ANGLO-CHINESE JUNIOR COLLEGE
JC2 PRELIMINARY EXAMINATION

Higher 2

/100

CANDIDATE NAME

TUTORIAL/ FORM CLASS

INDEX NUMBER

MATHEMATICS 9758/02

Paper 2

29 August 2023

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your index number, class and name on all the work you hand in.
Write in dark blue or black pen.

You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

Question	Marks
1	/5
2	/8
3	/8
4	/9
5	/10
6	/7
7	/10
8	/9
9	/9
10	/12
11	/13

This document consists of **26** printed pages and **2** blank pages.

Section A: Pure Mathematics [40 marks]

1 Using the substitution $u = \cos x$, find $\int \frac{\sin 2x}{\cos^2 x + \cos x + 2}\,dx$, giving your answer in terms of x. [5]

2 It is given that $y = \cot(x+a)$ for some constant a.

(i) Show that $\frac{dy}{dx} = -(1+y^2)$ and hence find $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ in terms of y. [3]

(ii) Explain why the Maclaurin series of y cannot be found when $a = 0$. [1]

In the case where $a = \frac{\pi}{2}$,

(iii) use your answers in **(i)** to find the series expansion of y in ascending powers of x, up to and including the term in x^3. [2]

(iv) Using your answer in **(iii)** and the standard series from the List of Formulae (MF26), find the series expansion of $\frac{y}{x+2}$ in ascending powers of x, up to and including the term in x^3. [2]

3 The Stoke's Law states that when a spherical object of radius R is falling at velocity v in a fluid, the resistance force F acting on the sphere is given by

$$F = -6\pi\eta R v,$$

where η is the coefficient of viscosity of the fluid. The following table shows the coefficients of viscosity of various fluids at $25\ °\text{C}$.

Fluid	Coefficient of viscosity η, $\text{kg m}^{-1}\text{s}^{-1}$
Acetone	3.06×10^{-4}
Benzene	1.81×10^{-5}
Castor Oil	0.985
Corn Syrup	1.3806
Ethanol	1.074×10^{-3}
Methanol	5.44×10^{-4}
Olive Oil	8.1×10^{-2}

A spherical ball of radius 0.05 m and mass 0.2 kg is released from rest in a huge container filled with corn syrup at $25\,^{\circ}\mathrm{C}$. It can be shown that as the ball falls through the corn syrup, the velocity v ms^{-1} at time t seconds after release satisfies the differential equation

$$\frac{\mathrm{d}v}{\mathrm{d}t}+\frac{6\pi\eta R}{m}v=g\,,$$

where $g=9.780\ \mathrm{m\,s}^{-2}$ is the acceleration due to gravity and m kg is the mass of the ball.

(i) Find v in terms of t. [5]

(ii) State the velocity of the ball after a long time, assuming that it does not reach the bottom of the container. [1]

(iii) The distance travelled by the ball from the point it was released is denoted by x m. Using the fact that $v=\dfrac{\mathrm{d}x}{\mathrm{d}t}$, find the distance covered by the ball after 30 seconds. [2]

4 **(a)** **(i)** Write $\dfrac{4}{4r^2+12r+5}$ in partial fractions. [1]

(ii) Find $\displaystyle\sum_{r=2}^{n}\left(\frac{4}{4r^2+12r+5}\right)$. [3]

A sequence $u_1, u_2, u_3, \ldots$ is such that $u_1=3$ and $u_r=u_{r-1}+\dfrac{4}{4r^2+12r+5}$ for $r\geq 2$.

(iii) By considering, $\displaystyle\sum_{r=2}^{n}\left(u_r-u_{r-1}\right)$, express u_n in terms of n. [2]

(b) The root test is a test for convergence of infinite series of the general form $\displaystyle\sum_{r=0}^{\infty}a_r$. The test states that the series converges when $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|}<1$, and diverges when $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|}>1$. When $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|}=1$, the test is inconclusive. Use the test to explain why the series $\displaystyle\sum_{r=0}^{\infty}\left(\frac{-3r^3+rx}{5r^3+7}\right)^r$ converges for all values of x. [3]

5 A curve C is defined by the parametric equations

$$x=\cos\theta-\frac{1}{\cos\theta},\ y=\cos\theta+\frac{1}{\cos\theta},\ \ -\frac{\pi}{2}<\theta<\frac{\pi}{2}.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\cos^2\theta-1}{\cos^2\theta+1}$. [2]

The line L is the normal to C at the point with parameter p.

(ii) Show that the equation of L is $y=\dfrac{1+\cos^2 p}{1-\cos^2 p}x+2\left(\dfrac{1+\cos^2 p}{\cos p}\right)$. [2]

(iii) The line L cuts the x- and y- axes at the points P and Q respectively. Find the area of triangle OPQ in terms of p. [3]

(iv) Given that p increases at a constant rate of 0.1 units per second, find the rate of increase of the area OPQ when $p=\dfrac{\pi}{3}$. [3]

Section B: Probability and Statistics [60 marks]

6 A factory manufactures a large number of chips in a variety of colours and packs them into boxes. Each box contains 36 randomly chosen chips. On average, 30% of the chips are yellow. You may assume that the number of yellow chips in a box follows a binomial distribution.

(i) A randomly chosen box is found to have at most 9 yellow chips. Find the probability that it contains more than 4 yellow chips. [2]

75 randomly chosen boxes are packed into a carton for shipment.

(ii) Find the probability that the 60th box chosen is the 14th box with at most 9 yellow chips. [2]

(iii) 30 cartons are sold to an arcade. Using a suitable approximation, find the probability that the mean number of boxes containing at most 9 yellow chips in each carton is more than 25. [3]

7 An electrical appliance supplier is investigating the potential impact of monthly advertisement expenditure, denoted as x (in thousands of dollars), on the popular social media platform SnapGram, on their monthly sales profits, denoted as y (in thousands of dollars). The supplier has collected the following data over the course of the past year.

Month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
Advertisement Expenditure, x	5	6.5	7	6	8	5.5	10	7.5	8.5	9	4	9.5
Sales Profits, y	16	19	19.5	17	k	16.5	32.5	22	26	28	15.5	30

It is thought that this set of data can be modelled by one of the following equations:

Model A: $y = mx + c$

Model B: $\ln y = ax + b$

(i) Explain the meaning of the value of m in the context of the data for Model A. [1]

(ii) It is given that the least squares regression line for Model A is $y = 0.3604 + 3.0194x$, correct to 4 decimal places. Show that $k = 23.5$, correct to 1 decimal place. [2]

(iii) Draw the scatter diagram for the data, labelling the axes clearly. [1]

(iv) By calculating the product moment correlation coefficients and using the scatter diagram from part **(iii)**, explain whether Model A or Model B is a more appropriate model for the given set of data. [3]

Use the more appropriate model selected in part **(iv)** for the rest of this question.

(v) Use a regression line to obtain the estimate for the sales profit if the advertisement expenditure is $11,000 for a particular month. [2]

(vi) Comment on the reliability of the estimate obtained in **(v)**. [1]

8 Kelly has a box of spherical beads labelled with the 26 letters from A to Z, all of which have the same size. For each letter, there are exactly 3 beads of different colours, red, blue and gold. By stringing the labelled beads together in a specific order, Kelly plans to make a keychain. A possible design of a keychain with 3 labelled beads is shown in Fig. 1.

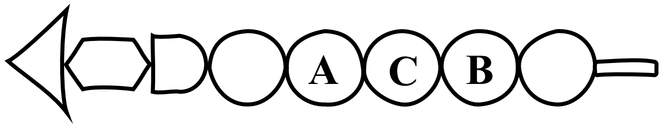

Fig. 1

(a) Find the number of ways that the labelled beads in the keychain can be arranged if it is to have

(i) 6 beads with distinct letters but are of the same colour. [1]

(ii) 6 beads with 3 distinct letters. [3]

Kelly would like to add in beads with different shapes to the keychain. She took out another box of beads and the number of each type of beads that she has is listed in the table below:

Shape \ Colour	Red	Gold	Blue
Star	c	6	2
Butterfly	2	5	5
Flower	6	3	2
Heart	5	2	3

(b) Kelly chooses two beads at random from this new box of beads. The probability that both chosen beads are star-shaped but neither of those two beads are gold is $\frac{1}{66}$.

(i) Find c. [2]

(ii) Find the probability that two randomly chosen beads include exactly one red bead and at least one bead with a flower shape. [3]

9 A bag contains one green disc, r red discs and $2r$ blue discs, where $r > 1$.

(a) Tim and Sim play a game by taking turns to draw a disc from the bag at random. The drawn disc is replaced after each turn. The first person to draw the green disc wins. If Sim started the game first, show that the probability of Sim winning the game is $\frac{3r+1}{6r+1}$. [2]

(b) In another game, a player draws 2 discs from the bag at random. If he draws a red disc, he scores 5 points, if he draws a blue disc, he scores 2 points and if he draws a green disc he scores 0. The player's score, denoted by T, is found by taking the difference in the number of points he scores for the two discs.

(i) Find $\mathrm{P}(T = t)$ for all possible values of t, leaving your answers in terms of r. [3]

(ii) Show that the expectation of the player's score is $\dfrac{2(2r+3)}{3r+1}$. [1]

(c) In a third game, a player draws a disc from the bag at random. If he draws a green disc, he wins \$0.15, if he draws a blue disc, he wins \$0.10 and if he draws a red disc, he loses \$0.25. May and Jun each draws a disc from the bag once, with the disc replaced in the bag after the first player drew. The probability that May ends up with more money than Jun is $\dfrac{27}{112}$. Find the value of r. [2]

Is the game fair for a player? Explain. [1]

10 In this question you should clearly state the values of the appropriate parameters of any distributions used.

(a) The diameters of footballs are normally distributed with mean 22.0 cm and standard deviation 0.3 cm, and the diameters of basketballs are normally distributed with mean μ cm and standard deviation 0.4 cm.

The probability that the diameter of a randomly chosen basketball is at least 10% larger than the diameter of a randomly chosen football is 0.35.

(i) Show that $\mu = 24.0$. [3]

(ii) Draw a sketch to show the distribution of the diameter of a randomly chosen basketball, including the main features of the curve. On your sketch, shade the area representing the probability that a randomly chosen basketball has diameter between 22.8 cm and 25.2 cm. [2]

(iii) Find the probability that the difference between the average diameters of 10 randomly chosen footballs and the diameter of a randomly chosen basketball is less than 1.5 cm. [3]

(b) A computer game simulates how a basketball player scores by shooting the ball through an elevated hoop. The player aims by using the mouse to change the angle of the shot. The probability of a player scoring on his first attempt is $\dfrac{5}{11}$ and the probability of him scoring on his second attempt is $\dfrac{5}{8}$.

(i) Show that the probability of the player scoring on both attempts is at least $\dfrac{7}{88}$. [2]

(ii) The probability of the player scoring on his first attempt but not on his second attempt is $\dfrac{15}{88}$. Find the probability of the player scoring on both attempts. Are the events "player scores on his first attempt" and "player scores on his second attempt" independent? Justify your answer. [2]

11 Energy bars are a convenient source of nutrition for athletes. A typical energy bar weighs between 30-50 g and is likely to supply about 200-300 calories. An American company sells energy bars and a nutritionist working in the company claims that the mean number of calories in a product, Perfect Protein bars, is 350 calories.

A random sample of 30 bars is collected and the number of calories, x cal, in the 30 bars is summarised as follows.

$$\sum x = 10446\ ,\ \sum x^2 = 3638000\,.$$

(i) Test at the 3% level of significance whether the mean number of calories in Perfect Protein bars is less than 350 calories, defining any symbols that you use. [7]

(ii) Explain why the test would be inadmissible if the nutritionist had taken a random sample of 15 energy bars and hence state an assumption needed for him to carry out the test. [1]

(iii) If the test was for whether the mean number of calories in Perfect Protein bars is 350 calories, find the smallest level of significance for which the nutritionist's claim will be rejected. [2]

The company launched a new product called Dynamic Protein bars which are advertised to be higher in calories than Perfect Protein bars. A random sample of 45 Dynamic Protein bars is collected to test this and the mean number of calories of the sample is found to be 352.2.

(iv) The result of this investigation shows that there is no reason to believe that the mean number of calories in the new product is more than 350 calories at the 4% level of significance. Find the range of possible values of the variance used in calculating the test statistic. [3]