Q1

When $x=1$,

$0=p(1)+\sqrt{q}-\ln(r)$

$p+\sqrt{q}-\ln r=0$ --------(1)

When $x=2$,

$0=p(2)+\sqrt{2q}-\ln(2r)$

$2p+\sqrt{2}\sqrt{q}-\ln r=\ln 2$ --------(2)

When $x=5$,

$0=p(5)+\sqrt{5q}-\ln(5r)$

$5p+\sqrt{5}\sqrt{q}-\ln r=\ln 5$ --------(3)

Using G.C.,

$p=-0.4518351361\approx-0.452$ (to 3 d.p.)

$\sqrt{q}=2.764231837 \Rightarrow q=2.764231837^2=7.640977649\approx7.641$ (to 3 d.p.)

$\ln r=2.312396701 \Rightarrow r=\mathrm{e}^{2.312396701}=10.098599\approx10.099$ (to 3 d.p.)

Q2

$$\frac{3x-7}{(x+3)(x-2)}-1=\frac{3x-7-(x+3)(x-2)}{(x+3)(x-2)}$$

$$=\frac{-x^2-x+6+3x-7}{(x+3)(x-2)}$$

$$=\frac{-x^2+2x-1}{(x+3)(x-2)}$$

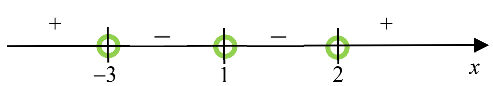
$$\frac{3x-7}{(x+3)(x-2)}<1$$

$$\frac{3x-7}{(x+3)(x-2)}-1<0$$

$$\frac{-x^2+2x-1}{(x+3)(x-2)}<0$$

$$\frac{x^2-2x+1}{(x+3)(x-2)}>0$$

$$\frac{(x-1)^2}{(x+3)(x-2)}>0$$



$\therefore\ x<-3$ or $x>2$

Replace x by e^x,

$e^x<-3$ (reject $\because e^x>0$) or $e^x>2$

$x>\ln 2$

Q3

(a) $$\sum_{r=5}^{n+1}\frac{1}{r(r+1)}=\sum_{r=1}^{n+1}\frac{1}{r(r+1)}-\sum_{r=1}^{4}\frac{1}{r(r+1)}$$
$$=1-\frac{1}{(n+1)+1}-\left[1-\frac{1}{4+1}\right]$$
$$=\frac{1}{5}-\frac{1}{n+2}$$

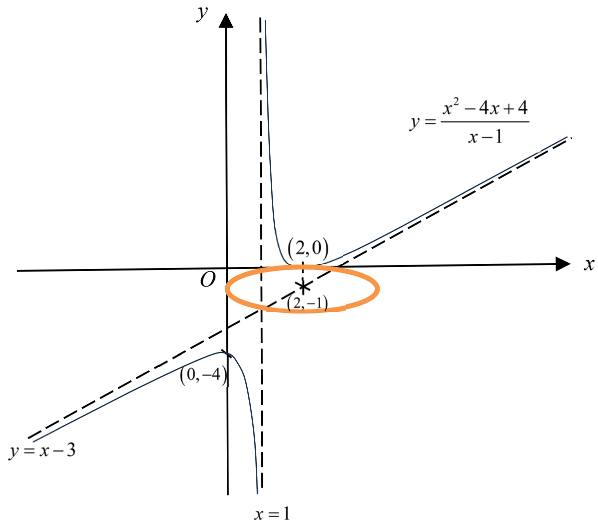
(b) As $n\to\infty$, $\frac{1}{n+2}\to 0$.

$$\therefore\sum_{r=5}^{\infty}\frac{1}{r(r+1)}=\frac{1}{5}$$

Q4

(a) Using Long Division: $y=\dfrac{x^2-4x+4}{x-1}=x-3+\dfrac{1}{x-1}$

Asymptotes: $x=1,\ \ y=x-3$



(b) $(x-2)^2+\dfrac{1}{b^2}\left(\dfrac{x^2-4x+4}{x-1}+1\right)^2=4$

$\dfrac{(x-2)^2}{2^2}+\dfrac{1}{(2b)^2}\left(\dfrac{x^2-4x+4}{x-1}+1\right)^2=1$

Ellipse centre $(2,-1)$

$0<2b<1$

$0<b<\dfrac{1}{2}$

Q5

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\sqrt{2+x^3}\right)=\frac{1}{1+\left(\sqrt{2+x^3}\right)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{2+x^3}\right)$$
$$=\frac{1}{3+x^3}\cdot\frac{1}{2}\left(2+x^3\right)^{-\frac{1}{2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3\right)$$
$$=\frac{1}{3+x^3}\cdot\frac{1}{2}\left(2+x^3\right)^{-\frac{1}{2}}\cdot 3x^2$$
$$=\frac{3x^2}{2\left(3+x^3\right)\sqrt{2+x^3}}$$

(b) **Method ①:**

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x}{\sqrt{1-x}}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)-\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{2}\ln(1-x)\right]$$
$$=\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{1-x}\cdot(-1)$$
$$=\frac{1}{x}+\frac{1}{2(1-x)}$$

Method ②:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x}{\sqrt{1-x}}\right)=\frac{1}{\frac{x}{\sqrt{1-x}}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sqrt{1-x}}\right)$$
$$=\frac{\sqrt{1-x}}{x}\cdot\frac{\sqrt{1-x}\cdot 1-x\cdot\frac{1}{2}(1-x)^{-\frac{1}{2}}\cdot(-1)}{1-x}$$
$$=\frac{\sqrt{1-x}}{x}\cdot\frac{\sqrt{1-x}+\frac{x}{2\sqrt{1-x}}}{1-x}$$
$$=\frac{(1-x)+\frac{x}{2}}{x(1-x)}$$
$$=\frac{1-\frac{x}{2}}{x(1-x)}$$
$$=\frac{2-x}{2x(1-x)}$$

Q6

(a) $y=\sin\left(\mathrm{e}^{x}-1\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\cos\left(\mathrm{e}^{x}-1\right)\cdot\mathrm{e}^{x}$$

$$\frac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}=\mathrm{e}^{x}\left[-\sin\left(\mathrm{e}^{x}-1\right)\cdot\mathrm{e}^{x}\right]+\cos\left(\mathrm{e}^{x}-1\right)\cdot\mathrm{e}^{x}$$

$$=-\mathrm{e}^{2x}\sin\left(\mathrm{e}^{x}-1\right)+\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$=\frac{\mathrm{d}y}{\mathrm{d}x}-y\mathrm{e}^{2x} \qquad \text{(shown)}$$

(b) When $x=0$,

$$y=\sin\left(\mathrm{e}^{0}-1\right)=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\cos\left(\mathrm{e}^{0}-1\right)\cdot\mathrm{e}^{0}=1$$

$$\frac{\mathrm{d}^{2}y}{\mathrm{d}x^{2}}=1$$

$$\mathrm{f}\left(x\right)=0+x+\frac{1}{2!}x^{2}+\ldots$$

$$=x+\frac{1}{2}x^{2}+\ldots$$

(c) $\sqrt{1+\sin\left(\mathrm{e}^{x}-1\right)}=\left[1+\sin\left(\mathrm{e}^{x}-1\right)\right]^{\frac{1}{2}}$

$$\approx\left[1+x+\frac{x^{2}}{2}\right]^{\frac{1}{2}}$$

$$=1+\frac{1}{2}\left(x+\frac{x^{2}}{2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(x+\frac{x^{2}}{2}\right)^{2}+\ldots$$

$$=1+\frac{1}{2}x+\frac{1}{4}x^{2}-\frac{1}{8}\left(x^{2}+\ldots\right)+\ldots$$

$$=1+\frac{1}{2}x+\frac{1}{8}x^{2}+\ldots$$

Q7

(a)

$$3\underset{\sim}{p}\times\underset{\sim}{q}=\underset{\sim}{r}\times\underset{\sim}{p}$$

$$\left(3\underset{\sim}{p}\times\underset{\sim}{q}\right)-\left(\underset{\sim}{r}\times\underset{\sim}{p}\right)=\underset{\sim}{0}$$

$$\left(3\underset{\sim}{p}\times\underset{\sim}{q}\right)+\left(\underset{\sim}{p}\times\underset{\sim}{r}\right)=\underset{\sim}{0}$$

$$\underset{\sim}{p}\times\left(3\underset{\sim}{q}+\underset{\sim}{r}\right)=\underset{\sim}{0}$$

$$\Rightarrow\ \underset{\sim}{p}\ //\ 3\underset{\sim}{q}+\underset{\sim}{r}$$

$$\Rightarrow \underset{\sim}{p}=k\left(3\underset{\sim}{q}+\underset{\sim}{r}\right),\ k\in\mathbb{R}\setminus\{0\}$$

(b)

$$\overrightarrow{OB}=\frac{3\overrightarrow{OA}+\overrightarrow{OC}}{4}$$

A 1 B 3 C

$$\overrightarrow{OC}=4\underset{\sim}{b}-3\underset{\sim}{a}$$

$$\overrightarrow{OD}=m\overrightarrow{OB}$$
$$=m\underset{\sim}{b}$$

O 1 B $m-1$ D

$$\overrightarrow{AB}=\underset{\sim}{b}-\underset{\sim}{a}$$

$$\overrightarrow{CD}=m\underset{\sim}{b}-\left(4\underset{\sim}{b}-3\underset{\sim}{a}\right)$$
$$=3\underset{\sim}{a}+(m-4)\underset{\sim}{b}$$

Given that *AB* is perpendicular to *CD*,

$$\overrightarrow{AB}\cdot\overrightarrow{CD}=0$$

$$\left(\underset{\sim}{b}-\underset{\sim}{a}\right)\cdot\left[3\underset{\sim}{a}+(m-4)\underset{\sim}{b}\right]=0$$

$$3\underset{\sim}{a}\cdot\underset{\sim}{b}+(m-4)\underset{\sim}{b}\cdot\underset{\sim}{b}-3\underset{\sim}{a}\cdot\underset{\sim}{a}-(m-4)\underset{\sim}{a}\cdot\underset{\sim}{b}=0$$

$$3\underset{\sim}{a}\cdot\underset{\sim}{b}+(m-4)\left|\underset{\sim}{b}\right|^2-3\left|\underset{\sim}{a}\right|^2-(m-4)\underset{\sim}{a}\cdot\underset{\sim}{b}=0$$

$$(m-4)-3-(m-7)\underset{\sim}{a}\cdot\underset{\sim}{b}=0$$

$$m\left(1-\underset{\sim}{a}\cdot\underset{\sim}{b}\right)=3-7\underset{\sim}{a}\cdot\underset{\sim}{b}+4$$

$$m=\frac{3-7\underset{\sim}{a}\cdot\underset{\sim}{b}+4}{1-\underset{\sim}{a}\cdot\underset{\sim}{b}}$$

$$=\frac{7\left(1-\underset{\sim}{a}\cdot\underset{\sim}{b}\right)}{1-\underset{\sim}{a}\cdot\underset{\sim}{b}}$$

$$=7$$

Q8

(a)

$$S_{36} \geq 500$$

$$\frac{36}{2}\left[2a+(36-1)(0.5)\right] \geq 500$$

Method ①: Algebraic

$$\frac{36}{2}\left[2a+(36-1)(0.5)\right] \geq 500$$

$$18(2a+17.5) \geq 500$$

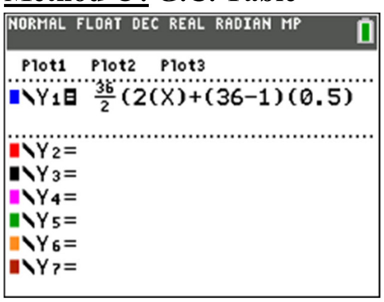
$$2a+17.5 \geq \frac{500}{18}$$

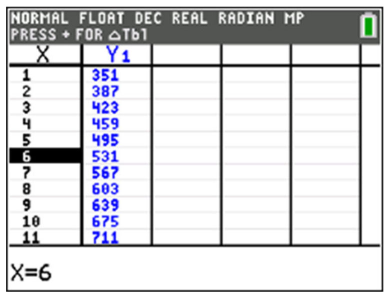
$$2a \geq \frac{500}{18}-17.5$$

$$a \geq 5.138888889$$

Hence, smallest value of a is 6.

Method ②: G.C. Table





a	$\frac{36}{2}\left[2a+(36-1)(0.5)\right]$
5	495
6	531
7	567

Hence, smallest value of a is 6.

(b)(i) **Method ①:**

$$u_n = 1.01u_{n-1} - x$$
$$= 1.01(1.01u_{n-2} - x) - x$$
$$= 1.01^2 u_{n-2} - 1.01x - x$$
$$= 1.01^2(1.01u_{n-2} - x) - 1.01x - x$$
$$= 1.01^3 u_{n-3} - 1.01^2 x - 1.01x - x$$
$$= \ldots$$
$$= 1.01^n u_0 - 1.01^{n-1}x - \ldots - 1.01^2 x - 1.01x - x$$
$$= 1.01^n(400) - x(1 + 1.01 + 1.01^2 + \ldots + 1.01^{n-1})$$
$$= 1.01^n(400) - x\left[\frac{1(1.01^n - 1)}{1.01 - 1}\right]$$
$$= 1.01^n(400) - 100x(1.01^n - 1) \quad (\text{shown})$$

Method ②:

$$u_1 = 1.01u_0 - x$$
$$u_2 = 1.01u_1 - x$$
$$= 1.01(1.01u_0 - x) - x$$
$$= 1.01^2 u_0 - 1.01x - x$$
$$u_3 = 1.01u_2 - x$$
$$= 1.01(1.01^2 u_0 - 1.01x - x) - x$$
$$= 1.01^3 u_0 - 1.01^2 x - 1.01x - x$$
$$u_n = 1.01^n u_0 - 1.01^{n-1}x - \ldots - 1.01^2 x - 1.01x - x$$
$$= 1.01^n(400) - x(1 + 1.01 + 1.01^2 + \ldots + 1.01^{n-1})$$
$$= 1.01^n(400) - x\left[\frac{1(1.01^n - 1)}{1.01 - 1}\right]$$
$$= 1.01^n(400) - 100x(1.01^n - 1) \quad (\text{shown})$$

(b)(ii) $y = 1.01\left(1.01^k(400) - 100(16)(1.01^k - 1)\right)$

$$= 1.01^{k+1}(400) - 1616(1.01^k - 1)$$

Since $0 < y < 16$,

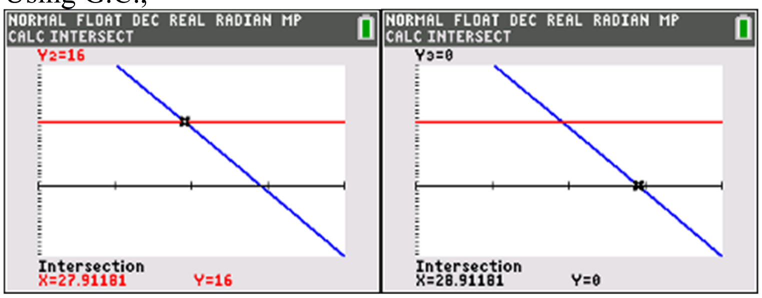
$$0 < 1.01^{k+1}(400) - 1616(1.01^k - 1) < 16$$

Method ①:
Using G.C.,

k	$1.01^{k+1}(400)-1616(1.01^{k}-1)$
27	30.451
28	14.595
29	−1.419

Hence, $k = 28$.

Method ②:
Using G.C.,



$27.91181 < k < 28.91181$
Hence, $k = 28$.

Method ③:

$$0 < 1.01^{k+1}(400) - 1616(1.01^{k}-1) < 16$$

$$0 < 1.01^{k}(404) - 1616(1.01^{k}) + 1616 < 16$$

$$0 < -1212(1.01^{k}) + 1616 < 16$$

$$-1616 < -1212(1.01^{k}) < -1600$$

$$\frac{-1600}{-1212} < 1.01^{k} < \frac{-1616}{-1212}$$

$$\frac{400}{303} < 1.01^{k} < \frac{4}{3}$$

$$\ln\left(\frac{400}{303}\right) < k\ln 1.01 < \ln\left(\frac{4}{3}\right)$$

$27.91180974 < k < 28.91180974$
Hence, $k = 28$.

$\therefore y = 14.6 \quad (\text{to 3 s.f.})$

Q9

(a) **Method ①:**

$4x^2-8x+4=4\left[x^2-2x\right]+4$

$=4\left[(x-1)^2-1\right]+4$

$=4(x-1)^2$

Method ②:

$4x^2-8x+4=A(x-B)^2$

$4x^2-8x+4=Ax^2-2ABx+AB^2$

Comparing coefficient of x^2: $A=4$

Comparing coefficient of x: $8=2(4)(B)$

$B=1$

$\therefore 4x^2-8x+4=4(x-1)^2$

Method ①:

$y=3\mathrm{e}^{x^2}$

$\downarrow$ Scale parallel to x-axis by scale factor $\frac{1}{2}$

$y=3\mathrm{e}^{4x^2}$

$\downarrow$ Translate 1 unit in the positive x-direction

$y=3\mathrm{e}^{4(x-1)^2}$

$\downarrow$ Translate 10 units in the negative y-direction

$y=3\mathrm{e}^{4(x-1)^2}-10$

Method ②:

$y=3\mathrm{e}^{x^2}$

$\downarrow$ Translate 2 units in the positive x-direction

$y=3\mathrm{e}^{(x-2)^2}$

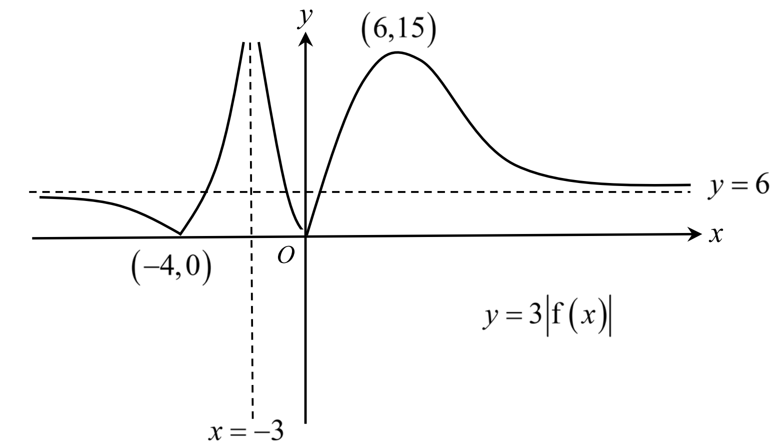
$\downarrow$ Scale parallel to x-axis by scale factor $\frac{1}{2}$

$y=3\mathrm{e}^{(2x-2)^2}$

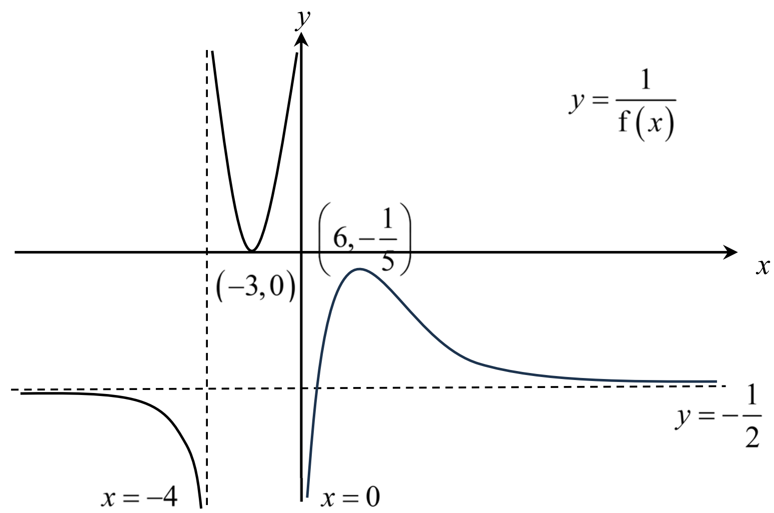
$\downarrow$ Translate 10 units in the negative y-direction

$y=3\mathrm{e}^{4(x-1)^2}-10$

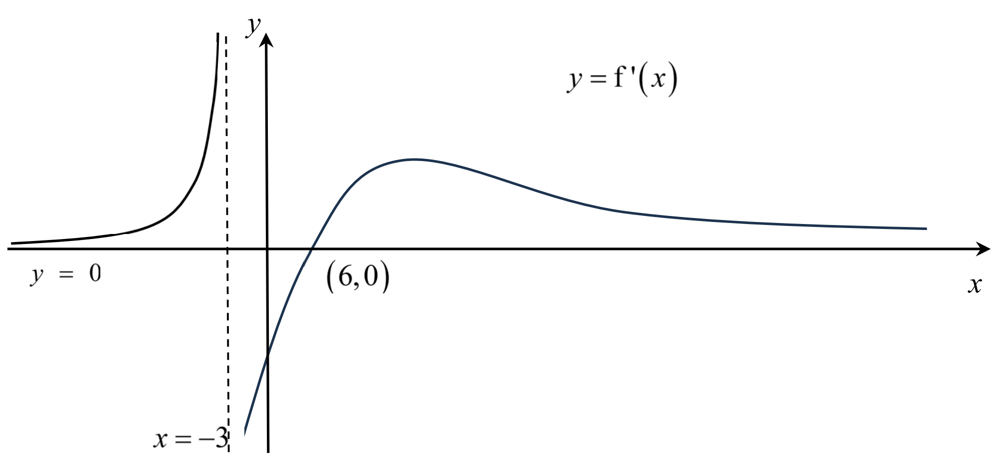
(b)(i)



(b)(ii)



(b)(iii)



Q10

(a) Volume of cone $= \frac{1}{3}\pi r^2 h = 20\pi$

$$\frac{r^2 h}{3} = 20$$

$$h = \frac{60}{r^2}$$

(b) **Method ①: Considering** A^2

$$\begin{aligned} A &= \pi r l \\ &= \pi r\sqrt{r^2 + h^2} \\ &= \pi r\sqrt{r^2 + \left(\frac{60}{r^2}\right)^2} \\ &= \pi r\sqrt{r^2 + \frac{3600}{r^4}} \end{aligned}$$

$$\begin{aligned} A^2 &= \pi^2 r^2\left(r^2 + \frac{3600}{r^4}\right) \\ &= \pi^2\left(r^4 + \frac{3600}{r^2}\right) \\ &= \pi^2\left(r^4 + 3600r^{-2}\right) \end{aligned}$$

Differentiating implicitly with respect to r:

$$2A\frac{\mathrm{d}A}{\mathrm{d}r} = \pi^2\left(4r^3 - 7200r^{-3}\right)$$

$$A\frac{\mathrm{d}A}{\mathrm{d}r} = \pi^2\left(2r^3 - \frac{3600}{r^3}\right) \qquad \text{(shown)}$$

Method ②: "Otherwise" Method

$$\begin{aligned} A &= \pi r l \\ &= \pi r\sqrt{r^2 + h^2} \\ &= \pi r\sqrt{r^2 + \left(\frac{60}{r^2}\right)^2} \\ &= \pi r\sqrt{r^2 + \frac{3600}{r^4}} \\ &= \pi r\left(r^2 + 3600r^{-4}\right)^{\frac{1}{2}} \end{aligned}$$

Differentiating with respect to r:

$$\frac{\mathrm{d}A}{\mathrm{d}r} = \pi\left[r\cdot\frac{1}{2}\left(r^2+3600r^{-4}\right)^{-\frac{1}{2}}\left(2r-4\cdot3600r^{-5}\right)+\left(r^2+3600r^{-4}\right)^{\frac{1}{2}}\right]$$

$$= \pi\left(r^2+3600r^{-4}\right)^{-\frac{1}{2}}\left[r^2-7200r^{-4}+r^2+3600r^{-4}\right]$$

$$= \pi\left(r^2+3600r^{-4}\right)^{-\frac{1}{2}}\left[2r^2-3600r^{-4}\right]$$

$$= \frac{\pi}{\sqrt{r^2+3600r^{-4}}}\left(2r^2-\frac{3600}{r^4}\right)$$

$$A\frac{\mathrm{d}A}{\mathrm{d}r} = \pi r\sqrt{r^2+3600r^{-4}}\cdot\frac{\pi}{\sqrt{r^2+3600r^{-4}}}\left(2r^2-\frac{3600}{r^4}\right)$$

$$= \pi^2 r\left(2r^2-\frac{3600}{r^4}\right)$$

$$= \pi^2\left(2r^3-\frac{3600}{r^3}\right) \quad \text{(shown)}$$

(c) At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0$

$$\pi^2\left(2r^3-\frac{3600}{r^3}\right)=0$$

$$2r^3-\frac{3600}{r^3}=0$$

$$2r^3=\frac{3600}{r^3}$$

$$r^6=1800$$

$$r=(1800)^{\frac{1}{6}}$$

To prove minimum:

<u>Method ①</u>: Using 1st derivative test

r	$\left((1800)^{\frac{1}{6}}\right)^-$	$(1800)^{\frac{1}{6}}$	$\left((1800)^{\frac{1}{6}}\right)^+$
Sign of $\frac{\mathrm{d}A}{\mathrm{d}r}$	<0	0	>0
Tangent	\	—	/

$r=(1800)^{\frac{1}{6}}$ gives the minimum A.

<u>Method ②:</u> Using 2nd derivative test

From the result $A\frac{\mathrm{d}A}{\mathrm{d}r}=\pi^2\left(2r^3-\frac{3600}{r^3}\right)$, using implicit differentiation to obtain:

$$A\frac{\mathrm{d}^2A}{\mathrm{d}r^2}+\left(\frac{\mathrm{d}A}{\mathrm{d}r}\right)^2=\pi^2\left(6r^2+\frac{10800}{r^4}\right)$$

At $\frac{\mathrm{d}A}{\mathrm{d}r}=0$ and $r=(1800)^{\frac{1}{6}}$, $\frac{\mathrm{d}^2A}{\mathrm{d}r^2}=\frac{\pi^2}{A}\left(6r^2+\frac{10800}{r^4}\right)>0$ (minimum)

$r=(1800)^{\frac{1}{6}}$ gives the minimum A.

<u>Method ①:</u>

$$\begin{aligned}\frac{r}{h}&=\frac{r}{\left(\frac{60}{r^2}\right)}\\&=\frac{r^3}{60}\\&=\frac{\left(1800^{\frac{1}{6}}\right)^3}{60}\\&=\frac{\sqrt{1800}}{60}\\&=\frac{30\sqrt{2}}{60}\\&=\frac{\sqrt{2}}{2},\ \text{where } k=\frac{1}{2}\end{aligned}$$

<u>Method ②:</u>

$$\begin{aligned}h&=\frac{60}{r^2}\\&=\frac{60}{\left(1800^{\frac{1}{6}}\right)^2}\\&=\frac{60}{1800^{\frac{1}{3}}}\end{aligned}$$

$$\frac{r}{h}=\frac{1800^{\frac{1}{6}}}{\frac{60}{1800^{\frac{1}{3}}}}$$

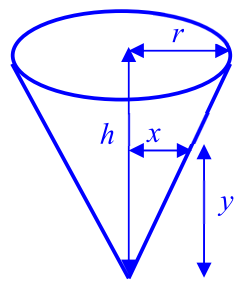
$$=\frac{\sqrt{1800}}{60}$$

$$=\frac{30\sqrt{2}}{60}$$

$$=\frac{\sqrt{2}}{2},\text{ where } k=\frac{1}{2}$$

(d) Let V be the volume of water in the cone when radius of water surface is x cm and depth is y cm.
By similar triangles,

$$\frac{x}{y}=\frac{r}{h}=\frac{\sqrt{2}}{2}$$



$$x=\frac{\sqrt{2}}{2}y$$

$$V=\frac{1}{3}\pi x^2 y$$

$$=\frac{1}{3}\pi\left(\frac{\sqrt{2}}{2}y\right)^2 y$$

$$=\frac{1}{6}\pi y^3$$

$$\frac{dV}{dy}=\frac{1}{2}\pi y^2$$

$$\frac{dV}{dt}=\frac{dV}{dy}\times\frac{dy}{dt}$$

$$=\frac{1}{2}\pi y^2\times\frac{dy}{dt}$$

When $y=2$,

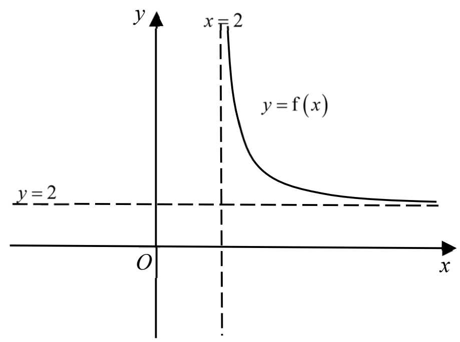
$$-3=\frac{1}{2}\pi(2)^2\times\frac{dy}{dt}$$

$$\frac{dy}{dt}=-\frac{3}{2\pi}$$

Rate of decrease of depth of water is $\frac{3}{2\pi}$ cm/s or 0.477 cm/s (3 s.f.)

Q11

(a)



Since **any horizontal line** $y = k, k \in \mathbb{R}$ cuts the graph **at most once**, f is **one-one function**, f inverse exist.

Alternative:
Since **any horizontal line** $y = k$, where $k > 2, k \in \mathbb{R}$ cuts the graph **exactly once**, f is **one-one function**, f inverse exist.

(b) Let $y = \mathrm{f}(x) = 2 + \dfrac{1}{x-2}$

$$y - 2 = \frac{1}{x-2}$$

$$x = 2 + \frac{1}{y-2}$$

$$\mathrm{f}^{-1}(x) = 2 + \frac{1}{x-2} = \mathrm{f}(x)$$

$$\mathrm{D}_{\mathrm{f}^{-1}} = \mathrm{R}_{\mathrm{f}} = (2, \infty)$$

Since $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$ for all $x > 2$, f is self-inverse.

Method ①:

$$\mathrm{f}(x) = \mathrm{f}^{-1}(x)$$

$$\mathrm{ff}(x) = \mathrm{ff}^{-1}(x)$$

$$\mathrm{f}^2(x) = x$$

Method ②:

$$\begin{aligned}\text{ff}(x) &= \text{f}\left(2+\frac{1}{x-2}\right)\\ &= 2+\frac{1}{\left(2+\frac{1}{x-2}\right)-2}\\ &= 2+\frac{1}{\left(\frac{1}{x-2}\right)}\\ &= 2+(x-2)\\ &= x\end{aligned}$$

(c) Since $\text{f}(x)=\text{f}^{-1}(x)$, $\text{f}^2(x)=x$

$$\begin{aligned}\text{f}^3(x) &= \text{f}^2\left[\text{f}(x)\right]\\ &= \text{f}^2\left(2+\frac{1}{x-2}\right)\\ &= 2+\frac{1}{x-2}\end{aligned}$$

$$\begin{aligned}\text{f}^4(x) &= \text{f}^2\left[\text{f}^2(x)\right]\\ &= \text{f}^2(x)\\ &= x\\ &\vdots\end{aligned}$$

$$\begin{aligned}\text{f}^{2025}(4) &= \text{f}\left(\text{f}^{2024}(4)\right)\\ &= \text{f}\left(\text{f}^2(4)\right)\\ &= \text{f}(4)\\ &= 2+\frac{1}{4-2}\\ &= \frac{5}{2}\end{aligned}$$

(d) gf exists if $\text{R}_\text{f}=(2,\infty)\subseteq \text{D}_\text{g}=(1,\infty)$

Thus we will be using the function $\text{g}(x)=(x-1)^2+2$, for $x>1$

$$\begin{aligned}\text{gf}(x) &= \text{g}\left(2+\frac{1}{x-2}\right)\\ &= \left[2+\frac{1}{x-2}-1\right]^2+2\\ &= \left[1+\frac{1}{(x-2)}\right]^2+2\end{aligned}$$

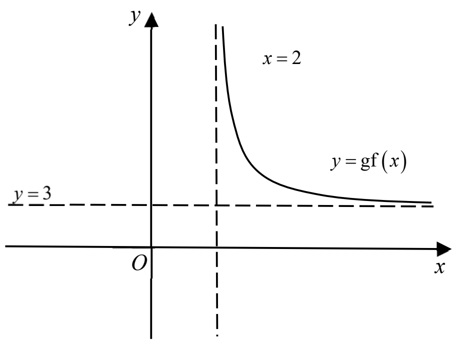
$D_{gf} = D_f = (2, \infty)$

(e) <u>**Method ①:**</u>

$D_f = (2, \infty) \xrightarrow{f} (2, \infty) \xrightarrow{g} (3, \infty)$

$R_{gf} = (3, \infty)$

<u>**Method ②:**</u>



$R_{gf} = (3, \infty)$

Q12

(a) $\overrightarrow{OA}=\begin{pmatrix}1\\2\\3\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}2\\-1\\4\end{pmatrix}$, $\overrightarrow{OC}=\begin{pmatrix}4\\5\\-1\end{pmatrix}$, $\overrightarrow{OD}=\begin{pmatrix}-2\\0\\-5\end{pmatrix}$

$$\overrightarrow{AB}=\begin{pmatrix}2\\-1\\4\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}1\\-3\\1\end{pmatrix}$$

$$\overrightarrow{AC}=\begin{pmatrix}4\\5\\-1\end{pmatrix}-\begin{pmatrix}1\\2\\3\end{pmatrix}=\begin{pmatrix}3\\3\\-4\end{pmatrix}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{pmatrix}1\\-3\\1\end{pmatrix}\times\begin{pmatrix}3\\3\\-4\end{pmatrix}=\begin{pmatrix}(-3)(-4)-(1)(3)\\-\left[(1)(-4)-(1)(3)\right]\\(1)(3)-(-3)(3)\end{pmatrix}=\begin{pmatrix}9\\7\\12\end{pmatrix}$$

$$\begin{aligned}\Pi:\ \underset{\sim}{r}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix}&=\begin{pmatrix}1\\2\\3\end{pmatrix}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix}\\&=9+14+36\\&=59\end{aligned}$$

$\therefore\ \ \alpha=59$

(b) $\overrightarrow{BD}=\begin{pmatrix}-2\\0\\-5\end{pmatrix}-\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}-4\\1\\-9\end{pmatrix}$

<u>Method ①:</u>

Let θ be the angle between the line BD and Π.

$$\left|\begin{pmatrix}-4\\1\\-9\end{pmatrix}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix}\right|=\left|\begin{pmatrix}-4\\1\\-9\end{pmatrix}\right|\left|\begin{pmatrix}9\\7\\12\end{pmatrix}\right|\sin\theta$$

$$137=\sqrt{98}\sqrt{274}\sin\theta$$

$$\sin\theta=\frac{137}{\sqrt{98}\sqrt{274}}$$

$$\theta=56.7^\circ\left(\text{to 1 d.p.}\right)\text{ or }0.990\left(\text{to 3 s.f.}\right)$$

Method ②:

Let θ be the angle between the line BD and normal of $\prod$.

$$\left|\begin{pmatrix}-4\\1\\-9\end{pmatrix}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix}\right| = \left|\begin{pmatrix}-4\\1\\-9\end{pmatrix}\right|\left|\begin{pmatrix}9\\7\\12\end{pmatrix}\right|\cos\theta$$

$$137 = \sqrt{98}\sqrt{274}\cos\theta$$

$$\cos\theta = -\frac{137}{\sqrt{98}\sqrt{274}}$$

$$\theta = 33.275^\circ \text{ or } 0.58075$$

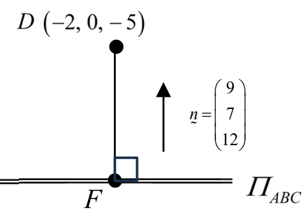
$\therefore$ angle between the line BD and $\prod$ is $90^\circ - 33.275^\circ = 56.7^\circ$ or $\frac{\pi}{2} - 0.58075 = 0.990\,(\text{to 3 s.f.})$.

(c) Let F be the foot of perpendicular from the point D to $\prod$.

$$l_{DF} : r = \begin{pmatrix}-2\\0\\-5\end{pmatrix} + \lambda\begin{pmatrix}9\\7\\12\end{pmatrix}, \lambda \in \mathbb{R}$$

Since F lies on line DF, $\overrightarrow{OF} = \begin{pmatrix}-2+9\lambda\\7\lambda\\-5+12\lambda\end{pmatrix}$ for some λ

$\prod$: $\underset{\sim}{r}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix} = 59$



$$\begin{pmatrix}-2+9\lambda\\7\lambda\\-5+12\lambda\end{pmatrix}\cdot\begin{pmatrix}9\\7\\12\end{pmatrix} = 59$$

$$9(-2+9\lambda) + 49\lambda + 12(-5+12\lambda) = 59$$

$$-18 + 81\lambda + 49\lambda - 60 + 144\lambda = 59$$

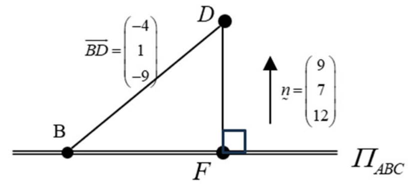
$$274\lambda = 137$$

$$\lambda = \frac{1}{2}$$

$$\overrightarrow{OF} = \begin{pmatrix}-2+9(\tfrac{1}{2})\\7(\tfrac{1}{2})\\-5+12(\tfrac{1}{2})\end{pmatrix} = \begin{pmatrix}5/2\\7/2\\1\end{pmatrix}$$

Alternative Method

$$\overrightarrow{BD} = \begin{pmatrix} -4 \\ 1 \\ -9 \end{pmatrix}$$



$$\overrightarrow{FD} = (\overrightarrow{BD} \cdot \underset{\sim}{\hat{n}})\underset{\sim}{\hat{n}}$$

$$\overrightarrow{FD} = \left[\begin{pmatrix} -4 \\ 1 \\ -9 \end{pmatrix} \cdot \frac{\begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}}{\left| \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix} \right|} \right] \frac{\begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}}{\left| \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix} \right|}$$

$$\overrightarrow{FD} = \left[\begin{pmatrix} -4 \\ 1 \\ -9 \end{pmatrix} \cdot \frac{\begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}}{\sqrt{274}} \right] \frac{\begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}}{\sqrt{274}}$$

$$\overrightarrow{FD} = \frac{1}{274} \left[\begin{pmatrix} -4 \\ 1 \\ -9 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix} \right] \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}$$

$$\overrightarrow{FD} = \frac{1}{274}(-137) \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}$$

$$\overrightarrow{FD} = -\frac{1}{2} \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}$$

$$\overrightarrow{FD} = \overrightarrow{OD} - \overrightarrow{OF}$$

$$-\frac{1}{2} \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} - \overrightarrow{OF}$$

$$\overrightarrow{OF} = \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}$$

$$\overrightarrow{OF} = \begin{pmatrix} 5/2 \\ 7/2 \\ 1 \end{pmatrix}$$

(d) Area of triangle ABC

$$= \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\left|\begin{pmatrix} 9 \\ 7 \\ 12 \end{pmatrix}\right|$$

$$= \frac{1}{2}\sqrt{9^2 + 7^2 + 12^2}$$

$$= \frac{1}{2}\sqrt{274}\ \text{units}^2$$

$$\overrightarrow{DF} = \begin{pmatrix} 5/2 \\ 7/2 \\ 1 \end{pmatrix} - \begin{pmatrix} -2 \\ 0 \\ -5 \end{pmatrix} = \begin{pmatrix} 9/2 \\ 7/2 \\ 6 \end{pmatrix}$$

$$\left|\overrightarrow{DF}\right| = \left|\begin{pmatrix} 9/2 \\ 7/2 \\ 6 \end{pmatrix}\right| = \sqrt{\left(\frac{9}{2}\right)^2 + \left(\frac{7}{2}\right)^2 + 6^2} = \sqrt{\frac{137}{2}}$$

$$\text{Volume of pyramid} = \frac{1}{3}\left(\frac{1}{2}\sqrt{274}\right)\left(\sqrt{\frac{137}{2}}\right) = \frac{137}{6}\ \text{unit}^3$$