**1** In a particular soccer league of 20 teams, each team plays a total of 38 games over the course of a season. Teams are awarded 3 points for a win, 1 point for a draw, and no point for a loss.

Lucy's favorite team scored a total of 54 points this season and she is especially happy because the sum of games won or drawn this season exceeded twice the number of games lost by 8 games.

**(a)** Find the number of games won by Lucy's favorite team this season. [3]

Mark's favorite team won 2 games fewer and drew 5 games more than Lucy's favorite team this season.

**(b)** Explain, with clear workings, which team performed better this season. [1]

**2 (a)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$ [2]

**(b)** Show that $\sin 2x \cot x = 2\cos^2 x$ [1]

**(c)** Hence, find the exact value of $\int_{\frac{\pi}{30}}^{\frac{\pi}{15}} \operatorname{cosec}(10x)\tan(5x)\,\mathrm{d}x$. [4]

**3 (a)** The cross section *PQRS* of a water passageway of depth *d* is as shown in the figure below. It is given that $PQ + QR + RS = 2a$ and with *PQ* and *RS* each inclined to the line *QR* at an angle $\theta$.


Show that the area of the cross-section *PQRS* is $A = 2ad + d^2(\cot\theta - 2\operatorname{cosec}\theta)$. [3]

**(b)** Find, in terms of *a* and *d*, the maximum value of *A*, as $\theta$ varies. [4]

**4** The equations of planes $p_1$ and $p_2$ are respectively given by

$$\mathbf{r} = s\begin{pmatrix}0\\1\\1\end{pmatrix} + t\begin{pmatrix}3\\-1\\2\end{pmatrix},\ s,t \in \mathbb{R} \quad \text{and} \quad \mathbf{r}.\begin{pmatrix}\alpha\\3\\\beta\end{pmatrix} = 7,$$

where $\alpha$ and $\beta$ are real constants.

**(a)** Given that the line $l_1: \dfrac{x+1}{4} = \dfrac{2-y}{3} = z$ lies in $p_2$, find $\alpha$ and $\beta$. [3]

It is now given that $\alpha = -3$ and $\beta = 2$.

**(b)** Find the equation of the line of intersection of $p_1$ and $p_2$. [3]

**(c)** The plane $p_3$ with equation $y + 3z + 4 = 0$ is parallel to $l_1$. Find the distance between $l_1$ and $p_3$. [2]

**5(a)** Show that $\dfrac{3}{2r-1} - \dfrac{4}{2r+1} + \dfrac{1}{2r+3} = \dfrac{8r+20}{(2r-1)(2r+1)(2r+3)}$. [2]

**(b)** Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2r+5}{(2r-1)(2r+1)(2r+3)} = \frac{2}{3} - \frac{n+2}{(2n+1)(2n+3)}$. [4]

**(c)** Find the smallest possible integer $k$ such that $\displaystyle\sum_{r=1}^{k} \frac{2r+5}{(2r-1)(2r+1)(2r+3)}$ is within 0.004 of the sum to infinity. [3]

**6 (a)** Find $\int \cot^2 3x \, \mathrm{d}x$. [2]

**(b)** Find the exact value of $\displaystyle\int_2^4 \frac{x^2 - 4x + 3}{x^2 - 4x + 8} \, \mathrm{d}x$. [3]

**(c)** Use the substitution $x = 2\cot\theta$ to find the exact value of $\displaystyle\int_{\frac{2\sqrt{3}}{3}}^{2} \frac{4 - x^2}{(4 + x^2)^2} \, \mathrm{d}x$. [5]

**7 (a)** The parametric equations of a curve $C$ are $x = 3\sec t$ and $y = -2\tan t$, where $0 < t < \frac{\pi}{2}$.

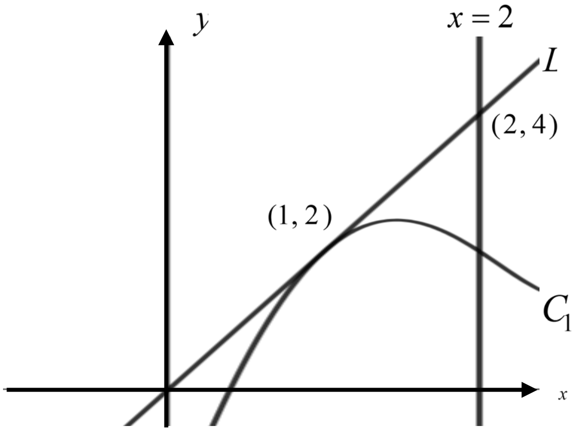
**(i)** Find the cartesian equation of the curve, stating any restriction on the values of $x$ and $y$. [2]

**(ii)** Sketch the curve $C$, indicating the equation of the asymptote if any. [2]

**(b)** A curve $C_1$ has parametric equations

$$x = 2\sin t + 1,\ \ y = 2\cos 3t + 4\sin t,\ -\frac{\pi}{3} \le t \le \frac{\pi}{3}.$$

The tangent to the curve $C_1$ at the point (1, 2) is given by $L$ as shown in the diagram below.




The line $L$ cuts the line $C_1$ at $(2,4)$. Show that the area bounded by the curve $C_1$, the line $L$ and the line $x = 2$ is given by $a - \int_0^{\theta} 4\cos 3t\cos t + 8\sin t\cos t \,\mathrm{d}t$, where $a$ and $\theta$ are constants to be determined in exact form. Hence find the exact value of this area. [6]

**8** It is given that $y = \dfrac{1}{3 + \sin 2x}$.

**(a)** Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} - 4y^2\sin 2x + 4y\dfrac{\mathrm{d}y}{\mathrm{d}x}\cos 2x = 0$. Hence find the Maclaurin series for $y$, up to and including the term in $x^2$. [4]

**(b)** Given further that $x$ is small so that $x^3$ and higher powers of $x$ can be neglected, use appropriate expansions from the List of Formulae (**MF26**) to verify the correctness of the series of $y$ in part **(a)**. [3]

**(c)** Use the series expansion found in part **(a)** and integration to find an approximate value for $\int_0^{1.5} \dfrac{1}{3 + \sin 2x}\,\mathrm{d}x$. [2]

**9** **Do not use a calculator in answering this question.**

**(a)** Three complex numbers are $z_1 = 1+\mathrm{i}$, $z_2 = \sqrt{3}-\mathrm{i}$ and $z_3 = \cos\frac{\pi}{3} + \mathrm{i}\sin\frac{\pi}{3}$. Find $\frac{z_1 z_2}{z_3{}^2}$ in the form $r(\cos\theta + \mathrm{i}\sin\theta)$, where $r > 0$ and $-\pi < \theta \le \pi$. [4]

**(b)** A fourth complex number $z_4$ is given by $\cos\theta + \mathrm{i}\sin\theta$.

**(i)** Show that $\frac{1+z_4}{1-z_4}$ can be expressed as $k\cot\frac{\theta}{2}$, where $k$ is a complex number to be determined. [4]

**(ii)** For the case $\theta = \frac{\pi}{4}$, show that $\frac{1+z_4}{1-z_4} = \left(1+\sqrt{2}\right)\mathrm{i}$. [2]

**(iii)** Hence or otherwise, find the exact value of $\tan\frac{\pi}{8}$. [2]

**10** Glucose is a type of sugar in our blood and our body uses it for energy. Having low sugar in the blood for long periods of time can cause health problems if it is not treated.

At 10:00 am, a patient is given glucose via an intravenous drip at a constant rate of $r$ units of glucose per hour. It is known that the rate at which the glucose in the blood stream is absorbed by the body, is proportional to the amount of glucose present in the blood stream. At the start of the treatment, the patient has $x_0$ units of glucose in his blood stream.

It is given that $x$ denotes the amount of glucose present in the blood stream of the patient at time $t$ hours after he starts the treatment.

**(a)** Form a differential equation in $x$ and show that $x = \left(x_0 - \frac{r}{k}\right)\mathrm{e}^{-kt} + \frac{r}{k}$ where $k$ is a positive constant. [4]

**(b)** State the theoretical limiting value of $x$ if the patient is given the treatment over a long period. [1]

**(c)** Sketch the graph of $x$ for $t \ge 0$. [2]

**(d)** At what time (to the nearest minute), would the glucose in his blood stream first exceed 90% of the value found in part **(b)**? [5]

**11** On 1st Jan 2023, Mr Kim borrows \$40 000 for his business from a bank. The bank provides two options for repaying the loan. Under Plan A, the bank imposes a one-time administrative fee of \$4660, which is added to the loan amount. Mr Kim needs to pay a monthly installment of \$700 at the start of each month, and the installment amount increases by \$60 for each subsequent month, until the loan amount is fully paid.

**(i) (a)** Show that he would have paid a total amount of $\$\left(30k^2+670k\right)$ after the $k^{\text{th}}$ payment. [2]

**(b)** How many payments will it take for Mr Kim to fully repay his loan? [2]

Under Plan B, an interest rate of 1.5% per month will be charged to the outstanding amount at the end of every month, starting from 31st Jan 2023. Mr Kim needs to pay a fixed monthly instalment of $\$p$ at the end of every month after the interest has been charged.

**(ii) (a)** Find the amount he owes at on 1st Mar 2023. [1]

**(b)** Show that the amount he owes at the end of $n^{\text{th}}$ month after payment of instalment is given by $40000\alpha^n-\beta p\left(\alpha^n-1\right)$, where $\alpha$ and $\beta$ are constants to be determined. [3]

**(c)** Mr Kim intends to pay off his loan in $k$ months. He decides to pay \$1585 per month for the first ($k$ – 1) months and to pay the outstanding amount $\$m$ (where $0<m\leq 1585$) left in the loan at the end of the $k^{\text{th}}$ month after interest has been charged. Find the values of $k$ and $m$. [4]