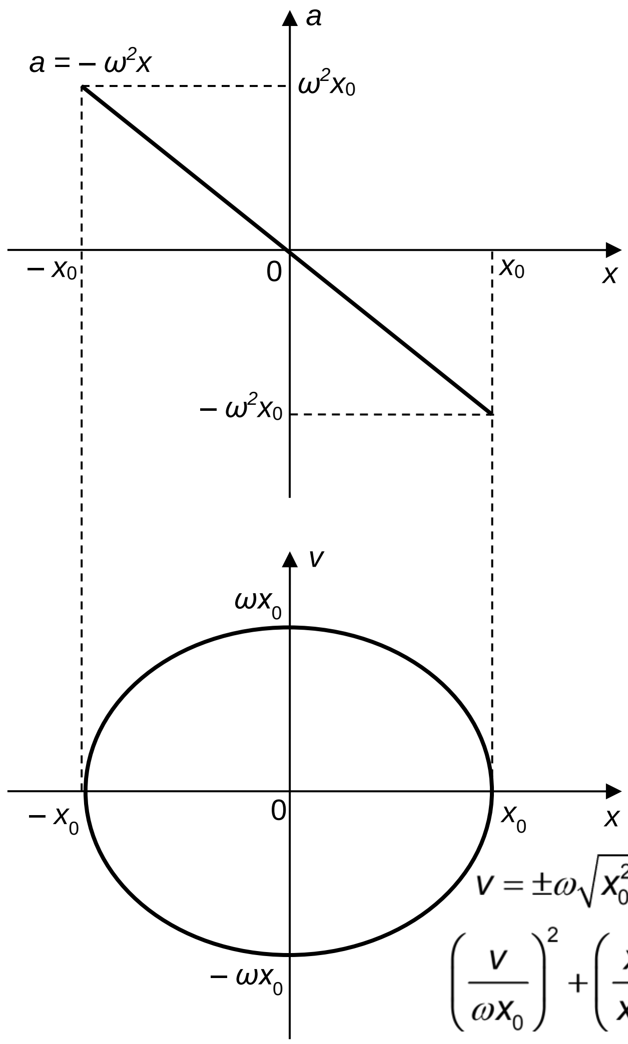
$$x = -0.0050\sin\left(\frac{2\pi}{0.180}(0.210)\right) = -0.00433 \text{ m}$$

$$v = -0.175\cos\left(\frac{2\pi}{0.180}(0.210)\right) = -0.0875 \text{ m s}^{-1}$$

$$a = 6.09\sin\left(\frac{2\pi}{0.180}(0.210)\right) = 5.27 \text{ m s}^{-2}$$

10.3.1 Graphical Representation of Variation of Quantities with Displacement

Acceleration a and velocity v against the same displacement axes will yield:



$$v = \pm\omega\sqrt{x_0^2 - x^2}$$

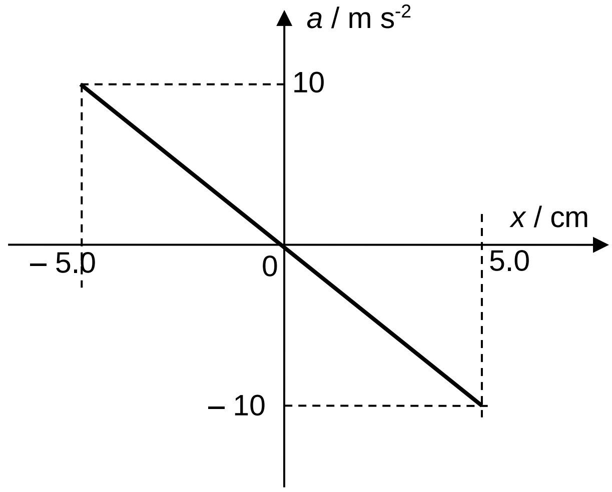
$$\left(\frac{v}{\omega x_0}\right)^2 + \left(\frac{x}{x_0}\right)^2 = 1$$

Note:
When sketching graphs involving oscillations, **do not extend beyond the amplitudes** (i.e. x_0 and $-x_0$).

Example 2

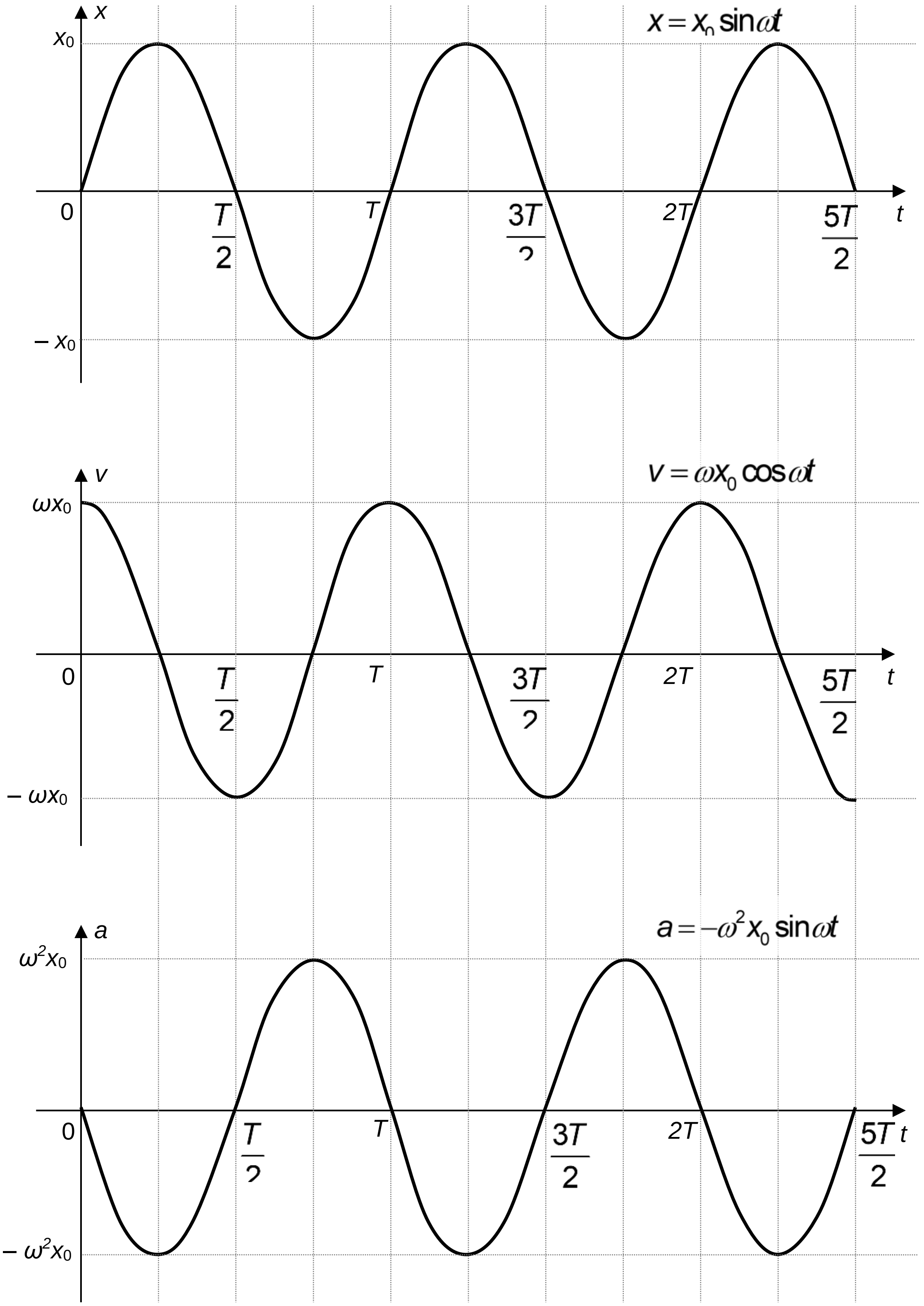
The graph shows the variation with displacement x of acceleration a of a body exhibiting simple harmonic motion. What is the amplitude and period of motion?

	amplitude / cm	period / s
A	5.0	0.44
B	5.0	14
C	10	0.44
D	10	14



10.3.2 Graphical Representation of Variation with Time

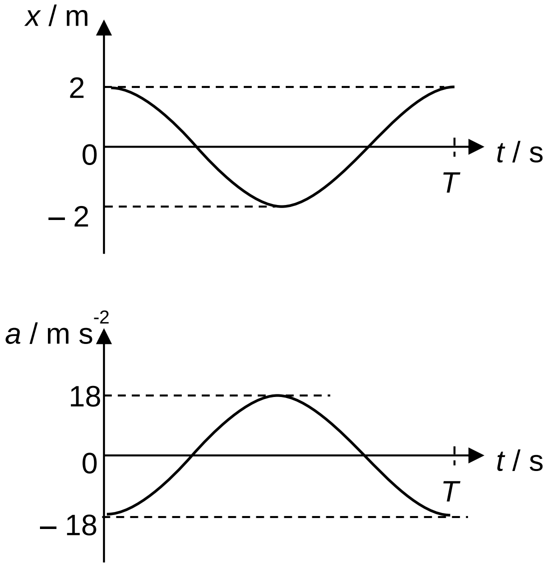
The variations with time for displacement x, velocity v and acceleration a:



<table>
<tr><td rowspan="3">phase difference</td><td>x-t</td><td>v-t</td><td>a-t</td><td>x-t</td></tr>
<tr><td colspan="2">$\Delta\phi = \frac{\pi}{2}$</td><td colspan="2">$\Delta\phi = \pi$</td></tr>
<tr><td></td><td colspan="2">$\Delta\phi = \frac{\pi}{2}$</td><td></td></tr>
</table>

Example 3

The variations of displacement x and the acceleration a with time t of a body performing simple harmonic motion are as shown. Determine the period of the oscillation T.



Solution

at $t = 0$,

$x = 2.0$ m

$a = 18$ m s^{-2}

$$a = -\omega^2 x$$

$$-18 = -\left(\frac{2\pi}{T}\right)^2 (2.0)$$

$$T = 2.09 \text{ s}$$

Example 4

The diagram shows the displacement-time graph for a body performing simple harmonic motion.

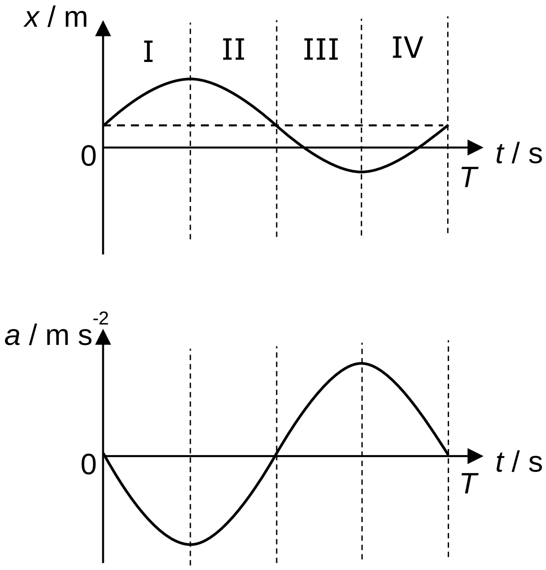
(a) List the regions in which the velocity and acceleration are in the same directions.

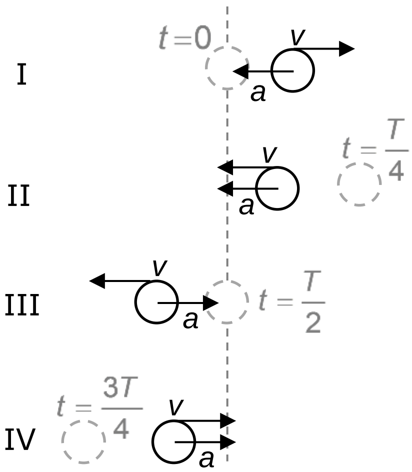
(b) List the regions in which the velocity and acceleration are in the opposite directions.

Solution

(a) II, IV

(b) I, III





Example 5

A simple harmonic oscillator has a time period of 10 seconds. Which equation relates its acceleration *a* and displacement *x*?

A $a = -\left(\frac{20}{2\pi}\right)^2 x$

B $a = (20\pi)^2 x$

C $a = \left(\frac{20}{2\pi}\right)^2 x$

D $a = -\left(\frac{2\pi}{10}\right)^2 x$

Solution

$$a = -\omega^2 x = -\left(\frac{2\pi}{T}\right)^2 x$$

$$= -\left(\frac{2\pi}{10}\right)^2 x$$

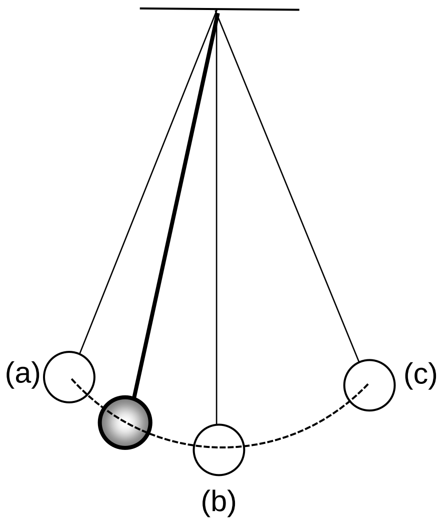
10.4 Energy of Oscillations

Considering the energies of an SHM can help to simplify the analysis of the oscillatory motion.

When a body is performing free oscillations, there is continuous and periodic interchange of its kinetic energy and potential energies. The body also experiences no energy loss or gain due to the absence of resistive or driving forces acting on the system.

For a pendulum bob undergoing free oscillations:

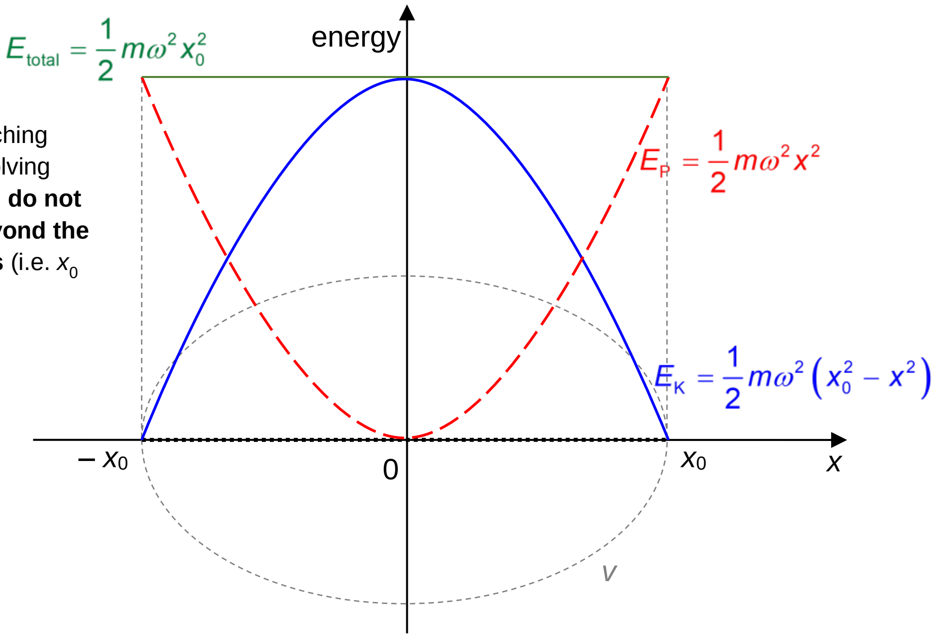
position	kinetic energy	gravitational potential energy
amplitude (a)	0	max
moving from (a) to (b)	GPE converted to KE	
equilibrium position (b)	max	0
moving from (b) to (c)	KE converted to GPE	
amplitude (c)	0	max



Note: There are similar energy conversions for an oscillating vertical spring-mass system. Depending on how the question is set up, we may need to sum up gravitational potential energy and elastic potential energy for total potential energy.

10.4.1 Variation of Energy with Displacement

kinetic energy with displacement	potential energy with displacement
$E_K = \frac{1}{2}mv^2$ $= \frac{1}{2}m\left[\pm\omega\sqrt{x_0^2 - x^2}\right]^2$ $= \frac{1}{2}m\omega^2\left(x_0^2 - x^2\right)$	designate total potential energy $E_P = 0$ at equilibrium such that all the KE is converted to PE at amplitude positions: $E_{total} = E_K + E_P$ $= E_{K,max} + 0$ (when KE is a max, PE is zero) $E_{total} = E_{K,\,max}$ $= \frac{1}{2}m\omega^2 x_o^2$ then $E_P = E_{total} - E_K$ $= \frac{1}{2}m\omega^2 x_0^2 - \frac{1}{2}m\omega^2\left(x_0^2 - x^2\right)$ $= \frac{1}{2}m\omega^2 x^2$

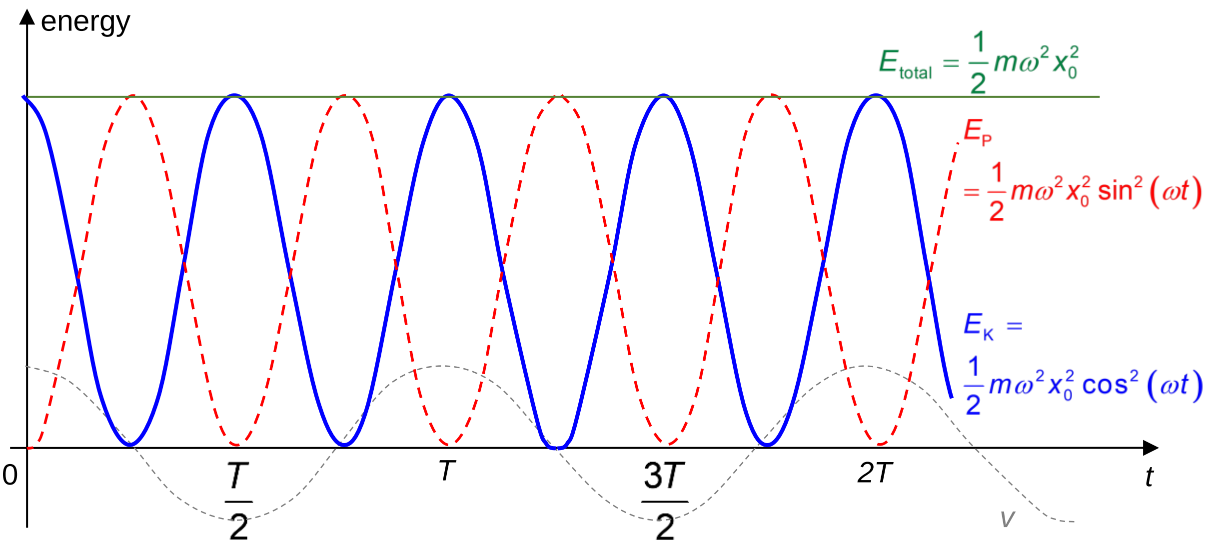


Note:
When sketching graphs involving oscillations, **do not extend beyond the amplitudes** (i.e. x_0 and $-x_0$).

10.4.2 Variation of Energy with Time

kinetic energy with time	potential energy with time
$E_K = \frac{1}{2}mv^2$ $= \frac{1}{2}m\left[\omega x_0 \cos\omega t\right]^2$ $= \frac{1}{2}m\omega^2 x_0^2 \cos^2(\omega t)$	$E_P = E_{total} - E_K$ $= \frac{1}{2}m\omega^2 x_0^2 - \frac{1}{2}m\omega^2 x_0^2 \cos^2(\omega t)$ $= \frac{1}{2}m\omega^2 x_0^2\left[1-\cos^2(\omega t)\right]$ $= \frac{1}{2}m\omega^2 x_0^2 \sin^2(\omega t)$

The variation of total energy E_{total}, total potential energy E_P and kinetic energy E_K with time:



Note: (i) There are two complete cycles of energy conversion in one period of oscillation.
(ii) The total energy of an oscillating system is directly proportional to the square of amplitude $E_{total} = \left[\frac{1}{2}m\omega^2\right]x_0^2$

Example 6
A mass of 8.0 g oscillate in simple harmonic motion with an amplitude of 5.0 mm at a frequency of 40 Hz. Find the total energy of this simple harmonic oscillator.

Solution

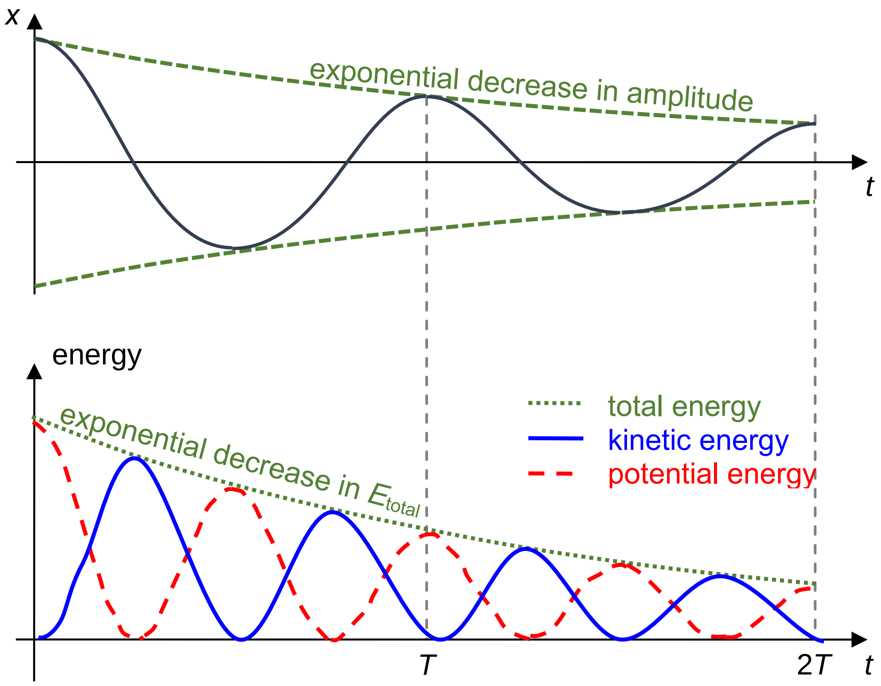
$$E_{total} = E_{K,\,max} = \frac{1}{2}mv_{max}^2$$
$$= \frac{1}{2}m\omega^2 x_0^2$$
$$= \frac{1}{2}m(2\pi f)^2 x_0^2$$
$$= \frac{1}{2}(8.0\times10^{-3})(2\pi(40))^2(5.0\times10^{-3})^2$$
$$= 6.31\text{ mJ}$$

10.5 Damped Oscillations

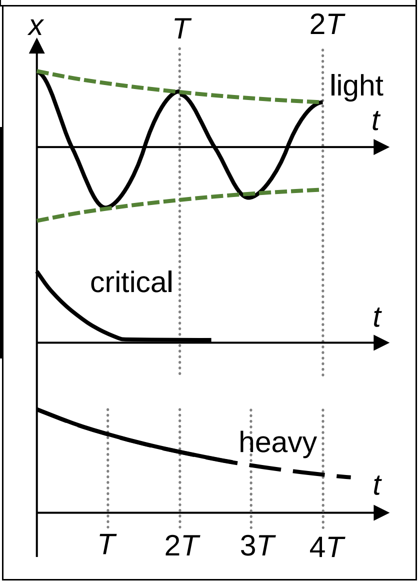
> **Damped oscillations** are oscillations in which
>
> the <u>amplitude decreases</u> with time
> as a result of <u>dissipative forces</u> that reduces the total energy of the oscillations.

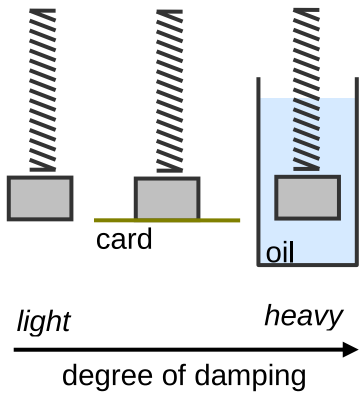
Unlike *undamped* oscillations (free oscillations), a body with damped oscillations will lose energy until it eventually stops.

Work is done against the resistive forces in a damped oscillation. The *degree of damping* will determine the rate of energy loss from the oscillator.



The *degree of damping* refers to the amount of resistance to motion the oscillator is subjected to:

degree of damping	description	variation with time
Light	• Undergoes a number of complete oscillations • Amplitude decreases exponentially with time	
Critical	• No oscillation takes place • Shortest time needed for the oscillator to come to a state of equilibrium	
Heavy	• No oscillation takes place • Displacement decreases exponentially with time • Time needed for the oscillator to come to a state of equilibrium is longer than critical damping	

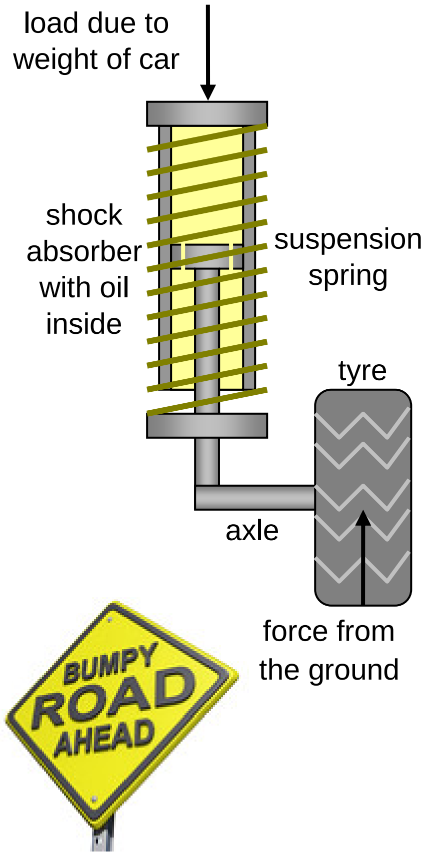


10.5.1 Application of Damped Oscillations: Car suspension system

A car suspension supports the chassis (body) of the car onto the axles of the car wheels. A good suspension system should

1. *partially absorb the impact of a bump* in the road for the comfort of the passengers
2. *return the car to equilibrium quickly* so that it is ready to respond to other bumps on the road.

The desired degree of damping is near critical damping.



other degrees of damping	effects
none / too light	car will bounce after a bump and continue oscillating for a long time uncomfortable for passengers
too heavy	car takes a long time to return to equilibrium cannot respond to further bumps

Modern car suspensions consist of a spring and a shock absorber, which typically uses oil as a damper.

10.6 Forced Oscillations

Forced oscillations are the opposite of free oscillations.

In free oscillations, a once-off input of energy initiates the periodic oscillations at the system's natural frequency.

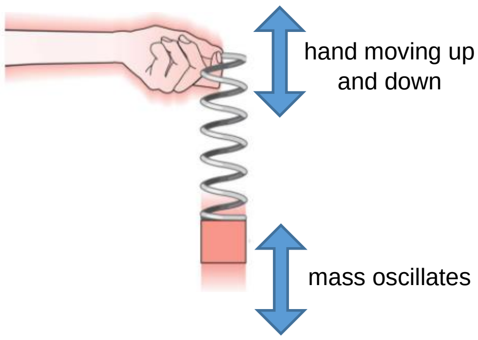
In contrast, energy is continuously supplied and the system is *forced* to vibrate at the frequency of the external *driver*.

Forced oscillations are oscillations where there is a

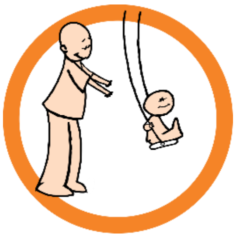
continuous input of energy
by an external periodic force that
maintains the oscillation amplitude.

As a simple example, a hand moving up and down repeatedly while holding onto a spring with a mass attached to it will cause the mass to oscillate.

The hand is providing an external periodic force (resulting in a continuous energy input) to the spring-mass system.



Some other examples of forced oscillations:



- A child on a swing being pushed by a parent standing at one end.
- A car driven over a series of evenly spaced humps at a constant speed.
- The body of a washing machine vibrating due to the spinning of its drum.

In an ideal situation in which there is no damping, the total energy of the oscillator of a forced oscillation keeps adding up and so the total energy (Correspondingly the amplitude x_0 since $E_{total} = \frac{1}{2}m\omega^2 x_0^2$) approaches infinity.

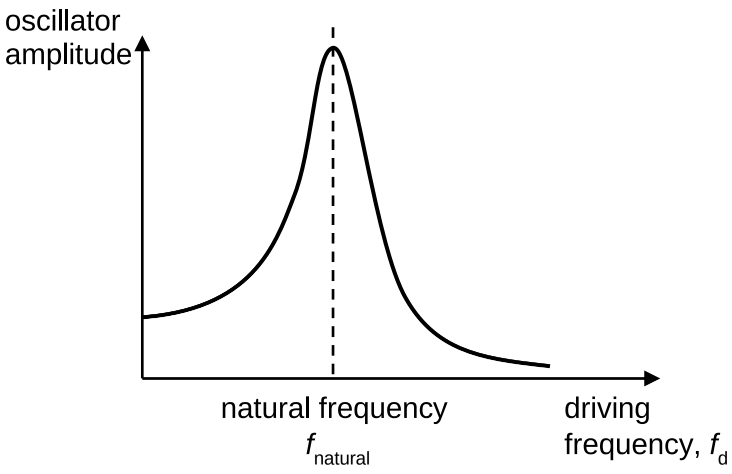
However, with damping, the total energy (and amplitude) of a body initially at rest, will increase until a steady state of constant amplitude. This is when

rate of transfer of energy from external driver to oscillating system	=	rate of loss of energy from oscillating system as work done against resistive forces

The *driving frequency* provided by the external periodic force determines the frequency that the system oscillates at. The driving frequency need not be the same as the natural frequency.

10.6.1 Frequency Response of Forced Oscillations

A *frequency response graph* shows how the amplitude of oscillations vary with the frequency of the external periodic force. A typical frequency response of a lightly damped oscillator is as shown.



As the frequency of the external driver increases from zero, the amplitude of the forced oscillation increases.

The amplitude reaches a maximum when the external driving frequency matches that of the natural frequency of the system.

Beyond that point, the amplitude decreases with further increases in driving frequency.

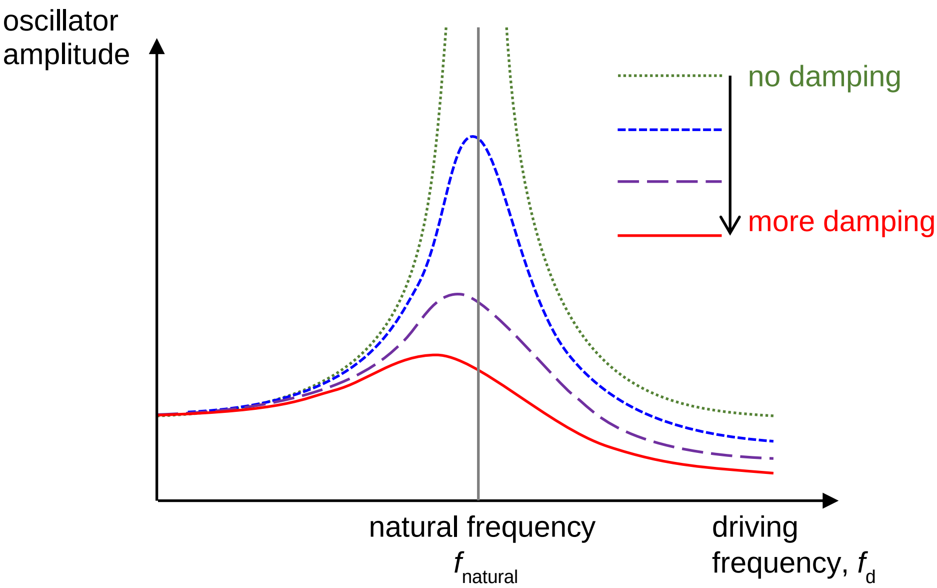
When external driving frequency matches the natural frequency of the system, *resonance* occurs.

10.6.2 Resonance

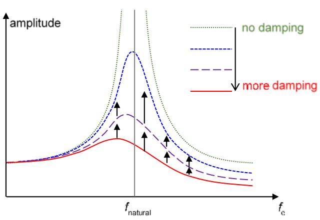
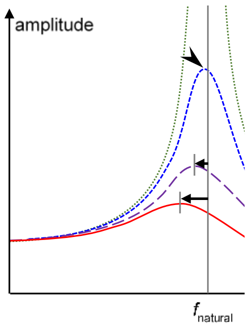
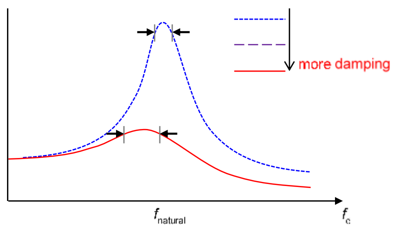
Resonance is a phenomenon in which

the amplitude of an oscillatory motion is at maximum because
there is maximum rate of transfer of energy
from the external driver to the oscillating system.

This occurs when the
driving frequency of external periodic force equals to natural frequency of the system.



The degree of damping determines the *frequency response and sharpness* of the resonance.

degree of damping	amplitude	frequency response	sharpness
lighter damping	consistently higher	Maximum amplitude very near (or equal) natural frequency	sharper
heavier damping	consistently lower	Maximum amplitude shifts lower than natural frequency (period T is *longer*)	flatter
			

10.6.3 Practical Uses of Resonance

(a) *Microwave Cooking*

The driving frequency of the microwave matches the natural frequency of vibration of water molecules. When food containing water molecules is placed in the oven, there is maximum transfer of energy to the water molecules which is set into resonance. This results in the water molecules oscillating with maximum amplitude and greater vibrational kinetic energy and is indicated with an increase in temperature. The heated water transfer heat to other parts of the food.

(b) *FM radio reception*

The driving frequency of the radio tuner is adjusted to match the natural frequency of the broadcast in order to hear it. There is maximum transfer of energy and resonance occurs. Resonance amplifies the signals contained in the selected frequency while the radio waves of other frequencies are diminished.

(c) *Magnetic Resonance Imaging*

A strong electromagnetic field is made to operate at driving frequencies which matches that of atomic nuclei of the molecules in a human body. As a result, maximum energy is transferred to the body by the field. When resonance occurs, the atomic nuclei of the molecules oscillate with maximum amplitude. By analyzing the pattern of energy absorption, a computer-generated image can be produced.

(d) *Musical instruments*

The external driving force is applied at a frequency that matches the natural frequencies of the strings, or columns of air or physical shapes/conditions of string, wind and percussion instruments respectively. Maximum energy is transferred to instruments which in turn oscillate with maximum amplitude.

10.6.4 Destructive Resonance

Buildings and Structures

All physical structures, by virtue of their shape, dimensions, material and other physical properties, have natural frequencies. The London Millennium Footbridge was closed after 2 days of opening because the frequency of motion from pedestrians matched the natural frequency of the bridge, causing it to sway with alarming amplitude. The bridge was closed for 2 years of remediation works.

During earthquakes, buildings are forced to oscillate by the seismic waves. If the frequency of the seismic waves approaches the natural frequency of the buildings, resonance occurs. This results in large amplitude oscillations of the buildings which increases the risks of collapse.

Wine glasses, have natural frequencies too. You can hear the natural frequency by lightly tapping on a glass, or running a damp finger around the lip of the glass to set it into oscillation.

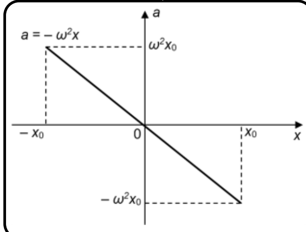
10.7 Summary

oscillations
complete to-and-fro motion between two limits

initiated by one perturbation → **free oscillations**
constant amplitude w/o energy loss or gain as no driving or resistive force acts on system

driven by external driver → **forced oscillations**
continuous input of energy by external periodic force that maintains oscillation amplitude

simplest type to study → **simple harmonic motion**
oscillatory motion where acceleration is directly proportional to displacement from equilibrium position and directed opposite to displacement
$a = -\omega^2 x$



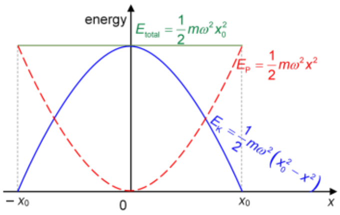
free oscillations – will vibrate at → **natural frequency**
frequency that system vibrates at in absence of driving or resistive forces

forced oscillations – if external driving frequency matches → **resonance**
driving frequency of external periodic force equals to natural frequency of the system, resulting amplitude is maximum as there is maximum rate of transfer of energy from the external driver to the oscillating system

cons
structural failures

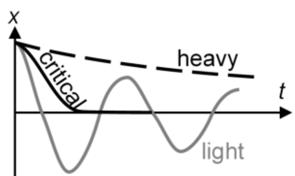
pros
microwave, MRI, radio broadcast

energy ∝ amplitude²



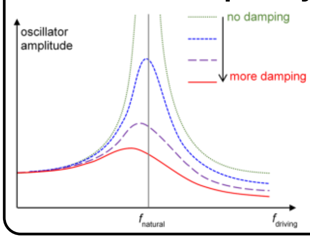
add resistive force(s) → **damped oscillations**
amplitude decreases with time as a result of dissipative forces that reduces the total energy of the oscillations.

variation with time



critical damping
no oscillations occur and displacement is brought to zero in shortest possible time

frequency response



heavier damping:
lower amplitude,
less-sharp peak and
lower resonant frequency

solution to equation → **displacement**
$x = x_0 \sin(\omega t)$

$\frac{d}{dt}$

velocity
$v = (\omega x_0)\cos(\omega t)$

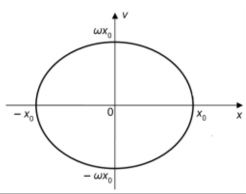
acceleration
$a = (-\omega^2 x_0)\sin(\omega t)$
$= -\omega^2 x$

$\sin^2\theta + \cos^2\theta = 1$
eliminate time variable

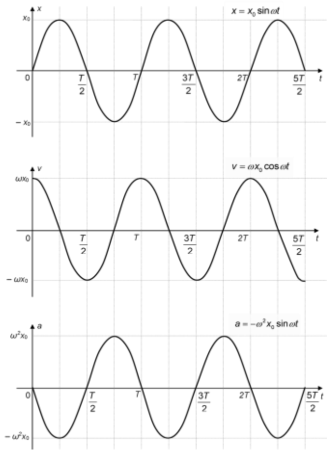
velocity
$v = \pm\omega\sqrt{x_0^2 - x^2}$

$\left(\frac{v}{\omega x_0}\right)^2 + \left(\frac{x}{x_0}\right)^2 = 1$
equation for ellipse

velocity-displacement
v-x



variation with time



phase difference

$\Delta\phi = \pi$	*a-t* vs *x-t*
$\Delta\phi = \frac{\pi}{2}$	*x-t* vs *v-t*
	v-t vs *a-t*

10.8 Ending Notes

We embark on another family of topics dealing with waves and oscillations in this topic. The simple harmonic motion of a single, individual body is the building block of "bulk" phenomenon such as waves and interference.

You may use the space below for your own mindmaps and summaries.