2023 NJC H2 Mathematics Preliminary Examination Paper 1

1 The equations of three planes p_1, p_2 and p_3 are

$$4x+6y-z=3$$
$$2x+5y-2z=1$$
$$\alpha x-3y+4z=\beta$$

respectively, where α and β are constants.

(i) Given that $\alpha=5$ and $\beta=9$, find the coordinates of the point that lies on all three planes. [1]

(ii) Find the values of α and β such that the line where p_1 and p_2 meet lies in p_3. [5]

2 In this question, k is a positive constant.

(a) Using an algebraic method, solve the inequality $\dfrac{2x^2-2k^2}{x}<x+k$. Express your answer in terms of k. [4]

(b) Hence, solve exactly the inequality $\dfrac{2x^2-10}{|x|}<|x|+\sqrt{5}$. [3]

3 It is given that $y=(1+\tan^{-1}x)^2$.

(a) Prove that $(1+x^2)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+2x(1+x^2)\dfrac{\mathrm{d}y}{\mathrm{d}x}=2.$ [3]

(b) Obtain an equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [1]

(c) Find the Maclaurin's series for y, up to and including the terms in x^3. [2]

(d) Deduce the equation of the normal to the curve $y=(1+\tan^{-1}x)^2$ at $x=0$. [1]

4 **(a)** Explain why $\int_0^{2a}\sqrt{4a^2-x^2}\,\mathrm{d}x=a^2\pi$, where a is a positive real value. [1]

It is given that

$$\mathrm{f}(x)=\begin{cases}\sqrt{4a^2-x^2} & \text{for } 0\le x\le 2a,\\ 3x-6a & \text{for } 2a<x<3a,\end{cases}$$

and that $\mathrm{f}(x+3a)=\mathrm{f}(x)$ for all real values of x.

(b) Find $\mathrm{f}(8.5a)$. [2]

(c) Sketch the graph of $y=\mathrm{f}(x)$ for $-3a\le x\le 8.5a$. [2]

(d) Find $\int_0^{\frac{599}{2}a}\mathrm{f}(x)\,\mathrm{d}x$ in the exact form. [3]

5 **(a)** A curve C has equation $y=\dfrac{4x-9}{5-2x}$. Describe a sequence of three transformations which transforms the graph of C onto the graph of $y=-\dfrac{1}{x}$. [3]

(b) It is given that $\mathrm{f}\left(x\right)=\sqrt{1+\dfrac{x^2}{q^2}}$, where q is a positive constant. State the shape of $y=\mathrm{f}\left(x\right)$. [1]

On separate diagrams, sketch the curves with equations

(i) $y=\dfrac{1}{\mathrm{f}\left(x\right)}$,

(ii) $y=\mathrm{f}'\left(x\right)$,

stating the equations of any asymptotes and the coordinates of the points where the curve crosses the axes. [4]

6 A sequence u_1, u_2, u_3, … is given by the relation

$$u_{n+1}=\frac{u_n}{10}+k\text{, where } k \text{ is a constant and } n\geq 1.$$

It is given that $u_1=-3$.

(a) Write down u_2 and u_3 in terms of k. [2]

It is further given that $u_n=\beta+\alpha\left(\dfrac{1}{10}\right)^n$ for any positive integer.

(b) Show that $\alpha=-\dfrac{100}{9}k-30$ and find β in terms of k. [3]

(c) State $\lim\limits_{n\to\infty}u_n$ in terms of k. [1]

(d) Given that $k=9$, find $\sum\limits_{n=1}^{N}u_n$ in terms of N. [3]

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7 **(a)** The complex numbers, z_1, z_2 and z_3 are $\sqrt{2}-\mathrm{i}\sqrt{2}$, $2+\mathrm{i}\left(2\sqrt{3}\right)$ and $\dfrac{\mathrm{e}^{\mathrm{i}\frac{\pi}{12}}}{5}$ respectively.

The complex number w is given by $w=\dfrac{z_2}{\left(z_3{}^*\right)z_1}$.

(i) Find w in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r>0$ and $-\pi<\theta\le\pi$. [4]

(ii) In the same Argand diagram, sketch the points representing w, $w+5$ and w^*. [3]

(b) Another complex number v is such that $\dfrac{v+c}{v-c}$ is purely imaginary, where c is a positive real number. Show that $|v|=c$. [3]

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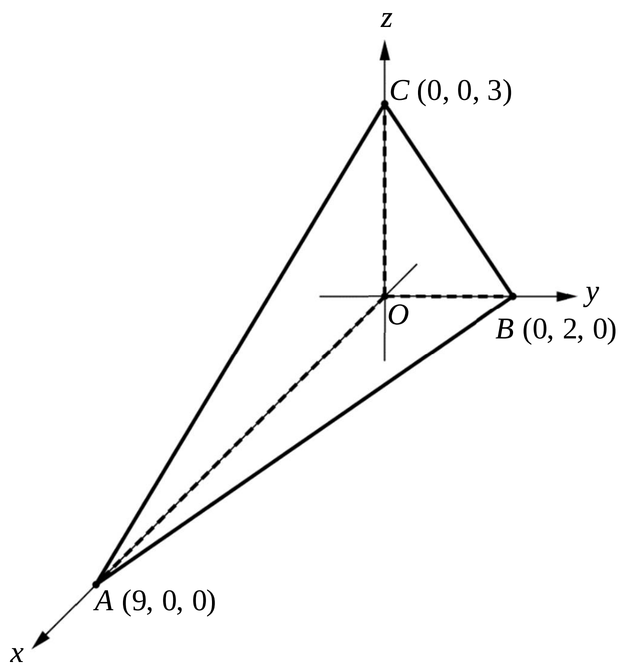

A tetrahedron $OABC$ has four triangular faces with a vertex O at the origin. The coordinates of A, B and C are $(9, 0, 0)$, $(0, 2, 0)$ and $(0, 0, 3)$ respectively (see diagram).

(a) Find a cartesian equation of face ABC. [3]

(b) Find the acute angle between face ABC and the z-axis. [2]

(c) Find the coordinates of the foot of perpendicular from O to face ABC. [2]

(d) The line s passes through O and the midpoint of AC. Show that a vector equation of the of reflection of s in face ABC, is $\mathbf{r}=\begin{pmatrix}\frac{9}{2}\\0\\\frac{3}{2}\end{pmatrix}+\mu\begin{pmatrix}a\\-216\\-23\end{pmatrix}$ where a is a constant to be determined. [3]

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9 A curve C is defined by the parametric equations

$$x = 4\sin t,$$
$$y = b\cos t - 1,$$

where $0 \le t \le \pi$ and $1 < b < 4$.

(a) Sketch C, labelling the points where it meets the y-axis. [2]

(b) Show that the tangent to C that is parallel to $y = \frac{1}{4}bx$ has equation

$$y = \frac{bx}{4} - \sqrt{2}b - 1.$$ [4]

Let $b = 2$.

(c) Find the exact area of the region bounded by C, the tangent found in **(b)** and the y-axis. [5]

10 Jeremy plans to invest \$100 000 at the beginning of year 2024 into a current account which pays **compound** interest at a rate of 3.2% per annum. The interest is paid on the last day of each year based on the amount in the account. He also puts a further \$$X$ into the current account on the first day of each subsequent year.

(a) Find the amount that Jeremy will have in the current account at the end of the year 2025 in terms of X. [2]

(b) Find a simplified expression for the amount that Jeremy will have in the current account at the end of the nth year. [3]

Let $X = 8000$ for the remaining parts of the question.

Jeremy plans to buy a car which costs \$170 000.

(c) In which year will Jeremy have sufficient money in the current account to buy the car? Explain whether this occurs on the first or last day of the year. [3]

Car financing loans are based on **simple** interest on the principal amount loaned. A bank offers a car loan at an annual interest rate of 2.8% for m years to be repaid by equal instalments at the end of every year.

(d) If Jeremy takes a loan of \$170 000, show that the annual instalment will be $\frac{1}{m}(170000 + 4760m)$. [1]

Jeremy is considering the option to take this car loan for 10 years so that he can own the car earlier. From year 2025, instead of putting \$8000 into the current account at the beginning of every year, he will use it to partially pay the annual instalment. The rest of the annual instalment will be deducted from the current account. Justify whether this arrangement is possible. [3]

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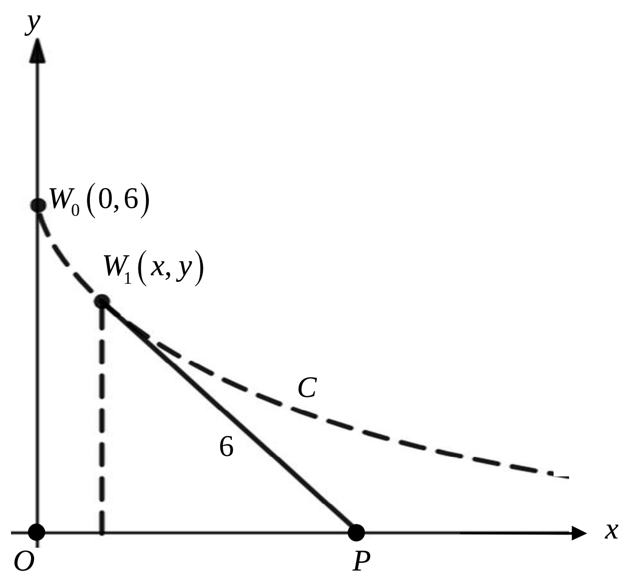
11 Baby Peter is pulling a toy wagon on a flat ground which is modelled by the *xy*-coordinate plane. Peter is initially at the origin O and the toy wagon is located on the y-axis at $W_0(0,6)$. The toy wagon is tethered to him with a rope of length 6 units. As he moves in the direction of the positive x-axis (given by point P), he pulls the toy wagon (given by the point W_1) using the rope W_1P which is kept taut throughout the motion. The path of the toy wagon is given by curve C. It is known that the rope is always tangent to C.



(a) Show that a differential equation for C can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\alpha y}{\sqrt{36-y^2}},$$

where α is a constant to be determined. [2]

(b) Using the substitution $w^2=36-y^2$ where $w>0$, show that the differential equation in **(a)** can be reduced to

$$\frac{\mathrm{d}w}{\mathrm{d}x}=\frac{36-w^2}{w^2}.$$ [2]

(c) Find x in terms of y. [6]

(d) Determine the behaviour of x as y approaches 0. [2]

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