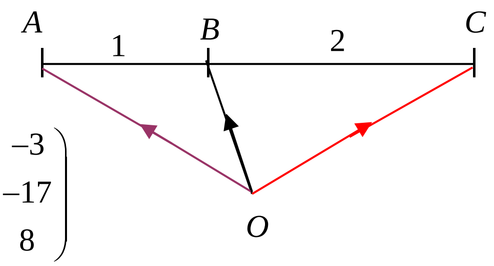
H2 Mathematics (9758)
Chapter 5 Vectors
Extra Practice Solutions

Qn 1	2014/IJC Promo/4
(i)	$\overrightarrow{AO}=\begin{pmatrix}-1\\-3\\2\end{pmatrix}$ and $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\begin{pmatrix}-7\\1\\-5\end{pmatrix}$ $\cos\measuredangle OAB=\dfrac{\begin{pmatrix}-1\\-3\\2\end{pmatrix}\bullet\begin{pmatrix}-7\\1\\-5\end{pmatrix}}{\sqrt{14}\sqrt{75}}=\dfrac{7-3-10}{\sqrt{14}\sqrt{75}}=\dfrac{-6}{\sqrt{1050}}$ $\measuredangle OAB=\cos^{-1}\left(\dfrac{-6}{\sqrt{1050}}\right)=100.7^{\text{o}}$ (1d.p)
(ii)	Vector perpendicular to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$ $=\overrightarrow{OA}\times\overrightarrow{OB}$ $=\begin{pmatrix}1\\3\\-2\end{pmatrix}\times\begin{pmatrix}-6\\4\\-7\end{pmatrix}$ $=\begin{pmatrix}-13\\19\\22\end{pmatrix}$ $\therefore$ Required unit vector $=\dfrac{\begin{pmatrix}-13\\19\\22\end{pmatrix}}{\left\|\begin{pmatrix}-13\\19\\22\end{pmatrix}\right\|}=\dfrac{1}{\sqrt{1014}}\begin{pmatrix}-13\\19\\22\end{pmatrix}$ (**alternatively**, $=\dfrac{\overrightarrow{OA}\times\overrightarrow{BO}}{\left\|\overrightarrow{OA}\times\overrightarrow{BO}\right\|}=\ldots\ldots=\dfrac{1}{\sqrt{1014}}\begin{pmatrix}13\\-19\\-22\end{pmatrix}$)

Qn 2	2016/MI Prelim/I/Q3

Using ratio theorem,

$$\overrightarrow{OB}=\frac{2\overrightarrow{OA}+\overrightarrow{OC}}{3}$$



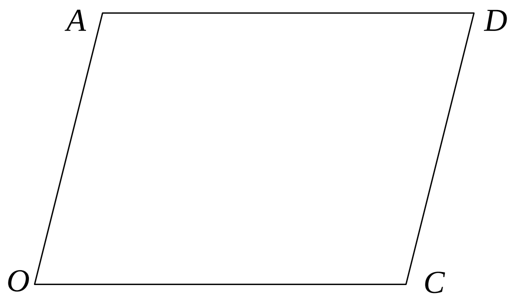
$$\overrightarrow{OC}=3\overrightarrow{OB}-2\overrightarrow{OA}=3\begin{pmatrix}1\\-5\\4\end{pmatrix}-2\begin{pmatrix}3\\1\\2\end{pmatrix}=\begin{pmatrix}-3\\-17\\8\end{pmatrix}$$

$$\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{AD}$$

$$=\overrightarrow{OA}+\overrightarrow{OC}\quad (\because \text{ equal vectors } \overrightarrow{OC}=\overrightarrow{AD})$$

$$=\begin{pmatrix}3\\1\\2\end{pmatrix}+\begin{pmatrix}-3\\-17\\8\end{pmatrix}$$

$$=\begin{pmatrix}0\\-16\\10\end{pmatrix}$$



Area of $OADC$

$$=\left|\overrightarrow{OA}\times\overrightarrow{OC}\right|$$

$$=\left|\begin{pmatrix}3\\1\\2\end{pmatrix}\times\begin{pmatrix}-3\\-17\\8\end{pmatrix}\right|=\left|\begin{pmatrix}42\\-30\\-48\end{pmatrix}\right|=6\left|\begin{pmatrix}7\\-5\\-8\end{pmatrix}\right|$$

$$=6\sqrt{7^2+(-5)^2+(-8)^2}$$

$$=6\sqrt{138}$$

Qn 3	2017/CJC Prelim/II/Q2

(i) Length of projection of $\mathbf{a}$ on to $\mathbf{b}$ $\quad(\because \mathbf{b}$ is a unit vector)

(ii)

$$|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$$

$$=(2)(1)\sin\frac{\pi}{4}$$

$$=\sqrt{2}$$

(iii)

$$\mathbf{p}\times\mathbf{q}$$

$$=\left[3\mathbf{a}+(\mu+2)\mathbf{b}\right]\times\left[(\mu+3)\mathbf{a}+\mu\mathbf{b}\right]$$

$$=3(\mu+3)(\mathbf{a}\times\mathbf{a})+3\mu(\mathbf{a}\times\mathbf{b})+(\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})+\mu(\mu+2)(\mathbf{b}\times\mathbf{b})$$

$$=(-3\mu+\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})\ [\because \mathbf{a}\times\mathbf{a}=\mathbf{0} \text{ and } \mathbf{b}\times\mathbf{b}=\mathbf{0}]$$

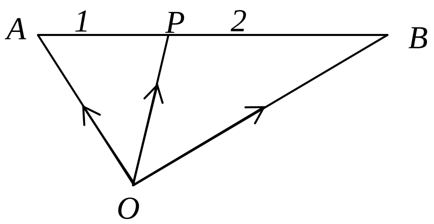
$$=(\mu^2+2\mu+6)(\mathbf{b}\times\mathbf{a})$$

(iv)	Area OPQ = $\frac{1}{2}\left\|\left(\mu^2+2\mu+6\right)\right\|\left\|(\mathbf{b}\times\mathbf{a})\right\|$ $=\frac{1}{2}\left\|\left(\mu^2+2\mu+6\right)\right\|\sqrt{2}$ $=\frac{\sqrt{2}}{2}\left\|\left(\mu+1\right)^2+5\right\|$ Smallest Area OPQ = $=\frac{5\sqrt{2}}{2}\text{unit}^2$ Minimum value of $\left\|\left(\mu+1\right)^2+5\right\|$ is 5. (Minimum point of quadratic equation)

Qn 4	**2014/TPJC Promo/4**
(i)	By Ratio Theorem: $\overrightarrow{OC}=\frac{1}{4}(3\mathbf{a}+\mathbf{b})$
(ii)	$\|\mathbf{b}\|=\sqrt{5}\|\mathbf{a}\|=\sqrt{5}\sqrt{10}=\sqrt{50}$ $\therefore\|\mathbf{b}\|=\sqrt{3^2+5^2+k^2}=\sqrt{50}$ $\Rightarrow\sqrt{34+k^2}=\sqrt{50}$ $\Rightarrow 34+k^2=50$ $\Rightarrow k^2=16$ $\Rightarrow k=4$ ($\because k$ is a positive constant)
(iii)	$\overrightarrow{OC}=\frac{1}{4}\left[\begin{pmatrix}3\\9\\0\end{pmatrix}+\begin{pmatrix}3\\5\\4\end{pmatrix}\right]=\begin{pmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{pmatrix}$ Area of triangle $OAC=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{OC}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}1\\3\\0\end{pmatrix}\times\begin{pmatrix}\frac{3}{2}\\\frac{7}{2}\\1\end{pmatrix}\right\|$ $=\frac{1}{2}\left\|\begin{pmatrix}3\\-1\\-1\end{pmatrix}\right\|$ $=\frac{\sqrt{11}}{2}$

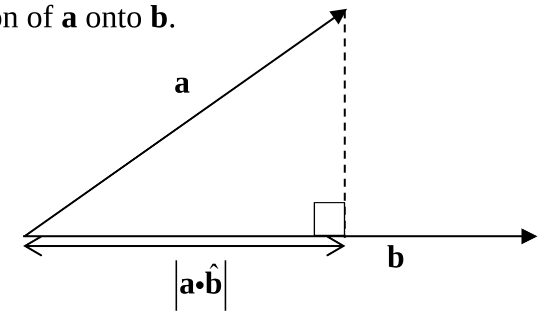
(iv)	Length of projection of $\overrightarrow{OC}$ onto the line OB $=\left\lvert \overrightarrow{OC}\bullet\frac{\overrightarrow{OB}}{\lvert\overrightarrow{OB}\rvert}\right\rvert$ $=\frac{1}{5\sqrt{2}}\left\lvert \begin{pmatrix}\frac{3}{2}\\ \frac{7}{2}\\ 1\end{pmatrix}\bullet\begin{pmatrix}3\\5\\4\end{pmatrix}\right\rvert$ $=\frac{1}{5\sqrt{2}}\left\lvert\frac{9}{2}+\frac{35}{2}+4\right\rvert=\frac{26}{5\sqrt{2}}=\frac{13\sqrt{2}}{5}$

Qn 5	**2015/DHS Promo/2**
(i)	$\overrightarrow{AP}=\lambda\overrightarrow{AB}$ $\overrightarrow{OP}-\overrightarrow{OA}=\lambda\left(\overrightarrow{OB}-\overrightarrow{OA}\right)$ $\overrightarrow{OP}=\begin{pmatrix}2\\-1\\1\end{pmatrix}+\lambda\begin{pmatrix}-4\\6\\0\end{pmatrix}$
(ii)	$\overrightarrow{OP}\perp\overrightarrow{AB}\Rightarrow\overrightarrow{OP}\bullet\overrightarrow{AB}=0$ $\begin{pmatrix}2-4\lambda\\-1+6\lambda\\1\end{pmatrix}\bullet\begin{pmatrix}-4\\6\\0\end{pmatrix}=0$ $-8+16\lambda-6+36\lambda=0$ $52\lambda=14$ $\lambda=\frac{7}{26}$
(iii)	Area of triangle OPA = $\frac{1}{2}\left\lvert\overrightarrow{OP}\times\overrightarrow{OA}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}\frac{2}{3}\\1\\1\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}\right\rvert$ $=\frac{1}{2}\left\lvert\begin{pmatrix}2\\ \frac{4}{3}\\ -\frac{8}{3}\end{pmatrix}\right\rvert=\frac{\sqrt{29}}{3}$ units^2 or 1.80 units^2

(iv)	 Since the two triangles share the same height, Area of triangle *OPB* : Area of triangle *OPA* $= PB : PA = 2:1$

Qn 6	**2015/NYJC Promo/4**
(i)	$\mathbf{a}\bullet(2\mathbf{a}+5\mathbf{b})=0$ $2\|\mathbf{a}\|^2+5\mathbf{a}\bullet\mathbf{b}=0$ $2+5\mathbf{a}\bullet\mathbf{b}=0$ since $\mathbf{a}$ is a unit vector $\mathbf{a}\bullet\mathbf{b}=-\frac{2}{5}$ Since angle between **a** and **b** is $\frac{2\pi}{3}$, $\cos\left(\frac{2\pi}{3}\right)=\frac{\mathbf{a}\bullet\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|}$ $-\frac{1}{2}=\frac{-2/5}{(1)\|\mathbf{b}\|}$ $\|\mathbf{b}\|=\frac{4}{5}$ (shown)
(ii)	By Ratio theorem, $\overrightarrow{OM}=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$ $\overrightarrow{ON}=\overrightarrow{MB}$ $=\mathbf{b}-[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}]$ $=(1-\lambda)\mathbf{b}-(1-\lambda)\mathbf{a}$ Area of triangle *OAN* $=\frac{1}{2}\left\|\overrightarrow{OA}\times\overrightarrow{ON}\right\|$ $=\frac{1}{2}\left\|\mathbf{a}\times[(1-\lambda)\mathbf{b}-(1-\lambda)\mathbf{a}]\right\|$

	$=\frac{1}{2}\left\| (1-\lambda)\mathbf{a}\times\mathbf{b}-(1-\lambda)\mathbf{a}\times\mathbf{a}\right\|$ $=\frac{1}{2}(1-\lambda)\|\mathbf{a}\times\mathbf{b}\|$ $=\frac{1}{2}(1-\lambda)\|\mathbf{a}\|\|\mathbf{b}\|\sin\left(\frac{2\pi}{3}\right)=\frac{(1-\lambda)}{2}\left(\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right)$ $=\frac{(1-\lambda)\sqrt{3}}{5}$ Note that $\|\mathbf{a}\times\mathbf{b}\|=\|\mathbf{a}\|\|\mathbf{b}\|\sin\left(\frac{2\pi}{3}\right)=\left(\frac{4}{5}\right)\left(\frac{\sqrt{3}}{2}\right)=\frac{2\sqrt{3}}{5}$

Qn 7	**2012/DHS/I/7**
(i)	$\overrightarrow{OD}=\frac{3\mathbf{a}+2p\mathbf{b}}{5}$ $\overrightarrow{OE}=\frac{3\mathbf{a}+\mathbf{b}}{4}$
(ii)	$\overrightarrow{OD}=q\overrightarrow{OE}$, where q is a constant $\frac{3\mathbf{a}+2p\mathbf{b}}{5}=q\left(\frac{3\mathbf{a}+\mathbf{b}}{4}\right)$ $\Rightarrow \frac{3}{5}=\frac{3}{4}q \Rightarrow q=\frac{4}{5}$ $\Rightarrow \frac{2}{5}p=\frac{1}{4}q \Rightarrow p=\frac{1}{2}$
(iii)	Shortest distance from the point E to OB $=\left\|\overrightarrow{OE}\times\frac{\overrightarrow{OB}}{OB}\right\|$ $=\left\|\left(\frac{3\mathbf{a}+\mathbf{b}}{4}\right)\times\frac{\mathbf{b}}{5}\right\|$ $=\frac{1}{20}\left\|(3\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{b})\right\|$ $=\frac{3}{20}\|\mathbf{a}\times\mathbf{b}\| \quad (\because \mathbf{b}\times\mathbf{b}=\mathbf{0})$ $k=\frac{3}{20}$
(iv)	It is the length of projection of **a** onto **b**. 

Qn 8	2008/HCI/I/12a
(i)	$(-\mu-1)\,\mathbf{a}+\mu\mathbf{b}+\mathbf{c}=0 \quad \because \quad \lambda=-\mu-1$ $\mu(\mathbf{b}-\mathbf{a})+(\mathbf{c}-\mathbf{a})=0$ $\mu\overrightarrow{AB}+\overrightarrow{AC}=0$ $\overrightarrow{AB}=k\,\overrightarrow{AC}$ where $k=-\frac{1}{\mu}$ $\Rightarrow A, B, C$ are collinear
(ii)	$\mathbf{p}=4\mathbf{a}-3\mathbf{b}$ $\Rightarrow 4\mathbf{a}=\mathbf{p}+3\mathbf{b}$ $\Rightarrow \mathbf{a}=(\mathbf{p}+3\mathbf{b})/4$ A divides PB in the ratio 3 : 1 $\therefore$ P lies on BA produced with ratio $PA:PB=3:4$ **Alternatively** $\overrightarrow{PA}=\mathbf{a}-\mathbf{p}=\mathbf{a}-4\mathbf{a}+3\mathbf{b}=-3\mathbf{a}+3\mathbf{b}$ $\overrightarrow{PB}=\mathbf{b}-\mathbf{p}=\mathbf{b}-4\mathbf{a}+3\mathbf{b}=-4\mathbf{a}+4\mathbf{b}$ $\therefore$ P lies on BA produced with ratio $PA:PB=3:4$

Qn 9	2019/TJC/Prelim 9758/01/Q9
(a)	Since $\mathbf{u}\times\mathbf{v}+\mathbf{u}$ is perpendicular to $\mathbf{u}\times\mathbf{v}+\mathbf{v}$, we have $((\mathbf{u}\times\mathbf{v})+\mathbf{u}).((\mathbf{u}\times\mathbf{v})+\mathbf{v})=0$ $\Rightarrow \quad (\mathbf{u}\times\mathbf{v}).(\mathbf{u}\times\mathbf{v})+(\mathbf{u}\times\mathbf{v}).\mathbf{v}+(\mathbf{u}\times\mathbf{v}).\mathbf{u}+\mathbf{u}.\mathbf{v}=0$ $\Rightarrow \quad \lvert\mathbf{u}\times\mathbf{v}\rvert^2+0+0-1=0$ since $(\mathbf{u}\times\mathbf{v})\perp\mathbf{v}$ and $(\mathbf{u}\times\mathbf{v})\perp\mathbf{u}$ $\Rightarrow \quad \lvert\mathbf{u}\times\mathbf{v}\rvert=1$ (shown) Let θ be the angle between **u** and **v**. $\mathbf{u}.\mathbf{v}=-1 \quad \Rightarrow \quad \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta=-1$ ---(1) $\lvert\mathbf{u}\times\mathbf{v}\rvert=1 \quad \Rightarrow \quad \lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin\theta=1$ ---(2) $\frac{(2)}{(1)}: \quad \tan\theta=-1 \quad \Rightarrow \quad \theta=135^\circ$
(b)	$\overrightarrow{OF}=\overrightarrow{OA}+\overrightarrow{AF}=\mathbf{a}-\mathbf{b}$ $\overrightarrow{OX}=\overrightarrow{OB}+\overrightarrow{BX}=\mathbf{b}-\frac{1}{2}\mathbf{a}$ **<u>Method 1</u>** Let $AY:YC=\lambda:1-\lambda$ and $FY:YX=\mu:1-\mu$ $\therefore \quad \overrightarrow{OY}=\lambda\overrightarrow{OC}+(1-\lambda)\overrightarrow{OA}=\mu\overrightarrow{OX}+(1-\mu)\overrightarrow{OF}$ $\Rightarrow \quad \lambda(\mathbf{b}-\mathbf{a})+(1-\lambda)\mathbf{a}=\mu\left(\mathbf{b}-\frac{1}{2}\mathbf{a}\right)+(1-\mu)(\mathbf{a}-\mathbf{b})$ $\Rightarrow \quad (\lambda-\mu+1-\mu)\mathbf{b}=\left(\lambda-1+\lambda-\frac{1}{2}\mu+1-\mu\right)\mathbf{a}$ Since **a** and **b** are non-zero and non-parallel,

$$\left.\begin{aligned} \lambda - 2\mu + 1 = 0 \\ 2\lambda - \frac{3}{2}\mu = 0 \end{aligned}\right\} \text{ solving gives } \lambda = \frac{3}{5}, \quad \mu = \frac{4}{5}$$

$$\therefore AY : YC = \frac{3}{5} : 1 - \frac{3}{5} = 3 : 2$$

<u>Method 2</u>

Line AC: $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - 2\mathbf{a})$, $\lambda \in \mathbb{R}$

Line FX: $\mathbf{r} = \mathbf{a} - \mathbf{b} + \mu\left(2\mathbf{b} - \frac{3}{2}\mathbf{a}\right)$, $\mu \in \mathbb{R}$

When the lines intersect at Y,

$$\mathbf{a} + \lambda(\mathbf{b} - 2\mathbf{a}) = \mathbf{a} - \mathbf{b} + \mu\left(2\mathbf{b} - \frac{3}{2}\mathbf{a}\right)$$

Since **a** and **b** are non-zero and non-parallel,

$$\left.\begin{aligned} \lambda - 2\mu + 1 = 0 \\ 2\lambda - \frac{3}{2}\mu = 0 \end{aligned}\right\} \text{ solving gives } \lambda = \frac{3}{5}, \quad \mu = \frac{4}{5}$$

$$\overrightarrow{OY} = \mathbf{a} + \frac{3}{5}(\mathbf{b} - 2\mathbf{a}) = -\frac{1}{5}\mathbf{a} + \frac{3}{5}\mathbf{b}$$

$$\overrightarrow{AY} = -\frac{6}{5}\mathbf{a} + \frac{3}{5}\mathbf{b} \quad \text{and} \quad \overrightarrow{YC} = -\frac{4}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$$

$$\therefore \quad AY : YC = 3 : 2$$

Qn 10	2012/HCI/II/Q4
(i)	$AB \perp OP \Rightarrow \overrightarrow{AB} \cdot \overrightarrow{OP} = 0$ $\therefore (\mathbf{b}-\mathbf{a}) \cdot (\mathbf{a}+5\mathbf{b}) = 0$ $\mathbf{b}\cdot\mathbf{a} + 5\mathbf{b}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{a} - 5\mathbf{a}\cdot\mathbf{b} = 0$ $5\|\mathbf{b}\|^2 - \|\mathbf{a}\|^2 - 4\mathbf{a}\cdot\mathbf{b} = 0$ $5 - \|\mathbf{a}\|^2 - 4\|\mathbf{a}\|\|\mathbf{b}\|\cos 60° = 0$ $\|\mathbf{a}\|^2 + 2\|\mathbf{a}\| - 5 = 0$ $\|\mathbf{a}\| = \frac{-2 \pm \sqrt{24}}{2} = \sqrt{6}-1$ or $-\sqrt{6}-1$ (rejected as $\|\mathbf{a}\| > 0$) $\therefore \|\mathbf{a}\| = \sqrt{6}-1$
(ii)	$\overrightarrow{OC} = \frac{1}{2}\mathbf{a}$ By Ratio Theorem, $\overrightarrow{OE} = \frac{3\overrightarrow{OB} + 4\overrightarrow{OC}}{7} = \frac{3\mathbf{b}+2\mathbf{a}}{7}$ Let $AD : AB = \lambda : 1$ By Ratio Theorem, $\therefore \overrightarrow{OD} = \lambda\overrightarrow{OB} + (1-\lambda)\overrightarrow{OA} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ Since *O*, *E*, *D* are collinear, $\overrightarrow{OE} = \mu\overrightarrow{OD}$ for some $\mu \in \mathbb{R} \setminus \{0\}$. $\therefore \mu\left(\frac{3\mathbf{b}+2\mathbf{a}}{7}\right) = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$ $\left(\frac{3}{7}\mu - \lambda\right)\mathbf{b} = \left(1-\lambda-\frac{2}{7}\mu\right)\mathbf{a}$ Since **a** and **b** are non-zero and non-parallel, $\frac{3}{7}\mu - \lambda = 0$ --- (1) $1-\lambda-\frac{2}{7}\mu = 0$ --- (2) Using GC to solve (1) and (2), $\lambda = \frac{3}{5}$ $\therefore AD : AB = 3:5$