A4: POLYNOMIALS AND PARTIAL FRACTIONS

- Multiplication and division of polynomials
- Use of remainder and factor theorems, including factorising polynomials and solving cubic equations
- Use of:
  - $a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$
  - $a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$
- Partial fractions with cases where the denominator is no more complicated than:
  - $(ax + b)(cx + d)$
  - $(ax + b)(cx + d)^2$
  - $(ax + b)\left(x^2 + c^2\right)$

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1. Express $\frac{2+x^2}{(x+2)^2(x+4)}$ in partial fractions.	[4]
2. The function $f(x)$ is defined by $f(x) = 3x^2 + hx^2 + kx - 4$ for all real values of x. Given that $3x - 1$ is a factor of $f(x)$ and that when $f(x)$ is divided by $x + 1$, the remainder is $- 4$, find the value of each of the constants h and k.	[5]
3. (a) Factorise $27x^3 + 125$.	[2]
3. (b) Explain why $x =- \frac{5}{3}$ is the only real root of the operation $27x^3 + 125 = 0$.	[2]
4. Express $\frac{18x+7}{(x-1)(4x+1)}$ in partial fractions.	[4]
5. The polynomial $f(x)$ is given by $f(x) = 2x^3 + 5x^2 - 4x - 3$.	
(a) Divide $f(x)$ by $x + 3$.	[2]
(b) What can you deduce about $x + 3$?	[1]
(c) Solve the equation $f(x) = 0$.	[2]
6. Factorise $x^3 - 8$ and explain why $x = 2$ is the only real root of the equation $x^3 - 8 = 0$.	[4]

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7. By using long division, find the remainder when $9x^4 - 13x^2 + 4x - 2$ is divided by $3x^2 + 2x - 1$.	[3]
8. Given that $20x^2 + 13x + 5 = Ax(1 + 2x) + Bx + 5$ for all real values of x, find the value of A and of B.	[3]
9. Express $\frac{6x}{(x+1)(x-1)^2}$ in partial fractions.	[5]
10. Factorise $125a^3 - 8b^3$.	[3]
11. The polynomial $f(x) = 6x^3 + ax^2 + 11x - 6$ has a factor $(x + 2)$.	
(a) Show that $a = 19$.	[2]
(b) Solve the equation $f(x) = 0$. Show your working clearly.	[4]
(c) Hence solve the equation $\frac{6}{y^3} + \frac{19}{y^2} + \frac{11}{y} - 6 = 0$.	[2]
12. Factorise $8x^3 + 125$.	[3]
13. Express $\frac{3x^2-2}{(x+1)^2(2x-1)}$ in partial fractions.	[5]
14. The polynomial $p(x)$ is given by $p(x) = 5x^3 + ax^2 + bx - 2$, where a and b are constants. It is given that $(x - 2)$ is a factor of $p(x)$ and when $p(x)$ is divided by $(x - 1)$, the remainder is 5.	
(a) Show that $a =- 21$ and find the value of b.	[4]
(b) Using the values from part (a), find the remainder when $p(x)$ is divided by $(2x + 1)$.	[2]
15. (a) Divide $2x^3 - x^2 + 8x - 4$ by $2x - 1$.	[1]
15. (b) Express $\frac{2+5x-x^2}{2x^3-x^2+8x-4}$ in partial fractions.	[5]

16. The polynomial $p(x)$ is given by $p(x) = x^3 + 8x^2 + 21x + 9a$, where a is a constant. It is given that $(x + 2)$ is a factor of $p(x)$.	
(a) Find the value of the constant a.	[2]
(b) Using the value from (a), find the remainder when $p(x)$ is divided by $(x - 1)$.	[1]
17. (a) Show that $2x - 1$ is a factor of $f(x) = 2x^3 + 3x^2 - 8x + 3$.	[1]
17. (b) Hence, factorise $f(x)$ completely.	[3]
18. (a) Factorise $8x^3 - 1$.	[2]
18. (b) Explain why $x = \frac{1}{2}$ is the only real root of the equation $8x^3 - 1 = 0$.	[2]
19. The polynomial f(x) is given by $f(x) = x^3 + ax^2 + bx + 24$, where a and b are constants. $x + 4$ is a factor of $f(x)$ and when $f(x)$ is divided by $x - 2$, the remainder is 36.	
(a) Find the value of a and b.	[4]
(b) Using the values from part (a), find the remainder when $f(x)$ is divided by $2x + 1$.	[2]
20. Express $\frac{3x+8}{(x-1)^2(x+2)}$ in partial fractions.	[4]
21. Express $\frac{x^2+18x-50}{x(x+5)^2}$ in partial fractions.	[5]

A4: POLYNOMIALS AND PARTIAL FRACTIONS (MARKING SCHEME)

1. Express $\frac{2+x^2}{(x-2)^2(x+4)}$ in partial fractions. [4]

$\frac{2+x^2}{(x-2)^2(x+4)} = \frac{A}{(x-2)} + \frac{B}{(x+2)^2} + \frac{C}{x+4}$

$2 + x^2 = A(x - 2)(x + 4) + B(x + 4) + C(x - 2)^2$

★Sub $x = 2$ (A and C = 0, to find B)

$2 + x^2 = A(2 - 2)(x + 4) + B(x + 4) + C(2 - 2)^2$

$2 + 4 = 0 + 6B + 0$

$B = 1$

★ Sub $x =- 4$

$2 + x^2 = A(x - 2)(x + 4) + B(x + 4) + C(x - 2)^2$

$2 + 16 = 0 + 0 + 36C$

$18 = 36C$

$C = \frac{1}{2}$

★ Sub $x = 0$

$2 + x^2 = A(x - 2)(x + 4) + B(x + 4) + C(x - 2)^2$

$2 + 0 = A(- 2)(4) + 4B + 4C$

$2 =- 8A + 4B + 4C$

Sub $B = 1,\ C = \frac{1}{2}$

$2 - 4(1) - 4\left(\frac{1}{2}\right) =- 8A$

$- 4 =- 8A$

$A = \frac{1}{2}$

$\frac{2+x^2}{(x-2)^2(x+4)} = \frac{1}{2(x-2)} + \frac{1}{(x+2)^2} + \frac{1}{2(x+4)}$

2. The function $f(x)$ is defined by $f(x) = 3x^2 + hx^2 + kx - 4$ for all real values of x. Given that $3x - 1$ is a factor of $f(x)$ and that when $f(x)$ is divided by $x + 1$, the remainder is -4, find the value of each of the constants h and k.

[5]

Let $f(x) = 3x^3 + hx^2 + kx - 4$

$(3x - 1)$ is a factor of $f(x)$ ★ No remainder since $(3x - 1)$ is a factor

$f\left(\frac{1}{3}\right) = 0$

$3\left(\frac{1}{3}\right)^3 + h\left(\frac{1}{3}\right)^2 + k\left(\frac{1}{3}\right) - 4 = 0$

$\frac{1}{9} + \frac{1}{9}h + \frac{1}{3}k - 4 = 0$

$1 + h + 3k - 36 = 0$

$h + 3k = 35 --- (1)$

When $f(x)$ is divided by $x + 1$, $R = -4$

$f(-1) = -4$

$3(-1)^3 + h(-1)^2 + k(-1) - 4 = -4$

$-3 + h - k - 4 + 4 = 0$

$h - k = 3 --- (2)$

$(1) - (2)$:

$h + 3k - (h - k) = 35 - 3$

$h + 3k - h + k = 32$

$4k = 32$

$k = 8$

$h = 11$

3. (a) Factorise $27x^3 + 125$. [2]

$27x^3 + 125$ ★ Factorise cubic equation: $a^3 + b^3 = (a + b)\left(a^2 - ab + b^2\right)$

$= (3x)^3 + (5)^3$

$= (3x + 5)\left(9x^2 - 15x + 25\right)$

3. (b) Explain why $x =- \frac{5}{3}$ is the only real root of the operation $27x^3 + 125 = 0$. [2]

Consider $\left(9x^2 - 15x + 25\right)$

$b^2 - 4ac$

$= (-15)^2 - 4(9)(25)$

$=- 675$

Since $b^2 - 4ac < 0$, therefore $9x^2 - 15x + 25$ cannot be factorised.

Thus, $x =- \frac{5}{3}$

4. Express $\frac{18x+7}{(x-1)(4x+1)}$ in partial fractions. [4]

$\frac{18x+7}{(x-1)(4x+1)} = \frac{A}{x-1} + \frac{B}{4x+1}$

$18x + 7 = A(4x + 1) + B(x - 1)$

Let $x = 1$

$18 + 7 = 5A$

$A = 5$

Let $x =- \frac{1}{4}$

$18\left(-\frac{1}{4}\right) + 7 = B\left(-\frac{1}{4} - 1\right)$

$\frac{5}{2} =- \frac{5}{4}B$

$B =- 2$

$\frac{18x+7}{(x-1)(4x+1)} = \frac{5}{x-1} - \frac{2}{4x+1}$

5. The polynomial $f(x)$ is given by $f(x) = 2x^3 + 5x^2 - 4x - 3$.

(a) Divide $f(x)$ by $x + 3$. [2]

★ Do long division

$$\begin{array}{r} 2x^2-x-1 \\ x+3\overline{)\;2x^3+5x^2-4x-3} \\ -\left(2x^3+6x^2\right) \\ \hline -x^2-4x \\ -\left(-x^2-3x\right) \\ \hline -x-3 \\ -\left(-x^2-3x\right) \\ \hline 0 \end{array}$$

(b) What can you deduce about $x + 3$? [1]

$x + 3$ is a factor of $f(x)$.

(c) Solve the equation $f(x) = 0$. [2]

$(x + 3)\left(2x^2 - x - 1\right) = 0$

$(x + 3)(2x + 1)(x - 1) = 0$

$x =- 3$ or $-\frac{1}{2}$ or 1

6. Factorise $x^3 - 8$ and explain why $x = 2$ is the only real root of the equation $x^3 - 8 = 0$. [4]

★ Factorise cubic equation: $a^3 - b^3 = (a - b)\left(a^2 + ab + b^2\right)$

$x^3 - 8 = (x - 2)(x^2 + 2x + 4)$

Consider $(x^2 + 2x + 4)$

$b^2 - 4ac$

$= (2)^2 - 4(1)(4)$

$=- 12$

$b^2 - 4ac < 0$, therefore $x^2 + 2x + 4$ cannot be factorised.

Thus, $x = 2$

7. By using long division, find the remainder when $9x^4 - 13x^2 + 4x - 2$ is divided by $3x^2 + 2x - 1$. [3]

$$\begin{array}{r} 3x^2-2x-2 \\ 3x^2+2x-1\overline{)9x^4+0x^3-13x^2+4x-2} \\ -(9x^4+6x^3-3x^2) \\ \hline -6x^3-10x^2+4x-2 \\ -(-6x^3-4x^2+2x) \\ \hline -6x^2+2x-2 \\ -(-6x^2-4x+2) \\ \hline 6x-4 \end{array}$$

Remainder is $6x - 4$

8. Given that $20x^2 + 13x + 5 = Ax(1 + 2x) + Bx + 5$ for all real values of x, find the value of A and of B. [3]

Sub $x =- \frac{1}{2}$

$20\left(-\frac{1}{2}\right)^2 + 13\left(-\frac{1}{2}\right) + 5 = 0 + B\left(-\frac{1}{2}\right) + 5$

$\frac{7}{2} - 5 =- \frac{1}{2}B$

$-\frac{3}{2} =- \frac{1}{2}B$

$B = 3$

Sub $x = 1$

$20(1)^2 + 13(1) + 5 = 3A + 3 + 5$

$20 + 13 + 5 - 8 = 3A$

$3A = 30$

$A = 10$

9. Express $\frac{6x}{(x+1)(x-1)^2}$ in partial fractions. [5]

$\frac{6x}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$

$6x = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$

Sub $x = 1$

$6 = 2C$

$C = 3$

Sub $x =- 1$

$-6 = 4A$

$A =- \frac{3}{2}$

Sub $x = 0$

$0 = A - B + C$

$B = A + C$

$B = 3 - \frac{3}{2}$

$B = \frac{3}{2}$

$\frac{6x}{(x+1)(x-1)^2} =- \frac{3}{2\left(x+1\right)} + \frac{3}{2\left(x-1\right)} + \frac{3}{(x-1)^2}$

10. Factorise $125a^3 - 8b^3$. [3]

$(5a)^3 - (2b)^3 = (5a - 2b)\left(25a^2 + 10ab + 4b^2\right)$

11. The polynomial $f(x) = 6x^3 + ax^2 + 11x - 6$ has a factor $(x + 2)$.	
(a) Show that $a = 19$. $f(-2) = 0$ $6(-2)^3 + a(-2)^2 + 11(-2) - 6 = 0$ $4a - 76 = 0$ $a = 19$ (shown)	[2]
(b) Solve the equation $f(x) = 0$. Show your working clearly. $6x^3 + 19x^2 + 11x - 6 = (x + 2)\left(6x^2 + bx - 3\right)$ $6x^3 + 19x^2 + 11x - 6 = 6x^3 + bx^2 - 3x + 12x^2 + 2bx - 6$ $6x^3 + 19x^2 + 11x - 6 = 6x^3 + (12 + b)x^2 + (-3 + 2b)x - 6$ Compare coefficient of x: $11 =- 3 + 2b$ $14 = 2b$ $b = 7$ Therefore, $(x + 2)\left(6x^2 + 7x - 3\right) = 0$ $(x + 2)(3x - 1)(2x + 3) = 0$ $x =- 2$ or $\frac{1}{3}$ or $-\frac{3}{2}$	[4]
(c) Hence solve the equation $\frac{6}{y^3} + \frac{19}{y^2} + \frac{11}{y} - 6 = 0$. Replace x with $\frac{1}{y}$: $\frac{1}{y} =- 2$ or $\frac{1}{3}$ or $-\frac{3}{2}$ $y =- \frac{1}{2}$ or 3 or $-\frac{2}{3}$	[2]

12. Factorise $8x^3 + 125$. [3]

$(2x)^3 + (5)^3 = (2x + 5)\left(4x^2 - 10x + 25\right)$

13. Express $\frac{3x^2-2}{(x+1)^2(2x-1)}$ in partial fractions. [5]

$\frac{3x^2-2}{(x+1)^2(2x-1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{2x-1}$

$3x^2 - 2 = A(x + 1)(2x - 1) + B(2x - 1) + C(x + 1)^2$

Sub $x =- 1$

$3 - 2 = 0 - 3B + 0$

$1 =- 3B$

$B =- \frac{1}{3}$

Sub $x = \frac{1}{2}$

$- \frac{5}{4} = 0 + 0 + \frac{9}{4}C$

$C =- \frac{5}{9}$

Sub $x = 0$

$- 2 =- A - B + C$

$- 2 =- A - \left(- \frac{1}{3}\right) - \frac{5}{9}$

$- 2 + \frac{2}{9} =- A$

$- \frac{16}{9} =- A$

$A = \frac{16}{9}$

$\frac{3x^2-2}{(x+1)^2(2x-1)} = \frac{16}{9(x+1)} - \frac{1}{3(x+1)^2} - \frac{5}{9\left(2x-1\ \right)}$

14. The polynomial $p(x)$ is given by $p(x) = 5x^3 + ax^2 + bx - 2$, where a and b are constants. It is given that $(x - 2)$ is a factor of $p(x)$ and when $p(x)$ is divided by $(x - 1)$, the remainder is 5.

(a) Show that $a =- 21$ and find the value of b. [4]

$p(2) = 0$

$5(2)^3 + a(2)^2 + b(2) - 2 = 0$

$40 + 4a + 2b - 2 = 0$

$4a + 2b + 38 = 0 --- (1)$

$p(1) =- 5$

$5(1)^3 + a(1)^2 + b(1) - 2 = 5$

$5 + a + b - 2 = 5$

$a + b - 2 = 0$

$a = 2 - b --- (2)$

Sub (2) to (1):

$4(2 - b) + 2b + 38 = 0$

$8 - 4b + 2b + 38 = 0$

$- 2b =- 46$

$b = 23$

$a =- 21$

(b) Using the values from part (a), find the remainder when $p(x)$ is divided by $(2x + 1)$. [2]

$p(x) = 5x^3 - 21x^2 + 23x - 2$

$p\left(-\frac{1}{2}\right) = 5\left(-\frac{1}{2}\right)^3 - 21\left(-\frac{1}{2}\right)^2 + 23\left(-\frac{1}{2}\right) - 2$

$p\left(-\frac{1}{2}\right) =- \frac{155}{8}$

15. (a) Divide $2x^{3} - x^{2} + 8x - 4$ by $2x - 1$. [1]

$$
\begin{array}{rl}
 & \quad\quad\quad\; x^{2} \; + \; 4 \\
2x-1 & \overline{)\,2x^{3}-x^{2}+8x-4} \\
- & \;\; 2x^{3}-x^{2} \\
\hline
 & \quad\quad\quad\quad 8x-4 \\
 & \quad\quad\quad\quad \underline{8x-4} \\
 & \quad\quad\quad\quad\quad\; 0
\end{array}
$$

15. (b) Express $\frac{2+5x-x^{2}}{2x^{3}-x^{2}+8x-4}$ in partial fractions. [5]

$$\frac{2+5x-x^{2}}{(2x-1)\left(x^{2}+4\right)} = \frac{A}{2x-1} + \frac{Bx+C}{x^{2}+4}$$

$$2 + 5x - x^{2} = A(x^{2} + 4) + (Bx + C)(2x - 1)$$

Sub $x = \frac{1}{2}$

$$2 + \frac{5}{2} - \frac{1}{4} = \frac{17}{4}A$$

$$\frac{17}{4} = \frac{17}{4}A$$

$$A = 1$$

Comparing coefficient of x^{2}

$$- x^{2} = Ax^{2} + 2Bx^{2}$$

$$- x^{2} = x^{2} + 2Bx^{2}$$

$$- 2x^{2} = 2Bx^{2}$$

$$B =- 1$$

Comparing constants

$$2 = 4A - C$$

$$2 = 4 - C$$

$$- 2 =- C$$

$$C = 2$$

$$\frac{2+5x-x^{2}}{(2x-1)\left(x^{2}+4\right)} = \frac{1}{2x-1} + \frac{2-x}{x^{2}+4}$$

16. The polynomial $p(x)$ is given by $p(x) = x^3 + 8x^2 + 21x + 9a$, where a is a constant. It is given that $(x + 2)$ is a factor of $p(x)$.	
(a) Find the value of the constant a. $p(x) = x^3 + 8x^2 + 21x + 9a$ $p(-2) = 0$ $(-2)^3 + 8(-2)^2 + 21(-2) + 9a = 0$ $-18 + 9a = 0$ $a = 2$	[2]
(b) Using the value from (a), find the remainder when $p(x)$ is divided by $(x - 1)$. $p(x) = x^3 + 8x^2 + 21x + 18$ $p(1) = (1)^3 + 8(1)^2 + 21(1) + 18$ $p(1) = 48$ Remainder $= 48$	[1]
17. (a) Show that $2x - 1$ is a factor of $f(x) = 2x^3 + 3x^2 - 8x + 3$. $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + 3\left(\frac{1}{2}\right)^2 - 8\left(\frac{1}{2}\right) + 3$ $f\left(\frac{1}{2}\right) = 0$ (shown)	[1]
17. (b) Hence, factorise $f(x)$ completely. $2x^3 + 3x^2 - 8x + 3 = (2x - 1)\left(x^2 + bx - 3\right)$ Comparing x^2 coefficient $3x^2 = 2bx^2 - x^2$ $3 = 2b - 1$ $4 = 2b$ $b = 2$ $2x^3 + 3x^2 - 8x + 3 = (2x - 1)\left(x^2 + 2x - 3\right)$ $2x^3 + 3x^2 - 8x + 3 = (2x - 1)(x + 3)(x - 1)$	[3]

18. (a) Factorise $8x^3 - 1$. $(2x)^3 - (1)^3 = (2x - 1)(4x^2 + 2x + 1)$	[2]
18. (b) Explain why $x = \frac{1}{2}$ is the only real root of the equation $8x^3 - 1 = 0$. Consider $4x^2 + 2x + 1$ $b^2 - 4ac$ $= (2)^2 - 4(4)(1)$ $=- 12$ Since $b^2 - 4ac < 0$, $4x^2 + 2x + 1$ cannot be factorised and has no real roots. $2x - 1 = 0$ Hence, $x = \frac{1}{2}$ is the only real root.	[2]

19. The polynomial f(x) is given by $f(x) = x^3 + ax^2 + bx + 24$, where a and b are constants. $x + 4$ is a factor of $f(x)$ and when $f(x)$ is divided by $x - 2$, the remainder is 36.

(a) Find the value of a and b. [4]

$f(-4) = 0$

$(-4)^3 + a(-4)^2 + b(-4) + 24 = 0$

$-64 + 16a - 4b + 24 = 0$

$16a - 4b = 40 --- (1)$

$f(2) = 36$

$(2)^3 + a(2)^2 + b(2) + 24 = 36$

$8 + 4a + 2b + 24 = 36$

$4a + 2b = 4$

$16a + 8b = 16 --- (2)$

$(2) - (1):$

$16a + 8b - (16a - 4b) = 16 - 40$

$8b + 4b =- 24$

$12b =- 24$

$b =- 2$

$a = 2$

(b) Using the values from part (a), find the remainder when $f(x)$ is divided by $2x + 1$. [2]

$f(x) = x^3 + 2x^2 - 2x + 24$

$f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) + 24$

$f\left(-\frac{1}{2}\right) = 25\frac{3}{8}$

Remainder $= 25\frac{3}{8}$

20. Express $\frac{3x+8}{(x-1)^2(x+2)}$ in partial fractions. $\frac{3x+8}{(x-1)^2(x+2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x+2}$ $3x + 8 = A(x - 1)(x + 2) + B(x + 2) + C(x - 1)^2$ Sub $x =- 2$ $2 = 0 + 0 + 9C$ $C = \frac{2}{9}$ Sub $x = 1$ $11 = 0 + 3B + 0$ $B = \frac{11}{3}$ Sub $x = 0$ $8 =- 2A + 2B + C$ $8 =- 2A + 2\left(\frac{11}{3}\right) + \frac{2}{9}$ $\frac{4}{9} =- 2A$ $A =- \frac{2}{9}$ $\frac{3x+8}{(x-1)^2(x+2)} =- \frac{2}{9(x-1)} + \frac{11}{3(x-1)^2} + \frac{2}{9(x+2)}$	[4]

21. Express $\frac{x^2+18x+50}{x(x+5)^2}$ in partial fractions. [5]

$$\frac{x^2+18x+50}{x(x+5)^2} = \frac{A}{x} + \frac{B}{x+5} + \frac{C}{(x+5)^2}$$

$x^2 + 18x + 50 = A(x + 5)^2 + B(x)(x + 5) + C(x)$

Sub $x = 0$

$50 = 25A + 0 + 0$

$A = 2$

Sub $x =- 5$

$25 - 90 + 50 = 0 + 0 - 5C$

$- 15 =- 5C$

$C = 3$

Comparing coefficient of x^2

$x^2 = Ax^2 + Bx^2$

$1 = 2 + B$

$B =- 1$

$$\frac{x^2+18x+50}{x(x+5)^2} = \frac{2}{x} - \frac{1}{x+5} + \frac{3}{(x+5)^2}$$