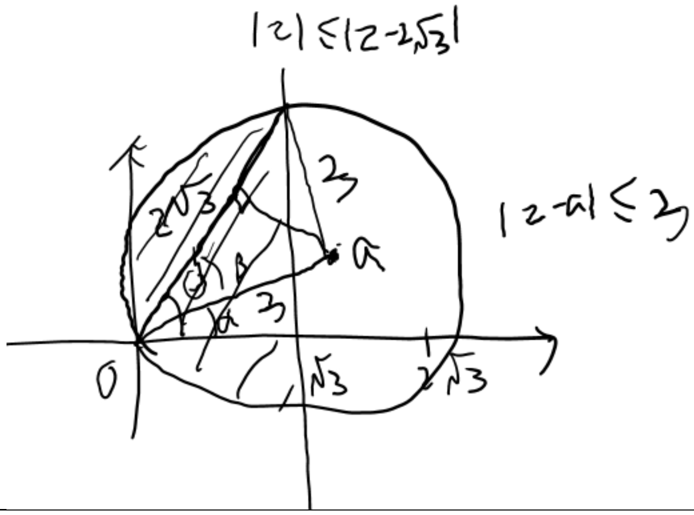
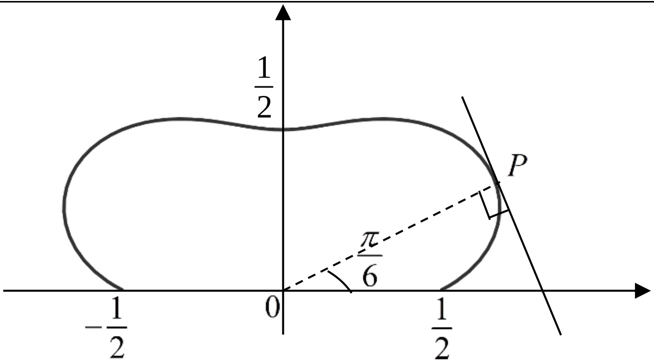
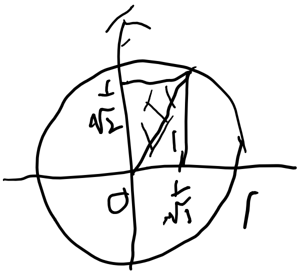
	2023 ASRJC H2 FM Prelim Paper 1 Solutions
1	
	$\cos\theta = \frac{\sqrt{3}}{2\sqrt{3}} = \frac{1}{2} \Rightarrow \theta = 60^0$
	$\cos\beta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \Rightarrow \beta = 54.736^0$
	$\therefore \quad \alpha = 60^0 - 54.736^0 = 5.264^0$
	$\therefore \quad a = 3e^{0.0919i} = 3\left(\cos 0.0919 + i\sin 0.0919\right) = 2.99 + 0.275i$
2	Using the reflective property of ellipse, $\gamma = \theta$
	$(2c)^2 = (FP)^2 + (F'P)^2 - 2(FP)(F'P)\cos(\pi - 2\theta)$
	$(2c)^2 = (FP)^2 + (F'P)^2 + 2(FP)(F'P)\cos 2\theta$
	$4c^2 = (FP)^2 + (F'P)^2 + 2(FP)(F'P)\left(1 - 2\sin^2\theta\right)$
	$4c^2 = (FP)^2 + (F'P)^2 + 2(FP)(F'P) - 4(FP)(F'P)\sin^2\theta$
	$4c^2 = (FP + F'P)^2 - 4(FP)(F'P)\sin^2\theta$
	$4c^2 = (2a)^2 - 4(FP)(F'P)\sin^2\theta$
	$(FP)(F'P)\sin^2\theta = a^2 - c^2$
	$(FP)(F'P) = \frac{b^2}{\sin^2\theta}$, shown

3(i)	$u = y^{1-n}$
	$\frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{(1-n)}y^n\frac{du}{dx}$
	$\frac{1}{(1-n)}y^n\frac{du}{dx} + \frac{y}{x^2} = \frac{1}{(1-n)x^3}y^n$
	$\frac{du}{dx} + \frac{1-n}{x^2}y^{1-n} = \frac{1}{x^3}$
	$\therefore \frac{du}{dx} + \frac{1-n}{x^2}u = \frac{1}{x^3}$, shown
(ii)	$\frac{du}{dx} - \frac{2}{x^2}u = \frac{1}{x^3}$
	Integrating factor = $e^{\int -\frac{2}{x^2}\,dx} = e^{\frac{2}{x}}$
	$e^{\frac{2}{x}}u = \int \frac{1}{x^3}e^{\frac{2}{x}}\,dx = \int -\frac{1}{2x}\left(-\frac{2}{x^2}e^{\frac{2}{x}}\right)dx$
	$= -\frac{1}{2x}e^{\frac{2}{x}} - \int \frac{1}{2x^2}e^{\frac{2}{x}}\,dx$
	$= -\frac{1}{2x}e^{\frac{2}{x}} + \frac{1}{4}e^{\frac{2}{x}} + C$
	$u = -\frac{1}{2x} + \frac{1}{4} + Ce^{-\frac{2}{x}}$
	$y^{-2} = -\frac{1}{2x} + \frac{1}{4} + Ce^{-\frac{2}{x}}$
	$y = \left(-\frac{1}{2x} + \frac{1}{4} + Ce^{-\frac{2}{x}}\right)^{-\frac{1}{2}}, (\because y > 0)$

4(i)	$r = \sin\theta + \frac{1}{2}\cos 2\theta$
	$= \sin\theta + \frac{1}{2}(1 - 2\sin^2\theta)$
	$= \frac{3}{4} - \left(\sin\theta - \frac{1}{2}\right)^2$
	$\therefore$ maximum value of r is $\frac{3}{4}$.
	Alternatively,
	$\frac{dr}{d\theta} = \cos\theta - \sin 2\theta = \cos\theta - 2\sin\theta\cos\theta = \cos\theta(1 - 2\sin\theta)$
	$\frac{dr}{d\theta} = 0 \Rightarrow \cos\theta(1 - 2\sin\theta) = 0$
	$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ or $\sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

	$\theta=\frac{\pi}{2}$, $r=\frac{1}{2}$
	$\theta=\frac{\pi}{6},\frac{5\pi}{6}$, $r=\frac{3}{4}$
	maximum value of r is $\frac{3}{4}$.
(ii)	
(iii)	$\sin\theta=\frac{1}{2}\Rightarrow\theta=\frac{\pi}{6}$
	Area of triangle $=\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{3}{4}\tan\frac{\pi}{6}\right)=\frac{1}{2}\left(\frac{3}{4}\right)\left(\frac{\sqrt{3}}{4}\right)=\frac{3}{32}\sqrt{3}$
	Area of sector $=\frac{1}{2}\int_0^{\frac{\pi}{6}} r^2\ \mathrm{d}\theta$
	$=\frac{1}{2}\int_0^{\frac{\pi}{6}}\left(\sin\theta+\frac{1}{2}\cos 2\theta\right)^2\ \mathrm{d}\theta$
	$=\frac{1}{2}\int_0^{\frac{\pi}{6}}\sin^2\theta+\frac{1}{4}\cos^2 2\theta+\sin\theta\cos 2\theta\ \mathrm{d}\theta$
	$=\frac{1}{2}\int_0^{\frac{\pi}{6}}\frac{1}{2}(1-\cos 2\theta)+\frac{1}{8}(1+\cos 4\theta)+\frac{1}{2}(\sin 3\theta-\sin\theta)\ \mathrm{d}\theta$
	$=\frac{1}{16}\int_0^{\frac{\pi}{6}}5-4\cos 2\theta+\cos 4\theta+4\sin 3\theta-4\sin\theta\ \mathrm{d}\theta$
	$=\frac{1}{16}\left[5\theta-2\sin 2\theta+\frac{1}{4}\sin 4\theta-\frac{4}{3}\cos 3\theta+4\cos\theta\right]_0^{\frac{\pi}{6}}$
	$=\frac{1}{16}\left[\left(\frac{5\pi}{6}-2\sin\frac{\pi}{3}+\frac{1}{4}\sin\frac{2\pi}{3}-\frac{4}{3}\cos\frac{\pi}{2}+4\cos\frac{\pi}{6}\right)-\left(-\frac{4}{3}+4\right)\right]$
	$=\frac{5\pi}{96}+\frac{9}{128}\sqrt{3}-\frac{1}{6}$
	Required area $=\frac{3}{32}\sqrt{3}-\left(\frac{5\pi}{96}+\frac{9}{128}\sqrt{3}-\frac{1}{6}\right)$
	$=\frac{3}{128}\sqrt{3}-\frac{5\pi}{96}+\frac{1}{6}$ where $a=\frac{3}{128},b=-\frac{5}{96},c=\frac{1}{6}$

5	Since $\int_0^1 \left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x$ is the area of the quarter circle $y=\sqrt{1-x^2}$ with centre at origin and radius 1, therefore $I=\int_0^1 \left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x=\frac{1}{4}\pi\left(1^2\right)=\frac{1}{4}\pi$.
	Using Simpson's Rule, $I \approx \frac{\left(\frac{1}{4}\right)}{3}(y_0+4y_1+2y_2+4y_3+y_4)$ $=\frac{1}{12}\left(1+4\sqrt{1-\left(\frac{1}{4}\right)^2}+2\sqrt{1-\left(\frac{1}{2}\right)^2}+4\sqrt{1-\left(\frac{3}{4}\right)^2}+\sqrt{1-(1)^2}\right)$ $=\frac{1}{12}\left(1+4\sqrt{\frac{15}{16}}+2\sqrt{\frac{3}{4}}+4\sqrt{\frac{7}{16}}+0\right)$ $=0.7708988$ $=0.7709 \quad (4\ dp)$
	 From symmetry , area under the curve from $\frac{1}{\sqrt{2}}$ to 1 is $\frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}\right)=\frac{\pi}{8}-\frac{1}{4}$ $2\int_0^{\frac{1}{\sqrt{2}}}\left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x=\frac{\pi}{2}-2\left(\frac{\pi}{8}-\frac{1}{4}\right)=\frac{\pi}{4}+\frac{1}{2}=I+\frac{1}{2}$
	Using Simpson's rule, $\int_0^{\frac{1}{\sqrt{2}}}\left(1-x^2\right)^{\frac{1}{2}}\,\mathrm{d}x \approx \frac{\left(\frac{1}{4\sqrt{2}}\right)}{3}\left(1+4\sqrt{1-\left(\frac{1}{4\sqrt{2}}\right)^2}+2\sqrt{1-\left(\frac{2}{4\sqrt{2}}\right)^2}+4\sqrt{1-\left(\frac{3}{4\sqrt{2}}\right)^2}+\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}\right)$ $=0.642648365$ $I \approx 2\left(0.642648365\right)-\frac{1}{2}=0.78529673=0.7853 \quad (4\ dp)$
	The second method gives a better approximation to the first method because the width of the interval is smaller and so the quadratic approximation for Simpson's Rule in the second method give a better approximation to the area under the curve.

6	(a) $u_{n+1} = (1 + \text{birth rate } - \text{ death rate})u_n - \text{cull}$ $= (1 + b - d)u_n - c,$
	(b) $u_n = pu_{n-1} + q,$ $= p(pu_{n-2} + q) + q$ $= p^2u_{n-2} + pq + q$ $= p^2(pu_{n-3} + q) + pq + q$ $= p^3u_{n-3} + p^2q + pq + q$ $= .....$ $= p^{n-1}u_1 + p^{n-2}q + p^{n-3}q + ... + pq + q$ $= p^{n-1}u_1 + \dfrac{q(1-p^{n-1})}{1-p}$
	(c) If x is the equilibrium population, then $u_{n+1} = u_n$, $x = 1.08x - 160$ $\Rightarrow x = 2000$
	(d) (i) $u_n = 1.08^{n-1}u_1 + 2000(1 - 1.08^{n-1})$ $= 1.08^{n-1}(u_1 - 2000) + 2000$ If $u_1 > 2000$, then $u_1 - 2000 > 0$ and $1.08^{n-1}(u_1 - 2000) \to \infty$ as $n \to \infty$, ie $u_n \to \infty$ The mathematical model which we have developed predicted unlimited growth in population.
	(ii) The most likely scenario is a reducing rate of growth as resources are depleted.

7(a)	When $u = \alpha$, $v = 2\alpha^2 + a \Rightarrow \alpha' = 2\alpha^2 + a$ When $u = \beta$, $v = 2\beta^2 + a \Rightarrow \beta' = 2\beta^2 + a$ $v = 2u^2 + a \Rightarrow \dfrac{dv}{du} = 4u$

	$\int_{\alpha}^{\beta} u^3\sqrt{u^4+au^2+b}\ du = \int_{\alpha}^{\beta} u^3\sqrt{\left(u^2+\frac{1}{2}a\right)^2-\frac{1}{4}a^2+b}\ du$ $=\int_{2\alpha^2+a}^{2\beta^2+a} u^3\sqrt{\left(u^2+\frac{1}{2}a\right)^2-\frac{1}{4}a^2+b}\ .\frac{1}{4u}dv$ $=\int_{2\alpha^2+a}^{2\beta^2+a} \frac{1}{4}\left(\frac{v-a}{2}\right)\sqrt{\left(\frac{1}{2}v\right)^2-\frac{1}{4}a^2+b}\ dv$ Hence $f(v)=\frac{1}{16}(v-a)\sqrt{v^2-a^2+4b}$
(b)	$C_1:\ x=3t^2,\quad y=t^2(2t-3)$ where $t\geq 0$. $y<0 \Rightarrow t^2(2t-3)<0 \Rightarrow 0<t<\frac{3}{2}$ $\therefore\quad r=\int_0^{\frac{3}{2}}\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $=\int_0^{\frac{3}{2}}\sqrt{(6t)^2+(6t(t-1))^2}\,dt$ $=\int_0^{\frac{3}{2}}6t\sqrt{1+(t-1)^2}\,dt$ Let w $=t-1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t}=1$ When $t=0 \Rightarrow w=-1$ When $t=\frac{3}{2} \Rightarrow w=\frac{1}{2}$ $\therefore\quad r=\int_{-1}^{\frac{1}{2}}6(w+1)\sqrt{1+w^2}\,\mathrm{d}w$ Or By translating the curve by 1 unit in the negative direction of the t-axis, $\int_0^{\frac{3}{2}}6t\sqrt{1+(t-1)^2}\,dt = \int_{-1}^{\frac{1}{2}}6(t+1)\sqrt{1+t^2}\,\mathrm{d}t = \int_{-1}^{\frac{1}{2}}6(w+1)\sqrt{1+w^2}\,\mathrm{d}w$ $C_2:\ 3x=t^3-3t,\quad 3y=t^3$ $\frac{\mathrm{d}x}{\mathrm{d}t}=t^2-1,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=t^2$

	$\therefore \quad s=\int_0^{-\frac{\sqrt{3}}{2}} 2\pi y\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$ $=\int_0^{-\frac{\sqrt{3}}{2}} 2\pi\left(\frac{t^3}{3}\right)\sqrt{\left(t^2-1\right)^2+t^4}\,dt$ $=\frac{2\pi}{3}\int_0^{-\frac{\sqrt{3}}{2}} t^3\sqrt{2t^4-2t^2+1}\,dt$ $=\frac{2\sqrt{2}\pi}{3}\int_0^{-\frac{\sqrt{3}}{2}} t^3\sqrt{t^4-t^2+\frac{1}{2}}\,dt$ Using part (a), $\therefore \quad s=\frac{2\sqrt{2}\pi}{3}\int_{-1}^{\frac{1}{2}}\frac{1}{16}(w+1)\sqrt{w^2-1+2}\,\mathrm{d}w$ $=\frac{\pi\sqrt{2}}{24}\int_{-1}^{\frac{1}{2}}(w+1)\sqrt{w^2+1}\,\mathrm{dw}$ $=\frac{\pi\sqrt{2}}{24}.\frac{r}{6}$ $=\frac{\sqrt{2}\pi r}{144}$

8(i)	$x=L\sin\theta\ ,\ y=-L\cos\theta$
	$\frac{\mathrm{d}x}{\mathrm{d}t}=L\cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}t},\ \frac{\mathrm{d}y}{\mathrm{d}t}=L\sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$
	$\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=L\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-L\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2,\ \frac{\mathrm{d}^2y}{\mathrm{d}t^2}=L\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+L\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$, shown
(ii)	$-T\sin\theta=m\left[L\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-L\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]$ ----- (1)
	Applying Newton's 2nd law, $T\cos\theta-mg=m\left[L\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+L\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]$
	$\Rightarrow T\cos\theta=mg+mL\left[\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2\right]$ ----- (2)
	$\frac{(2)}{(1)}\Rightarrow-\frac{\cos\theta}{\sin\theta}=\frac{g+L\sin\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+L\cos\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}{L\cos\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}-L\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2}$

	$-L\cos^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+L\cos\theta\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2=g\sin\theta+L\sin^2\theta\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}+L\cos\theta\sin\theta\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$
	$\left(\cos^2\theta+\sin^2\theta\right)L\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=-g\sin\theta$
	$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}=-\frac{g}{L}\sin\theta \quad \left(\because \cos^2\theta+\sin^2\theta=1\right)$
	If θ is small, $\sin\theta\approx\theta$
	$\therefore \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2}\approx-\frac{g}{L}\theta$, shown
(iii)	Characteristic equation: $m^2+\frac{g}{L}=0\Rightarrow m=\pm\sqrt{\frac{g}{L}}i$
	$\therefore \theta=A\sin\sqrt{\frac{g}{L}}t+B\cos\sqrt{\frac{g}{L}}t$ where A,B are arbitrary constants
	$\theta_0=\frac{\pi}{12} \Rightarrow \frac{\pi}{12}=B$
	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=A\sqrt{\frac{g}{L}}\cos\sqrt{\frac{g}{L}}t-B\sqrt{\frac{g}{L}}\sin\sqrt{\frac{g}{L}}t$
	$\frac{\mathrm{d}\theta}{\mathrm{d}t}=-\frac{\pi}{12}\sqrt{\frac{g}{L}}$, $t=\sqrt{\frac{L}{g}}\pi \Rightarrow -\frac{\pi}{12}\sqrt{\frac{g}{L}}=-A\sqrt{\frac{g}{L}}$
	$\Rightarrow A=\frac{\pi}{12}$
	$\therefore \theta=\frac{\pi}{12}\left(\sin\sqrt{\frac{g}{L}}t+\cos\sqrt{\frac{g}{L}}t\right)$
	$\theta=\frac{\sqrt{2}\pi}{12}\sin\left(\sqrt{\frac{g}{L}}t+\frac{\pi}{4}\right)$
	Maximum $\theta=\frac{\sqrt{2}\pi}{12}$ since $-1\le\sin\left(\sqrt{\frac{g}{L}}t+\frac{\pi}{4}\right)\le1$
(iv)	$\omega=\frac{\mathrm{d}\theta}{\mathrm{d}t}------(1)$
	$\frac{\mathrm{d}\omega}{\mathrm{d}t}=-\frac{g}{L}\sin\theta------(2)$
	$\theta_1=\theta_0+h\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\|_{\theta=\theta_0,\ \omega=\omega_0}=\theta_0$
	$\omega_1=\omega_0+h\left.\frac{\mathrm{d}\omega}{\mathrm{d}t}\right\|_{\theta=\theta_0,\ \omega=\omega_0}=0+1\left(-\frac{g}{L}\sin\theta_0\right)=-\frac{g}{L}\sin\theta_0$
	$\theta_2=\theta_1+h\left.\frac{\mathrm{d}\theta}{\mathrm{d}t}\right\|_{\theta=\theta_1,\ \omega=\omega_1}=\theta_0-\frac{g}{L}\sin\theta_0$

9	(i) For $\alpha = 5$, from GC, $rref\begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & 0 \\ -1 & -3 & 5 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $w - 2y - 3z = 0 \Rightarrow w = 2y + 3z$ $x - y + 2z = 0 \Rightarrow x = y - 2z$ $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2y+3z \\ y-2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ Basis for $K_1 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$ since $\begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \neq \beta \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ for any $\beta \in \mathbb{R}$ When $\alpha = 5$, $\dim(K_1) = 2 \Rightarrow$ rank of A = 4 - 2 = 2 $\begin{pmatrix} 1 & 2 & -4 & 1 \\ 2 & 3 & -7 & 5-\alpha \\ -1 & -3 & \alpha & -3 \end{pmatrix} \xrightarrow[R1+R3]{-2R1+R2} \begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 1 & 3-\alpha \\ 0 & -1 & \alpha-4 & -2 \end{pmatrix} \xrightarrow{-R2+R3} \begin{pmatrix} 1 & 2 & -4 & 1 \\ 0 & -1 & 1 & 3-\alpha \\ 0 & 0 & \alpha-5 & \alpha-5 \end{pmatrix}$ For $\alpha \neq 5$, $w + 2x - 4y + z = 0$ $-x + y + (3-\alpha)z = 0 \Rightarrow x = y + (3-\alpha)z$ $(\alpha-5)y + (\alpha-5)z = 0 \Rightarrow y + z = 0 \Rightarrow y = -z$ We have $w + 2(-z + (3-\alpha)z) + 4z + z = 0$ $w = 9z - 2\alpha z \Rightarrow w = (2\alpha - 9)z$ $\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (2\alpha-9)z \\ -z+(3-\alpha)z \\ -z \\ z \end{pmatrix} = z\begin{pmatrix} 2\alpha-9 \\ 2-\alpha \\ -1 \\ 1 \end{pmatrix}$

Basis for $K_2 = \left\{ \begin{pmatrix} 2\alpha - 9 \\ 2 - \alpha \\ -1 \\ 1 \end{pmatrix} \right\}$

When $\alpha \neq 5$, $\dim(K_2) = 1 \Rightarrow$ rank of A = 4 - 1 = 3

(ii)

From observation,

$$A\left(\lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right) = \lambda\begin{pmatrix}-1\\-2\\1\end{pmatrix}$$

$$A\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = A\left(\lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right)$$

$$A\begin{pmatrix}w\\x\\y\\z\end{pmatrix} - A\left(\lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right) = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix} - \left(\lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix}\right) = \mu\begin{pmatrix}2\\1\\1\\0\end{pmatrix} + \tau\begin{pmatrix}3\\-2\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}w\\x\\y\\z\end{pmatrix} = \lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\\0\end{pmatrix} + \tau\begin{pmatrix}3\\-2\\0\\1\end{pmatrix}$$

Therefore $S = \left\{ \lambda\begin{pmatrix}1\\1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\\0\end{pmatrix} + \tau\begin{pmatrix}3\\-2\\0\\1\end{pmatrix}, \lambda, \mu, \tau \in \mathbb{R} \right\}$

	Alternatively, $rref\left(\begin{array}{cccc\|c} 1 & 2 & -4 & 1 & -1 \\ 2 & 3 & -7 & 0 & -2 \\ -1 & -3 & 5 & -3 & 1 \end{array}\right) = \left(\begin{array}{cccc\|c} 1 & 0 & -2 & -3 & -1 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$ and obtain the answer $S = \left\{ \lambda \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \tau \begin{pmatrix} 3 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \lambda, \mu, \tau \in \mathbb{R} \right\}$ S is the column space of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -2 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, therefore S is a vector space.
	(iv) Range space of T = $\left\{ \theta \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \phi \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}, \theta, \phi \in \mathbb{R} \right\}$ Assume that $3x - y + z = 0$. Then $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 3x+z \\ z \end{pmatrix} = -x \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} - z \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ But $\begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = (-3) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} = (-2) \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -3 \end{pmatrix}$ Therefore $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in R,$ the range space of T. Therefore $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \notin U$. Hence if $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U,$ then $3x - y + z \neq 0$.

10(i)	Since the ellipse is a scaling of the circle parallel to the *y*-axis with a scale factor of $\frac{b}{a}$, $y=\frac{b}{a}(a\sin E)=b\sin E$,
(ii)	$r^2=x^2+y^2$
	$r^2=(a\cos E-ae)^2+(b\sin E)^2$
	$=a^2\cos^2 E-2a^2e\cos E+a^2e^2+b^2\sin^2 E$
	$=a^2\cos^2 E-2a^2e\cos E+a^2e^2+(a^2-c^2)\sin^2 E$
	$=a^2\cos^2 E-2a^2e\cos E+a^2e^2+(a^2-a^2e^2)\sin^2 E$
	$=a^2(\cos^2 E+\sin^2 E)-2a^2e\cos E+a^2e^2(1-\sin^2 E)$
	$=a^2-2a^2e\cos E+a^2e^2\cos^2 E$
	$\Rightarrow r^2=a^2(1-e\cos E)^2$
	$\therefore r=a(1-e\cos E)$ since $r\ge 0$, shown
	Alternative method
	$x=r\cos\theta=a\cos E-ae$ ----- (1)
	$r=\dfrac{a(1-e^2)}{1+e\cos\theta}$
	$r+er\cos\theta=a(1-e^2)$
	$r+e(a\cos E-ae)=a(1-e^2)$ (using (1))
	$r=a-ae^2-ae\cos E+ae^2$
	$r=a(1-e\cos E)$, shown
(iii)	$r\cos\theta=a\cos E-ae$
	$r\left(2\cos^2\frac{\theta}{2}-1\right)=a\cos E-ae$
	$2r\cos^2\frac{\theta}{2}=a\cos E-ae+a(1-e\cos E)$ (using (ii))
	$2r\cos^2\frac{\theta}{2}=a\left(2\cos^2\frac{E}{2}-1\right)-ae+a-ae\left(2\cos^2\frac{E}{2}-1\right)$
	$r\cos^2\frac{\theta}{2}=a\cos^2\frac{E}{2}-a-ae+a-ae\cos^2\frac{E}{2}+ae$
	$r\cos^2\frac{\theta}{2}=a\cos^2\frac{E}{2}(1-e)$ ----- (1)
	Similarly, $r\left(1-2\sin^2\frac{\theta}{2}\right)=a\cos E-ae$
	$2r\sin^2\frac{\theta}{2}=a-ae\cos E-a\cos E+ae$
	$2r\sin^2\frac{\theta}{2}=a-ae\left(1-2\sin^2\frac{E}{2}\right)-a\left(1-2\sin^2\frac{E}{2}\right)+ae$

	$r\sin^2\frac{\theta}{2}=a\sin^2\frac{E}{2}(1+e)$ ----- (2)
	$\frac{(1)}{(2)}\Rightarrow \tan^2\frac{\theta}{2}=\frac{1+e}{1-e}\tan^2\frac{E}{2}$
	$\therefore \tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2}$ when $0\le\theta\le\pi$, shown
(iv)	Let $\mathrm{f}(E)=E-\frac{1}{4}\sin E-\frac{4\pi}{19}$
	$\mathrm{f}'(E)=1-\frac{1}{4}\cos E$
	$E_{n+1}=E_n-\dfrac{E_n-\frac{1}{4}\sin E_n-\frac{4\pi}{19}}{1-\frac{1}{4}\cos E_n}$
	$E_0=\left(1+\frac{3}{4}\left(\frac{1}{4}\right)\right)\frac{4\pi}{19}=\frac{\pi}{4}$
	$E_1=\frac{\pi}{4}-\dfrac{\frac{\pi}{4}-\frac{1}{4}\sin\frac{\pi}{4}-\frac{4\pi}{19}}{1-\frac{1}{4}\cos\frac{\pi}{4}}\approx 0.849496$
	$E_2\approx 0.849051$
	$E_3\approx 0.849051\approx 0.8491$, correct to 4 dp
	$\tan\frac{\theta}{2}=\sqrt{\dfrac{1+\frac{1}{4}}{1-\frac{1}{4}}}\tan\frac{0.8491}{2}\Rightarrow\theta=1.06$ rad, correct to 3 s.f.