TAMPINES MERIDIAN JUNIOR COLLEGE

JC2 PRELIMINARY EXAMINATION

H2 FURTHER MATHEMATICS — 9649/02

Paper 2

17 SEPTEMBER 2024

3 hours

Additional material: Answer Booklet
List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your name and civics group on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

This document consists of **7** printed pages and **1** blank page.

[Turn over

Section A: Pure Mathematics [50 marks]

1 A common logistic model to describe how population P (in thousands) varies with time t (in months) is given as

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP\left(1-\frac{P}{k}\right) - h,$$

where r, k and h are positive real constants.

It is given that there are two distinct and positive equilibrium population values.

(a) Find, in terms of r and k, the range of values that h can take. [2]

(b) On a single diagram, sketch all possible cases of the behaviour of the population against time, as the initial population varies. You may assume that the initial population is positive. [4]

2 Adam and Bernard start with \$$k$ and \$($N - k$) respectively to play a game, where k and N are positive integers and $N > k$. Adam has a probability p, $0 < p < 1$, of winning each round. The winner of a round wins \$1 from the opponent and they repeat the rounds until one player wins all the money from the opponent.

R_k is the probability that Adam wins all the money given that he starts with \$$k$ and it satisfies the following recurrence relation

$$R_k = pR_{k+1} + (1-p)R_{k-1},$$

where $R_0 = 0$ and $R_N = 1$.

(a) By considering the cases $p = \frac{1}{2}$ and $p \neq \frac{1}{2}$, find the corresponding expressions of R_k in terms of p, k and N. [7]

(b) Discuss the implications if, instead of Bernard, Adam plays against an opponent who has infinite capital. [4]

3 **(a)** Given that w is a complex number, show that the loci $|w-1| = 2|w-3|$ describes a circle. [3]

(b) The loci L_1 and L_2 are given by

$$|z-2-3\text{i}| = |z+2-5\text{i}| \quad \text{and} \quad |z-5-\text{i}| = 3.$$

(i) The complex number z_1 lies on L_1 such that z_1 is the closest point of L_1 to L_2. Find z_1. [4]

(ii) The loci $\arg(z+2-5\text{i}) = \theta$, where $-\pi < \theta \le \pi$, meets tangentially with L_2. Find the possible values of θ. [3]

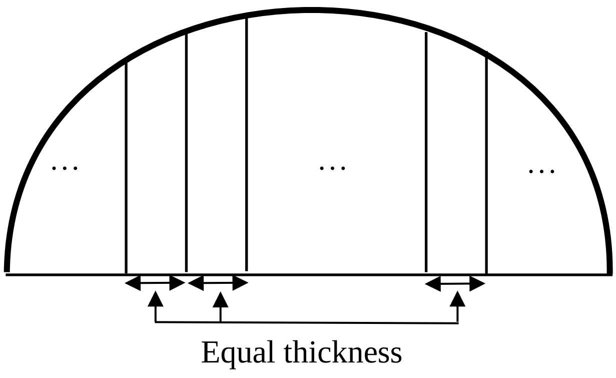
4 **(a)** **(i)** Let $I_n = \int_0^{\frac{\pi}{6}} \cos^n x \, \text{d}x$. Form a recurrence relation, expressing I_n in terms of I_{n-2} for $n \ge 2$. [3]

(ii) A polar equation is given by

$$r = 4\cos^3\left(\frac{\theta}{3}\right), \text{ for } 0 \le \theta < 3\pi.$$

Find the exact area of the region bounded by the curve and the half-lines $\theta = 0$ and $\theta = \frac{\pi}{2}$. [4]

(b) A student cuts a hemispherical cake into vertical slices of equal thickness. The diagram below shows a cross-sectional view of the cake.



By considering an equation of a circle centred at the origin with radius a units, show, using integration, that each slice of the cake has the same **curved** surface area. [4]

[Turn over

5 A differential equation is given as $\left(1+x^4\right)\frac{\mathrm{d}y}{\mathrm{d}x}+8x^3y=x$ where $y(1)=\frac{1}{3}$.

(a) Use the Euler's method with step size 0.1 to estimate the value of y when $x=1.2$. [3]

(b) Explain whether you would expect the value found in part **(a)** to be an under-estimate or an over-estimate of the true value. [2]

(c) Solve the differential equation and explain if the estimate found in part **(a)** is close to the actual value. [7]

Section B: Probability and Statistics [50 marks]

6 Eric and Jane plans to purchase chairs and tables for a College. To begin their research, Eric and Jane collect a range of responses from a random sample of 80 students in the College.

(a) Using the data collected for heights of students, Jane finds that the sample mean is 166 cm and an unbiased estimate of population variance is 5.2064 cm^2. Find a 95% confidence interval, correct to 1 decimal place, for the population mean height of students in the college. Explain what a 95% confidence interval means in the context of this question. [3]

(b) Eric finds that a k% confidence interval for the proportion of students who prefers blue-coloured tables and chairs is $(0.598, 0.801)$. Find the value of k. [3]

7 **(a)** In a particular laundry store, there is only one checkout counter. It is observed that an average of 10 customers will checkout per hour. It can be assumed that customer checkouts can be modelled by a Poisson distribution.

(i) Find the probability that 15 customers will checkout from the laundry store in the next hour. [1]

(ii) The store owner is interested in the amount of time a customer spends at the checkout counter. Find the probability that a customer will spend less than 9 minutes at the checkout counter. [2]

(iii) Given that a customer has spent 6 minutes at the checkout counter, find the probability that the customer will spend at least an additional 3 minutes at the checkout counter. [2]

(b) A random variable X follows an exponential distribution with mean λ, where $\lambda > 0$. Let random variable $Y = \lfloor X \rfloor + 1$, where $\lfloor X \rfloor$ gives the greatest integer less than or equal to X. Show that Y follows a geometric distribution. [3]

8 Albert was browsing some online articles on financial literacy and he came across an article which contained survey results on financial literacy of Singapore citizens. While recounting the article to his colleagues, Albert could only recall the following details:

- The survey was conducted on 1000 Singapore citizens to assess their financial literacy and the results are collated in the table below.

	Age below 25 years old	Age between 25 to 50 years old	Age above 50 years old
Financially literate	70	210	150
Not financially literate	130		

- The article noted that there were at least 10 respondents for each category and concluded, at 2.5% significance level, that the financial literacy among Singaporeans seems to be independent of the various age groups.

(a) Based on the information given above, find the minimum number of survey respondents who are aged above 50 years old and are not financially literate. [9]

(b) Assume that the answer in part **(a)** is indeed the actual number of survey respondents. By considering the two largest contributions to the test statistic, discuss if there is any potential concern. [2]

[Turn over

9 A researcher believes that an innovative teaching strategy can improve the Mathematics test scores of students. The researcher selects a random sample of 10 students to carry out the innovative teaching strategy. The pre-test and post-test scores of these students are as follows.

Student	A	B	C	D	E	F	G	H	I	J
Pre-test	36	27	25	45	51	38	20	37	26	32
Post-test	38	37	34	49	44	30	26	42	25	44

(a) The researcher carries out a Wilcoxon test in this situation and checks that the sum of the ranks is 55. Briefly explain why the sum of the ranks is 55. [1]

(b) Test at the 5% significance level, whether the innovative teaching strategy can improve the Mathematics test scores of students. [5]

(c) State an assumption needed in order to carry out the Wilcoxon test. [1]

(d) Another researcher claims that using a sign test on the above data would give the same conclusion as part **(b)**. Verify this claim. [3]

(e) State what other test might be used and comment on its validity. [2]

10 It is given that the number of days of hospital stay for patients who visit a hospital, X, can be modelled by a Poisson distribution with parameter λ. Patients who need at least one day of hospital stay are considered admitted patients and the number of days for admitted patients, Y, follows a zero-truncated Poisson distribution with probability distribution function as follows.

$$\mathrm{P}(Y=r)=\begin{cases} k\mathrm{P}(X=r), & r=1,2,3,\ldots \\ 0, & \text{otherwise,} \end{cases}$$

where $X \sim \mathrm{Po}(\lambda)$ and k is a constant to be determined.

(a) For $\lambda = 6.5$, find the probability that a randomly chosen admitted patient stays for more than a week. [4]

The hospital management would like to estimate the value of λ using records of the number of admitted patients and the number of days of hospital stay. It is known that the expected number of days of hospital stay for admitted patients is m.

(b) For $m = 4.5$, find the value of λ. [2]

For the rest of the question, use a general value of λ.

(c) The data analyst in the hospital claims that the expected number of days of hospital stay for admitted patients can be taken as expected number of days of hospital stay for patients who visit the hospital. How far do you agree with this statement? [2]

(d) Find, in terms of λ,

(i) $\mathrm{E}[Y(Y-1)]$ and [3]

(ii) $\mathrm{Var}(Y)$. [2]

End of Paper