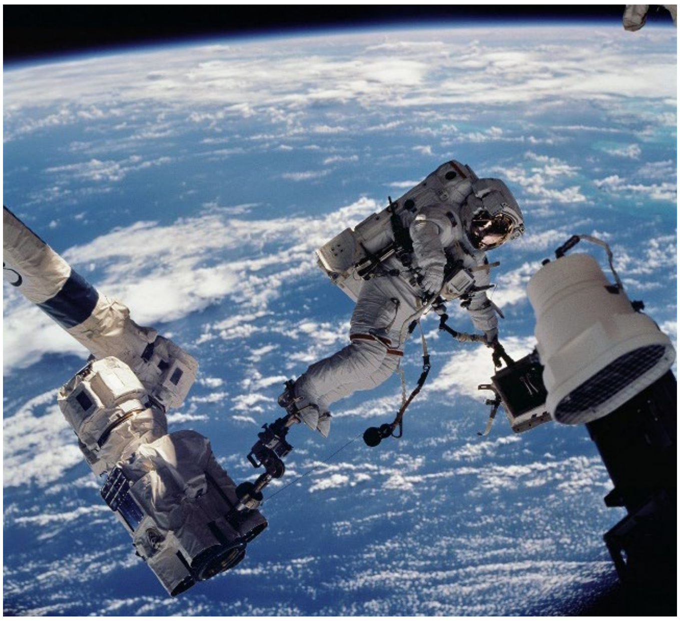
H2 Topic 07 – Gravitational Field



Despite the apparent "weightlessness" of astronauts at the International Space Station (ISS), they are still subject to a gravitational acceleration of 8.7 m s^{-1}, or about 89% of "1 g". If the Earth's gravity were to suddenly stop acting on ISS, the ISS will move away from Earth tangentially at a constant velocity instead. Find out more here in this topic.

Content:

- Gravitational field
- Gravitational force between point masses
- Gravitational field of a point mass
- Gravitational field near to the surface of the earth
- Gravitational potential
- Circular Orbits

Learning Objectives:

Candidates should be able to:

(a) show an understanding of the concept of a gravitational field as an example of field of force and define gravitational field strength at a point as the gravitational force exerted per unit mass place at that point.

(b) recognise the analogy between certain qualitative and quantitative aspect of gravitational and electric fields.

(c) recall and use Newton's law of gravitation in the form $F = \frac{Gm_1m_2}{r^2}$.

(d) derive, from Newton's law of gravitation and the definition of gravitational field strength, the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass.

(e) recall and apply the equation $g = \frac{GM}{r^2}$ for the gravitational field strength of a point mass to new situations or to solve related problems.

(f) show an understanding that on the surface of the Earth g is approximately constant and equal to the acceleration of free fall.

(g) define the gravitational potential at a point as the work done per unit mass in bringing small test mass from infinity to the point.

(h) solve problems using the equation $\phi = -\frac{GM}{r}$ for the potential in the field of a point mass.

(i) analyse circular orbits in inverse square law fields by relating the gravitational force to the centripetal acceleration it causes.

(j) show an understanding of geostationary orbits and their application.

(bold) items are needed for 8867 H1 Physics

7.1 Gravitational Field

A “field” is a *region of space* where a force is experienced by an “entity” without contact. A *mass* experiences a gravitational force when placed in a *gravitational field*.

> A **gravitational field** is
>
> a region of space
> where a mass
> experiences a gravitational force.

Forces are vector quantities so the direction has to be well-defined.

Gravitational force is a purely attractive force.

Gravity is an attractive force. *Unfortunately, the anti-gravity hover-board from the “Back to the Future” franchise is one piece of technology that will stay as science-fiction.*

An isolated mass generates its gravitational field in the region surrounding the mass and it permeates all of space (“to infinity”). When another mass is inside this region, that mass *interacts* with the existing gravitational field and *both* masses experience gravitational force.

7.1.1 Field Lines

Field lines represent a field visually. Arrows show the direction of force acting on a *test* mass.

A *test* particle is an idealized object which does not *alter* the behaviour of the rest of the system.

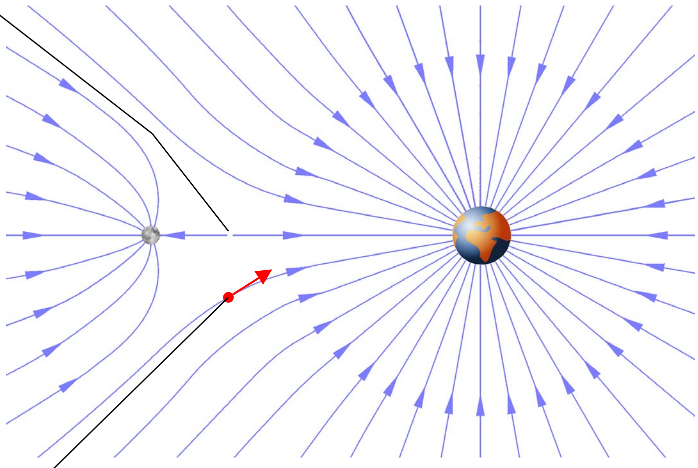
> A **line of force** in a gravitational field is
>
> the direction of the gravitational force
> acting on a small test mass.

The value of a test particle is shown when there is more than one body of significant mass. The diagram below shows the *resultant* field lines of the gravitational field *jointly* generated by both the Earth and the moon:

The moon is not a test particle. It itself generates a gravitational field of its own and alters the behaviour of Earth’s gravitational field. If we replace the moon with a test particle, the resulting field will be purely radial towards Earth’s centre.

At this point (location), a test mass is equally attracted to both the moon and the Earth:

Both the moon and Earth will attract any test masses in this left-hand-side region. So the field lines here generally point to the right.



In this right-hand-side region, the smaller mass of the moon results in negligible changes to the resultant field pattern; as if only Earth’s gravitational field is present.

At this point (location), a test mass will be attracted to both the moon (weaker) and the Earth (more strongly). The resultant direction is due to the vector sum of forces:

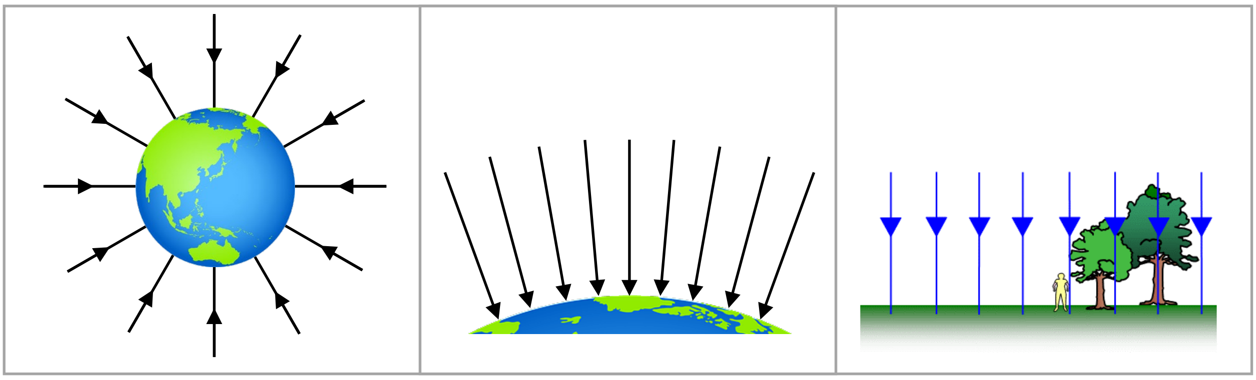
> The tangent at a point on a gravitational field line is the direction of gravitational force acting on a test mass placed at that point. Field lines will never cross each other.

Example 1

By reference to lines of gravitational force near the Earth's surface, explain why the acceleration of free fall near the Earth's surface is approximately constant.

Solution

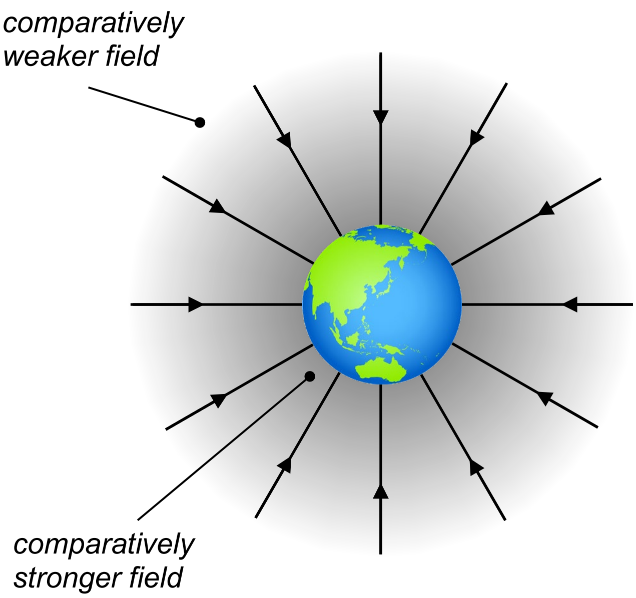
- The lines of force are radial and appear to converge at the centre of Earth.
- Earth is a large radius, relatively the heights near and above the surface is small so the lines are approximately parallel.
- The parallel field lines indicate uniform field strength hence constant acceleration of free fall.



Notes: (i) we represent uniform fields using equally-spaced parallel field lines pointing in the same direction. (ii) A uniform field is one where the field strength has the same magnitude and same direction everywhere within the field. (iii) Near Earth's surface, by Newton's 2nd Law, $F_{net} = ma = mg$ so the gravitational field strength g is equal to the acceleration of free fall.

There is a formal definition for *field strength* which we will introduce later. Visually, from the above example we can already see that gravitational field strength is

- stronger where gravitational field lines are closer together
- weaker where gravitational field lines are further apart from each other



Now that we are familiar with

1. the direction of gravitational force (mutual attraction between masses)
2. the general variation (the further away, the weaker the force of attraction)

We now look at how to *quantify* the gravitational force.

7.2 Newton's Law of Gravitation

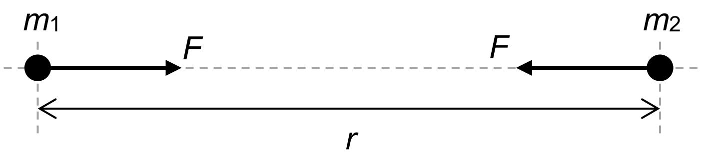
Newton's law of gravitation states that the

[type of force] gravitational force of attraction between two point masses

[magnitude] is directly proportional to the product of the masses and

inversely proportional to the square of separation between the masses

$$F = G\frac{m_1 m_2}{r^2}$$

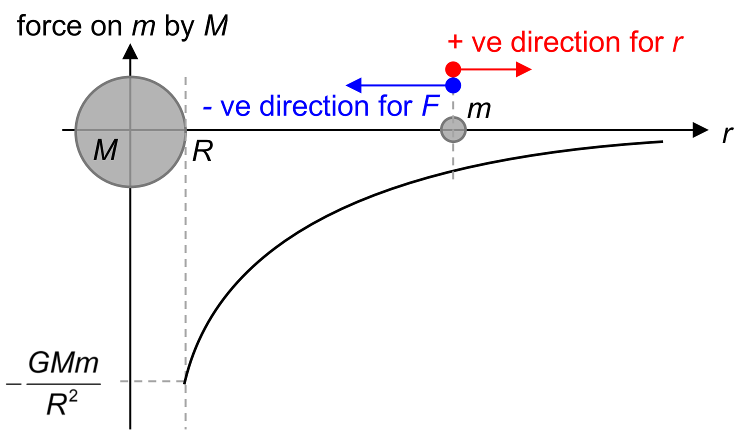


The gravitational constant G = 6.67 × 10^{-11} N m^2 kg^{-2}. m_1 and m_2 are the masses of the two point masses and r is the distance (separation) between the two point masses (i.e. the distance between the centres of the mases).

The direction of the gravitational force acts along the line joining the two point masses. By Newton's 3rd Law, the gravitational attraction that m_1 exerts on m_2 is equal in magnitude and opposite in direction as the gravitational force that m_2 exerts on m_1.

Newton's Law of Gravitation works between point masses. Even though planets and stars are massive, Newton's Law of Gravitation still applies as the *distances between planetary bodies are significantly much larger than their diameters* so the bodies can still be regarded as point masses.

Note: In some textbooks, Newton's Law of Gravitation is given as $F = -\frac{GMm}{r^2}$; the negative sign denotes that the direction of the force is opposite to the direction of r, the displacement (or position) vector.



Example 2

The Sun (mass of 1.99×10^{30} kg) is 1.50×10^{11} m away from the Earth (mass of 5.98×10^{24} kg).

(a) Find the gravitational force of Sun on Earth

(b) What is the force exerted by the Earth on the Sun?

(c) Estimate the gravitational force of attraction between a pair of students seated in class.

Solution

(a) $F=\dfrac{Gm_{\text{Sun}}m_{\text{Earth}}}{r^2}=\dfrac{(6.67\times10^{-11})(1.99\times10^{30})(5.98\times10^{24})}{(1.50\times10^{11})^2}=3.53\times10^{22}\text{ N}$

(b) By Newton's 3rd Law, the force exerted by Earth on Sun is of same magnitude but opposite direction to that of force of Sun on Earth; 3.53×10^{22} N

(c) $F=\dfrac{Gm_1m_2}{r^2}\approx\dfrac{(6.67\times10^{-11})(60)^2}{(1)^2}=2.40\times10^{-7}\text{ N}$

Note: Magnitudes of gravitational force between people and everyday objects are very small and almost unable to perceive at about 29 orders of magnitude less than that between Earth and Sun. Consequently, we only need to consider gravitational effects involving big masses.

Example 3

The Sun (of mass 1.99×10^{30} kg) is 1.50×10^{11} m away from the Earth (of mass 5.98×10^{24} kg). Base on the above data, determine the period of Earth's orbit around the Sun.

Solution

gravitational force on Earth by Sun provides centripetal force

Always identify the type of force that provides the centripetal force

$$\frac{Gm_{\text{Sun}}\cancel{m_{\text{Earth}}}}{r^2}=\cancel{m_{\text{Earth}}}\,r\omega^2$$

$$Gm_{\text{Sun}}=r^3\left(\frac{2\pi}{T}\right)^2$$

$$T^2=\frac{4\pi^2}{Gm_{\text{Sun}}}r^3 \qquad \text{\_\_\_(eqn)}$$

$$T=2\pi\sqrt{\frac{r^3}{Gm_{\text{Sun}}}}=2\pi\sqrt{\frac{(1.50\times10^{11})^3}{(6.67\times10^{-11})(1.99\times10^{30})}}$$

$$=3.17\times10^{7}\text{ s}$$

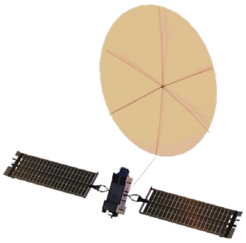
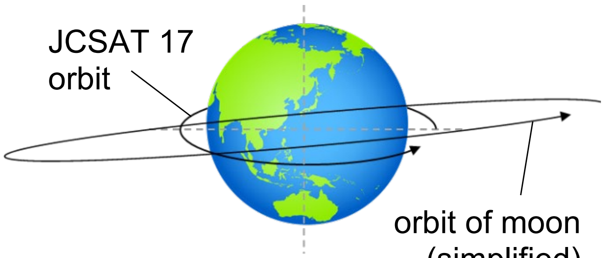
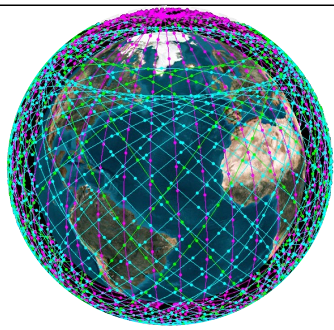
$$=367\text{ days}$$

Note: The equation $T^2\propto r^3$ is known as Kepler's 3rd law. This equation is not in syllabus and has to be derived (rather than quoted outright) when needed. It holds true in general for bodies orbiting around a mass.

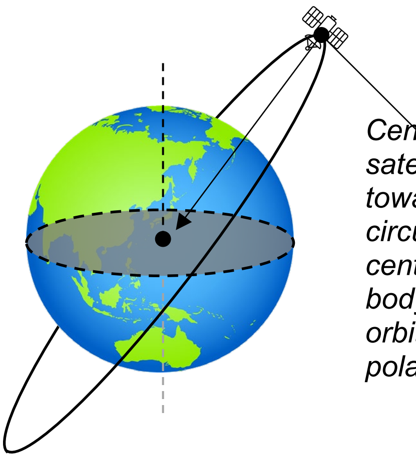
7.3 Satellites

A satellite is an object in space that orbits around a more-massive body. An example of *natural* satellite is the moon orbiting around Earth; an example of *artificial* satellite is the International Space Station also orbiting Earth.

The results of Example 3 show that the orbital properties of a satellite are not affected by the mass of the satellite; the period of revolution is dependent on the distance of the satellite from the centre of the more massive body, $T^2 \propto r^3$.

	Moon	JCSAT 17	Starlink
Satellite			
Type	Natural	Artificial; for cellular and broadcast networks owned by Japan's NTT Docomo	Artificial, part of a 12000-strong constellation owned by Elon Musk's Space X to provide global wireless broadband internet
Approximate radius of orbit	385 000 km	42 000 km	6800 - 7500 km
Period of orbit	30 days	24 hours	90 minutes
Orbit			

Because gravitational force provides the centripetal force, the centre of the circular orbit must coincide with the centre of the more-massive body:



Centripetal force on satellite points towards centre of circular path i.e. centre of massive body. This type of orbit is known as a polar orbit.

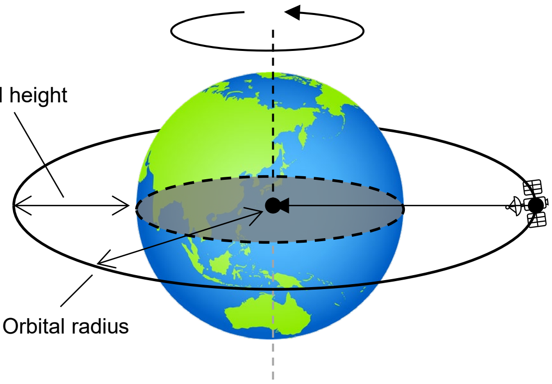
7.3.1 Geostationary Satellites

As the name suggests, a geostationary satellite appears to be stationary in the sky to an observer on Earth. Therefore, a geostationary satellite must:

- have a period of 24 hour
- be in circular orbit at a particular radius

$\left.\begin{matrix} \\ \end{matrix}\right\} T^2 \propto r^3$

- orbit directly above Equator
- move from west to east along same axis as Earth's rotation



Geostationary satellites allow transmission of signals between two regions at all times because the satellite will remain within "line-of-sight" of ground-based transmission and reception equipment.

They are useful for constant monitoring of weather especially in the equatorial region.

Example 4

Show that all geostationary satellites orbiting the Earth (of radius 6.4×10^6 m and mass 6.0×10^{24} kg) have the same the orbital radius which is independent of the mass of the satellite.

(a) Find the orbital radius.

(b) Find the height of a geostationary satellite above Earth's surface.

Solution

(a) gravitational force on satellite by Earth provides centripetal force

$$\frac{Gm_{\text{Earth}} \cancel{m_{\text{sat}}}}{r^2} = \cancel{m_{\text{sat}}} r\omega^2$$

$$Gm_{\text{Earth}} = r^3 \left(\frac{2\pi}{T}\right)^2$$

$$r = \sqrt[3]{\frac{Gm_{\text{Earth}}T^2}{4\pi^2}} = \sqrt[3]{\frac{(6.67 \times 10^{-11})(6.0 \times 10^{24})((24)60^2)^2}{4\pi^2}}$$

$$= 4.23 \times 10^7 \text{ m}$$

(b)

$$h = r - R_E$$

$$= 4.23 \times 10^7 \text{ m} - 6.4 \times 10^6$$

$$= 3.59 \times 10^7 \text{ m}$$

7.4 Gravitational Field Strength

In Physics and beyond, the use of ratios help us describe and characterize systems. For example, GDP *per* capita describes prosperity, number of people *per* unit area describes population density, and moles *per* cubic decimetre describes concentration.

Similarly, gravitational field strength describes how strong the gravitational force on a unit mass at a point could be. When a unit mass is placed at two points with different gravitational field strengths, a greater gravitational force acts on the unit mass at the point where there is a greater gravitational field strength.

The **gravitational field strength at a point in the gravitational field** is the

[force type]	gravitational force of attraction
[ratio]	*per* unit mass
[specifics]	acting on a small test mass placed at that point in the field.

$$\boxed{g = \frac{GM}{r^2}}$$

g : gravitational field strength (N kg^{-1}) or (m s^{-2})
G : gravitational constant (6.67×10^{-11} N m^2 kg^{-2})
M : mass of object providing the gravitational field
r : distance between the centre of mass of the mass providing the gravitational field and the point

Gravitational field strength at a point is a <u>vector quantity</u>. The resultant field strength at a point due to 2 or more masses will be the <u>vector sum</u> of the individual contributing field strengths.

Derivation

[Newton's Law of Gravitation]

The gravitational force of attraction between two point masses is directly proportional to the product of the masses and inversely proportional to the square of separation between the masses, $F = \frac{GMm}{r^2}$

where G: gravitational constant,
M,m: masses of the two point masses
R: distance between the two point masses

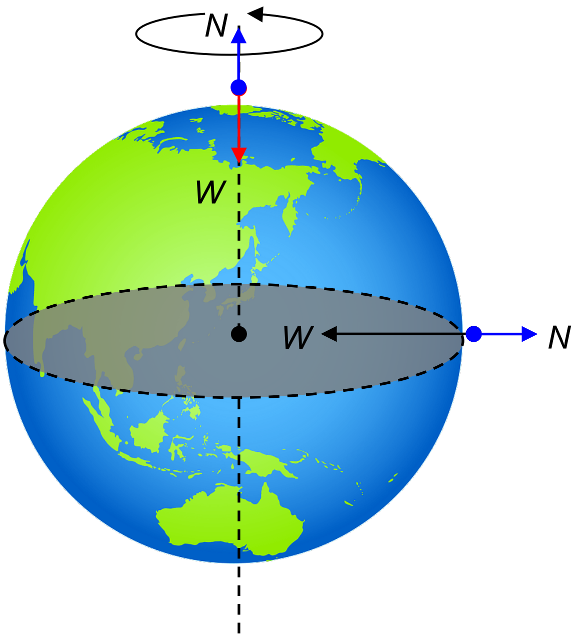
[Definition of gravitational field]

The gravitational field strength at a point in the gravitational field is the gravitational force of attraction *per* unit mass acting on a small test mass placed at that point in the field.

hence, $$g = \frac{F}{m} = \frac{\left(\frac{GM\cancel{m}}{r^2}\right)}{\cancel{m}} = \frac{GM}{r^2}$$

7.4.1 Apparent Weight

We read our "apparent weight" as the normal contact force on us via weighing scales.



At the poles:

actual weight = apparent weight

$$W = N$$

Along the equator:

vector sum of weight and normal force provides centripetal force

$$W - N = mr\omega^2$$
$$N = mg - mr\omega^2$$
$$= m\left[g - R_E\left(\frac{2\pi}{T}\right)^2\right]$$
$$= m\left[g - \left(6.4\times10^6\right)\left(\frac{2\pi}{24\times60^2}\right)^2\right]$$
$$= m(9.78)$$

Apparent weight (N) is less than actual weight

Example 5

For Earth (of radius 6.39×10^6 m and mass 5.98×10^{24} kg), find the gravitational field strength

(a) on Earth's surface, and

(b) at the International Space Station, orbiting at a height of 400 km above Earth's surface.

Solution

(a) $$g_{\text{surface}} = \frac{Gm_{\text{Earth}}}{r^2} = \frac{\left(6.67\times10^{-11}\right)\left(6.0\times10^{24}\right)}{\left(6.39\times10^6\right)^2} = 9.80 \text{ N kg}^{-1}$$

(b) $$g_{\text{ISS}} = \frac{Gm_{\text{Earth}}}{r_{\text{ISS}}^2} = \frac{Gm_{\text{Earth}}}{\left(h_{\text{ISS}} + R_E\right)^2} = \frac{\left(6.67\times10^{-11}\right)\left(6.0\times10^{24}\right)}{\left(400\times10^3 + 6.39\times10^6\right)^2} = 8.68 \text{ N kg}^{-1}$$

Example 6

Astronauts in the International Space Station (ISS) and the ISS itself are subject to the effects of Earth's gravity. Explain why astronauts experience weightlessness while on-board the ISS.

Solution

Gravitational force provides centripetal force for both astronaut and ISS.

Both astronaut and ISS are subject to the same gravitational field strength and so accelerates to the centre of earth at the same rate.

ISS cannot provide normal contact force to the astronaut.

Example 7

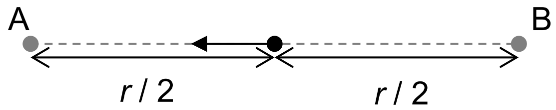
Two spherical planets A and B are of masses of 3.0×10^{20} kg and 1.4×10^{20} kg respectively. The distance between them is 2.5×10^{9} m. Determine the

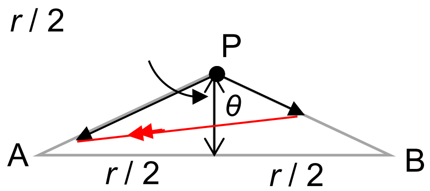
(a) **(i)** resultant field strength at the mid-point of a line joining A and B,

(ii) magnitude of field strength at P, located 0.5×10^{9} m up from midpoint f AB along a normal to line AB, and

(b) position of point *X* between A and B where the resultant gravitational field strength is zero,

Solution





$$(a)(i)\; g = g_A - g_B = \frac{GM_A}{\left(\frac{r}{2}\right)^2} - \frac{GM_B}{\left(\frac{r}{2}\right)^2}$$

$$= \frac{4G}{r^2}(M_B - M_A)$$

$$= \frac{4(6.67\times10^{-11})}{(2.5\times10^{9})^2}\left((3.0-1.4)\times10^{20}\right)$$

$$= 6.83\times10^{-9}\text{ N kg}^{-1}\text{ towards A}$$

$$(a)(ii)\; \tan\theta = \frac{r/2}{r/5} = \frac{5}{2}$$

$$|AP| = |BP| = \sqrt{\left(\frac{r}{5}\right)^2 + \left(\frac{r}{2}\right)^2} = r\sqrt{\frac{1}{25}+\frac{1}{4}} = r\sqrt{0.29}$$

$$g_y = g_B\cos\theta + g_A\cos\theta = \left[\frac{GM_B}{(BP)^2} + \frac{GM_A}{(AP)^2}\right]\cos\theta$$

$$= \frac{(6.67\times10^{-11})\cos\left(\tan^{-1}(2.5^\circ)\right)}{(0.29)(2.5\times10^{9})^2}\left[(3.0+1.4)\times10^{20}\right]$$

$$g_x = g_A\sin\theta - g_B\sin\theta = \left[\frac{GM_A}{(AP)^2} - \frac{GM_B}{(BP)^2}\right]\sin\theta$$

$$= \frac{(6.67\times10^{-11})\sin\left(\tan^{-1}(2.5^\circ)\right)}{(0.29)(2.5\times10^{9})^2}\left[(3.0-1.4)\times10^{20}\right]$$

$$g = \sqrt{g_y^2 + g_x^2} = 8.13\times10^{-9}\text{ N kg}^{-1}$$

(b) no resultant field

so $|g_A| = |g_B|$

$$\frac{\cancel{G}M_A}{x^2} = \frac{\cancel{G}M_B}{(r-x)^2}$$

$$\sqrt{\frac{M_B}{M_A}} = \frac{r-x}{x} = \frac{r}{x} - 1$$

$$x = \frac{r}{\sqrt{\frac{M_A}{M_B}}+1} = \frac{2.5\times10^{9}}{\sqrt{\frac{1.4}{3.0}}+1}$$

$$= 1.48\times10^{9}\text{ m}$$

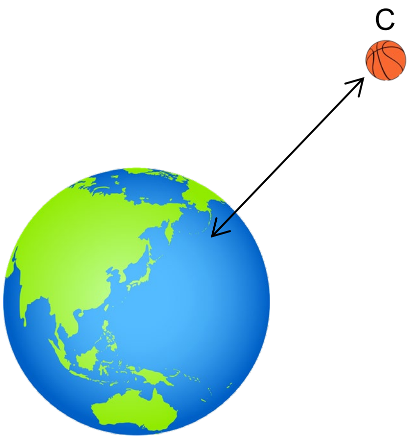
from A towards B

7.5 Gravitational Potential Energy

In the topic on Work Energy and Power, we revisited the equation $E_p = mgh$ for gravitational potential energy changes *near the Earth's surface*, something we are made familiar with in secondary school.

For solving questions, we are accustomed to defining arbitrary locations as zero height and working out the *change in gravitational potential energy* using the difference in heights.

In the diagram to the left, ball B has greater gravitational potential energy than ball A, because it is at a "higher height".



It is therefore logical that ball C has more gravitational potential energy than ball B because it is at a "higher height".

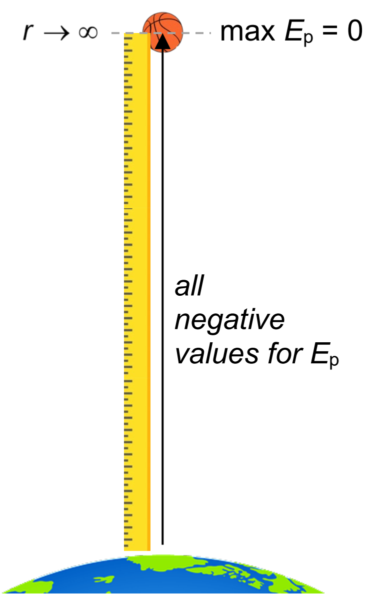
What will be the gravitational potential energy of a ball that is at infinity?

Mathematically it is an issue because infinity is a *concept* rather than a number that we perform operations on. For example, if we were to bring a ball that is "at infinity" by 10 km closer to the Earth, the distances involved will result in statements such as:

$$\text{“}\infty - (10 \times 10^3) = \infty\text{”}$$

Similarly, it becomes quite nonsensical to describe the amount of gravitational potential energy stored in this Earth-ball system as "infinity".

Instead, we (scientists) chose to define the *maximum* gravitational potential energy of massive objects at infinitely far away distances as … zero.



Another inherent issue is the difficult in comparison – the total gravitational potential energy does not only depend on the distance of separation but also both the masses involved.

Again, we turn to the *ratio* concept mentioned in Section 7.4 on Gravitational Field Strength to characterise only one massive body.

7.5.1 Gravitational Potential

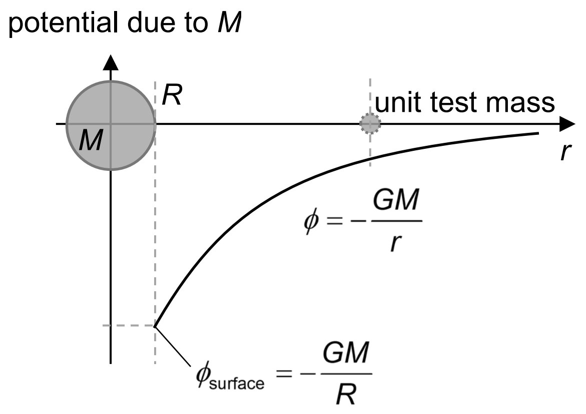
The **gravitational potential at a point in the gravitational field** is the

[process]	work done
[ratio]	*per* unit mass
[specifics]	in bringing a small test mass from infinity to that point in the field.

$$\boxed{\phi = -\frac{GM}{r}}$$

ϕ : gravitational potential (J kg^{-1}) or (m s^{-2})
G: gravitational constant (6.67×10^{-11} N m^2 kg^{-2})
M: mass that provides the gravitational field
r : distance between the centre of mass of the mass providing the gravitational field and the location inside the field

The negative sign in the equation indicates that as r increases, ϕ increases (becomes less negative) till a maximum of zero at $r \to \infty$. (See graph on the left)



Mathematically, gradient of the ϕ - r graph gives

$$\frac{d\phi}{dr} = \frac{d}{dr}\left(-\frac{GM}{r}\right) = \frac{GM}{r^2}$$

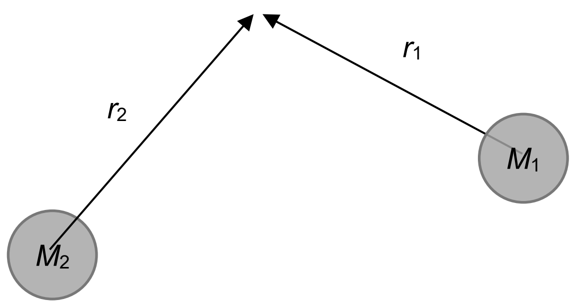
Hence $\boxed{g = -\frac{d\phi}{dr}}$

[magnitude] Gravitational field strength g at a point in the field is numerically equal to the gravitational potential gradient at that point

[direction] Negative sign shows that the field points towards direction of lower potential.

Gravitational potential at a point ϕ is a <u>scalar quantity</u>. The resultant gravitational potential at a point due to 2 or more masses is the <u>scalar sum</u> of gravitational potential at that point due to each individual mass.

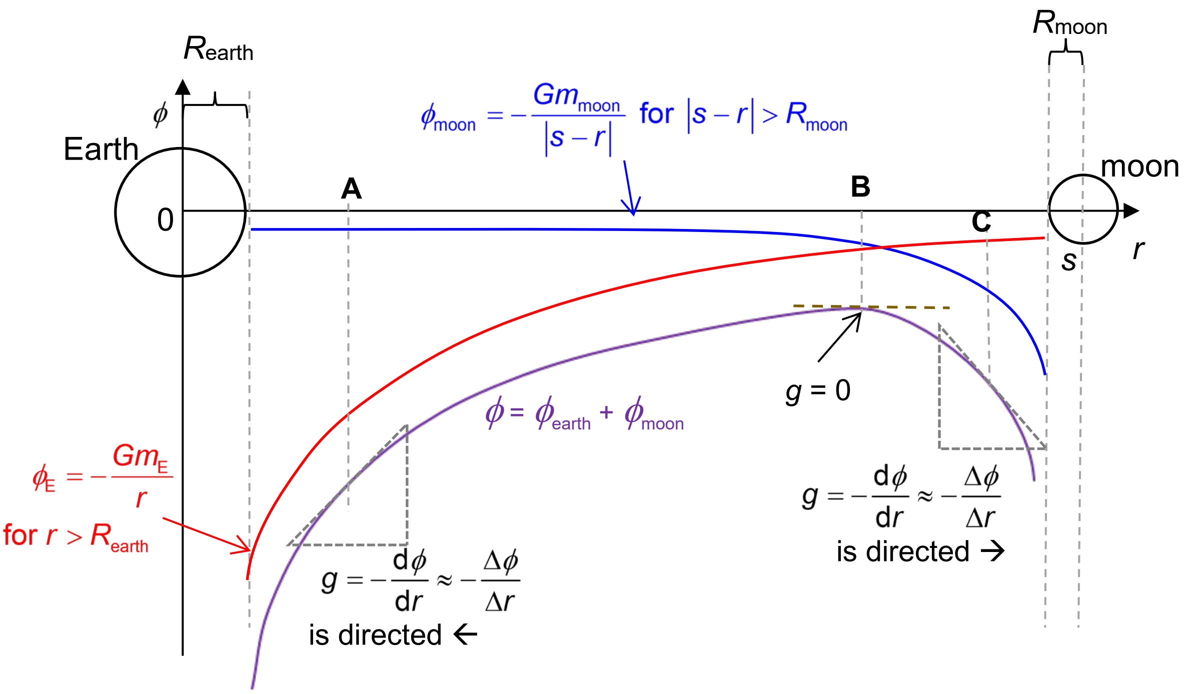
$$\phi_{total} = \left(-\frac{GM_1}{r_1}\right) + \left(-\frac{GM_2}{r_2}\right) + \ldots$$



7.5.2 Variation of Gravitation Potential with Displacement between Earth and Moon

First, consider the gravitational potential due to the Earth ϕ_E and Moon ϕ_M independently. The resultant gravitational potential ϕ due to both the Earth and Moon is the scalar sum of both the gravitational potentials of the Earth and the Moon.

The gravitational field strength at a point is the negative of the potential gradient at that point.



At Point A:

The gradient of the tangent drawn at the point on the resultant gravitational potential graph is positive. Hence the potential gradient is positive. Since $g = -\dfrac{d\phi}{dr}$, the gravitational field strength at that point in the field is directed in the negative-r direction (directed towards the Earth). Any mass that is nearer to the Earth than it is to the moon shall experience a net gravitational force directed towards the centre of the Earth.

At Point B:

The gradient of the tangent drawn at that point on the resultant gravitational potential graph is zero. Hence the potential gradient is zero. This implies that the gravitational field strength at that point is zero.

Any mass that is located at B experiences equal magnitudes of gravitational force of attraction from both the Earth and the moon. Hence the mass experiences a net gravitational force of zero.

Also note that Point B is NOT at the point of where the 2 gravitational potentials are equal in magnitude (where the individual potentials intersect).

At Point C

The gradient of the tangent drawn at the point on the resultant gravitational potential graph is negative. Hence the potential gradient is negative. Since $g = -\dfrac{d\phi}{dr}$, the gravitational field strength at that point in the field is positive (directed towards the Moon).

Any mass at Point C shall experience a net gravitational force directed towards the centre of the Moon, as it is located sufficiently near the Moon such that the moon's gravitational force of attraction is stronger than that of Earth's.

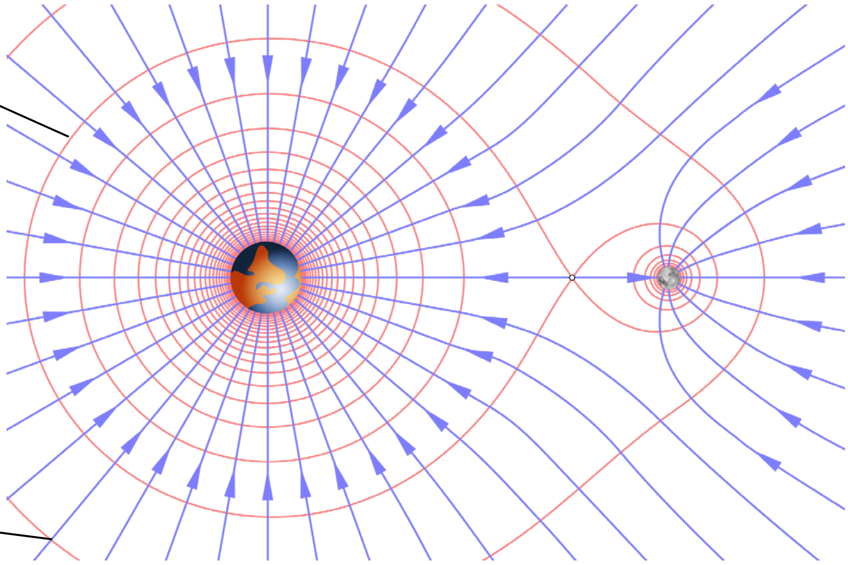
Equipotential Lines of the Earth- Moon System

Resultant gravitational potential is a simple scalar sum of contributing potentials.

In this left-hand-side region, the moon has negligible influence on the Earth's gravitational field.

All points at the same radius away from the centre of Earth has the same potential. These points of same gravitational potential form an equipotential line (2D) or equipotential surface (3D).

Direction of gravitational field lines is always normal to equipotential lines



Gravitational potential at a point ϕ describes a property at a particular location in the gravitational field. If we place a "third-party" mass at *r* distance away, the gravitational potential energy possessed by the "third-party" mass is given by:

$$\boxed{E_p = m\phi}$$

E_P : gravitational potential energy (J)
m : mass placed in gravitational field (kg)
ϕ: gravitational potential (J kg^{-1}) or (m s^{-2})

Example 8

The planet Mars has a radius *R* of 3400 km and a mass of 6.42×10^{23} kg. Ignoring the gravitational effects due to other celestial bodies, calculate

(a) the gravitational potential at a point X, *R* from the surface of Mars,
(b) the gravitational potential at a point Y, *2R* from the surface of Mars,
(c) the change in potential energy of a satellite of mass 2500 kg in moving from Y to X.

Solution

(a)
$$\phi_X = -\frac{GM}{r_X} = -\frac{(6.67 \times 10^{-11})(6.42 \times 10^{23})}{(3400 + 3400) \times 10^3} = -6.30 \times 10^6 \text{ J kg}^{-1}$$

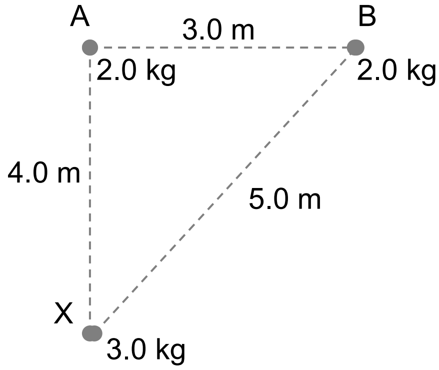
(b)
$$\phi_Y = -\frac{GM}{r_Y} = -\frac{(6.67 \times 10^{-11})(6.42 \times 10^{23})}{(3400 + 6800) \times 10^3} = -4.20 \times 10^6 \text{ J kg}^{-1}$$

(c)
$$\begin{aligned} \Delta E_P &= m\Delta\phi \\ &= m\left(\phi_{\text{final}} - \phi_{\text{initial}}\right) \\ &= m\left(\phi_X - \phi_Y\right) \\ &= 2500\left[\left(-6.30 \times 10^6\right) - \left(-4.20 \times 10^6\right)\right] \\ &= -5.25 \times 10^9 \text{ J} \end{aligned}$$

Example 9

Calculate the gravitational potential at point X due to masses A and B shown in the diagram below. Hence determine the gravitational potential energy of mass C, which is placed at point X.

Solution



gravitational potential at X due to A:

$$\phi_{\text{at X by A}} = -\frac{GM_A}{r} = -\frac{(6.67\times10^{-11})(2.0)}{4.0}$$
$$= -3.335\times10^{-11}\ \text{J kg}^{-1}$$

gravitational potential at X due to B:

$$\phi_{\text{at X by B}} = -\frac{GM_B}{r} = -\frac{(6.67\times10^{-11})(2.0)}{5.0}$$
$$= -2.668\times10^{-11}\ \text{J kg}^{-1}$$

total gravitational potential at X due to A and B:

$$\phi_{\text{at X}} = \phi_{\text{at X by A}} + \phi_{\text{at X by B}}$$
$$= -(3.335+2.668)\times10^{-11}$$
$$= -(6.00)\times10^{-11}\ \text{J kg}^{-1}$$

Gravitational potential energy of mass C:

$$U = m_C\phi_C$$
$$= (3.0)(-6.00\times10^{-11})$$
$$= -1.80\times10^{-10}\ \text{J}$$

Try: find the field strength at X due to A and B, and hence the force on C at X.

Example 10

The diagram below shows the Earth's gravitational field. The dashed lines represent equipotential lines. The gravitational potential is as shown for each of the equipotential lines.

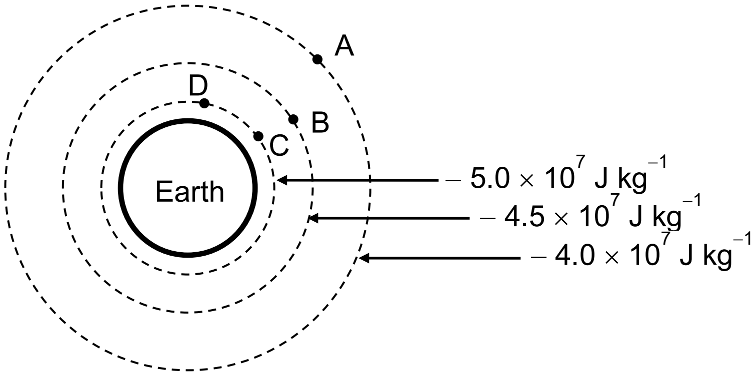
(a) Explain which of the 4 points is the gravitational potential the highest.

(b) Calculate the work done *by the gravitational field* in bringing a spacecraft of mass 5000 kg

(i) from A to C

(ii) from C to D

(c) Explain why the equipotential lines are not equally spaced.



Solution

(a) gravitational potential is the highest (least negative) at A because it is furthest from the Earth

$$\phi \propto -\frac{1}{r}$$

(b) (i) w.d. by gravitational force in bringing spacecraft from A to C

= loss in gravitational E_p of spacecraft

= initial E_p (at point A) – final E_p (at point C)

$= m(\phi_A - \phi_C) = (5000)\left[(-4.0-(-5.0))\times 10^7\right] = 5.0\times 10^{10}$ J

(ii) w.d. $= m(\phi_C - \phi_D) = 0$; C and D lie on the same equipotential line so same gravitational potential

(c) Gravitational potential is inversely proportional to the distance from Earth's centre, $\phi \propto -\frac{1}{r}$.

Hence each change in distance will be different for the same equal change in gravitational potential.

Example 11

By reference to the definition of gravitational potential, explain why gravitational potential is a negative quantity.

Solution

Gravitational potential at infinity is defined as zero.

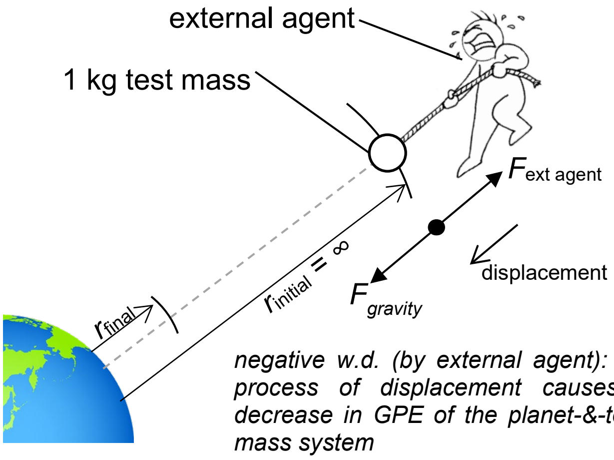
Gravitational force is attractive, so displacement of object moves from infinity to point is field is in opposite direction to applied force

Negative work is done, so net loss of energy from system.

It is useful to consider that the definition of gravitational potential involves an *external agent.*

This external agent exerts a force on the unit test mass that is always of same magnitude but opposite in direction to that of the gravitational force attracting the test mass.

The external agent brings the test mass closer to the more-massive body without a change in kinetic energy (i.e. no acceleration because no net force on the unit test mass).



negative w.d. (by external agent): the process of displacement causes a decrease in GPE of the planet-&-test-mass system

All of the work done by the external agent only changes the gravitational potential energy of the test mass; and not does change the kinetic energy

7.5.3 Escape Velocity

When an object has sufficient escape velocity, the object has enough initial kinetic energy to escape the influence of the gravitational field it is currently in.

This means moving to "infinite" distance away. Take note that this is an object projected with an initial velocity and is *not* like a rocket which continuously burns fuel to generate thrust. At infinity, the object's kinetic energy is zero and velocity is zero.

By Principle of Conservation of Energy,
for an object of mass *m* escaping from the surface of a planet of mass *M* and radius *R*:

$$\begin{aligned}
\text{loss in } E_K &= \text{gain in} E_P \\
E_{K,\text{ initial}} - E_{K,\text{ final}} &= m\Delta\phi \\
\frac{1}{2}\cancel{m}v_{esc}^2 - 0 &= \cancel{m}\left(\phi_{\text{final}} - \phi_{\text{initial}}\right) \\
\frac{1}{2}v_{esc}^2 &= 0 - \left(-\frac{GM}{R}\right) \\
v_{esc} &= \sqrt{\frac{2GM}{R}}
\end{aligned}$$

This allow us to explain why Earth's atmosphere can exist, whereas gases of "lighter" atoms or molecules are rare because given the same kinetic energy (temperature), their mean square speeds permit them to escape Earth's gravitational field.

Example 12

Determine the escape velocity of a rocket from the surface of Earth, ignoring effects of air resistance as the rocket leaves the Earth's atmosphere. Mass of the Earth is 5.98×10^{24} kg and the average radius of the Earth is 6.37×10^{6} m.

Solution

gain in gravitational potential energy = loss in kinetic energy

$$E_{\text{P, final}} - E_{\text{P, initial}} = E_{\text{K, initial}} - E_{\text{K, final}}$$

$$0 - \left(-\frac{Gm_E m_R}{R_E}\right) = \frac{1}{2}m_R v^2 - 0$$

$$v = \sqrt{\frac{2Gm_E}{R_E}}$$

$$= \sqrt{\frac{2(6.67\times10^{-11})(5.98\times10^{24})}{(6.37\times10^{6})}}$$

$$= 1.12\times10^{4}\ \text{m s}^{-1}$$

OR

initial total energy = final total energy

$$E_{\text{K, initial}} + E_{\text{P, initial}} = E_{\text{K, final}} + E_{\text{P, final}}$$

$$\frac{1}{2}m_R v^2 + \left(-\frac{Gm_E m_R}{R_E}\right) = 0 + 0$$

$$v = \sqrt{\frac{2Gm_E}{R_E}}$$

Further Challenge: Reduce v to be in terms of g and R_E only.

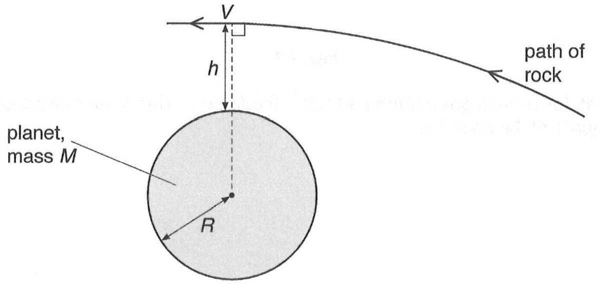
Example 13

A satellite in circular orbit at height *h* above the surface of a planet experiences no resistive forces. The planet has mass *M* and radius *R*.

(a) Express the speed of the satellite v_s in terms of *M, R, h* and the gravitational constant *G.*

(b) A rock, initially at rest at an infinite distance away, approaches the planet. It travels at speed v_{rock} when it passes by tangential to the planet, at height *h* above the planet's surface. Express the speed of the rock v_{rock} in terms of *M, R, h* and the gravitational constant *G.*

(c) Hence, explain if the rock will fall into the planet's surface **or** orbit around the planet **or** continue to travel off into space.



Solution

(a) gravitational force provides centripetal force

$$\frac{m_s v_s^2}{r_s} = \frac{GMm_s}{r_s^2}$$

$$\frac{\cancel{m_s} v_s^2}{R+h} = \frac{GM\cancel{m_s}}{(R+h)^2}$$

$$v_s = \sqrt{\frac{GM}{R+h}}$$

(b) by principle of conservation of energy,

Total Energy near planet= Total energy at infinity

$$E_{\text{K, final}} + E_{\text{P, final}} = E_{\text{K, initial}} + E_{\text{P, initial}}$$

$$\frac{1}{2} m_{\text{rock}} V_{\text{rock}}^2 + m_{\text{rock}} \phi_{\text{initial}} = 0 + 0$$

$$V_{\text{rock}}^2 = -2\phi_{\text{initial}} = (-2)\left(-\frac{GM}{R+h}\right)$$

$$V_{\text{rock}} = \sqrt{\frac{2GM}{R+h}}$$

(c) velocity of rock at distance *h* above planet <u>is always larger than</u> velocity of a satellite in orbit at same distance,

the rock will not enter orbit
and will travel off into space

Note: because $T^2 \propto r^3$, a satellite must have a well-defined speed to remain orbiting the planet. At that orbit, if a satellite "suddenly" has increased speed it will veer off the orbit and away from the planet. If a satellite "suddenly" loses speed, it will spiral towards the planet with increasing speed as its gravitational potential energy is converted into kinetic energy and thermal energy in the drop.

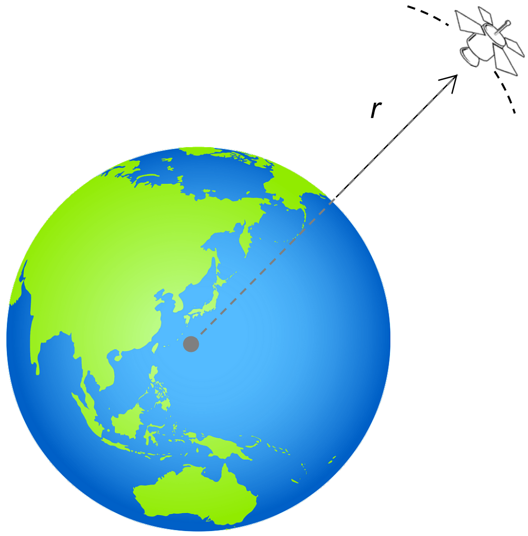
7.5.4 Total Energy of a Satellite in Circular Orbit

A satellite of mass m orbiting round the Earth has total energy (E_T) due to the sum of its kinetic energy E_K and potential energy E_P.

To derive E_K:

gravitational force provides centripetal force:

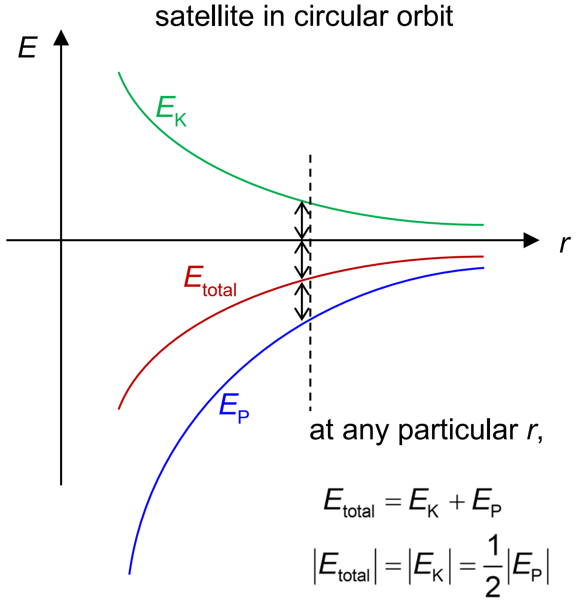
$$\frac{mv^2}{r} = \frac{GMm}{r^2}$$
$$\left(\frac{\cancel{r}}{2}\right)\frac{mv^2}{\cancel{r}} = \left(\frac{\cancel{r}}{2}\right)\frac{GMm}{r\cancel{^2}}$$
$$E_K = \frac{GMm}{2r}$$



To derive E_T:

$$\begin{aligned} E_{total} &= E_K + E_P \\ &= \frac{GMm}{2r} + \left(-\frac{GMm}{r}\right) \\ &= -\frac{GMm}{2r} \end{aligned}$$

satellite in circular orbit



at any particular r,

$$E_{total} = E_K + E_P$$
$$|E_{total}| = |E_K| = \frac{1}{2}|E_P|$$

When a satellite is "bound" to the planet, it has negative total energy (since maximum possible energy potential is defined to be zero at infinite distance away).

An object that has positive total energy has enough energy to escape the gravitational influence of the planet and still have KE!

$$E_K = \frac{GMm}{2r}$$
$$E_P = -\frac{GMm}{r}$$
$$E_{total} = -\frac{GMm}{2r}$$

$$E_{total} = -E_K$$
$$E_{total} = \frac{1}{2}E_P$$

Example 14

A satellite of mass m_S is in circular orbit around the Earth, with an orbital radius r. The mass and radius of the Earth are m_E and R_E respectively.

(a) State the expression of the gravitational potential energy possessed by the satellite when it is

(i) on the Earth's surface,

(ii) at its orbit.

(b) Express the kinetic energy of the orbiting satellite in terms of G, m_E, m_S and r.

(c) Using answers to **(a)(ii)** and **(b)**, write an expression the total energy of the orbiting satellite.

(d) Using answers to **(a)** and **(b)** or otherwise, determine the energy required by the satellite to move it from the Earth's surface to its orbit. Ignore the rotation of the Earth.

(e) Explain the impact of atmospheric drag on the subsequent trajectory of satellites.

Solution

(a)

$$E_P = -\frac{Gm_E m_S}{R_E} \qquad\qquad E_P = -\frac{Gm_E m_S}{r}$$

(b) gravitational force of Earth on the satellite provides the centripetal force for circular motion.

$$\frac{m_S v^2}{r} = \frac{Gm_E m_S}{r^2}$$

$$\left(\frac{\cancel{r}}{2}\right)\frac{m{v_S}^2}{\cancel{r}} = \left(\frac{\cancel{r}}{2}\right)\frac{Gm_E m_S}{r\cancel{^2}}$$

$$E_K = \frac{Gm_E m_S}{2r}$$

(c)

$$\begin{aligned} E_{total} &= E_K + E_P \\ &= \frac{Gm_E m_S}{2r} + \left(-\frac{Gm_E m_S}{r}\right) \\ &= -\frac{Gm_E m_S}{2r} \end{aligned}$$

(d) by principle of conservation of energy

$$\begin{aligned} \text{total energy required} &= \text{gain in } E_P + \text{gain in } E_K \\ &= (E_{P,\text{ final}} - E_{P,\text{ initial}}) + (E_{K,\text{ final}} - E_{K,\text{ initial}}) \\ &= \left[\left(-\frac{Gm_E m_S}{r}\right) - \left(-\frac{Gm_E m_S}{R_E}\right)\right] + \left[\frac{Gm_E m_S}{2r} - 0\right] \\ &= Gm_E m_S\left(\frac{1}{R_E} - \frac{1}{2r}\right) \end{aligned}$$

(e) negative work is done against drag so total energy decreases. GPE decreases, so satellite lowers in height $\left(E_{total} = \frac{1}{2}E_P\right)$. KE increases, so linear speed increases $\left(E_{total} = -E_K\right)$.

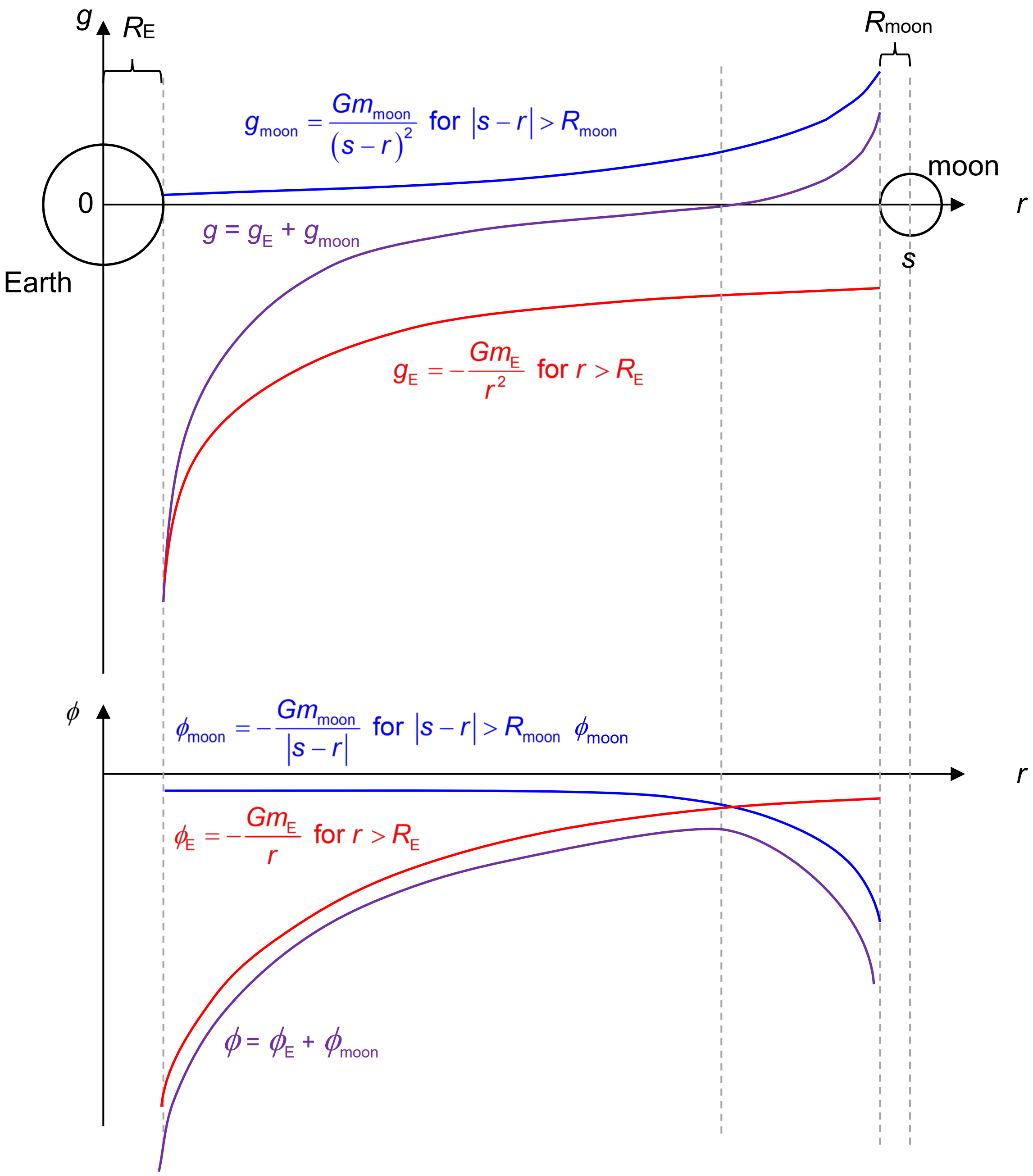
Satellite spirals towards planet surface with increasing speed.

7.6 Graph Sketching

When approaching graphs in general, consider the following:

1. Which body has greater mass?
2. Where is the point of zero field strength? (closer to the body with less mass)
3. Which body has more negative Gravitational Potential at its surface? (more massive body)
4. Which body has the stronger field strength at its surface? (more massive body)
5. Radii of bodies?
6. Distance between bodies' centre between A and B?

7.6.1 Graphs For Earth-Moon System



7.7 Comparisons with Gravitational Field

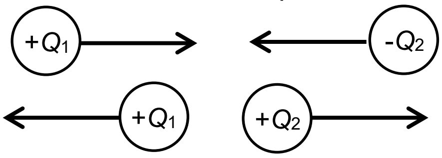
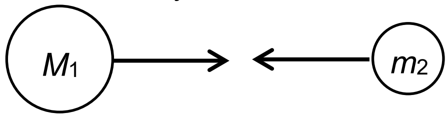
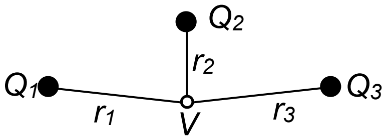
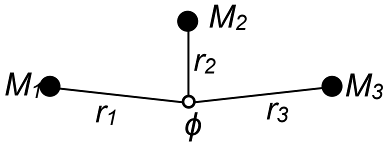
7.7.1 Similarities

(for isolated point mass/charge) variation of field strength with distance	field strength inversely proportional to square of distance from centre of mass/charge $$\text{(field strength)} \propto \frac{1}{r^2}$$
(for isolated point mass/charge) field lines	• greater separation of field lines with increasing distance from point mass/charge • field lines are normal to surface of mass/charge
energy considerations	Both electric force and gravitational force are conservative forces. (i.e. the work done is independent of the path taken, and depends only on the initial and final positions.)
relationship between field strength and potential / force and potential energy	**[magnitude]** field strength is numerically equal to the potential gradient at that point **[direction]** field points towards direction of lower potential $E = -\frac{dV}{dr}$ and $g = -\frac{d\phi}{dr}$ **[magnitude]** force is numerically equal to the rate of change of potential energy at that point **[direction]** force is directed towards lower potential energy $F = -\frac{dU}{dr} = -\frac{dE_P}{dr}$

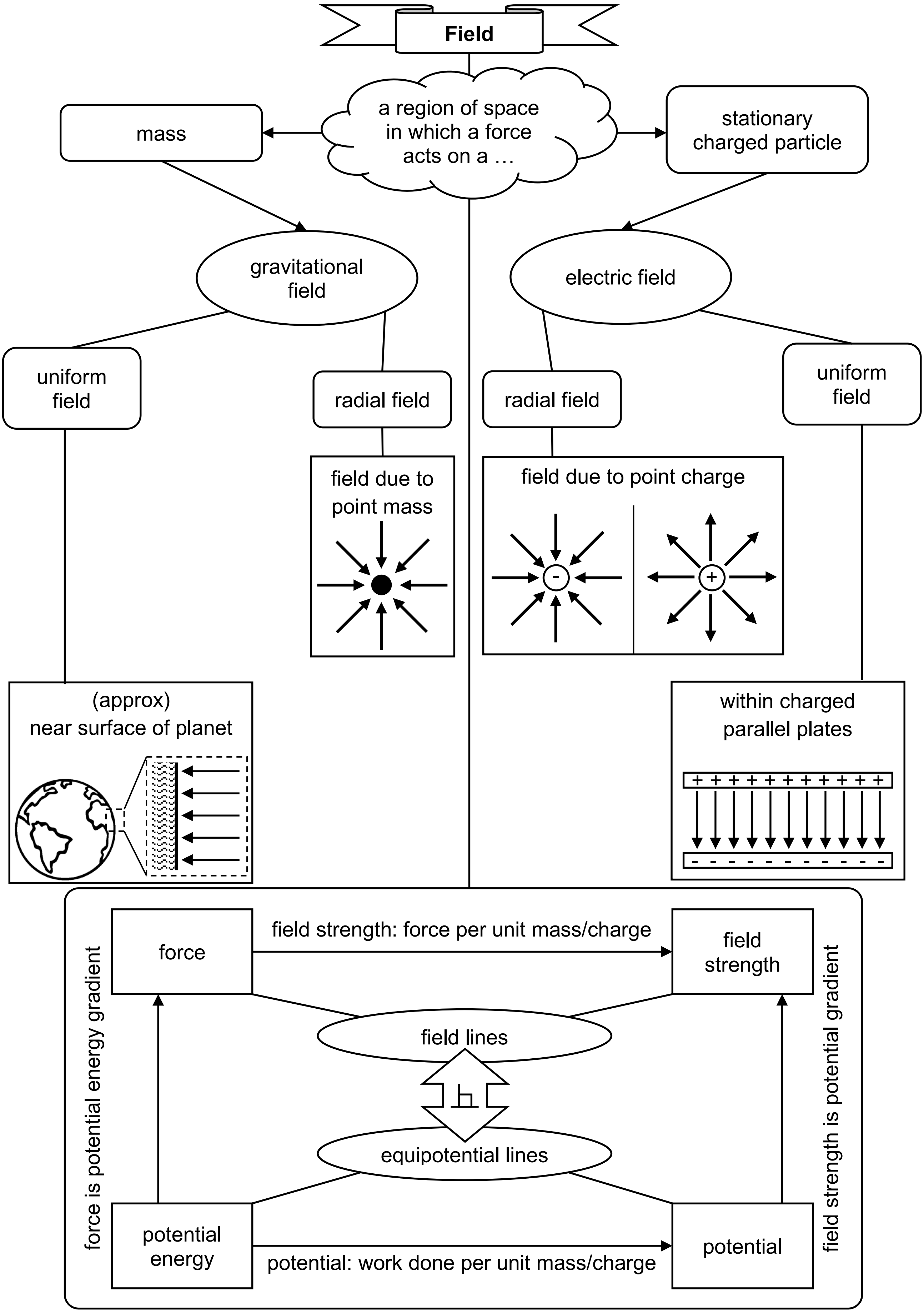
7.7.2 Differences

	Electric Field	Gravitational Field
direction of field lines	towards negative charge and away from positive charge	towards point mass
nature of force interactions	can be attractive or repulsive	always attractive

7.7.3 Analogies

	electric field	gravitational field
origin	charge interaction	mass interaction
nature	can be attractive or repulsive 	attractive only 
force law	Coulomb's Law $F=\left(\frac{1}{4\pi\varepsilon_0}\right)\frac{Q_1Q_2}{r^2}$	Newton's Law of Gravitation $F=G\frac{M_1m_2}{r^2}$
field strength	force per unit positive charge on a small stationary test charge at that point $E=\frac{F}{Q}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$ [N C^{-1}]	gravitational force per unit mass on a small test mass at that point $g=\frac{F}{m}=\frac{GM}{r^2}$ [N kg^{-1}]
potential	work done per unit positive charge in moving a small test charge from infinity to that point $V=\frac{U}{q}=\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$ [J C^{-1}]	work done per unit mass in bringing small test mass from infinity to that point $\phi=\frac{W}{m}=-\frac{GM}{r}$ [J kg^{-1}]
potential due to multiple masses or charges	$V=\frac{Q_1}{4\pi\varepsilon_0 r_1}+\frac{Q_2}{4\pi\varepsilon_0 r_2}+\ldots$ 	$\phi=\frac{-GM_1}{r_1}+\frac{-GM_2}{r_2}+\ldots$ 
potential energy	$U=QV$	$E_P=m\phi$
relationship between field and potential	$E=-\frac{dV}{dr}$ field strength numerically equal to potential gradient at that point negative sign indicates that the force acts in the direction of decreasing potential	$g=-\frac{d\phi}{dr}$
relationship between force and potential energy	$F=-\frac{dE_p}{dr}$	$F=-\frac{dU}{dr}$

7.7.4 Concept Map



7.8 Ending Notes

This is your first foray into the concepts of a *field*. Do take time to consider the idea that the properties of a field vary with different locations, mainly measured from the centre of massive bodies. The space below is for your own summary mind-map(s):