Inertial Frames of Reference (non-relativistic)

In the H3 syllabus, candidates should understand and apply concepts related to non-relativistic dynamics viewed from different inertial frames building on the understanding of collisions and the significance of the centre of mass in equilibrium situations.

Learning Outcomes for the Inertial Frames (non-relativistic):

Candidates should be able to:

(a) show an understanding of what is meant by a frame of reference
(b) recall and apply the Galilean transformation equations to solve problems relating observations in different frames of reference
(c) show an understanding of what is meant by an inertial frame of reference, in the context of Newton's laws of motion
(d) show an understanding that the centre of mass moves as though the total mass is concentrated at that point and is acted upon by the net external force on the system
(e) solve two-dimensional collision problems by considering velocities relative to the centre of mass of the system

It will also be useful for this topic to recall the learning outcomes about the linear momentum and collisions in H2 syllabus.

Students should be able to:

(d) define and use linear momentum as the product of mass and velocity
(e) define and use impulse as the product of force and time of impact
(f) relate resultant force to the rate of change of momentum
(g) recall and solve problems using the relationship $F = ma$, appreciating that resultant force and acceleration are always in the same direction
(h) state the principle of conservation of momentum
(i) apply the principle of conservation of momentum to solve simple problems including inelastic and (perfectly) elastic interactions between two bodies in one dimension
(j) show an understanding that, for a (perfectly) elastic collision between two bodies, the relative speed of approach is equal to the relative speed of separation
(k) show an understanding that, whilst the momentum of a closed system is always conserved in interactions between bodies, some change in kinetic energy usually takes place

The notes, figures and questions are the compilation from the following physics textbooks for the students who take H3 A level Physics 9814:

- Young and Freedman, University Physics
- Serway and Jerwett, Physics for Scientists and Engineers
- Eric Mazur, Principles and Practices of Physics

Introduction

Did you ever, sitting in a car at a red light and looking at the car next to you, slam on the brakes because you thought you were starting to roll but it turned out your car never moved? If you did, you experienced the relativity of motion, first described quantitatively by Galileo.

The velocity measured for any object depends on the motion of the observer (the person doing the measuring). For example, to a person sitting in a moving train, a suitcase on the overhead rack is at rest, but to a person standing on a station platform watching the train speed by, the suitcase is not at rest. According to the person in the train, the suitcase has zero momentum and zero kinetic energy. According to the person on the platform, the suitcase has nonzero momentum and nonzero kinetic energy.

In this topic, we investigate whether or not the laws of conservation of momentum and conservation of energy depend on the velocity of the observer. In other words, if these laws are valid for one observer, are they also valid for an observer who is moving relative to the first observer?

Frame of reference

Whenever we talk about motion, we must specify a reference axis along which the motion occurs and an origin.

A frame of reference is a combination of a reference axis that defines a direction in space and a reference point that defines the origin from which motion is measured.

A frame of reference can be described by a Cartesian coordinate system for which an observer is at rest with respect to the origin. Let us conceptualize a sample situation in which there will be different observations for different observers. Consider the two observers A and B along the number line as in Fig. 1.1.

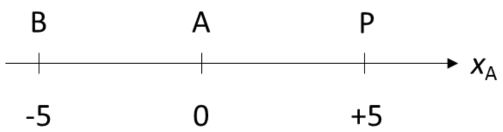


Fig. 1.1

Observer A is located at the origin of a one dimensional x axis, while observer B is at the position $x_A = -5$. Both observers measure the position of point P, which is located at $x_A = +5$.

Suppose observer B decides that he is located at the origin of an x_B axis as shown in Fig. 1.2.

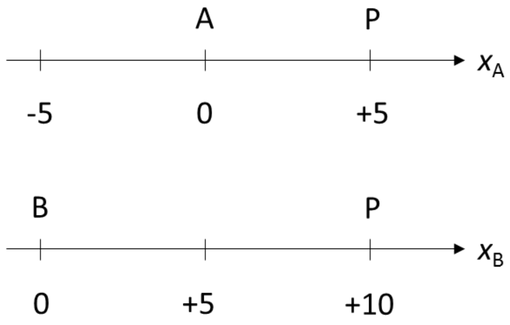


Fig. 1.2

Observer A claims point P is a located at a position with a value of +5, whereas observer B claims it is located at a position with a value of +10.

Both observers are correct, even though they make different measurements. Their measurements differ because they are making the measurement from different frames of reference.

Imagine now that observer B is moving to the right along *x*-axis. Now the two measurements are even more different. Observer A claims point P remains at rest a position with a value of +5, whereas observer B claims the position of P continuously changes with time, even passing him and moving behind him! Again, both observers are correct, with the difference in their measurements arising from their different frames of reference.

Consider two observers watching a man walking on a moving beltway at an airport in Fig. 1.3. The woman standing on the moving beltway sees the man moving a normal walking speed. The woman observing from the stationary floor sees the man moving with a higher speed because the beltway speed combines with this walking speed. Both observers look at the same man and arrive at different values for his speed. Both are correct; the difference in their measurements results from the relative velocity of their frames of reference.

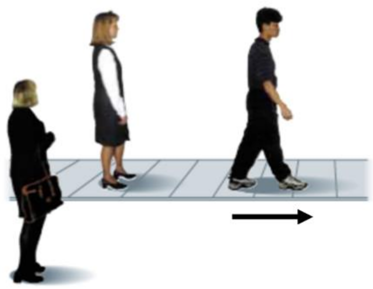


Fig. 1.3

You may watch the following video to have a better sense of about frames of reference.

Galilean transformation equations

In a more general situation, consider a particle located at point P as shown in Fig. 1.4. Imagine that the motion of this particle is being described by two observers, observer A in a reference frame S_A fixed relative to the Earth and a second observer B in a reference frame S_B moving to the right relative to S_A (and therefore relative to the Earth) with a constant velocity $\vec{v}_{AB}$. In this discussion of relative velocity, we use a double-subscript notation; the first subscript represents who is doing the observing, and the second represents what is being observed. Therefore, the notation $\vec{v}_{AB}$ means the velocity of observer B (and the attached frame S_B) as observed by observer A. With this notation, observer B measures A to be moving to the left with a velocity $\vec{v}_{BA} = -\vec{v}_{AB}$. Let us place each observer at his or her respective origin.

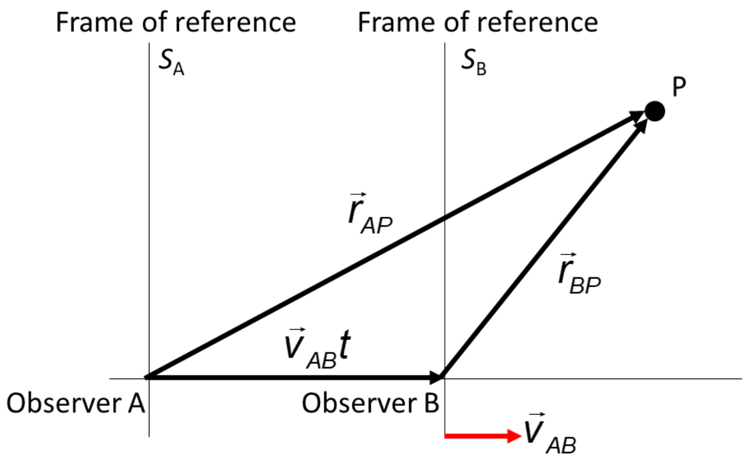


Fig. 1.4

We define the time $t = 0$ as the instant at which the origins of the two reference frames coincide in space. Therefore, at time t, the origins of the reference frames will be separated by a distance $\vec{v}_{AB}t$. We label the position P of the particle relative to observer A with the position vector $\vec{r}_{AP}$ and that relative to observer B with position vector $\vec{r}_{BP}$, both at time t.

From Fig. 1.4, we see that the vectors $\vec{r}_{AP}$ and $\vec{r}_{BP}$ are related to each other through the expression

$$\vec{r}_{AP} = \vec{r}_{BP} + \vec{v}_{AB}t \qquad (1.1)$$

By differentiating the Equation (1.1) with respect to time, noting that v is constant, we obtain

$$\begin{aligned} \frac{d\vec{r}_{AP}}{dt} &= \frac{d\vec{r}_{BP}}{dt} + \vec{v}_{AB} \\ \vec{u}_{AP} &= \vec{u}_{BP} + \vec{v}_{AB} \end{aligned} \qquad (1.2)$$

where $\vec{u}_{AP}$ is the velocity of the particle at P measured by observer A and $\vec{u}_{BP}$ is its velocity measured by B.

Equations (1.1) and (1.2) are known as **Galilean transformation equations.** They relate the position and velocity of a particle as measured by observers in relative motion. The Galilean transformation equations are predicated on the assumption that measurements of time intervals and lengths are not affected by motion. Even though this assumption sounds reasonable, it has been invalidated by experimental confirmation of the theory of special relativity. For most everyday phenomena, however, when speeds are much lower than the speed of light (about 3.0 x 10^8 m s^{-1}), the Equations (1.1) and (1.2) remain excellent approximations for non-relativistic motions.

Although observers in two frames measure different velocities for the particle, they measure the same acceleration when $\vec{v}_{AB}$ is constant. We can verify that by taking the time derivative of the Equation (1.2):

$$\frac{d\vec{u}_{AP}}{dt} = \frac{d\vec{u}_{BP}}{dt} + \frac{d\vec{v}_{AB}}{dt} \qquad (1.3)$$

Because $\vec{v}_{AB}$ is constant, $\frac{d\vec{v}_{AB}}{dt} = 0$. Therefore, we conclude that **changes in velocity are the same in any two reference frames moving at constant relative velocity**.

From (1.3), it is also concluded that $\vec{a}_{AP} = \vec{a}_{BP}$. That is, the acceleration of the particle measured by an observer on one frame of reference is the same as that measured by any other observer moving with constant velocity relative to the first frame.

EXAMPLE 1

You are driving at 25 m s^{-1} on a straight, horizontal road when a truck going 30 m s^{-1} in the same direction overtakes you. Let the positive *x* direction point in the direction of travel, and let the origins of the reference frames affixed to your car and the truck coincide at the instant the truck overtakes you. Use Galilean transformation equations to answer the following questions.

a) What is your car's velocity as measured by someone in the truck?
b) What is the velocity of the truck relative to your car?
c) What is your car's position as measured by someone in the truck 60 s after overtaking you?

EXAMPLE 2

In a train moving due north at 3.1 m s^{-1} relative to Earth, a passenger carrying a suitcase walks forward down the aisle at 1.2 m s^{-1} relative to the train. A spider crawls along the bottom of the suitcase at 0.5 m s^{-1} southward relative to the suitcase.

Using Galilean transformation equations, find the velocity of the spider relative to Earth.

Inertial frame of reference in the context of Newton's laws of motion

How does the choice of reference frame affect our methods for momentum and energy? To answer this question, consider two carts, cart 1 is at rest and cart 2 is moving to the right, from the point of view of two observers as shown in Fig. 1.5(a) and (b) below.

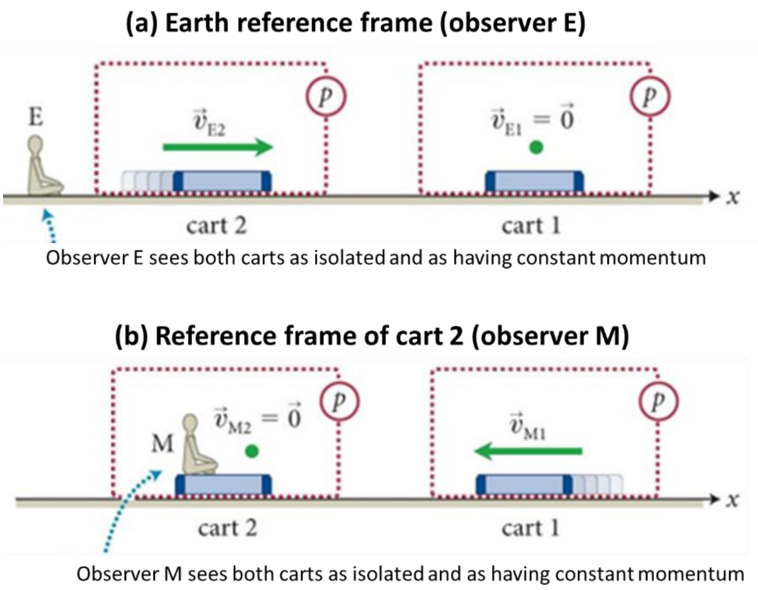


Fig. 1.5

(*The symbol* Ⓟ *denotes an isolated system and indicates that no momentum crosses the system's boundary.)*

To an observer E, who is at rest relative to Earth, cart 1 is at rest and cart 2 is moving at constant velocity as shown in Fig. 1.5a.

To an observer M, who is moving along with cart 2, cart 2 is at rest and cart 1 is moving at constant velocity as shown in Fig. 1.5b.

Even though the observers obtain different values for the carts' velocities, these values are constant, and therefore both observers conclude that the momentum of each cart is constant. Both observers also agree that the two carts are isolated, which means that the conservation of momentum yields $\Delta\vec{p} = 0$ for each cart, in agreement with their observations.

EXAMPLE 3

From the point of view of each observer in Fig. 1.5,

a) Is the energy of each cart constant? Explain with a reason.
b) Is the isolated system containing cart 1 closed? Explain with a reason.
c) Is the isolated system containing cart 2 closed? Explain with a reason.

When we are moving at constant velocity relative to Earth, things around us behave the same way they behave when we are at rest on the ground. For example, imagine being in a plane flying at a constant speed of 260 m s^{-1} in a straight line. The surface of the coffee in the cup

in front of you looks no different from the way it looks at home. If you drop your keys, they fall straight down just as they would if the plane were at rest on the ground. And when the flight attendant pours coffee into your cup while you all move at 260 m s^{-1}, you do not expect him to spill it on you any more than you do when waiter pours your coffee in a restaurant.

In fact, if the engines are very quiet, the ride is smooth, and the window shades are down, it is impossible for you to determine whether or not the plane is moving. Here is the simplest possible experiment: Put a marble on the floor of the plane. Provided the floor is level relative to the ground, the marble remains at rest where you placed it. This is true whether the plane is at rest or cruising at a constant 260 m s^{-1}. When the plane is sitting at the gate, inertia keeps the marble at rest relative to the floor (which is at rest relative to the ground), and so its acceleration is zero. In the air, inertia keeps the marble moving along with the plane at a constant velocity relative to the ground (or, put differently, inertia keeps the marble at rest relative to the floor), and so again its acceleration is zero.

The Earth reference frame or any reference frame moving at constant velocity relative to Earth is called **inertial reference frame**. We can tell whether or not a reference frame is inertial by testing whether or not the law of inertia (Newton's 1st Law) holds:

In an inertial reference frame, any isolated object that is at rest remains at rest, and any isolated object in motion keeps moving at a constant velocity.

In other words, when no force acts on an object, the acceleration of the object is zero. From the first law, we conclude that any isolated object (one that does not interact with its environment) is either at rest or moving with constant velocity. The tendency of an object to resist any attempt to change its velocity is called inertia. Given the statement of the first law above, we can conclude that an object that is accelerating must be experiencing a force. In turn, from the first law, we can define force as that which causes a change in motion of an object.

The law of inertia does not hold in a reference frame that is accelerating relative to Earth. For example, imagine sitting in the passenger seat of a car moving at constant velocity with a marble on the floor near your feet. As long as the car is on a horizontal surface and keeps going straight at a constant speed, the marble remains at rest. When the driver suddenly accelerates forward, however, everything in the car lurches backward, with the marble probably ending up somewhere in the back of the car. According to an observer at rest in the Earth reference frame (Fig. 1.6a), the marble resists being accelerated forward because of its inertia, and unless you have glued it to the floor or restrained it in some other way, it fails to keep up with the accelerating car. Seen from inside the car, the isolated marble suddenly accelerates backward, and so the law of inertia does not hold in the reference frame of the accelerating car (Fig. 1.6b). Reference frames in which the law of inertia does not hold are called *noninertial reference frames*.

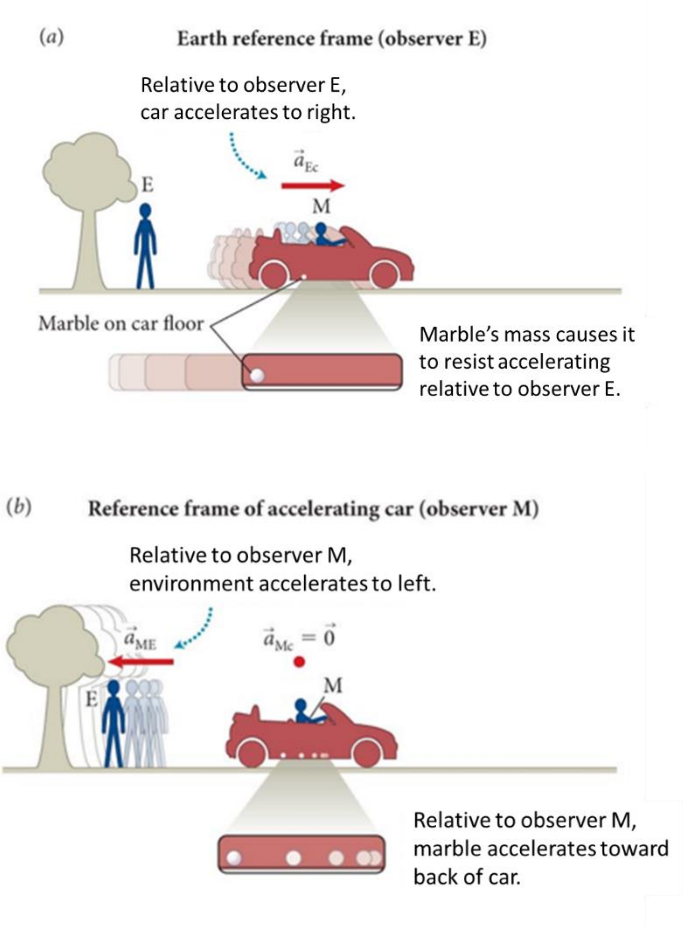


Fig. 1.6

We have singled out the Earth reference frame as the basic inertial reference frame. There is nothing fundamentally special about the Earth reference frame, however, other than that we perform most experiments in this frame. Strictly speaking, Earth is not an inertial reference frame because it revolves around a north-south axis and orbits the Sun in a nearly circular orbit. Because motion on a curved path means that the direction of the velocity is changing and so the velocity is not constant, Earth is accelerating and therefore is a noninertial reference frame. For most cases, the acceleration is too small to be noticeable, and so we may, for most practical purposes, consider the Earth reference frame to be inertial.

EXAMPLE 4

Which of these reference frames are inertial: one affixed to
a) A merry-go-round
b) An airplane taking off
c) A train moving at constant speed a long a straight track

Our accounting procedures for momentum and energy cannot be used in noninertial reference frames. Consider, for example, the two carts shown in Fig. 1.7.

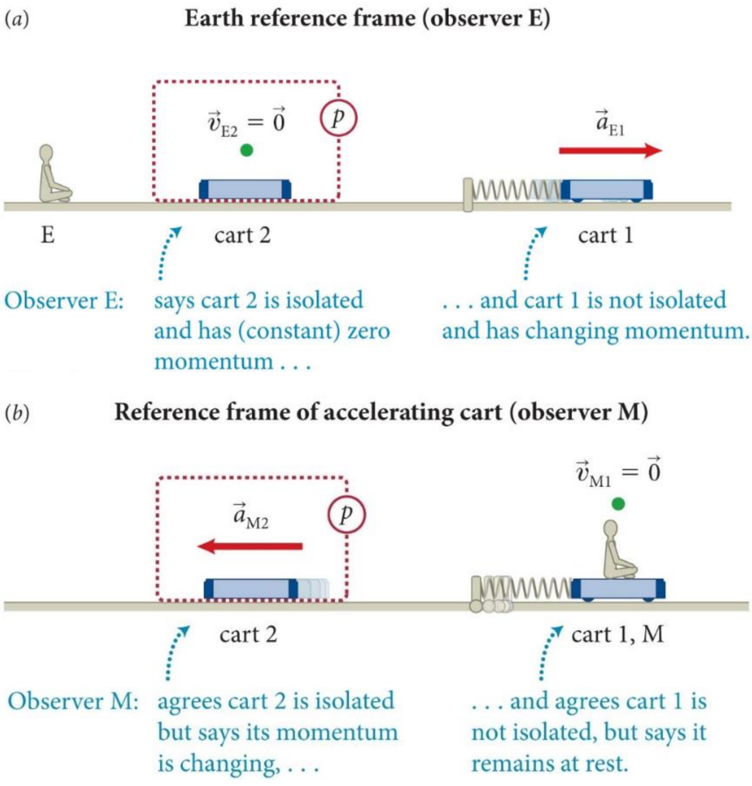


Fig. 1.7

Cart 1 is being accelerated by a spring, and cart 2 is at rest in the Earth reference frame. Cart 2 constitutes an isolated system, but cart 1 is not isolated because it interacts with the spring. To observer E in the Earth reference frame (Fig. 1.7a), the behaviour of both carts is in agreement with the momentum law: the momentum of nonisolated cart 1 changes, while the momentum of the isolated cart 2 is constant.

For observer M, however, who is accelerating along with cart 1 (Fig. 1.7b), things do not quite add up. From this observer’s perspective, cart 1 remains at rest even though it interacts with the spring, and the momentum of cart 2 changes even though that cart is isolated. The principle of conservation of momentum and that of energy, do not hold in the noninertial reference frame of observer M in Fig. 1.7b.

Are we going to run into problems because the laws of the universe are different in noninertial reference frames? No, because nothing prescribes the reference frame; we get to choose it. So for now we just avoid using noninertial reference frames. For the accelerating car, for instance, we would choose not a reference affixed to the car but the Earth reference frame.

Implications for conservation laws

Consider a system from two reference frames, A and B, that are moving at velocity $\vec{v}_{AB}$ relative to each other. The momentum of an object is different in two reference frames that are moving relative to each other but how about changes in momentum? For an object of mass m_A, we can write $\Delta\vec{p}_{A0} = m_0\Delta\vec{v}_{A0}$, because a change in velocity in reference frame A is the same as a change in reference frame B.

We have $\Delta\vec{p}_{A0} = m_0\Delta\vec{v}_{A0} = m_0\Delta\vec{v}_{B0} = \Delta\vec{p}_{B0}$ which tells us that the change in momentum of an object is the same in reference frames A and B. So, for the momentum of a system of objects, we have $\Delta\vec{p}_{Asys} = \Delta\vec{p}_{Bsys}$. In words, changes in the momentum of a system are the same in any two reference frames moving at constant velocity relative to each other.

As we will see in the next example, the changes in kinetic energy of the system in any inertial frames of reference are the same too. This brings us to the principle of relativity:

The laws of the universe are the same in all inertial reference frames moving at constant velocity relative to each other.

The principle of relativity provides a criterion for judging theories: For a theory to be valid, it must prescribe the same behaviour in all inertial reference frames.

EXAMPLE 5

Cart 1, mass of 0.36 kg, is at rest and Cart 2, mass of 0.12 kg, is moving at a velocity of 0.80 m s^{-1}. Both carts collide head-on elastically on a low-friction track. This collision is seen from the Earth reference frame. Determine the following for each cart

a) The change in velocity
b) The change in momentum
c) The change in kinetic energy

Another observer observes the same event from a reference frame M moving along the track at - 0.20 m s^{-1} relative to Earth.

d) Repeat (a), (b) and (c) for the reference frame M.

Now, consider the same conditions for the carts above. This time the collision is inelastic and after the collision, the velocity of cart 1 is +0.30 m s^{-1}.

e) Determine the change in kinetic energy of the carts in both reference frames.

No inertial reference frame is preferred over any other; it is not possible to determine the absolute velocity of any reference frame or object. The laws of the universe are the same in all inertial reference frames, and there is no reference frame that is “at rest” in some absolute sense.

When you are standing on Earth’s surface, are you “at rest” or moving at 30 km s^{-1} (the speed of Earth in its orbit around the Sun), or is your speed higher still – say, the speed of the Sun in it galactic orbit? None of these statements is meaningful because no experiments can be done to determine your speed in empty space. You can only speak of your velocity or speed relative to something else.

If you can choose whichever inertial reference frame you wish, is any choice better than another? The reference frame you are in (usually one at rest relative to Earth) is a logical choice. By choosing some other reference frame, however, you could adjust the value of the momentum of a system up or down by some fixed amount without violating the conservation laws. You could, for instance, adjust the velocity of your reference frame in such a way that the momentum of the system you are observing becomes zero. Such a reference frame is called the system’s **zero-momentum reference frame**. This reference is also called “centre of mass frame”.

EXAMPLE 6

Consider the collision in Example 5.

a) Determine the velocity of the zero-momentum reference frame relative to Earth.

b) Find the change in velocity, momentum and kinetic energy in the zero-momentum frame.

Centre of Mass

In this section, we describe the overall motion of a system in terms of a special point called the **centre of mass** of the system.

The system can be either a group of particles, such as a collection of atoms in a container, or an extended object, such as a gymnast leaping through the air. We shall see that the translational motion of the centre of mass of the system is the same as if all the mass of the system were concentrated at that point. That is, the system moves as if the net external force were applied to a single particle located at the centre of mass. This behaviour, the particle model, is independent of other motion, such as rotation or vibration of the system of deformation of system (for instance, when a gymnast folds her body).

Consider a system consisting of a pair of particles that have different masses and are connected by a light, rigid rod. The position of the centre of mass of a system can be described as being the *average position* of the system's mass. The centre of mass of the system is located somewhere on the line joining the two particles and is closer to the particle having the larger mass as shown in Fig. 1.8.

If a single force is applied at a point on the rod above the centre of mass, the system rotates clockwise (Fig. 1.8a).

If the force is applied at a point on the rod below the centre of mass, the system rotates in anticlockwise (Fig. 1.8b).

If the force is applied at the centre of mass, the system moves in the direction of the force without rotating (Fig. 1.8c).

The centre of mass of an object can be located with this procedure.

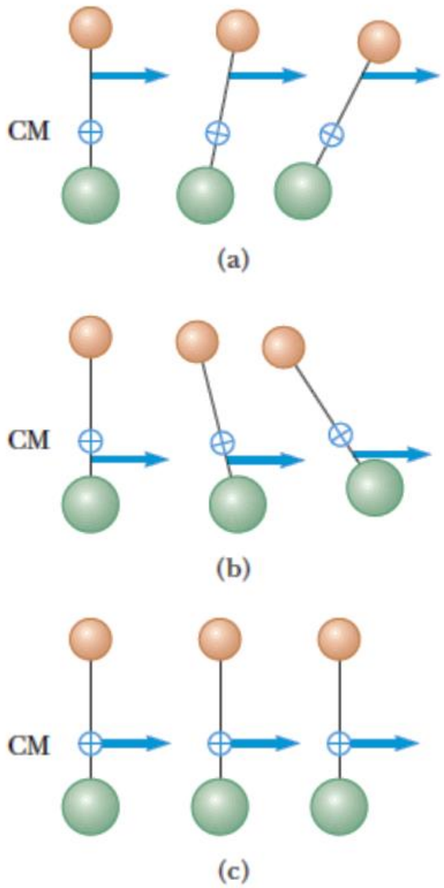


Fig. 1.8

The centre of mass of the pair of particles described in Fig. 1.9 is located on the *x* axis and lies somewhere between the particles.

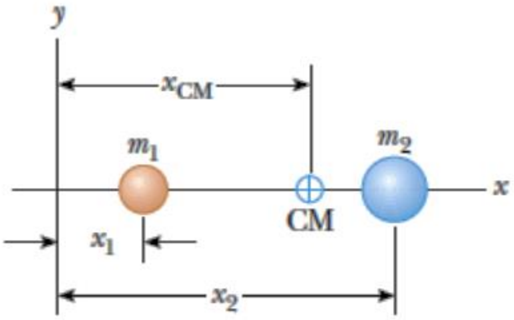


Fig. 1.9

Its x coordinate is given by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} \tag{1.4}$$

For example, if $x_1 = 0$, $x_2 = d$ and $m_2 = 2m_1$, we find that $x_{CM} = \frac{2}{3}d$. That is, the centre of mass lies closer to the more massive particle. If the two masses are equal, the centre of mass lies midway between the particles.

We can extend this concept to a system of many particles with masses m_1 in three dimensions. The x coordinate of the centre of mass of n particles is defined to be

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2 + \ldots\ldots + m_n x_n}{m_1 + m_2 + \ldots\ldots + m_n} = \frac{\sum_i m_i x_i}{\sum_i m_i} = \frac{\sum_i m_i x_i}{M} = \frac{1}{M}\sum_i m_i x_i \tag{1.5}$$

Where x_i is the x coordinate of the ith particle and the total mass is $M \equiv \sum_i m_i$ where the sum runs over all n particles. The y and z coordinates of the centre of mass are similarly defined by the equations

$$y_{CM} = \frac{1}{M}\sum_i m_i y_i \text{ and } z_{CM} = \frac{1}{M}\sum_i m_i z_i \tag{1.6}$$

The centre of mass can be located in three dimensions by its position vector $\vec{r}_{CM}$. The components of this vector are x_{CM}, y_{CM} and z_{CM} defined in Equations (1.5) and (1.6). Therefore,

$$\vec{r}_{CM} = x_{CM}\hat{i} + y_{CM}\hat{j} + z_{CM}\hat{k} = \frac{1}{M}\sum_i m_i x_i \hat{i} + \frac{1}{M}\sum_i m_i y_i \hat{j} + \frac{1}{M}\sum_i m_i z_i \hat{k}$$

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i \vec{r}_i \tag{1.7}$$

where $\vec{r}_i$ is the position vector of the ith particle, defined by

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

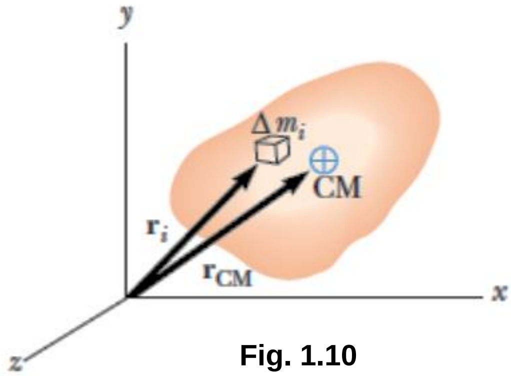


Fig. 1.10

Although locating the centre of mass for an extended object is somewhat more cumbersome than locating the centre of mass of a system of particles, the basic ideas we have discussed still apply. Think of an extended object as a system of containing a large number of small mass elements such as the cube in Fig. 1.10. Because the separation between elements is very small, the object can be considered to have a continuous mass distribution. By dividing the object into elements of mass Δm_i with coordinates x_i, y_i, z_i, we see that the *x* coordinate of the centre of mass is approximately

$$x_{CM} \approx \frac{1}{M}\sum_i x_i \Delta m_i$$

with similar expressions for y_{CM} and z_{CM}. If we let the number of elements *n* approach infinity, the size of each element approaches zero and x_{CM} is given precisely. In this limit, we replace the sum by an integral and Δm_i by the differential element *dm*:

$$x_{CM} = \lim_{\Delta m_i \to 0} \frac{1}{M}\sum_i x_i \Delta m_i = \frac{1}{M}\int x\, dm \qquad (1.8)$$

Likewise, for y_{CM} and z_{CM} we obtain

$$y_{CM} = \frac{1}{M}\int y\, dm \text{ and } z_{CM} = \frac{1}{M}\int z\, dm \qquad (1.9)$$

We can express the vector position of the centre of mass of an extended object in the form

$$\vec{r}_{CM} = \frac{1}{M}\int \vec{r}\, dm \qquad (1.10)$$

The centre of mass of any symmetric object of uniform density lies on an axis of symmetry and on any plane of symmetry. For example, the centre of mass of a uniform rod lies in the rod, midway between its ends. The centre of mass of a sphere or a cube lies at its geometric centre.

Because an extended object is a continuous distribution of mass, each small mass element is acted upon by the gravitational force. The net effect of all these forces s equivalent to the effect of a single force $M\vec{g}$ acting through a special point, called the centre of gravity. If $\vec{g}$ is constant over the mass distribution, the centre of gravity coincides with the centre of mass. If an extended object is pivoted at its centre of gravity, it balances in any orientation.

The centre of gravity of an irregularly shaped object such as a wrench can be determined by suspending the object first from one point and then from another. In Fig. 1.11, a wrench is hung from point A and a vertical line AB (which can be established with a plumb bob) is drawn when the wrench has stopped swinging. The wrench is then hung from point C, and a second vertical line CD is drawn. The centre of gravity is halfway through the thickness of the wrench, under the intersection of these two lines. In general, if the wrench is hung freely from any point, the vertical line through this point must pass through the centre of gravity.

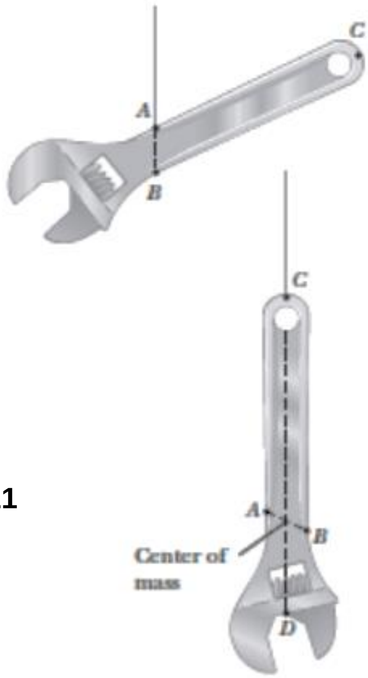
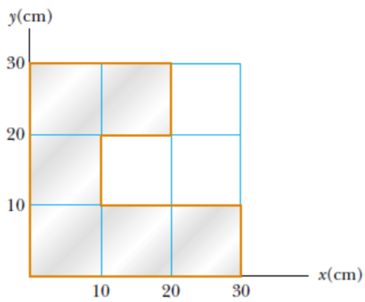


Fig. 1.11

EXAMPLE 7

A uniform piece of sheet metal is shaped as shown in the Figure. Compute the *x* and *y* coordinates of the centre of mass of the piece.



EXAMPLE 8

a) Show that the centre of mass of a rod of mass *M* and length *L* lies midway between its ends, assuming the rod has a uniform mass per unit length ($\lambda = M / L$).

b) Suppose a rod is non-uniform such that its mass per unit length varies linearly with *x* according to the expression $\lambda = \alpha x$, where α is a constant. Find the *x* coordinate of the centre of mass as a fraction of *L*.

Velocity of the Centre of Mass

Consider a system of two or more particles for which we have identified the centre of mass. We can begin to understand the physical significance and utility of the centre of mass concept by taking the time derivative of the position vector for the centre of mass given by

$$\vec{r}_{CM} = \frac{1}{M}\sum_i m_i \vec{r}_i$$

We know that the time derivative of a position vector is by definition the velocity vector. Assuming the total mass M remains constant for a system of particles – that is, no particles enter or leave the system – we obtain the following expression for the velocity of the centre of mass of the system:

$$\vec{v}_{CM} = \frac{d\vec{r}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{r}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{v}_i \qquad (1.11)$$

where $\vec{v}_i$ is the velocity of the ith particle. Rearranging the equation gives

$$M\vec{v}_{CM} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{total} \qquad (1.12)$$

Therefore, the total linear momentum of the system equals the total mass multiplied by the velocity of the centre of mass. In other words, the total linear momentum of the system is equal to that of a single particle of mass M moving with a velocity $\vec{v}_{CM}$.

Differentiating the velocity of the centre of mass of the system, $\vec{v}_{CM}$ with respect to time, we obtain the acceleration of the centre of mass of the system:

$$\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt} = \frac{1}{M}\sum_i m_i \frac{d\vec{v}_i}{dt} = \frac{1}{M}\sum_i m_i \vec{a}_i \qquad (1.13)$$

Rearranging this expression and using Newton's second law gives

$$M\vec{a}_{CM} = \sum_i m_i \vec{a}_i = \sum_i \vec{F}_i \qquad (1.14)$$

where $\vec{F}_i$ is the net force on particle i.

The forces on any particle in the system may include both external forces (from outside the system) and internal forces (from within the system). By Newton's third law, however, the internal force exerted by particle 1 on particle 2, for example, is equal in magnitude and opposite in direction to the internal force exerted by particle 2 on particle 1. Therefore, when we sum over all internal force vectors, they cancel in pairs and we find that the net force on the system is caused only by external forces. We can then re-write the equation above in the form

$$\sum \vec{F}_{net} = M\vec{a}_{CM} \text{ (body or collection of particles)} \qquad (1.15)$$

That is, the net external force on a system of particles equals the total mass of the system multiplied by the acceleration of the centre of mass. Comparing this equation with Newton's second law for a single particle, the particle model could be described in terms of the centre of mass for an extended body.

This result may not sound very impressive, but in fact it is central to the whole subject of mechanics. Without it, we would not be able to represent an extended body as a point particle when we apply Newton's laws. It explains why only external forces can affect the motion of an extended body.

Suppose a cannon shell traveling in a parabolic trajectory (neglecting air resistance) explodes in flight, splitting into two fragments with equal mass, as shown in Fig. 1.12. The fragments follow new parabolic paths, but the centre of mass continues on the original parabolic trajectory, just as though all the mass were still concentrated at that point.

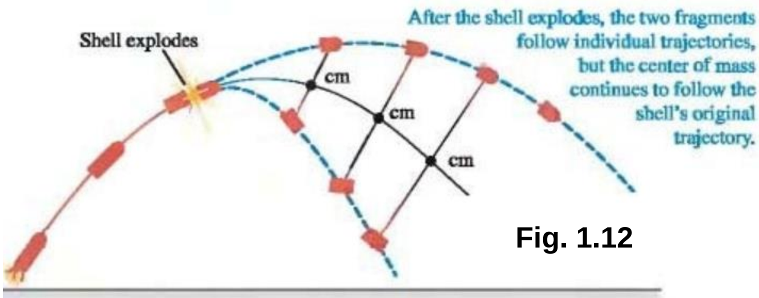


Fig. 1.12

There is one more useful way to describe the motion of a system of particles. Using $\vec{a}_{CM} = \frac{d\vec{v}_{CM}}{dt}$, we can rewrite (1.14)

$$M\vec{a}_{CM} = M\frac{d\vec{v}_{CM}}{dt} = \frac{d\left(M\vec{v}_{CM}\right)}{dt} = \frac{d\vec{P}}{dt} \qquad (1.16)$$

The total system mass M is constant, so we are allowed to take it inside the derivative. Substituting (1.16) into (1.15), we find

$$\sum \vec{F}_{net} = \frac{d\vec{P}}{dt} \quad \text{(extended body or system of particles)} \qquad (1.17)$$

The Eq. (1.17) describes a system of particles, such as an extended body. The interactions between the particles that make up the system can change the individual momenta of the particles, but the total momentum of the system can be changed only by external forces acting from outside the system.

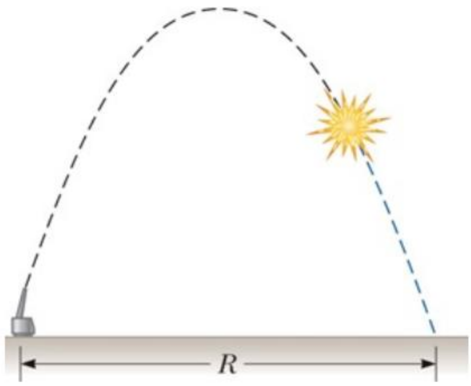
This property of the centre of mass is important when we analyse the motion of rigid bodies. We describe the motion of an extended body as a combination of translational motion of the centre of mass and rotational motion about an axis through the centre of mass. This property also plays an important role in the motion of astronomical objects. It is not correct to say that the moon orbits the earth; rather the earth and moon both move in orbits around their centre of mass.

EXAMPLE 9

A projectile fired into the air suddenly explodes into several fragments.

a) What can be said about the motion of the centre of mass of the system made up of all the fragments after the explosion?

b) If the projectile did not explode, it would land at a distance *R* from its launch point. Suppose the projectile explodes and splits into two pieces of equal mass. One piece lands at a distance 2*R* from the launch point. Where does the other piece land?



EXAMPLE 10

A ball of mass 0.200 kg with a velocity of 1.50 m s^{-1} meets a ball of mass 0.300 kg with a velocity of -0.400 m s^{-1} in a head-on, elastic collision.

a) Find their velocities after the collision.
b) Find the velocity of their centre of mass before and after the collision.
c) Determine the velocity of each ball before and after the collision from the reference frame of centre of mass.
d) Hence, calculate the total momentum of the system from the centre of mass frame.

TUTORIAL QUESTIONS

1) A boat crossing a wide river moves with a speed of 10.0 km h^{-1} relative to the water. The water in the river has a uniform speed of 5.00 km h^{-1} due east relative to the Earth.

a) If the boat heads due north, determine the velocity of the boat relative to an observer on Earth.

b) Determine the velocity of the boat relative to an observer on Earth if the boat heads only upstream and travels northward across the river.

2) Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet, who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0 kg boat move toward the shore it is facing?

3) **[9814/N2021/Q2]** A particle of mass 4*m* traveling with velocity *v* collides elastically with a stationary particle of mass *m* on a smooth horizontal surface.

a) In the first scenario, the particles collide head-on. Determine the percentage of kinetic energy that is transferred from the particle of mass 4*m* to the particle of mass *m*.

b) In the second scenario, the two particles have a glancing collision. After the collision, the particle of mass 4*m* is deflected through an angle θ.

Fig. 2.1 shows the collision in the laboratory frame of reference.

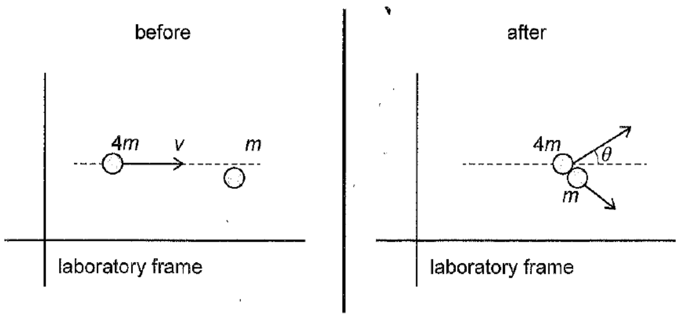


Fig. 2.1

Fig. 2.2 shows the collision transformed into the centre of mass frame.

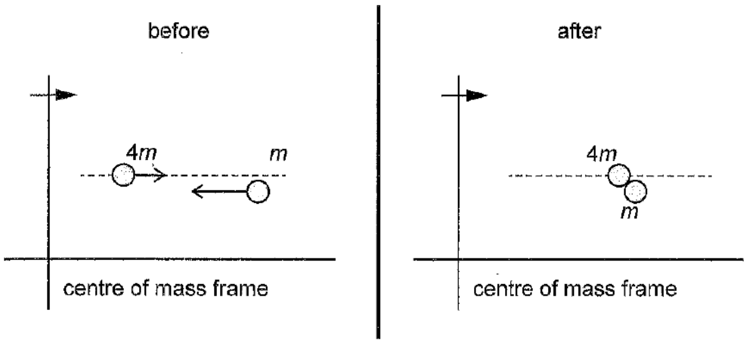


Fig. 2.2

Complete Fig. 2.2 by showing in terms of velocity *v*:

(i) The velocity of the centre of mass frame
(ii) The velocities of the particles in the centre of mass frame before the collision
(iii) The velocities of the particles in the centre of mass frame after the collision.

c) Determine the maximum possible of angle of deflection θ_{max} of the particle of mass 4*m*.

4) An object of mass of 1.0 kg moving with a velocity of +4.0 m s^{-1} collides inelastically with another object of mass 3.0 kg that is at rest. After the collision, the first object moves at a velocity of -0.50 m s^{-1} and the stationary object starts to move at a velocity of +1.5 m s^{-1}.

a) Determine the coefficient of restitution of this collision.

b) Determine the kinetic energy converted into internal energy both in the Earth reference frame and in an inertial reference frame moving at -1.0 m s^{-1} relative to Earth.

5) The 0.100 kg head of a golf club is moving at 45 m s^{-1} when it strikes a stationary 0.050 kg golf ball.

a) Determine the velocities of the club and the ball in the zero-momentum (centre of mass) reference frame.

b) What is their relative velocity in the zero-momentum reference frame?

c) How much of the kinetic energy of the system can be converted to internal energy?

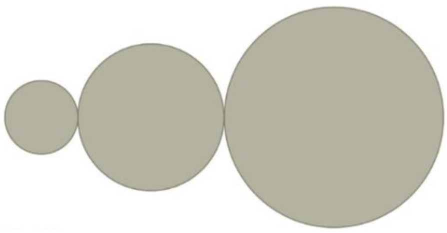
d) As measured from the Earth reference frame, the system's kinetic energy decreases by 20% because of the collision. Determine the velocities of the club and the ball after the collision.

6) Two identical carts A and B move toward each other on a low-friction track.

The speed of cart A is *u*, while the speed of cart B is 2*u* in the Earth reference frame. The system of the two carts has kinetic energy *K*.

State and explain another inertial frame of reference in which the *kinetic energy* of the two-cart system has the same kinetic energy *K*.

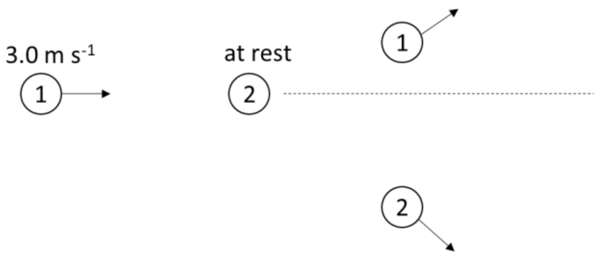
7) The figure shows 3 solid spheres of radii *a*, 2*a* and 3*a* made of materials with densities $\rho_1 = \rho$, $\rho_2 = 2\rho$ and $\rho_3 = 3\rho$ respectively. Find the position of the centre of mass of the system.



8) A rocket is fired vertically upward. At the instant it reaches an altitude of 1000 m and a speed of 300 m s^{-1}, it explodes into three fragments having equal mass. One fragment moves upward with a speed of 450 m s^{-1} following the explosion. The second fragment has a speed of 240 m s^{-1} and is moving east right after the explosion.

Determine the velocity of the third fragment immediately after the explosion.

9) A puck of mass of 0.200 kg moving at 3.0 m s^{-1} approaches and identical puck that is stationary on frictionless ice. After the collision, the first puck leaves with a speed v_1 and 30º relative to the original line of motion. The second puck leaves with speed v_2 at 60º.



a) Determine v_1 and v_2.

b) What are the relative speeds of the pucks before and after the collision? Is the collision elastic or inelastic?

10) **[9814/Specimen Paper/Q5]** Two perfectly elastic balls, A and B, are held above the ground at a height h, such that h is much greater than the diameter of each ball.

The balls are almost touching, one directly above the other.

The balls are released simultaneously. The upper ball A rebounds to a maximum height H above the ground.

a) Sketch a labelled diagram to show the velocities of the balls relative to the ground just before the they collide with each other.

Hence show that when the balls collide with each other they do so with a relative velocity of $v_{rel} = \sqrt{8gh}$

b) The mass of the lower ball m_B is greater than or equal to the mass of the upper ball m_A so that $m_B / m_A = n$ and $n \geq 1$.
Find an expression for the velocity of the centre of mass frame v_{CM} relative to the ground immediately before the two balls collide.

c) **(i)** Find an expression for the ratio of the rebound height H to the drop height h.

(ii) Deduce the maximum value of $\dfrac{H}{h}$ for $n \geq 1$.

NUMERICAL ANSWERS

Worked Examples

1 **(a)** - 5 m s^{-1}
(b) 5 m s^{-1}
(c) - 300 m
2 3.8 m s^{-1} towards North
5 **(a)** 0.40 m s^{-1} ; 1.20 m s^{-1}
(b) 0.144 kg m s^{-1}; - 0.144 kg m s^{-1}
(c) 0.0288 J ; - 0.0288 J
(d) 0.40 m s^{-1}; - 1.20 m s^{-1}; 0.144 kg m s^{-1}; - 0.144 kg m s^{-1}; 0.0576 J; -0.0576 J
(e) - 0.0216 J ; - 0.0216 J
6 **(a)** 0.20 m s^{-1}
(b) 0.40 m s^{-1}; - 1.20 m s^{-1}; 0.144 kg m s^{-1}; - 0.144 kg m s^{-1}; 0 J ; 0 J
7 (11.7 cm, 13.3 cm) outside of the metal sheet
8 **(a)** $L/2$
(b) $2L/3$
10 **(a)** - 0.78 m s^{-1}; 1.12 m s^{-1}
(b) 0.36 m s^{-1}
(c) 1.14 m s^{-1}; - 0.76 m s^{-1}; - 1.14 m s^{-1}; 0.76 m s^{-1}
(d) 0 kg m s^{-1}

Tutorial Questions

1 **(a)** 26.6°
(b) 30.0°
2 0.70 m
3 **(a)** 64%
(c) 14.47°
4 **(a)** 0.5
(b) - 4.5 J; - 4.5 J
5 **(a)** 15 m s^{-1} ; - 30 m s^{-1}
(b) 45 m s^{-1}
(c) 33.8 J
(d) 48.97 m s^{-1} ; 11.03 m s^{-1} or 20.52 m s^{-1} ; 39.49 m s^{-1}
7 8.1a
8 510 m s^{-1}, 61.9° above the horizontal (clockwise from the west)
9 **(a)** 1.5 m s^{-1}, 2.6 m s^{-1}
(b) 3.0 m s^{-1} (elastic collision)
10 **(b)** $\sqrt{2gh}\,\frac{(n-1)}{(n+1)}$ (upwards)
(c)(i) $\left(\frac{3n-1}{n+1}\right)^2$
(c)(ii) 9

HCI H3 Physics 2024

A2 - Rotational Motion

Mr. Pang BiaoJin

1 General Information

1.1 Content: A2

- Kinematics of angular motion
- Dynamics of motion
- Rigid body rotation about an axis of fixed orientation

1.2 Learning Outcomes

Candidates should be able to:

1. show an understanding of and use the terms angular displacement, angular velocity, and angular acceleration of a rigid body with respect to a fixed axis.
2. solve problems using the equations of motion for uniform angular acceleration that are analogous to the equations of motion for uniform linear acceleration.
3. show an understanding of and use the terms angular momentum and moment of inertia of a rotating rigid body.
4. calculate the moment of inertia about an axis for simple objects by using calculus and the parallel-axis theorem or otherwise (knowledge of the perpendicular-axis theorem is not required).
5. show an understanding of torque produced by a force relative to a reference point, and apply the principle that torque is related to the rate of change of angular momentum to solve problems, such as those involving point masses, rigid bodies, or bodies with variable moment of inertia e.g. an ice-skater.
6. derive from the equations of motion, and apply the formula $K_{rot} = \frac{1}{2}I\omega^2$ for the rotational kinetic energy of a rigid body.
7. recall and apply the result that the motion of a rigid body can be regarded as translational motion of its centre of mass with rotational motion about an axis through the centre of mass to solve related problems, including situations where the frictional force between surfaces heuristically takes a limiting value governed by a coefficient of friction and the normal contact force (no distinction is made between the coefficient of static and kinetic friction).

2 Kinematics of Rotational Motion

In H2 Physics, we have learnt about some angular quantities for a point object in circular motion (e.g. angular velocity). In H3 Physics, we will extend our discussion to analyzing the rotation of an extended object.

We will constrain our analysis to the motion of a rigid body (a perfectly definite body in which the relative locations of all particles of which the object is composed remain constant).

2.1 Angular Quantities

Consider a point on a rigid body.

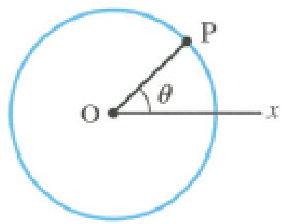


Figure 1: Angular Position of an arbitrary point P on a rotating body.

We define the *angular position* θ of this point.
This defines the angular position of the body. **(Qn: Why can we do this?)**

The point travels from point A to position B as shown in Figure 2 in a time Δt.

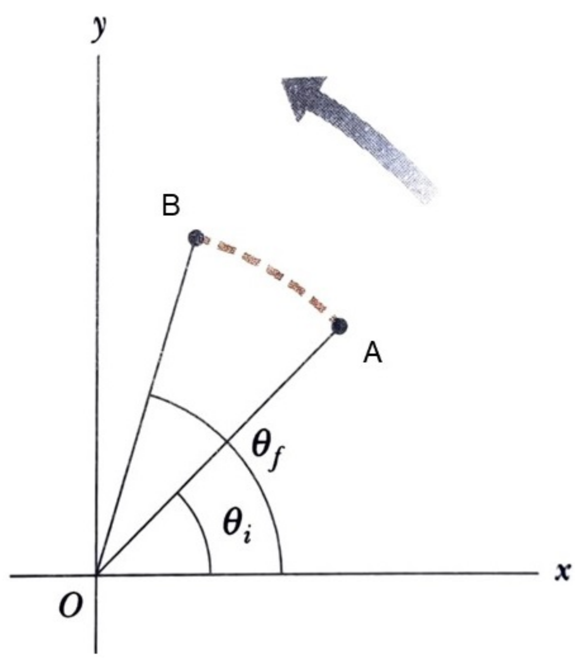


Figure 2: Motion of a particle on a rotating rigid object.

The *angular displacement* $\Delta\theta$ of the rigid body is then given by:

$$\Delta\theta = \theta_f - \theta_i \tag{1}$$

The *average angular velocity* of the body is given by

$$\omega_{ave} = \frac{\theta_f - \theta_i}{\Delta t} = \frac{\Delta\theta}{\Delta t} \tag{2}$$

The *instantaneous angular velocity* of the body is given by

$$\omega = \frac{d\theta}{dt} \tag{3}$$

If the instantaneous angular speed of an object changes from ω_i to ω_f over a time interval Δt, the object has an angular acceleration.

We can define the *average angular acceleration* of the body as

$$\alpha_{ave} = \frac{\omega_f - \omega_i}{\Delta t} = \frac{\Delta\omega}{\Delta t} \tag{4}$$

Finally, we can define the *instantaneous angular acceleration* as

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \tag{5}$$

The velocity of the point is given by its tangential velocity (**Why so?**), which we recall from H2 Physics as:

$$v = r\omega \tag{6}$$

Further, we can write expressions for the tangential and radial components of the acceleration as follows:

$$a_t = r\alpha \tag{7}$$

$$a_r = r\omega^2 \tag{8}$$

Lecture Example 1 (Uni Phy by YnF, Q9.7)

The angle θ through which a disk turns is given by $\theta = a + bt - ct^3$, where

- a, b and c are constants,
- t is in seconds,
- θ is in radians.

When $t = 0s$, $\theta = \pi/4$ and the angular velocity is 2.00 rad s^{-1}.
When $t = 1.5s$, the angular acceleration is 1.25 rad s^{-1}.

(a) Find the values of a, b and c

(b) What is the angular acceleration when $\theta = \pi/4$?

2.2 Rotational EOM with Constant Angular Acceleration

Noting the similarity between the expressions for linear motion ($a = \frac{dv}{dt}$) and rotational motion ($\alpha = \frac{d\omega}{dt}$, we can write down analogous expressions for the Equations of Motion for rotational motion with *constant angular acceleration*.

$$\omega = \omega_0 + \alpha t \tag{9}$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2 \tag{10}$$

$$\omega = \omega_0^2 + 2\alpha(\theta - \theta_0) \tag{11}$$

$$\theta = \theta_0 + \frac{1}{2}(\omega + \omega_0)t \tag{12}$$

Lecture Example 2 (Uni Phy by YnF, Q9.19)

At $t = 0s$, a grinding wheel has an angular velocity of 24.0 $rads^{-1}$. It experiences a constant angular acceleration of 30.0 $rads^{-2}$ until a circuit breaker trips at $t = 2.00s$. From this moment onwards, it turns through 432 rad as it coasts to a stop at a constant angular acceleration.

(a) Through what total angle did the wheel turn between $t = 0s$ and the time it stopped?

(b) At what time did it stop?

(c) What was its acceleration as it slowed down?

<u>Tutorial</u>: D1, **D2*** (starred questions are compulsory)

3 Kinetic Energy of a Rotating Body

3.1 Rotational Kinetic Energy

A rotating rigid body has kinetic energy.
Let us analyze a rigid object rotating about some axis with some angular velocity ω.

Consider a particle of mass m_i on the object, at some distance r_i from the rotation axis, shown in Figure 3. This particle is moving with some velocity v_i in the COM frame.

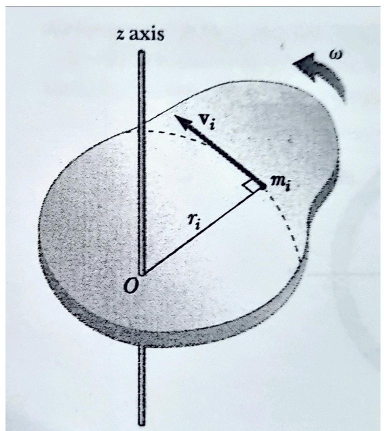


Figure 3: A rigid object rotating about the z axis with some angular speed ω.

Applying Equation 6, we can deduce that the kinetic energy of the point K_i is given by:

$$K_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega_i^2 \tag{13}$$

Recalling further that all particles share the same ω, and summing over all particles, we get an expression for the total kinetic energy K_{rot} of the rotating rigid object:

$$\begin{aligned} K_{rot} &= \sum_i (\frac{1}{2} m_i r_i^2 \omega_i^2) \\ &= \sum_i (\frac{1}{2} m_i r_i^2 \omega^2) \\ &= \frac{1}{2} (\sum_i m_i r_i^2) \omega^2 \\ &= \frac{1}{2} I \omega^2 \end{aligned} \tag{14}$$

Here, we define the quantity I as the *moment of inertia* of the body.

$$I = \sum_i m_i r_i^2 \tag{15}$$

Lecture Example 3 (Serway, Example 10.3)

Consider an oxygen molecule (O_2) rotating in the xy plane about the z axis. The rotation axis passes through the center of the molecule, perpendicular to its length.

The mass of each oxygen atom is 2.66×10^{-26} kg, and at room temperature the average separation between the two atoms is $d = 1.21 \times 10^{-10}$ m.

You can treat the atoms as point particles.

(a) Calculate the moment of inertia of the molecule about the z axis.

(b) The angular speed of the molecule about the z axis is $4.60 \times 10^{12}\ rads^{-1}$.
Calculate its rotational kinetic energy.

3.2 Moment of Inertia

We now expand on the expression for moment of inertia defined in Equation 15. If we consider the rigid object to consist of infinitely many particles, each individual $m_i \to 0$. In this limit, the expression can be written as an integral over the object:

$$I = \lim_{m_i \to 0} \sum_i m_i r_i^2 = \int r^2 dm \tag{16}$$

Lecture Example 4 (Serway, Example 10.6)

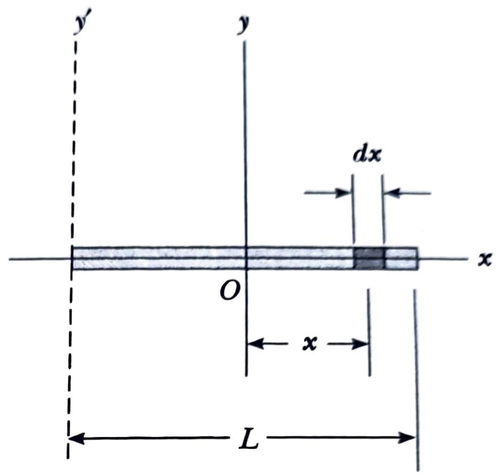


Figure 4: Uniform rigid rod of length L

Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis perpendicular to the rod (the y axis) and passing through its center of mass (Figure 4).
(Hint: How did you treat dm when handling Center of Mass in Chapter A1?)

3.3 Applying the Parallel Axis Theorem to find Moment of Inertia

Suppose the moment of inertia of a body of mass M about an axis passing through the center of mass of the body is given by I_{CM}.

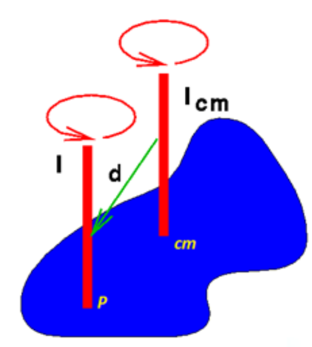


Figure 5: Parallel Axis Theorem

The parallel-axis theorem then states that the moment of inertia I about any axis **parallel to and a distance D from this axis** is given by:

$$I = I_{CM} + MD^2 \tag{17}$$

Lecture Example 5 (Serway, Example 10.6(modified))

Refer back to Figure 4. Calculate the moment of inertia of a uniform rigid rod of length L and mass M about an axis passing through one end of the rod (the y' axis)

(a) through direct integration, **without** using your answer in Example 4

(b) by using your answer in Example 4 and the parallel-axis theorem

<u>Tutorial</u>: **D3***, **D4***, D5, D6 (starred questions are compulsory)

4 Torque

4.1 Fundamentals and Recap of H2 Content

We recall from H2 Physics that a force exerted on a rigid extended object pivoted about an axis tends to cause the object to rotate about that axis.

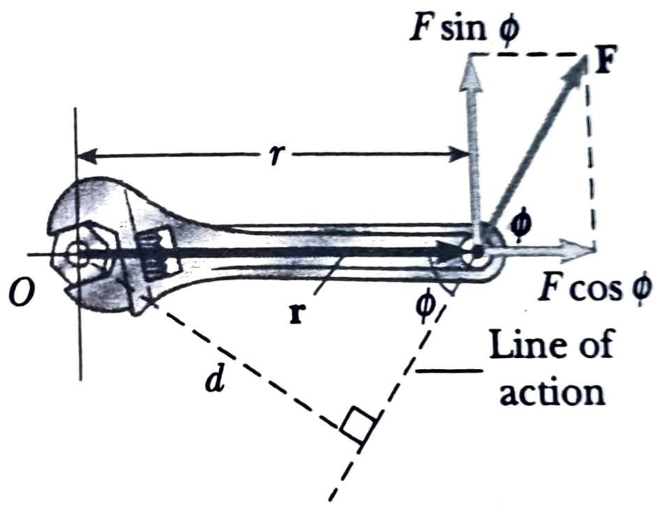


Figure 6: A force F applied on a wrench. $d = rsin\theta$ is known as the moment arm.

This tendency of the object to rotate is quantified by *torque* τ, of which magnitude is given by the product of the magnitude of the force exerted on it F and its moment arm d (See Figure 6 above for an example).

$$\tau = Fd \tag{18}$$

Note that torque is a **vector**.

Being slightly pedantic and considering the rotation direction, we notice that an object rotating about a fixed axis has two possible directions of rotation.

Typically, we define torque to be **positive** if it tends to produce **counter-clockwise** rotation and **negative** if it tends to produce **clockwise** rotation.

We thus have our final equation:

$$\tau = \pm Fd \tag{19}$$

This will suffice for the H3 Physics course.

4.2 Relating Torque to Angular Acceleration

We know from Newton's Second Law that a net force acting on an object will cause it to accelerate. Similarly, what happens when there is a net torque on an object?

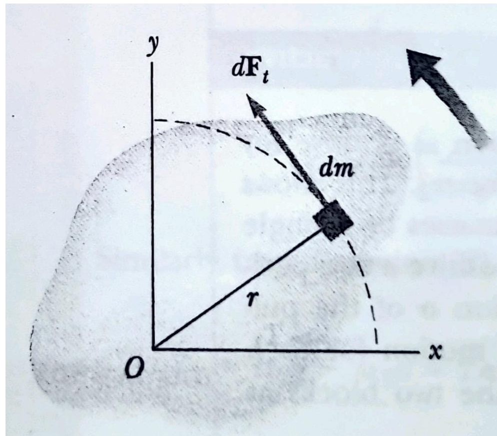


Figure 7: A rigid object rotating about an axis through O.

Consider a rigid object rotating about a fixed axis, such as in Figure 7 above.
Breaking the object down into infinitesimal mass elements dm, each has a tangential acceleration $\mathbf{a}_t$ caused by an external tangential force $\mathrm{d}\mathbf{F}_t$ acting on it.

By Newton's Second's Law,

$$dF_t = (dm)a_t \tag{20}$$

The torque $d\tau$ associated with the force $d\mathbf{F}_t$ can then be expressed by:

$$d\tau = r(dF_t) = r(dm)a_t \tag{21}$$

Invoking our expression in Equation 7 for a_t, we thus have an expression for $d\tau$:

$$d\tau = \alpha r^2 dm \tag{22}$$

Note that every dm here has the same angular acceleration α. This allows us to obtain an expression for the net torque through integration.

$$\sum \tau = \int \alpha r^2 dm = \alpha \int r^2 dm = I\alpha \tag{23}$$

Finally, we simplify the expression by invoking the expression for the Moment of Inertia defined in Equation 16.

$$\sum \tau = I\alpha \tag{24}$$

Lecture Example 6

A light cable is wrapped several times around a uniform solid cylinder that can rotate freely about its axis. The cylinder has a diameter of 0.120 m and mass of 50 kg.

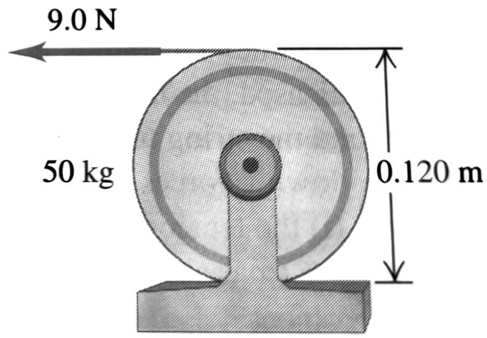


Figure 8: Cylinder and cable

The cable is now pulled with a force of 9.0 N, as shown in the Figure 8.

Assuming that the cable unwinds without stretching or slipping, what is its acceleration?

Lecture Example 7 (Serway, Example 10.10)

A uniform rod of length L and mass M is attached at one end to a frictionless pivot and is free to rotate about the pivot in the vertical plane, as shown in Figure 9.

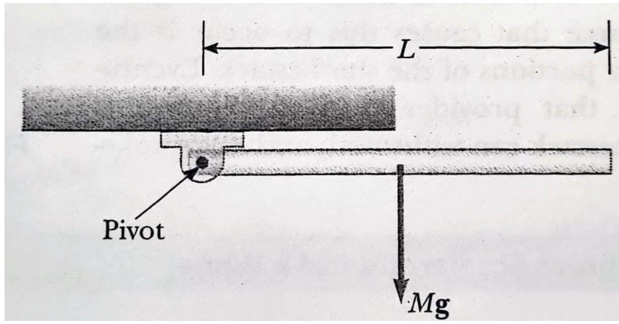


Figure 9: Rod rotating freely about a pivot

The rod is then released from rest in the horizontal position.

In terms of the gravitational acceleration g, what is the initial linear acceleration of the right end of the rod?

Tutorial: D7, D8, **D9***, D10

5 Angular Momentum

5.1 Fundamental Definition

We first establish the rotational counterpart to linear momentum: *angular momentum L*.
Similar to how linear momentum is given by the product of mass and velocity, the form of angular momentum should not surprise you:

$$L = I\omega \tag{25}$$

5.2 Conservation of Angular Momentum

Revisiting what we know about conservation of linear momentum in H2 Physics, we recall:

$$\sum F = \frac{dp}{dt} \tag{26}$$

This then suggests that if $\sum F = 0$, $p_i = pf$.

Similarly, we note that:

$$\sum \tau = \frac{dL}{dt} \tag{27}$$

We then observe that if $\sum \tau = 0$, $L_i = L_f$.
(To be pedantic however, note that $\sum \tau$ and L have to be measured about the same origin.)

Lecture Example 8 (Uni Phy by YnF, Q10.37)

Find the magnitude of the angular momentum of the second hand on a clock about an axis through the centre of the face of the clock. The clock hand has a length of 15.0 cm and a mass of 6.00 g. Assume the second hand is a slender rod rotating with a constant angular velocity about one end.

Lecture Example 9 (Uni Phy by YnF, Q10.42)

A diver comes off a board with arms straight up and legs straight down, giving her a moment of inertia of 18 kgm^2 about her axis of rotation. She tucks into a small ball, decreasing her moment of inertia to 3.6 kgm^2. While tucked, she makes two complete revolutions in 1.0 s from board to water. If she has not tucked at all, how many revolutions would she have made in 1.5 s from board to water?

Tutorial: **D11***, D12

6 Rolling Motion of a Rigid Object

To conclude our discussion, we extend our discussion to the rolling of a rigid object along a flat surface.

Typically, such analysis is not easy.
We can (and will, in this case), simplify our discussion by

1. focussing on the **center of mass** of the object
2. considering the case of *pure rolling*, where the object rolls without slipping.

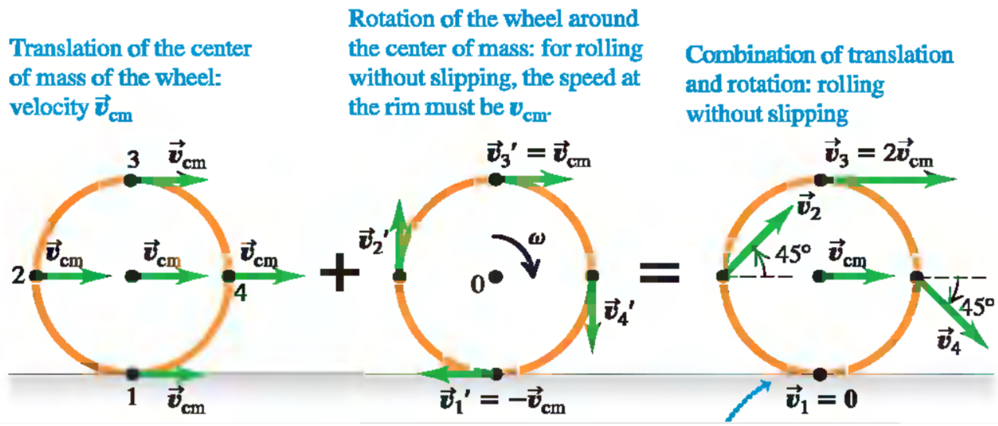


Figure 10: Pure Rolling Motion

We can break down pure rolling motion into a combination of translational motion and rotational motion (Figure 10 above).

For pure translational motion, the cylinder does not rotate, so each point moves to the right with speed v_{CM}.

For pure rotational motion, the rotation axis through the center of mass is stationary, with each point having the same angular speed ω.

The combination of both these two motions represents pure rolling motion.

In the next page, we will now derive expressions for the velocity and acceleration, as well as the kinetic energy of such a rotating system.

Consider a uniform cylinder of radius R rolling without slipping (Figure 11).

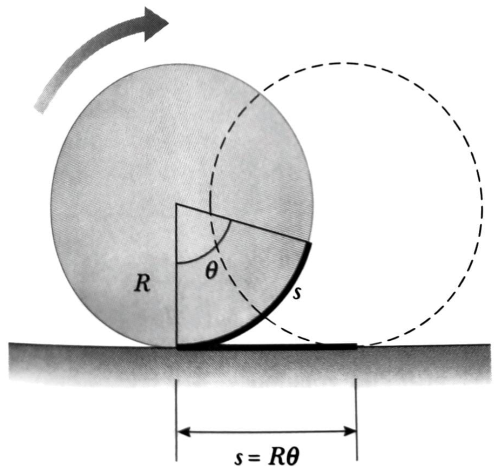


Figure 11: Pure Rolling Motion of a cylinder

Its center of mass moves a distance of $s = R\theta$ as it rotates through an angle θ. The linear speed of the center of mass v_{CM} is thus given by:

$$v_{CM} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \tag{28}$$

Consequently, the linear acceleration of the center of mass a_{CM} is thus given by:

$$a_{CM} = \frac{dv_{CM}}{dt} = R\frac{d\omega}{dt} = R\alpha \tag{29}$$

Finally, by invoking the approach shown in Figure 10, we can write down the expression for the total kinetic energy K of the object:

$$K = \frac{1}{2}I_{CM}\omega^2 + \frac{1}{2}Mv_{CM}^2 \tag{30}$$

Tutorial: **D13***, D14

Syllabus 9814

A3 - Planetary and Satellite Motion 2024

Content A3 :

- Kepler's laws of planetary motion
- Gravitational potential energy of a spherical shell
- Elliptical orbits and orbital transfers
- Concept of an effective radial potential

Learning Outcomes

Candidates should be able to:

(a) show an understanding of Kepler's laws of planetary motion, and

I. recall and apply Kepler's first law that the planets move in elliptical orbits with the Sun at one focus of the ellipse (knowledge of the eccentricity parameter is not required).

II. show an understanding of how Kepler's second law (that an imaginary line from the Sun to a moving planet sweeps out equal areas in equal intervals of time) is related to the conservation of angular momentum, and apply this law to solve related problems.

III. recall and apply Kepler's third law that the ratio of the square of a planet's period of revolution to the cube of the semi-major axis of its orbit around the Sun is a constant, and this constant is the same for all planets.

(b) derive expressions for the gravitational potential energy of a point mass inside and outside a uniform spherical shell of mass, and relate these expressions to the justification for treating large spherical objects as point masses.

(c) solve problems involving elliptical orbits and orbital transfers e.g. when a satellite fires its thrusters (knowledge of parabolic and hyperbolic trajectories is not required).

(d) derive, from energy considerations, an expression for the effective radial potential $U_{eff} = -GMm/r + L^2/(2mr^2)$ for a mass m interacting gravitationally with a large mass $M \gg m$ whose own motion is negligible, where L is the angular momentum of the mass m relative to the stationary mass M.

(e) discuss how the effective radial potential allows the determination of bound and unbound states, as well as turning points in the motion, and apply this to solve related problems.

2 Kepler's Laws of Planetary Motion

Kepler's laws describing the orbits of planets around the Sun are the following:

1. planets move in elliptical orbits with the Sun at one focus of the ellipse;
2. an imaginary line from the Sun to a moving planet sweeps out equal areas in equal intervals of time;
3. the ratio of the square of a planet's period of revolution to the cube of the semi-major axis of its orbit around the Sun is a constant, and this constant is the same for all planets.

Let us now discuss each of these in more detail.

2.1 Kepler's First Law

Planets move in elliptical orbits with the Sun at one focus of the ellipse.

Figure 1 illustrates a typical elliptical orbit, although the orbits of the planets in our solar system are considerably less eccentric, i.e., more circular. Asteroids, dwarf planets and short-period comets also follow Kepler's Laws, and some of their orbits are highly elliptical.

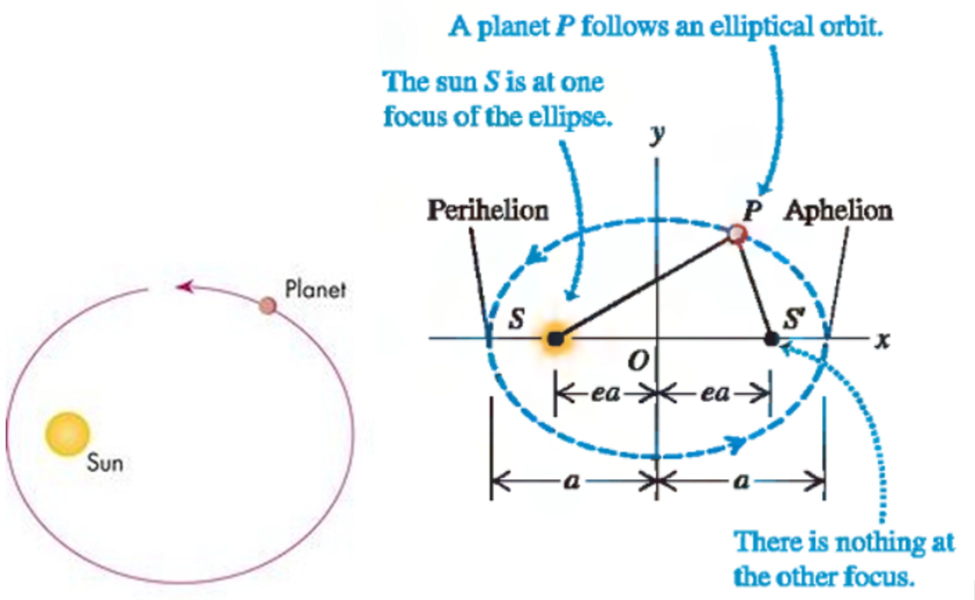


Figure 1: Kepler's First Law. e is the eccentricity of the orbit.

S and S' are known as the foci of the ellipse. The longest diameter of the ellipse is the major axis with half-length a and the shortest is the minor axis with half-length b. If $a = b$ then the ellipse is a circle. The half-lengths a and b are called the semi-major axis and semi-minor axis, respectively. The sum of the distance from S to P and from S' to P is the same for all points on the curve, which is the defining property of an ellipse.

The point in a planet's orbit closest to the sun is the *perihelion* and the point most distant from the sun is the *aphelion*. The general terms for the closest and farthest points in an elliptical orbit are *periapsis* and *apoapsis*, and we can for instance also talk about the *perigee* and *apogee* of a satellite in an elliptical orbit around the Earth.

2.2 Kepler's Second Law

A line from the Sun to a moving planet sweeps out equal areas in equal intervals of time.

Recall the definition of angular momentum, $\vec{L} = \vec{r} \times \vec{p}$. For the case of planetary motion, $\vec{L}$ is the angular momentum of the planet about the Sun, $\vec{r}$ is the position vector of the planet measured from the Sun, and $\vec{p} = m\vec{v}$ is the instantaneous linear momentum at any point in the orbit. Since the planet moves along the ellipse, $\vec{p}$ is always tangent to the ellipse.

We can resolve the linear momentum into two components: a radial component $\vec{p}_{\text{rad}}$ along the line to the Sun, and a component $\vec{p}_{\text{perp}}$ perpendicular to $\vec{r}$. The cross product for angular momentum can then be written as

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times (\vec{p}_{\text{rad}} + \vec{p}_{\text{perp}}) = \vec{r} \times \vec{p}_{\text{rad}} + \vec{r} \times \vec{p}_{\text{perp}}. \tag{1}$$

The first term on the right is zero because $\vec{r}$ is parallel to $\vec{p}_{\text{rad}}$, and in the second term $\vec{r}$ is perpendicular to $\vec{p}_{\text{perp}}$, so the magnitude of the cross product reduces to $L = rp_{\text{perp}} = rmv_{\text{perp}}$. Note that the angular momentum does *not* depend upon p_{rad}. Since the gravitational force is only in the radial direction, it can change only p_{rad} and not p_{perp}; hence, the angular momentum must remain constant.

Now consider Figure 2. A small triangular area ΔA is swept out in time Δt. The velocity is along the path and it makes an angle θ with the radial direction. Hence, the perpendicular velocity is given by $v_{\text{perp}} = v \sin\theta$. The planet moves a distance $\Delta s = v\Delta t \sin\theta$ projected along the direction perpendicular to r. Since the area of a triangle is one-half the base (r) times the height (Δs), for a small displacement, the area is given by $\Delta A = \frac{1}{2} r\Delta s$. Substituting for Δs, multiplying by m in the numerator and denominator, and rearranging, we obtain

$$\Delta A = \frac{1}{2} r\Delta s = \frac{1}{2} r\,(v\Delta t \sin\theta) = \frac{1}{2m} r\,(mv \sin\theta \Delta t) = \frac{1}{2m} r\,(mv_{\text{perp}}\Delta t) = \frac{L}{2m}\Delta t. \tag{2}$$

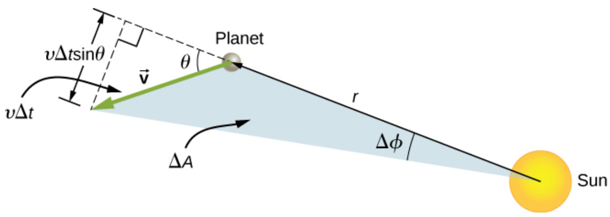


Figure 2: Kepler's Second Law.

The areal velocity is simply the rate of change of area with time, so we have

$$\text{areal velocity} = \frac{\Delta A}{\Delta t} = \frac{L}{2m}. \tag{3}$$

Since the angular momentum is constant, the areal velocity must also be constant, which is exactly Kepler's Second Law.

Figure 3 shows a visual interpretation of this result. The time it takes a planet to move from position A to B, sweeping out area A_1, is exactly the time taken to move from position C to D, sweeping area A_2, and to move from E to F, sweeping out area A_3. These areas are the same: $A_1 = A_2 = A_3$. Comparing the areas in the figure and the distance travelled along the ellipse in each case, we can see that in order for the areas to be equal, the planet must speed up as it gets closer to the Sun and slow down as it moves away. This behaviour is completely consistent with the principle of conservation of energy.

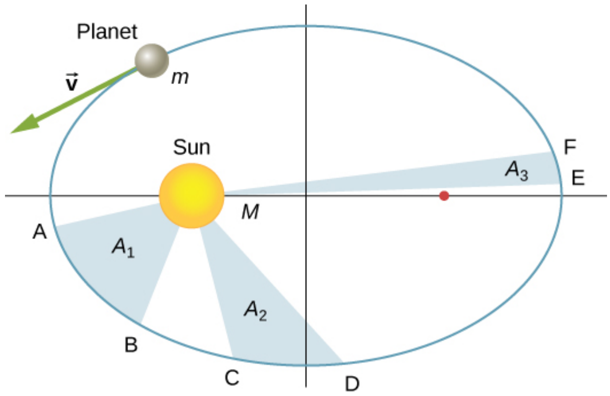


Figure 3: Kepler's Second Law.

2.3 Kepler's Third Law

The square of the period of any planet is proportional to the cube of the length of the major axis of its orbit.

The derivation for a circular orbit ($\frac{T^2}{4\pi^2} = \frac{r^3}{GM}$) is trivial and has been covered in H2 Physics.

Of course, Kepler's Third Law is not restricted to circular orbits. Although the derivation is not in the learning objectives of H3 Physics, you have to know the relationship between the period T of an elliptical orbit with semi-major axis a:

$$T^2 \propto a^3 \tag{4}$$

This equation holds true for any system where satellites orbit a central mass that is much more massive. For the planets in our solar system, we specifically have that

$$\frac{T^2}{4\pi^2} = \frac{a^3}{GM}, \tag{5}$$

where G is the universal gravitational constant and M is the mass of the Sun.

You will go through the formal proof of the above expression in your tutorial.

Example 1 (IOAA 2007, theoretical round, questions 6 and 7)

a) A Sun-orbiting periodic comet is farthest from the Sun at 31.5 AU and closest to the Sun at 0.5 AU. What is the orbital period of this comet?

b) For the comet in part (a) above, what is the area per unit time (in square AU per year) swept by the line joining the comet and the Sun?

Note: 1 AU = 1 astronomical unit, which is given by the average Sun-Earth distance, where 1 AU = 149.6×10^6 km. For part (b), you may use that the eccentricity of the orbit and the semi-minor axis b are related to the semi-major axis a by $b^2 = a^2(1 - e^2)$, and that the area of an ellipse A_{ellipse} is given by $A_{\text{ellipse}} = \pi ab$. Refer to Figure 1 for the eccentricity of an orbit.

Tutorial: Questions 1*, 2, 3* (Starred questions are compulsory)

3 Gravitational potential energy of simple mass distributions

In H2 Physics, you learnt to describe a gravitational field using 4 different quantities (the force F, the field strength g, the potential energy U, and the potential ϕ), the relationships between them, and specific expressions for point masses (and spherically symmetric mass distributions). We assumed that the gravitational field of a spherically symmetric mass distribution is equivalent to that of a point mass of the same mass, placed at the centre of the mass distribution.

We are now going to justify that large spherical objects may be treated as point masses, as long as their masses are distributed spherically symmetrically. This observation follows from the expressions for the gravitational potential energy of a point mass inside and outside a uniform spherical shell. The derivation of these expressions is challenging and requires us to break up the spherical shell into rings, and then integrate over those rings.

To begin, in the spirit of proving results taken for granted in H2 Physics, try Example 2 below. We previously assumed that the gravitational field strength is approximately constant near the Earth's surface. You can show this explicitly, using a Maclaurin expansion.

Example 2

The distance between Earth's surface and an object of mass m is changed by an amount Δx.

Show that when $x \approx R_E$ and $\Delta x << R_E$, the gravitational potential energy of the system reduces to the expression $\Delta U = mg\Delta x$.

Next, let's work those integration muscles through a (relatively) simple example.

Example 3 (University Physics by Young and Freedman, 12.40)

A thin uniform rod has length L and mass M. A small uniform sphere of mass m is placed a distance x from one end of the rod, along the axis of the rod, as shown in Figure 4.

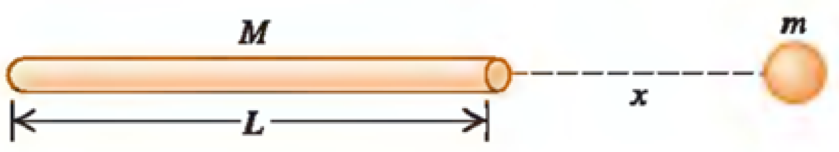


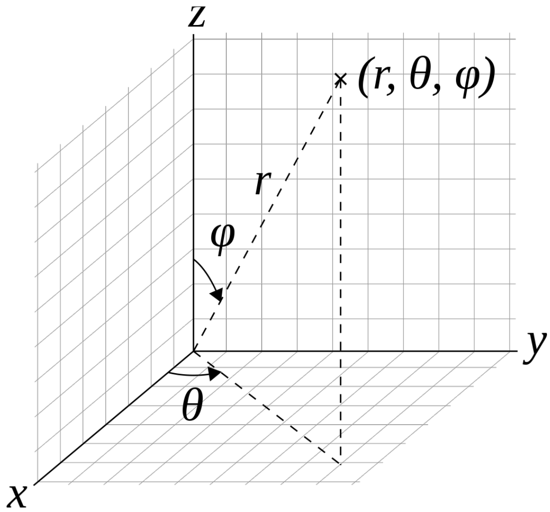
Figure 4: Thin uniform rod and small spherical mass

a) Calculate the gravitational potential energy of the rod-sphere system.
b) Show that your answer reduces to the expected result when $x >> L$.

We will now go on to formally derive expressions for the gravitational potential energy of a point mass inside and outside a uniform spherical shell of mass.

3.1 Gravitational potential energy of a point mass *outside* a uniform shell

As shells are spherically symmetric, it will be convenient to use a spherical coordinate system. A spherical coordinate system is a coordinate system for three-dimensional space where the position of a point is specified by three numbers: the *radial distance* of that point from a fixed origin, its *polar angle* measured from the positive Z axis, and the *azimuthal angle* from the positive X axis: (r, θ, ϕ) gives the radial distance, azimuthal angle, and polar angle.



By Dmcq - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=18913446

Take a point mass m, outside a uniform shell of mass M, radius R and area A. (Think of a shell as the outermost layer of a sphere.) Assume that m lies on the axis through the top and bottom of the shell a distance r from the shell's centre, so $r > R$. If m is *not* on that axis, we can always rotate our shell so that it is.

We will now break the shell up in rings of varying radius $R\sin\phi$, each of mass dM. Each ring is further broken up in even smaller parts, each of mass dm and area $(R\sin\phi\, d\theta)(R\, d\phi)$.

As the total mass of the shell is M and the total area of the shell is A, we thus have

$$dm = \frac{M}{A}(R\sin\phi\, d\theta)(R\, d\phi) \tag{6}$$

The gravitational potential energy of m and a small mass dm is then

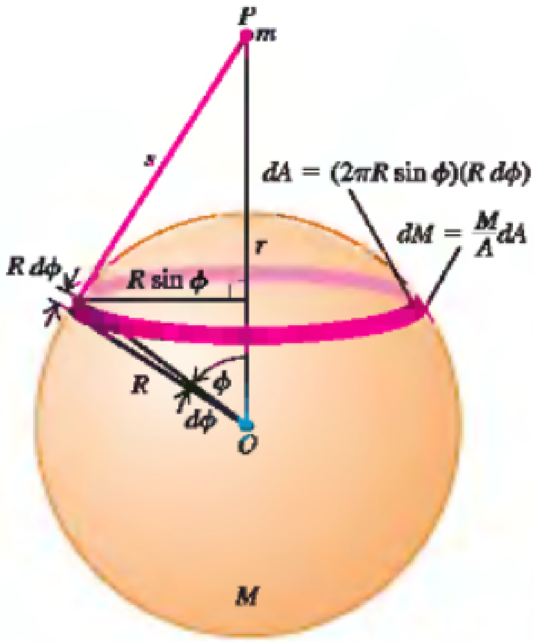
$$dU = -\frac{G(dm)m}{s}, \tag{7}$$

where s is the distance between m and dm. Using equation 6, this gives

$$dU = -\frac{Gm}{s}\frac{M}{A}(R\sin\phi\, d\theta)(R\, d\phi) \tag{8}$$

Integrating θ over the circumference of a ring of radius $R\sin\phi$ (see the diagram on the right), we get for the potential energy due to a ring

$$dU_{\text{ring}} = -\frac{GMm}{As}(2\pi R\sin\phi)(R\, d\phi). \tag{9}$$



Now we will be using a little trick. Referring again to the diagram above, by the cosine rule,

$s^2 = r^2 + R^2 - 2Rr\cos\phi$. Differentiating with respect to ϕ, with R and r being constants,

$$\begin{aligned} 2s\,ds &= 2rR(\sin\phi)\,d\phi \\ s\,ds &= rR(\sin\phi)\,d\phi \\ ds\,/\,rR &= (\sin\phi)\,d\phi\,/\,s \\ (\sin\phi)\,d\phi\,/\,s &= ds\,/\,rR \end{aligned} \tag{10}$$

Using this, we get

$$dU_{\text{ring}} = -\frac{GMm}{A}\frac{2\pi R}{r}\,ds \tag{11}$$

$A = 4\pi R^2$ is the whole area of the shell, so

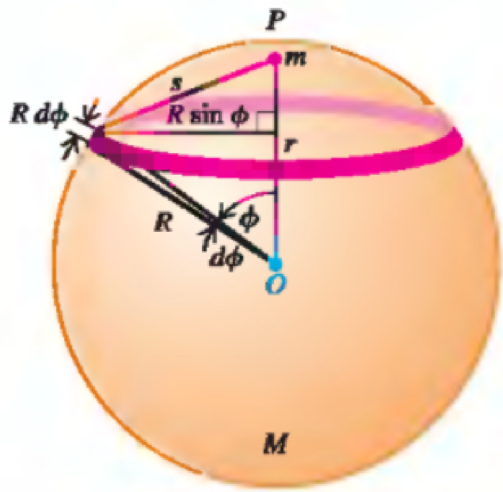
$$dU_{\text{ring}} = -\frac{GMm}{2rR}\,ds. \tag{12}$$

To get the potential energy due to the whole shell, we have to integrate s from $(r-R)$ at the top of the shell to $(r+R)$ at the bottom of the shell,

$$U = \int_{r-R}^{r+R} -\frac{GMm}{2rR}\,ds = -\frac{GMm}{2rR}\left[s\right]_{r-R}^{r+R} = -\frac{GMm}{r}. \tag{13}$$

Hence, the potential energy due to a small mass m outside a uniform spherical shell of mass M is equal to that due to a point mass M at a distance r away from m. Since a uniform sphere can be thought of as a number of concentric shells, it follows that the potential energy of a small mass m at a distance r from the centre of a sphere of mass M_{sphere} is equal to the potential energy of a small mass m at a distance r from a point mass of mass M_{sphere}.

3.2 Gravitational potential energy of a point mass *inside* a uniform shell



The derivation is equivalent to that for a point mass outside a spherical shell. Hence, we get for the potential energy due to a ring

$$dU_{\text{ring}} = -\frac{GMm}{2rR}\,ds. \tag{14}$$

However, we now have to integrate s from $(R - r)$ at the top of the shell to $(R + r)$ at the bottom of the shell,

$$\begin{aligned} U &= \int_{R-r}^{R+r} -\frac{GMm}{2rR}\,ds \\ &= -\frac{GMm}{2rR}\left[s\right]_{R-r}^{R+r} \\ &= -\frac{GMm}{R}. \end{aligned} \tag{15}$$

Hence, the potential energy of a small mass m in a spherical shell of mass M is a constant, and does not on the position of the mass m. As the force is equivalent to the negative of the gradient of the potential energy, and the gradient of a constant is zero, it follows that a mass inside a uniform shell does not experience a force due to that shell.

Note that the mass m may still experience a force due to any masses between the mass m and the centre of the shell, as derived above. Combining the two results, we find that the effective gravitational force on a small test mass inside a spherically symmetric mass distribution only depends on the mass *below* the test mass. If, for instance, we are interested in the gravitational forces acting on a part of the Earth's core, we can readily ignore the Earth's mantle and crust.

Tutorial: Questions 4, 5*, 6

3.3 Mechanical energy of a mass m interacting with a large mass M

Let us consider a satellite of mass m in an orbit about another mass M, where $m << M$. Note that this orbit does not have to be circular. The total mechanical energy E of the system is given by

$$E = \frac{1}{2}mv^2 + \left(-\frac{GMm}{r}\right). \tag{16}$$

Similar to the derivation of Kepler's Second Law before, we can resolve the velocity into two components: a radial component v_{rad} along the line to the Sun, and a component v_{perp} perpendicular to this line.

Since v_{rad} is perpendicular to v_{perp}, Pythagoras' theorem gives

$$E = \frac{1}{2}m\left(v_{\text{rad}}^2 + v_{\text{perp}}^2\right) - \frac{GMm}{r}. \tag{17}$$

We further have that L is the angular momentum of mass m relative to mass M (where M is assumed to be stationary). Using that $L = rmv_{\text{perp}}$ and taking that $\dot{r} = \frac{dr}{dt} = v_{\text{rad}}$, we get for the mechanical energy of a mass m interacting gravitationally with a large mass M,

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}. \tag{18}$$

In H3 Physics, you have to be able to derive the expression (18). As to why this expression may be useful, we will explore in the next subsection.

3.4 Effective radial potential of a mass m interacting with a large mass M

Rearranging equation 18, we can rewrite it as

$$\frac{1}{2}m\dot{r}^2 = E - \frac{L^2}{2mr^2} + \frac{GMm}{r}, \tag{19}$$

$$\frac{1}{2}m\dot{r}^2 = E - U_{\text{eff}}(r), \tag{20}$$

where

$$U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} \tag{21}$$

is the effective potential. The original problem depended on both the velocity and position of m; it has now been reduced to a problem that only depends on r, the instantaneous distance between M and m.

If we treat the system of M and m as isolated, both E and L are constant. By then comparing E and U_{eff}, we can classify the orbit of the system. We will discuss this in more detail in the next section, by considering the graph of $U_{\text{eff}}(r)$.

4 Orbits

4.1 Classification of orbits

The energy of a single body moving in two dimensions can be reinterpreted as the energy of a single body moving in one dimension, the radial direction r, in an effective potential given by two terms,

$$U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r}. \tag{22}$$

The energy is still the same, but our interpretation has changed,

$$E = K_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}m\dot{r}^2 + \frac{L^2}{2mr^2} - \frac{GMm}{r}, \tag{23}$$

where the effective kinetic energy K_{eff} associated with the one-dimensional motion is

$$K_{\text{eff}} = \frac{1}{2}m\dot{r}^2. \tag{24}$$

Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}} = 0$, the energy is equal to the effective potential energy,

$$E = U_{\text{eff}} = \frac{L^2}{2mr^2} - \frac{GMm}{r}. \tag{25}$$

Figure 8 shows a typical curve for the effective radial potential $U_{\text{eff}}(r)$, where r is the distance between M and m. Figure 8 further shows four energies corresponding to circular, elliptic, parabolic, and hyperbolic orbits.

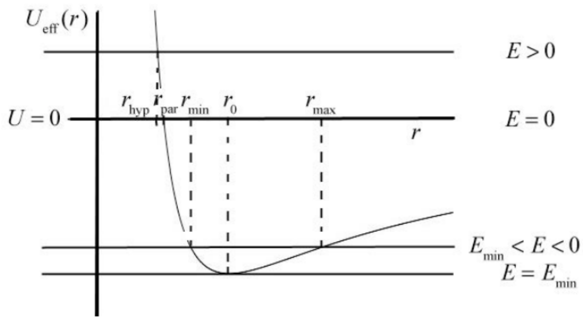


Figure 8: Plot of $U_{\text{eff}}(r)$ vs. r with four energies corresponding to circular, elliptic, parabolic, and hyperbolic orbits

4.1.1 Circular orbit, $E = E_{\text{min}}$

The lowest energy state, E_{min}, corresponds to the minimum of the effective radial potential. We can minimise the effective radial potential by taking the derivative and setting it to zero,

$$\frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{mr^3} + \frac{GMm}{r^2} = 0. \tag{26}$$

Solving for $r = r_0$ gives

$$r_0 = \frac{L^2}{GMm^2}, \tag{27}$$

which is a constant. In other words, for $E = E_{\text{min}}$, the distance r between M and m does not change, which corresponds to a **circular** orbit. Such an orbit is said to be *bound*, which means that the small mass m does not have enough energy to escape to infinity.

4.1.2 Elliptic orbit, $E_{\text{min}} < E < 0$

For $E_{\text{min}} < E < 0$, there are two points r_{min} and r_{max} such that $E = U_{\text{eff}}$. At these points $K_{\text{eff}} = 0$, so $\dot{r} = 0$, which corresponds to a point of closest of farthest approach (Figure 9). This condition corresponds to the minimum and maximum values of r for an **elliptic** orbit.

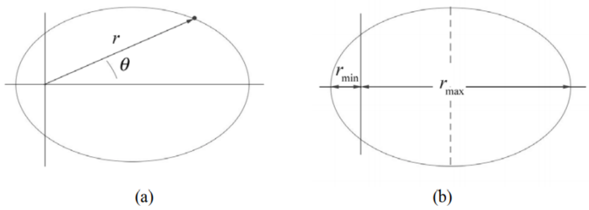


Figure 9: (a) elliptic orbit, (b) closest and farthest approach

Like the previous example, this orbit is also bound. This is consistent with what we found in H2 Physics, namely that an object with negative total energy is unable to escape. Here, the system oscillates between r_{min} and r_{max}. For planets, asteroids and periodic comets in orbit around the Sun, r_{min} and r_{max} correspond to the perihelion and the aphelion, respectively.

4.1.3 Parabolic orbit, $E = 0$

The effective radial potential, as given in Equation (22), approaches zero ($U_{\text{eff}} \to 0$) when the distance r approaches infinity ($r \to \infty$). When $E = 0$, the kinetic energy K_{eff} also approaches zero when r approaches infinity, $K_{\text{eff}} \to \infty$ as $r \to \infty$. This corresponds to a **parabolic** orbit.

At any point in its orbit, this object is moving at a speed $v = v_E$, known as the *escape speed*. Note that, just like the effective potential $U_{\text{eff}}(r)$, the escape speed v_E varies with distance from the central object r. This makes sense: if we are farther way from the central object, the gravitational pull on us is small, so a lower speed v allows us to escape to infinity. As the object can escape to infinity, this is an *unbound* orbit.

For a parabolic orbit, the body also has a distance of closest approach. This distance r_{par} can be found from the condition

$$E = U_{\text{eff}}(r) = \frac{L^2}{2mr^2} - \frac{GMm}{r} = 0. \tag{28}$$

Solving Equation (28) for $r = r_{\text{par}}$ yields

$$r_{\text{par}} = \frac{L^2}{2GMm^2}. \tag{29}$$

4.1.4 Hyperbolic orbit, $E > 0$

When $E > 0$, the kinetic energy remains positive when r approaches infinity, $K_{\text{eff}} > 0$ as $r \to \infty$. This corresponds to a **hyperbolic** orbit, which is also *unbound*.

In this case, there is again one turning point at $r = r_{\text{hyp}}$. At this point, the component of the velocity of mass m in the radial direction ($\dot{r}$) is instantaneously zero and the total energy of the system is equal to the effective radial potential.

If the object initially moves inwards, it turns around at $r = r_{\text{hyp}}$, then moves out to infinity forever. So-called *single-apparition comets* (comets that are only observable from the Earth once, before leaving, never to be seen again) may have such a hyperbolic orbit, which allows them to permanently exit the Solar System after a single pass of the Sun.

Example 4 (IOAA 2010, theoretical round, question 2)

If the escape velocity from a solar-mass object's surface exceeds the speed of light, what would be its radius?

The following values may be useful:
$c = 2.9979 \times 10^8$ m s^{-1}
$G = 6.6726 \times 10^{-11}$ N m^2 kg^{-2}
$M_{\text{sun}} = 1.9891 \times 10^{30}$ kg

Side note: What you have just calculated is the Schwarzschild radius, which is the radius of a (non-rotating, uncharged) black hole!

Example 5 (IOAA 2011, theoretical round, question 3)

On 9 March 2011, the Voyager probe was 116.406 AU from the Sun and moving at 17.062 km s^{-1}. Determine the type of orbit the probe is on: (a) elliptical, (b) parabolic, (c) hyperbolic.

The following values may be useful:
1 AU $= 1.4960 \times 10^{11}$ m
$G = 6.6726 \times 10^{-11}$ N m^2 kg^{-2}
$M_{\text{sun}} = 1.9891 \times 10^{30}$ kg

4.2 Transfer orbits

In orbital mechanics, we are often interested in moving an object from one orbit to another. Typically, we try to perform this while minimising the impulse on the system, i.e., we try to keep Δp to a minimum. The reasons for this are practical and financial: every change in momentum requires fuel, and the more fuel we have to carry, the heavier and more expensive our load.

A Hohmann transfer orbit is an elliptical orbit that is used to transfer between two circular and coplanar orbits of different radii, as shown in Figure 10. This orbital manoeuvre uses two engine impulses, once to put the object into the (elliptical) transfer orbit, and another to put the object into the new circular orbit.

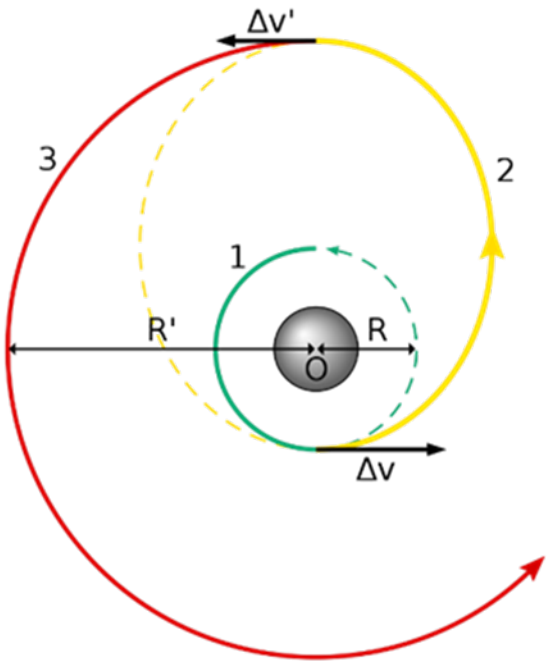


Figure 10: Hohmann transfer from a circular orbit of radius R to another circular orbit of radius R'

Referring to Figure 10, the object is initially in circular orbit 1 about a central massive object (e.g., the Sun). After an impulse, the speed of the object changes by Δv. This sets the object on the elliptical path 2. Note that this impulse is applied at the periapsis (perihelion) of the elliptical orbit. At the apoapsis (aphelion), another impulse $\Delta v'$ sets the object onto the new circular path 3. In Example 6 below, you will derive explicit expressions for Δv and $\Delta v'$.

Example 6

a) For the Hohmann transfer orbit in Figure 10, write down the expressions for v_1 and v_3, the velocity of the object in circular orbits 1 and 3. Denote the large central mass as M.

b) Write down expressions for v_a and v_p, the velocity of the object at apoapsis and periapsis of the elliptical orbits in terms of v_1, v_3, Δv and $\Delta v'$.

c) Using your understanding of elliptical orbits, write down an expression relating v_a and v_p. You may use any suitable lengths defined in the problem.

d) Applying energy considerations to the elliptical path, and using suitable results obtained in previous parts, show that Δv and $\Delta v'$ are given by Equations 30 and 31.

$$\Delta v = \sqrt{\frac{GM}{R}}\left(\sqrt{\frac{2R'}{R+R'}} - 1\right) \tag{30}$$

$$\Delta v' = \sqrt{\frac{GM}{R'}}\left(1 - \sqrt{\frac{2R}{R+R'}}\right) \tag{31}$$

Tutorial: Questions 7*, 8*, 9

Chapter B1

Electric and Magnetic Fields

Charles-Augustin de Coulomb

André-Marie Ampère

Carl Friedrich Gauss

James Clerk Maxwell

Charles-Augustin de Coulomb *(1736-1806) was a French military engineer turned scientist, famous for discovering Coulomb's Law, which describes the electrostatic attraction between charges.*

André-Marie Ampère *(1775-1836) was a French physicist and mathematician who discovered Ampere's Law, one of the links between electricity and magnetism.*

Carl Friedrich Gauss *(1777-1855) was a brilliant German mathematician and physicist who made many contributions in both fields. In this chapter we will study Gauss' Law, which relates electric flux to the enclosed charge.*

James Clerk Maxwell *(1831-1879) was a Scottish mathematician and physicist who synthesized earlier discoveries by Ampere, Gauss, and Faraday into Maxwell's Equations, which describe electricity, magnetism and light as manifestations of the same phenomenon – electromagnetic waves.*

Learning Outcomes

Candidates should be able to:

(a) show an understanding that ideal conductors form an equipotential volume and that the electric field within an ideal conductor is zero

(b) show an understanding that electric charge accumulates on the surfaces of a conductor and that the electric field at the surface of a conductor is normal to the surface

(c) recall and apply Gauss's law for electric and magnetic fields (knowledge of the differential form of Gauss's law is not required) and

  (i) solve problems involving symmetric charge distributions by relating the electric flux (in a vacuum) through a closed surface with the charge enclosed by that surface

  (ii) show an understanding of the non-existence of 'magnetic charge' expressed by Gauss's law for magnetism

(d) recall and apply Ampère's law relating the line integral of the magnetic field (in a vacuum) around a closed loop with the electric current enclosed by the loop to solve problems involving symmetric field configurations (knowledge of the differential form of Ampère's law is not required).
[Note further that candidates are not required to know Maxwell's generalisation of Ampère's law including the term related to the rate of change of electric flux, nor the Biot-Savart law]

(e) define the magnitude of the electric dipole moment as the product of the charge and the separation

(f) show an understanding of and use the torque on an electric dipole and the potential energy of an electric dipole to solve related problems

(g) define the magnitude of the magnetic dipole moment for a current loop as the product of the current and the area of the loop

(h) show an understanding of and use the torque on a magnetic dipole and the potential energy of a magnetic dipole to solve related problems

(i) appreciate that while electric and magnetic dipoles behave analogously, the theoretical framework at this level of study does not admit the possibility of magnetic monopoles

Recommended Reference Texts

- Knight, *Physics for Scientists and Engineers: A Strategic Approach.* 2016, 4th Ed. Pearson.
- Young & Freedman, *University Physics*. 2004, 13th Ed. Addison-Wesley.

Contents

1 Conductors in Electrostatic Equilibrium

Broadly speaking, all materials are either electrical conductors (charges are free to move throughout the material) or electrical insulators (charges are not free to move throughout the material).

Electrostatic equilibrium is the state where there is no *net* flow of charge. Note that this doesn't mean that the charges are not moving! You have learnt in chemistry in secondary school that metals have a "sea" of delocalised electrons, and these are definitely moving if the temperature is above 0 K. We usually only talk about electrostatic equilibrium for conductors, since charges "cannot" flow in insulators anyway.

The following is true for a conductor in electrostatic equilibrium:

	Result	Proof
1	The electric field inside the conductor is zero	If $E \neq 0$ somewhere inside the conductor, free charges will experience an electric force, and thus move. Then the conductor is not in electrostatic equilibrium.
2	The conductor is an equipotential volume	(using the result of 1): Along any path inside the conductor, $E = 0$. Thus, the potential difference between any two points in the conductor, $V_{ab} = 0$. Thus the electric potential $V = \text{constant}$ everywhere inside the conductor.
3	The electric field at the surface is normal to the surface	If there is a component of E along the surface, charges would experience an electric force along the surface, and thus flow. Then it is not at electrostatic equilibrium, which is a contradiction.
4	The surface of the conductor is an equipotential surface	(using the result of 3): For any path along the surface, the component of the electric field strength along the surface $E_{\parallel} = 0$. Thus, the potential difference between any two points on the surface, $V_{ab} = 0$. Thus the electric potential $V = \text{constant along the surface.}$

In order to achieve these results, electrical charge accumulates on the surface of the conductor (this can be shown using Gauss' Law, which we will study in Section 4). In general the surface charge density might not be constant, but under most circumstances encountered in H3 it will be (i.e. charged conductor not under the influence of an external field or another charge distribution).

In other words, the (net) charges in a conductor in electrostatic equilibrium will be a surface charge – the charges are arranged on the (inner and outer) surface(s) of the conductor. Therefore a charged cylinder will have a cylindrical shell of charge, a charged sphere will have a spherical shell of charge, etc.

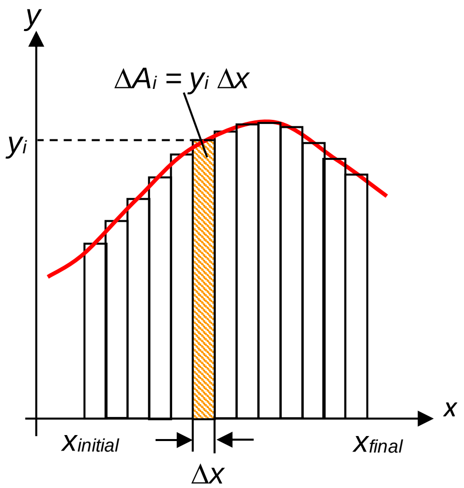
Of course, conductors are not always in electrostatic equilibrium – for example, the metal wires in electric circuits. And it is not that there are no charges at all inside the conductor – the positively-charged nuclei of atoms are still inside the volume! But there is no *net* charge inside the volume of the conductor.

2 Mathematical Preliminaries

Unfortunately (or fortunately?), we will be doing a lot of calculus in this chapter. Take some time to familiarise yourself with the following mathematics, but don't worry if it feels overwhelming at first – it will become a lot clearer once you start using it to solve problems.

2.1 Integrals as Riemann sums

Many quantities in physics are related to each other as rates or as areas under a curve. You'll have already learnt that we can calculate the area under a graph by approximating it with thin rectangles (of width Δx):



Then the area under the curve is approximately:

$$A = \sum_{i=1}^{N} y_i \Delta x_i = y_1 \Delta x_1 + y_2 \Delta x_2 + \cdots + y_N \Delta x_N$$

This method of calculating an integral is known as a Riemann sum, after the mathematician Bernhard Riemann. It is a *discrete* sum, in that you are adding a finite number of terms.

The approximation gets better and better as you use more rectangles (i.e. as Δx gets smaller) – the jagged "steps" get less and less obvious, until it appears to be as smooth as the curve itself. Then, the number of terms in the sum becomes infinite – we call this a *continuous* sum (because the curve is a continuous thing, as opposed to a series of chunks that you can hold and count).

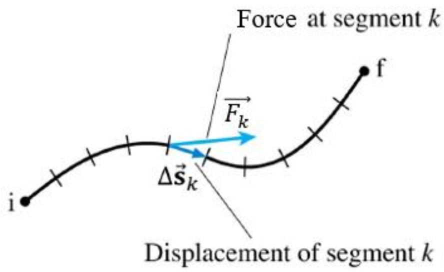
In fact, a definite integral is *defined* as the Riemann sum when Δx is made infinitely small:

$$\int_a^b y\, dx \coloneqq \lim_{\Delta x \to 0} \sum_{i=1}^{N} y_i\, \Delta x_i$$

An integral is therefore a *continuous, infinite sum* of infinitesimal terms.[1] Thus when moving from a discrete sum ($\sum$) to a continuous sum ($\int$), we replace Δx_i with dx. This idea of integration as a sum of many small numbers will be very important in this chapter. This Youtube video by 3Blue1Brown explains it very well (especially the first 10 minutes): https://www.youtube.com/watch?v=WUvTyaaNkzM

[1] In fact, the mathematician Gottfried Leibniz invented $\int$ as the symbol for integration because it resembles a long "s" (for "sum") See https://en.wikipedia.org/wiki/Integral_symbol

Even when there is no "graph" to take the area of, **integration can be used to represent the sum of very small quantities**. For example, the *work done* W by a force F moving an object over a displacement s is defined as $W = \vec{F} \cdot \vec{s} = Fs\cos\theta$. This is straightforward if F is a constant force. But what if the magnitude and direction of F changes with s?



One solution is to consider the work done W_k by F over short distances Δs (because F would be approximately constant over that interval), and sum that over the entire distance:

$$W = \sum_k^N W_k = \sum_{k=1}^N \overrightarrow{F_k} \cdot \overrightarrow{\Delta s_k} = \sum_{k=1}^N F_k \Delta s_k \cos\theta_k$$

In general we would get a better approximation as we consider smaller and smaller intervals Δs, i.e. let $\Delta s \to 0$. Then we have a continuous sum:[2]

$$W = \int dW = \int \vec{F} \cdot d\vec{s} = \int F\cos\theta \; ds$$

(Remember that here, F and θ are functions of s). This is the same as saying that the work done is the area under the force-displacement graph (which is what we teach you in H2 Physics)

We can also apply the same idea to areas and volumes – in fact you have already done this. Let's say we want to find the mass of an object of variable density ρ, where ρ is a function of position (e.g. it depends on x, y, z). First we "cut up" the mass into small volumes ΔV_i which have a density ρ_i, calculate the mass m_i of each piece, and add it all up:

$$m = \sum_i^N m_i = \sum_i^N \rho_i \, \Delta V_i$$

As $\Delta V_i \to 0$,

$$m = \int dm = \int \rho \, dV$$

In fact, you have done this already in chapter A1 and A2, where you found the centre of mass or moment of inertia of a continuous mass distribution! (Although in those lecture examples, ρ was a constant and could simply be factored out of the integral.)

dV may seem like an odd variable to be integrating over, since ρ is typically a function of coordinates (e.g. x, y, z). In that case, we can replace dV with the appropriate volume element (explained in the next section) and specify the limits of the integral accordingly.

[2] Integrating over a particular path is also known as a *line integral* or *path integral*. (More details in section 4)

2.2 Coordinate systems

Physics involves a lot of mathematical modelling of reality. You need to set up a coordinate system (or at least a coordinate origin – the "reference point") before you can express the objects you are studying in mathematical notation.

When modelling objects in three dimensions, here are three common coordinate systems you will encounter in physics (though there are many more):

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The **Cartesian coordinate system** or **rectangular coordinate system** (x, y, z) is the one you're probably the most familiar with. Usually, the x-y plane is taken as the "floor" and the z-axis points vertically upwards.

The volume element is the volume of a cuboid with dimensions dx, dy, dz: $dV = dx\ dy\ dz$. By integrating over dV, we get the volume of a cuboid.

The possible area elements are the area of any of the sides of the volume element:
$dA = dx\ dy$ or $dy\ dz$ or $dx\ dz$

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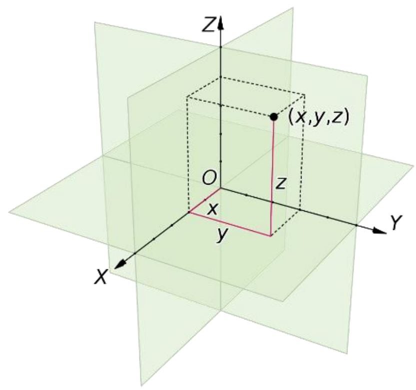

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The **cylindrical coordinate system** (r, ϕ, z) is obtained by adding a vertical z-axis on top of the 2D polar coordinate plane. In other words, you can convert this back to Cartesian coordinates using:

$$x = r\cos\phi$$
$$y = r\sin\phi$$
$$z = z$$

Where $r = \sqrt{x^2 + y^2}$ is the distance along the x-y plane

ϕ is the **polar angle:**[3] the angle with the x-axis. If you form a "thumbs up" with your right hand, aligning the thumb to the z-axis, the direction your fingers curl is the direction of increasing ϕ. Note that ϕ ranges from 0 to 2π.

The volume element is the volume of the shaded region in the diagram: $dV = (r\ d\phi)(dr)(dz)$. By integrating over dV, we get the volume of a cylinder or cylindrical shell of a certain thickness.

The possible area elements are the area of any of the sides of the volume element:
$dA = (r\ d\phi)(dr)$ or $(dr)(dz)$ or $(r\ d\phi)(dz)$

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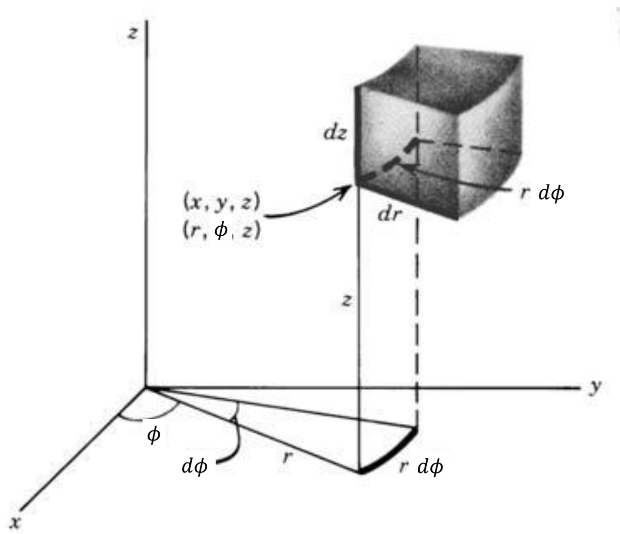

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[3] Sometimes θ is used as the symbol instead of ϕ, but it means the same thing – the angle made with the x-axis.

The **spherical coordinate system** (r, ϕ, θ) goes one step further by replacing the z-coordinate with an azimuthal angle (angle from the positive z-axis). r is now the straight-line distance from the origin.

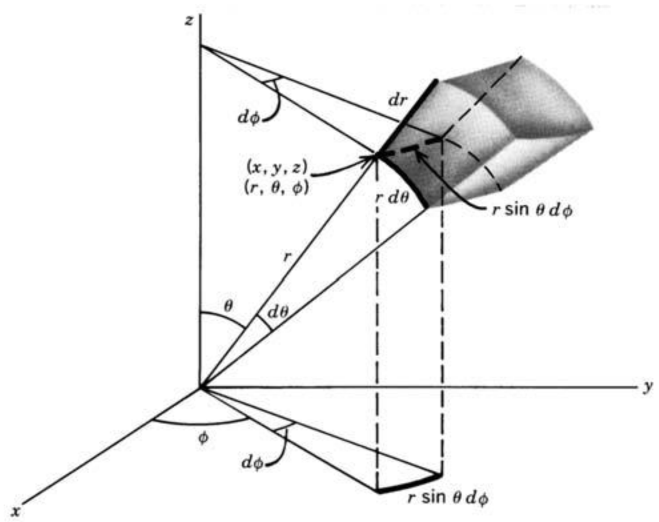
In terms of Cartesian coordinates:

$$x = r\cos\phi\sin\theta$$
$$y = r\sin\phi\sin\theta$$
$$z = r\cos\theta$$

Where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin.
ϕ is the polar angle: the angle with the x-axis, and ranges from 0 to 2π as explained above.
θ is the **azimuthal angle:** the angle with the z-axis. It ranges from 0 to π.

The volume element is the volume of the shaded region in the diagram: $dV = (r\,d\theta)(r\sin\theta\ d\phi)(dr)$. By integrating over dV, we get the volume of a sphere or spherical shell of a certain thickness.

The possible area elements are the area of any of the sides of the volume element:
$dA = (r\,d\theta)(r\sin\theta\ d\phi)$ or $(r\sin\theta\ d\phi)(dr)$ or $(r\,d\theta)(dr)$



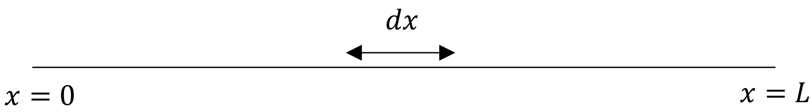
For more details on how the area and volume elements can be used to sum over an area or volume respectively, see Appendix 1.

2.3 Charges in a line, on a surface, and in a volume

Let's apply the ideas in Section 2.1 and 2.2 to charge distributions.

The **linear charge density,** or charge per unit length, is usually denoted with the symbol λ. If charge Q is uniformly distributed along length L, we have:

$$\lambda = \frac{Q}{L} \Longrightarrow Q = \lambda L$$



If the charge is not distributed uniformly, but λ is a function of x (i.e. $\lambda(x)$), let's say that in a small length dx the amount of charge is dQ. Then the total charge is:

$$Q = \int_0^L \lambda(x)\, dx \quad \text{or} \quad \lambda = \frac{dQ}{dx}$$

Lecture Example 1

Find the total charge in the following distributions:

(a) A line of charge of length L, with a uniform linear charge density λ_0

(b) A line of charge of length L, with linear charge density $\lambda = ax$, where a is a positive constant and x ranges from 0 to L.

Solution

(a) Since the charges are uniformly distributed, $Q = \lambda_0 L$

(b)

$$Q = \int_0^L \lambda\, dx = \int_0^L ax\, dx = \left[\frac{ax^2}{2}\right]_0^L = \frac{aL^2}{2}$$

The **surface charge density,** or charge per unit area, is usually denoted with the symbol σ. If charge Q is uniformly distributed across area A, we have:

$$\sigma = \frac{Q}{A} \Longrightarrow Q = \sigma A$$

If the charge is not uniformly distributed, we will need to cut up the area into small area elements dA. Each area element will have charge dQ. By summing up (integrating), we will get the total charge:

$$Q = \int dQ = \int \sigma\, dA \quad \text{or} \quad \sigma = \frac{dQ}{dA}$$

dA will then be replaced with the appropriate area element in Section 2.2, resulting in a double integral (see Appendix 1 for more details). Alternatively, if σ only depends on one variable (e.g. x), we can express dA in terms of dx, resulting in a single integral.

Lecture Example 2

Find the total charge in the following distributions:

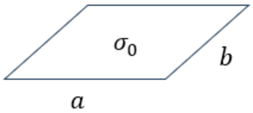


Fig 1

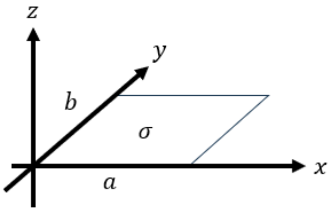


Fig 2

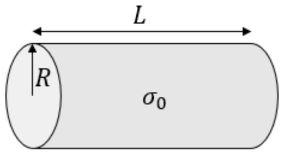


Fig 3

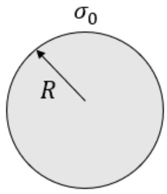


Fig 4

(a) A thin rectangular sheet of charge of sides a and b, with a uniform surface charge density σ_0 (Fig 1)
(b) A thin rectangular sheet of charge of sides a (aligned to the x-axis) and b (aligned to the y-axis), and surface charge density $\sigma = kx$ (Fig 2)
(c) A thin cylindrical shell of charge of radius R and length L, with a uniform surface charge density σ_0 (Fig 3)
(d) A thin spherical shell of charge of radius R, with a uniform surface charge density σ_0 (Fig 4)

Solution

(a) Since the charges are uniformly distributed, $Q = \sigma \times A = \sigma_0 ab$

(b)

Method 1: Double integral

$$Q = \int \sigma \, dA$$

Since area element $dA = dx\,dy$,

$$Q = \int_{y=0}^{y=b} \int_{x=0}^{x=a} kx \, dx \, dy$$
$$= \int_{x=0}^{x=a} bkx \, dx$$
$$= \frac{1}{2} ka^2 b$$

Double integrals can be evalulated in any order.

Method 2: Single integral (*preferred)

$$Q = \int \sigma \, dA$$

Cut the rectangle into thin strips of length b and width dx, so that $dA = b\,dx$. Within each strip, charge is uniformly distributed (but each strip has a different amount of charge as σ increases with x).

$$\therefore Q = \int_0^a \sigma \, (b \, dx) = \int_0^a bkx \, dx = \frac{1}{2} ka^2 b$$

* Note that any double integral (method 1) in H3 Physics can be reduced into a single integral (method 2) by slicing the area into suitable strips. Cambridge likes to set questions like this, and will typically guide you in selecting a suitable way to slice the region. For example, see question D1(b) in part A of the tutorial.

(c) Since the charges are uniformly distributed, $Q = \sigma \times A = \sigma_0 (2\pi RL)$

(d) Since the charges are uniformly distributed, $Q = \sigma \times A = 4\sigma_0 \pi R^2$

Surface area of cylinder $= 2\pi rh$

Surface area of sphere $= 4\pi r^2$

The **volume charge density,** or charge per unit volume, is usually denoted with the symbol ρ. If charge Q is uniformly distributed across volume V, we have:

$$\rho = \frac{Q}{V} \Rightarrow Q = \rho V$$

If the charge is not uniformly distributed, we will need to cut up the volume into small volume elements dV. Each volume element will have charge dQ. By summing up (integrating), we will get the total charge:

$$Q = \int dQ = \int \rho \, dV \quad \text{or} \quad \rho = \frac{dQ}{dV}$$

dV will then be replaced with the appropriate volume element in Section 2.2, resulting in a triple integral (see Appendix 1). Alternatively, if ρ only depends on one variable (e.g. x), we can express dV in terms of dx, resulting in a single integral.

Lecture Example 3

Find the total charge in the following distributions:

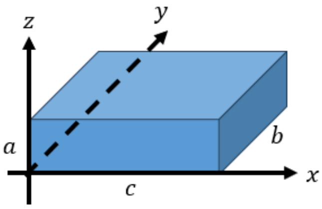


Fig 1

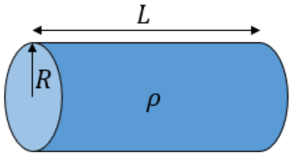


Fig 2

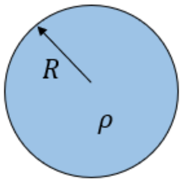


Fig 3

(a) A cuboid of sides a, b, c with a uniform volume charge density ρ_0.

(b) A cuboid of sides a, b, c with side c aligned to the x-axis, with volume charge density $\rho = kx$, where x ranges from 0 to c. (Fig 1)

(c) A cylinder of radius R and length L, with uniform volume charge density ρ_0

(d) A cylinder of radius R and length L, with volume charge density $\rho = kr$, where k is some constant. (Fig 2)

(e) A sphere of radius R with uniform volume charge density ρ_0

(f) A sphere of radius R with volume charge density $\rho = kr$, where k is some constant. (Fig 3)

Solution

(a) Since the charges are uniformly distributed, $Q = \rho \times V = \rho_0 abc$

(b)

Method 1: Triple integral

$$Q = \int \rho\, dV$$

Since volume element $dV = dx\,dy\,dz$,

$$Q = \int_{x=0}^{x=c} \int_{y=0}^{y=b} \int_{z=0}^{z=a} kx\, dx\, dy\, dz$$

$$= \int_{x=0}^{x=c} \int_{y=0}^{y=b} akx\, dx\, dy$$

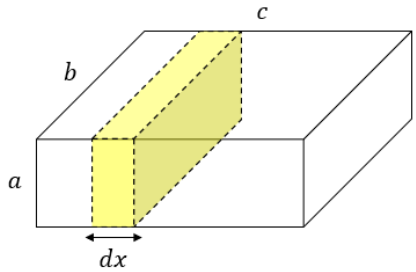
$$= \int_{x=0}^{x=c} abkx\, dx$$

$$= \frac{1}{2} abkc^2$$

Method 2: Single integral (*preferred)

$$Q = \int \rho\, dV$$

Cut the cuboid into thin slices of thickness dx, so that $dV = ab\,dx$. Within each slice, charge is uniformly distributed (but each slice has a different amount of charge as ρ increases with x).



$$\therefore Q = \int_0^c \rho\,(ab\,dx) = \int_0^c abkx\, dx = \frac{1}{2} abkc^2$$

(c) Since the charges are uniformly distributed, $Q = \rho \times V = \rho_0(\pi R^2 L)$

Volume of cylinder $= \pi r^2 h$

(d)

Method 1: Triple integral

$$Q = \int \rho\, dV$$

Since volume element $dV = r\,d\theta\,dr\,dz$,

$$Q = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} \int_{z=0}^{z=L} kr\, r\, d\theta\, dr\, dz$$

$$= \int_{r=0}^{r=R} \int_{z=0}^{z=L} 2\pi k r^2\, dr\, dz$$

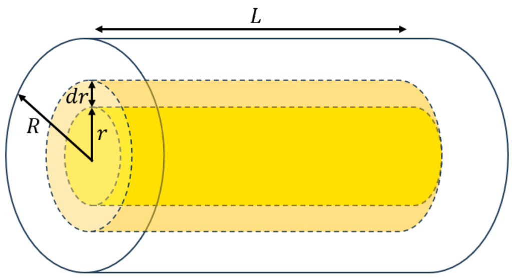
$$= \int_{r=0}^{R=a} 2\pi k L r^2\, dr$$

$$= \frac{2}{3} \pi k L R^3$$

Method 2: Single integral (*preferred)

$$Q = \int \rho\, dV$$

Cut the cylinder into thin cylindrical shells of radius r and thickness dr:



Then $dV = \pi(r+dr)^2L - \pi r^2 L = \pi L[(r+dr)^2 - r^2] = \pi L[r^2 + 2r\,dr + (dr)^2 - r^2] = 2\pi r L\,dr$ (we drop the $(dr)^2$ term as it is negligible). Within each shell, charge is uniformly distributed (but each shell has a different amount of charge as ρ increases with r).

$$\therefore Q = \int_0^a \rho\,(2\pi r L\,dr) = \int_0^a 2\pi k L r^2\,dr = \frac{2}{3}\pi k L R^3$$

(e) Since the charges are uniformly distributed, $Q = \rho \times V = \rho_0\left(\frac{4}{3}\pi R^3\right)$

Volume of sphere $= \frac{4}{3}\pi r^3$

(f)

Method 1: Triple integral

$$Q = \int \rho\,dV$$

Since volume element $dV = (r\,d\theta)(r\sin\theta\,\,d\phi)(dr)$,

$$Q = \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=R}\int_{\phi=0}^{\phi=2\pi} kr\,r^2\sin\theta\,\,d\theta\,dr\,d\phi$$

$$= \int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=R} 2\pi k r^3\sin\theta\,\,d\theta\,dr$$

$$= \int_{r=0}^{r=R} 4\pi k r^3\,dr = \pi k R^4$$

(Note: $\int_0^\pi \sin\theta\,\,d\theta = 2$)

Method 2: Single integral (*preferred)

$$Q = \int \rho\,dV$$

Cut the sphere into thin spherical shells of radius r and thickness dr, so that $dV = \frac{4}{3}\pi(r+dr)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi(r^3 + 3r^2(dr) + 3r(dr)^2 + (dr)^3 - r^3) = 4\pi r^2\,dr$. (The $(dr)^2$ and $(dr)^3$ terms are negligible). Within each shell, charge is uniformly distributed (but each shell has a different amount of charge as ρ increases with r).

$$\therefore Q = \int_0^R \rho\,(4\pi r^2\,dr) = \int_0^R kr\,(4\pi r^2)dr = \pi k R^4$$

For cylindrical and spherical systems, the charge distribution will usually be radially symmetric, so you should slice it in the way this lecture example did.

3 Continuous Charge Distributions

3.1 Electric Field Strength

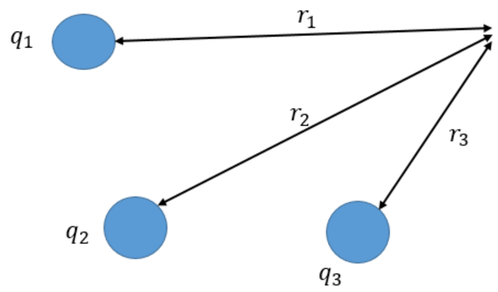
In H2 Physics, we learnt that the electric force between two point charges Q and q is given by Coulomb's Law,

$$F = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}$$

and that the electric field of a point charge Q is therefore:

$$E = \frac{F}{q} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

If there are multiple charges, the net electric field at a particular point is given by the **vector sum** of the electric fields due to each charge at the point. This is sometimes called the **Principle of Superposition** for electric charges.[4]



For N charges, where the ith charge has magnitude q_i and has a distance r_i from the point of interest,

$$E(r) = \sum_{i=1}^{N} E_i(r_i) = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0}\frac{q_i}{r_i^2}$$

Often, the distance between charges in a group of charges is so small compared to the point of interest ($\boldsymbol{r}$) that the charges appear "smeared out" or continuous. **To calculate the electric field of a continuous charge distribution, we need to replace the sum** ($\sum$) **with an integral** ($\int$), which also means making the following replacements:

- $q_i \rightarrow dq$
- $r_i \rightarrow r$

Thus the electric field of a continuous charge distribution is:

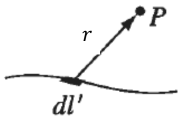
$$\vec{E}(r) = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$

Which is basically saying that the electric field strength at a point is found by summing up the contributions from the little charges dq. Note that the variable to be integrated is q, not r! This doesn't usually help us, so to proceed further we'll try to express q in terms of r or some coordinate of our coordinate system (e.g. x, y, z or r, θ, ϕ), and integrate over that instead.

[4] Not to be confused with the Principle of Superposition you learnt about in the context of waves. Although the idea is the same, there is no "diffraction" or "interference" of a *static* electric field.

Once again, here are three common ways to express q in terms of other coordinates:

For a line of charge:

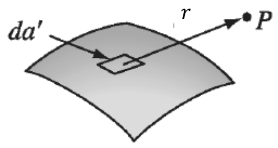


(b) Line charge, λ

Let λ be the charge per unit length

$$\boxed{dq = \lambda\, d\ell}$$

For a surface of charge:

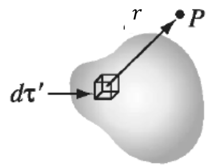


(c) Surface charge, σ

Let σ be the charge per unit area

$$\boxed{dq = \sigma\, dA}$$

For a volume of charge:



(d) Volume charge, ρ

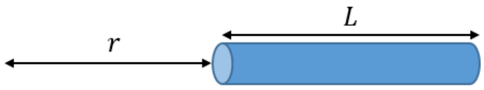
Let ρ be the charge per unit volume

$$\boxed{dq = \rho\, dV}$$

From here, express $d\ell, dA, dV$ in terms of dimensions from your coordinate system (e.g. using the area or volume elements), and then integrate!

<u>Lecture Example 4</u>

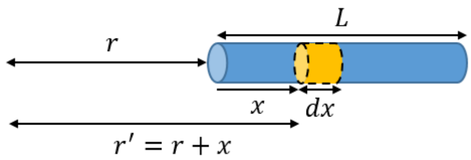
The figure shows a positively charged rod with a total charge of Q. The charges are uniformly distributed across the rod.



a) Calculate the electric field strength at a distance r from the end of the rod.
b) Show that at sufficiently large values of r, the charged rod behaves like a point charge.

<u>Solution:</u>

Choose the coordinate origin to be the left end of the rod, and let x be the distance along the rod from the origin:



(a) The magnitude of the electric field dE due to the small charge dq is given by:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{(r+x)^2}$$

Note: the distance from dq to the point of interest is $r' = r + x$

Let the linear charge density be λ, where $\lambda = Q/L$. Then $dq = \lambda\, dx$.

Note: the electric field dE due to all the dq all have the same direction (leftwards), so we can simply add them up without worrying about direction.

$$\begin{aligned}
\therefore E(r) &= \int_0^L \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{(r+x)^2} \\
&= \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{1}{(r+x)^2}\, dx \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{r+x}\right]_0^L \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{1}{r+L}+\frac{1}{L}\right) \\
&= \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{L}{r(r+L)}\right) \\
&= \frac{1}{4\pi\varepsilon_0}\frac{Q}{r(r+L)} \quad \text{where } E \text{ points to the left}
\end{aligned}$$

Note: λ is not in the question, so we must replace it with $\lambda = Q/L$

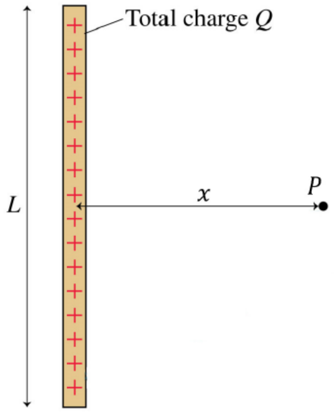
(b) When $r \gg L$, $r + L \approx r$

$$E(r) \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

(Note: anything looks like a point if you get sufficiently far away from it. This can be a quick way to check that your answer is correct.)

Lecture Example 5

The figure shows a positively charged rod aligned to the y-axis. The charges are uniformly distributed across the rod with linear charge density λ.

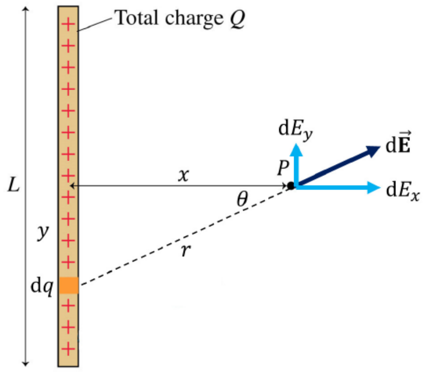


(a) Calculate the electric field strength at point P at a distance x from the mid-point of the rod

(b) Hence, calculate the electric field strength at a distance r from a *long wire* also with linear charge density λ.

Solution:

The coordinate origin is the middle of the rod:



Note: E is a vector quantity! Need to account for direction! Often there is a symmetry we can exploit to simplify our calculations.

(a) The magnitude of the electric field dE due to the small charge dq is given by:

$$dE = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2+y^2}$$

Note: the distance from dq to the point of interest is $r = \sqrt{x^2+y^2}$

Since the rod is symmetric about the x-axis, all the y-components of dE will cancel out, so we only need to sum up the x-components of dE:

$$dE_x = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2+y^2}\cos\theta = \frac{1}{4\pi\varepsilon_0}\frac{dq}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}$$

Given $dq = \lambda\, dy$,

$$E(x) = \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dy}{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}}$$

Note: we are integrating over y so x is a constant.

$$= \frac{\lambda\, x}{4\pi\varepsilon_0}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{1}{(x^2+y^2)^{3/2}}\, dy$$

$$= \frac{\lambda\, x}{4\pi\varepsilon_0}\left[\frac{y}{x^2\sqrt{x^2+y^2}}\right]_{y=-\frac{L}{2}}^{y=\frac{L}{2}}$$

This integral took some effort! In Section 4 we will learn a simpler method to solve this problem ☺

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{x\sqrt{4x^2+L^2}}$$ where E points to the right

(b) Rearranging our answer in (a) by factoring L out,

$$E(x) = \frac{\lambda}{4\pi\varepsilon_0}\frac{2L}{Lx\sqrt{\frac{4x^2}{L^2}+1}} = \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{x\sqrt{4\left(\frac{x}{L}\right)^2+1}}$$

For a long wire, $L \to \infty$ or $x \ll L \Longrightarrow \frac{x}{L} \approx 0$.

$$\therefore E(x) \approx \frac{\lambda}{4\pi\varepsilon_0}\frac{2}{x} = \frac{\lambda}{2\pi\varepsilon_0 x}$$

(This formula is given in the formula sheet of the A-level exam, and can be used in solving related problems)

So, the electric field strength at a distance r from an infinitely long wire is:

$$\boxed{E = \frac{\lambda}{2\pi\varepsilon_0 r}}$$

It turns out that the electric field strength due to a large (infinite) flat plane of charge surface charge density σ is:

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}$$

Note that:

- The field strength is independent of the distance from the plane!
- This only applies to *large* or *infinite* flat planes – in other words, the edges are very far away from the point of consideration, relative to the nearest point on the plane.

This formula will also be given in the formula sheet of the A-level exam. We will derive this formula later, in the tutorial, and again in Section 4 (on Gauss' Law).

To summarise, to sum up the electric field due to a continuous charge distribution:

1. Set up a coordinate system
2. Divide the charge into small segments dq
   a. Line of charge, $dq = \lambda\, d\ell$
   b. Surface of charge, $dq = \sigma\, dA$
   c. Volume of charge, $dq = \rho\, dV$
3. Express $d\ell, dA, dV$ in terms of variables in your coordinate system
4. Write down the electric field dE due to dq
5. Perform the integral: $E = \int dE$

3.2 Electric Potential

In H2 Physics, we learnt the following relationship between the electric field strength E and the electric potential V:

$$E = -\frac{dV}{dr}, \qquad \Delta V = -\int E\, dr$$

This offers us another way to calculate the magnitude of E: sum the *potentials* of a continuous charge distribution, then differentiate to find the field strength.

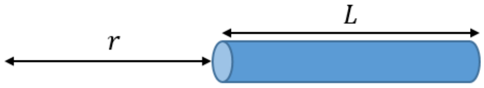
Alternatively, this also offers us another way to calculate V: if the electric field strength E is easily determined, we can find the electric potential by integrating. If V_a and V_b are the potentials at $r = a$ and $r = b$, we get:

$$\Delta V = V_b - V_a = -\int_a^b E\, dr$$

By choosing one of the points (e.g. $r = a$) as a reference point (e.g. $V_a = 0$), we can determine the potential at the other.

Lecture Example 6

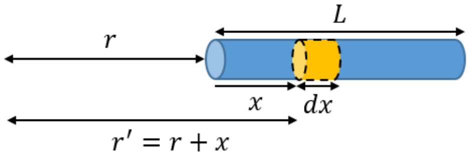
The figure shows a positively charged rod with a uniform charge per unit length λ. It has a total charge of Q.



a) Calculate the electric potential V at a distance r from the end of the rod.
b) Find the gradient of V, and hence show that you get the same result for the electric field strength E as Lecture Example 4.
c) Now suppose $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r(r+L)}$ calculate V and check your answer in (a)

Solution:

Choose the coordinate origin to be the left end of the rod, and let x be the distance along the rod from the origin:



(a) The magnitude of the electric potential dV due to the small charge dq is given by:

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r+x}$$

Since $dq = \lambda\, dx$,

$$\begin{aligned}
\therefore V &= \int_0^L \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx}{r+x} \\
&= \frac{\lambda}{4\pi\varepsilon_0}\int_0^L \frac{1}{r+x}\, dx \\
&= \frac{\lambda}{4\pi\varepsilon_0}[\ln(r+x)]_0^L \\
&= \frac{\lambda}{4\pi\varepsilon_0}(\ln(r+L) - \ln r) \\
&= \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{r+L}{r}\right)
\end{aligned}$$

(b)

$$\frac{dV}{dr} = \frac{\lambda}{4\pi\varepsilon_0}\left(-\frac{L}{r(r+L)}\right)$$

$$-\frac{dV}{dr} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda L}{r(r+L)} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r(r+L)} = E$$

Which is the result we got in part (a) of Lecture Example 4.

(c) Given $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r(r+L)}$ we can find the potential V_r at an arbitrary point r with respect to a reference point a:

$$V_r - V_a = -\int_a^r E\, dr' = -\int_{r'=a}^{r'=r} \frac{1}{4\pi\varepsilon_0}\frac{Q}{r'(r'+L)}dr' = -\frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{L}\ln r' - \frac{1}{L}\ln(r'+L)\right]_{r'=a}^{r'=r}$$

$$= \frac{Q}{4\pi\varepsilon_0 L}\left[\ln\frac{r'+L}{r'}\right]_{r'=a}^{r'=r} = \frac{\lambda}{4\pi\varepsilon_0}\left(\ln\frac{r+L}{r} - \ln\frac{a+L}{a}\right)$$

If we pick a to be infinity, $\lim_{a\to\infty} \ln\frac{a+L}{a} = \lim_{a\to\infty} \ln\left(1+\frac{L}{a}\right) = 0$

$$V_r - V_\infty = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r+L}{r} - 0$$

By comparing terms, $V_\infty = 0$ as we expect. Thus we are left with:

$$V = \frac{\lambda}{4\pi\varepsilon_0}\ln\frac{r+L}{r}$$

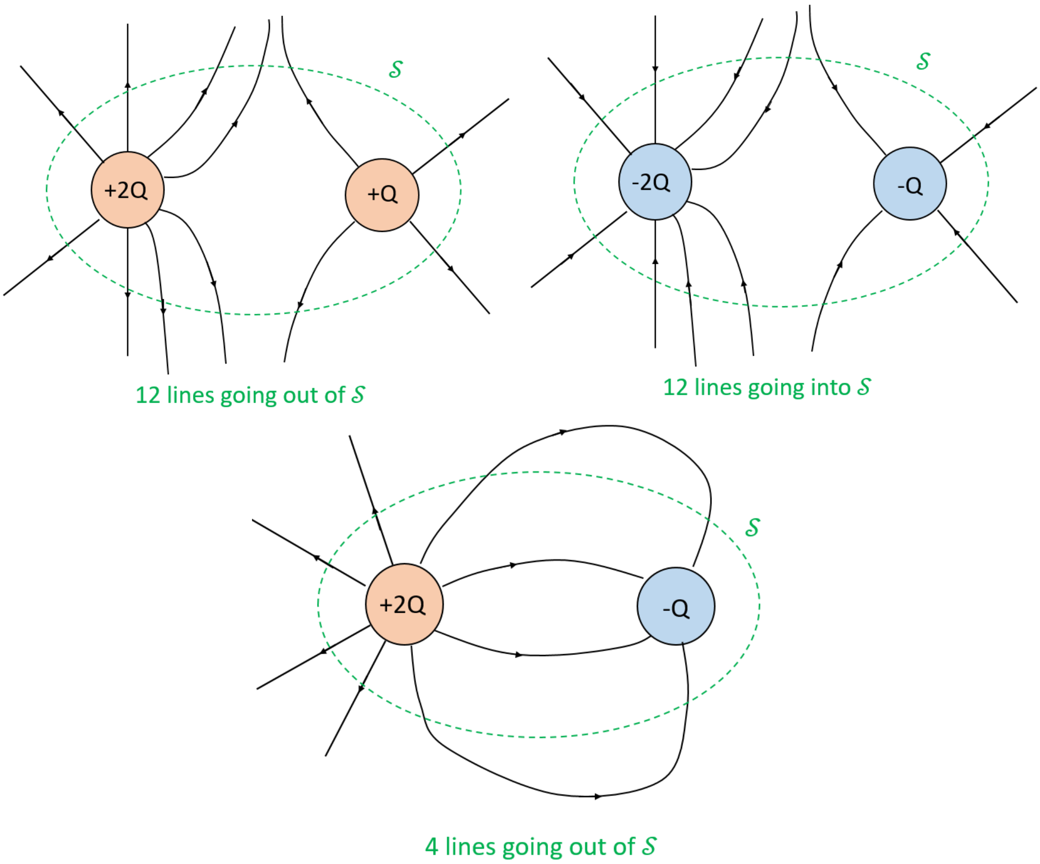
Now, you are ready to try part A of the tutorial.

4 Electric Flux and Gauss' Law

4.1 From Electric Flux to Gauss' Law

So far, given a system of charges and information about where they are, we have performed calculations to determine the electric field a certain distance away. Now, what about the reverse? How would you determine the amount of charge in an enclosed volume, just by measuring the electric field?

One handwaving solution would be to "count the field lines escaping or entering the volume." Recall from H2 Physics that the number of field lines is proportional to magnitude of charge, so this will give you an idea of the total magnitude of charge in the volume. Since field lines point from positive charges to negative charges, you'll also have a rough idea of the polarity of charges in the volume.



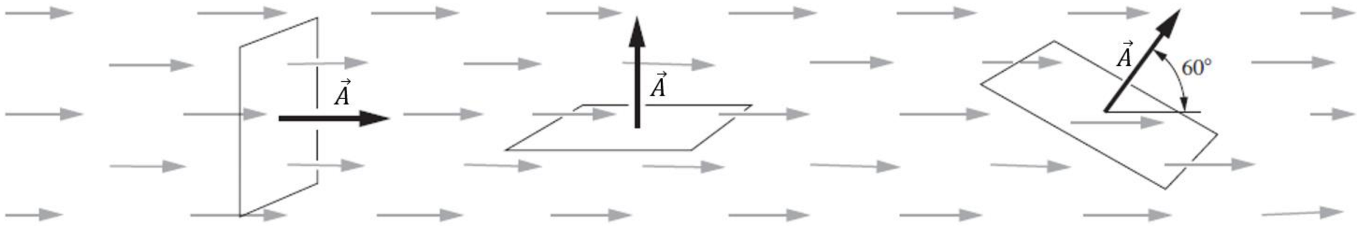
In the diagrams above, a surface $\mathcal{S}$ encloses a volume containing charges. We count the number of field lines entering or leaving the surface $\mathcal{S}$, and find that it is proportional to the net charge inside $\mathcal{S}$.

However, field lines are imaginary constructs, right? How can we do it in "real life"?

Define the **electric flux**[5] through a plane area as the (dot) product of the electric field strength and the area. The area vector points normal to the surface:

$$\boxed{\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta}$$

[5] The word "flux" comes from the Latin "fluxus" which means "flow". You could imagine it as a flow of water, for example. An early use of the concept of flux in physics was in describing how heat flows through a material. See https://en.wikipedia.org/wiki/Flux



$\Phi = EA$ $\Phi = 0$ $\Phi = EA\cos 60°$

Now, enclose the charges you're interested in with a surface, and calculate the total flux through the surface. As we've seen, this is proportional to the net enclosed charge Q_{enc} in the volume! Therefore:

$$\Phi = \int \vec{E} \cdot d\vec{A} \propto Q_{enc}$$

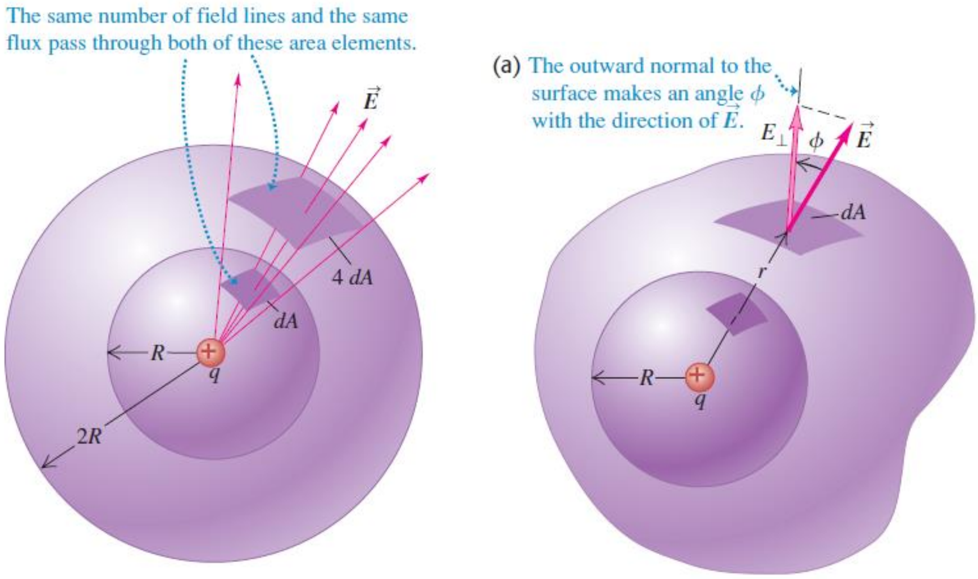
To be precise, we should write the integral as a double integral ($\iint$) because we are integrating over an area[6], and we add a circle in the middle ($\oiint$) to be explicit that this is a closed surface (see Appendix 1).

Gauss' Law states that the net electric flux through any closed surface is equal to the net electric charge inside the surface divided by ε_0. The closed surface is called a Gaussian surface.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

- ε_0 is the electric permittivity of free space, given in the formula sheet as $8.85 \times 10^{-12}\ \mathrm{F\ m^{-1}}$.
- Gauss' Law is always true, but it may not always be useful – it is only useful when there is symmetry that can be exploited (e.g. spherical symmetry, cylindrical symmetry, etc). This is pretty much always the case in H3 Physics.

The interesting thing is that no matter how you draw your Gaussian surface, the total flux will be the same! In the diagrams below, all the flux that passes through the inner surface also passes through the outer surface, and vice versa.

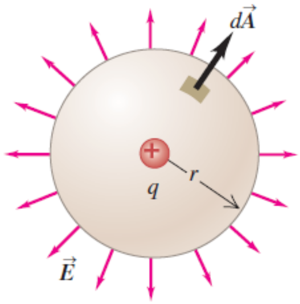


Thus we are free to choose any shape for our surface. If we pick nice surfaces such that the flux is always parallel or perpendicular to the area, the integral reduces to $E \cdot A$ and we no longer need to do any actual integration! (Hence *symmetry* is important!)

[6] See Appendix 1 for more details on double integrals.

3.2 Applying Gauss' Law

But does this really work? Let's check. Consider a point charge q at the centre of a spherical Gaussian surface $\mathcal{S}$ of radius r:



Gauss' Law says that:

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Remember that the left hand side of Gauss' Law is the electric flux. The electric flux at every point on $\mathcal{S}$ is perpendicular to $\mathcal{S}$ (so $\vec{E} \cdot d\vec{A} = E\,dA$) and has the same magnitude (due to spherical symmetry of the field due to point charges). Thus, E is the same at every area element dA, and it can be factored out from the integral:

$$\oiint_{\mathcal{S}} \vec{E} \cdot d\vec{A} = \oiint_{\mathcal{S}} E\,dA = E \oiint_{\mathcal{S}} dA = EA$$

The magnitude of the electric field due to a point charge q is given by $E = \frac{q}{4\pi\varepsilon_0 r^2}$ and the area of a sphere is $4\pi r^2$:

$$EA = \left(\frac{q}{4\pi\varepsilon_0 r^2}\right)(4\pi r^2) = \frac{q}{\varepsilon_0}$$

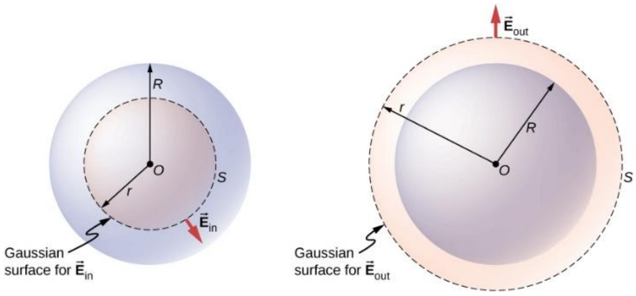
Which is the right hand side of Gauss' Law! So Gauss' Law does indeed work.

Lecture Example 7

Calculate the electric field strength E at a distance r from the centre of a thin spherical shell of charge of radius R, with uniform surface charge density σ. Sketch a graph of E against r.

Solution:

Set the coordinate origin to be the centre of the spherical shell or sphere respectively. First consider the case where $r > R$.



Draw a spherical Gaussian surface $\mathcal{S}$ of radius $r > R$ enclosing the charged shell, centred at the origin.

From Gauss' Law,

$$\oiint_{\mathcal{S}_1} \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

Since the flux is uniform along the surface (spherical symmetry),

$$\oiint_{\mathcal{S}_1} \vec{E} \cdot d\vec{A} = E \cdot A = E(4\pi r^2)$$

Given surface charge density σ,

$$Q_{enc} = \int_{shell} \sigma\, dA = \sigma \cdot A = \sigma(4\pi R^2)$$

Because the distribution is spherically symmetric, we draw a spherical Gaussian surface.

Subbing these results into Gauss' Law,

$$E(4\pi r^2) = \frac{\sigma(4\pi R^2)}{\varepsilon_0}$$

$$E = \frac{\sigma R^2}{\varepsilon_0 r^2}, \qquad r > R$$

Note: our answer gives $E \propto 1/r^2$, which is what we'd expect for a spherically symmetric charge distribution.

Now, consider the case where $r < R$. Draw a spherical Gaussian surface $\mathcal{S}_2$ of radius $r < R$ centred at the origin. Since $Q_{enc} = 0$,

$$\oiint_{\mathcal{S}_2} \vec{E} \cdot d\vec{A} = 0$$

Since the flux is uniform along the surface (spherical symmetry),

$$\oiint_{\mathcal{S}_2} \vec{E} \cdot d\vec{A} = E \cdot A$$

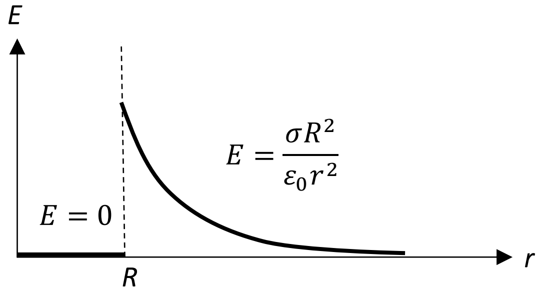
$$\therefore E \cdot A = 0$$

$$\therefore E = 0$$

This is essentially the same situation as the field inside a charged conductor

$$\therefore E = \begin{cases} \dfrac{\sigma R^2}{\varepsilon_0 r^2}, & r > R \\ 0, & \text{otherwise} \end{cases}$$

Sketch of E against r:

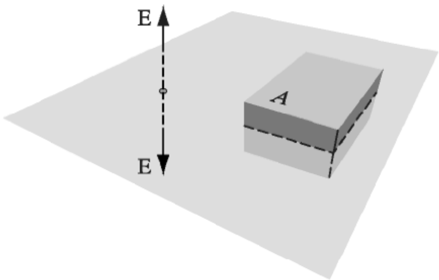


Lecture Example 8

Calculate the electric field strength at a distance d from an infinitely large plane of charge with uniform surface charge density σ.

Solution:

Draw a Gaussian "pillbox" (cuboid) that sandwiches the plane, with upper and lower surface of area A.



Because the distribution is rectangularly symmetric, we draw a cuboid Gaussian surface.

By symmetry, the electric field points vertically away from the plane (upward for points above, downward for points below). So the flux through the sides is zero, and flux through the top and bottom surfaces is each EA.

$$\therefore \oiint_S \vec{E} \cdot d\vec{A} = 2EA$$

Thus, applying Gauss' Law,

$$2EA = \frac{\sigma A}{\varepsilon_0}$$

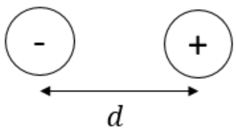
$$E = \frac{\sigma}{2\varepsilon_0}$$

which is the formula that was given earlier.

(This result is given in the formula sheet of the A-level exam, and can be used in solving related problems)

$$\therefore E = \begin{cases} \dfrac{\sigma}{2\varepsilon_0} & \text{above the plane} \\ -\dfrac{\sigma}{2\varepsilon_0} & \text{below the plane} \end{cases}$$

Let's consider one last example. An **electric dipole** is a system of two opposite charges of *equal magnitude* with some *small* separation (d):



Practical examples of such systems are polar molecules like water molecules, in which the side with the hydrogen atoms is slightly positive and the other side is slightly negative.

What would Gauss' Law tell you about the flux from a dipole? Since the net charge of the system of both charges is 0, $Q_{enc} = 0$ so we'd get:

$$\oiint \vec{E} \cdot d\vec{A} = 0$$

We get the same result for quadrupoles, octupoles, and so on. We'll study dipoles in more detail in Section 6.

Now, you are ready to try part B of the tutorial.

5 Magnetic Flux and Ampere's Law

5.1 Magnetic Flux

Just as we defined flux for electric fields, we can do the same for magnetic fields: the **magnetic flux** through a plane area is the (dot) product of the magnetic flux density[7] ($\vec{B}$) and the area:

$$\boxed{\Phi_B = \vec{B} \cdot \vec{A} = BA\cos\theta}$$

Then by drawing a Gaussian surface we could determine the net "magnetic charge" in the enclosed volume. We could define a magnetic North pole as being "positive" and a magnetic South pole as being "negative" (or vice versa), and carry on with our physics.

But magnets always come with a North pole *and* a South pole of equal magnitude[8] – in other words, any magnet is a dipole. So **Gauss' Law for magnetism** is just:

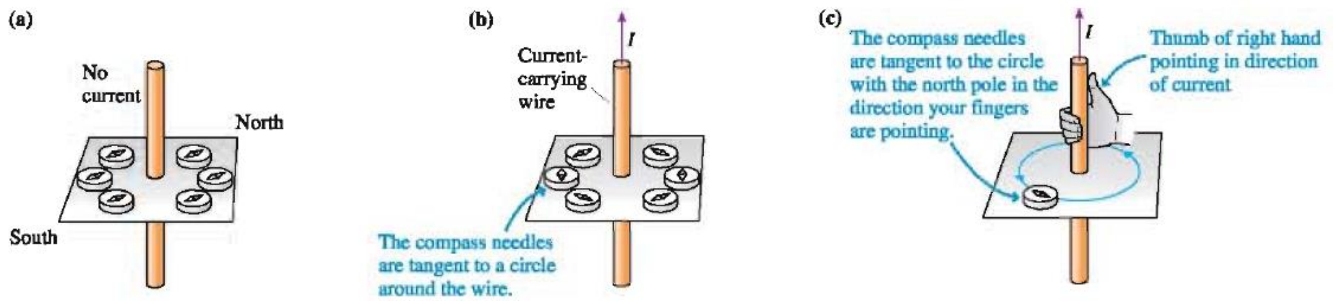
$$\boxed{\oiint \vec{B} \cdot d\vec{A} = 0}$$

How boring! Fortunately, there is something more interesting that we can do with magnetic fields.

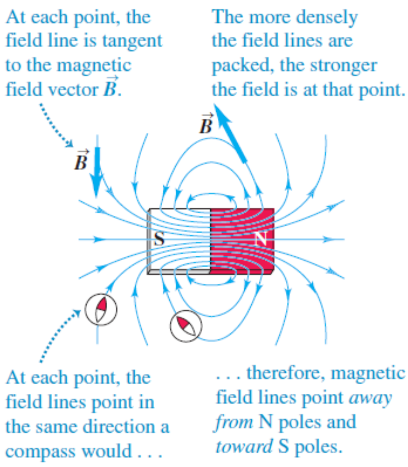
5.2 Line integrals and Ampere's Law

In 1819, the Danish physicist Hans Christian Oersted was conducting a classroom demonstration when he noticed that the electric current flowing in a wire was causing a nearby compass to deflect – in effect, <u>electric currents set up magnetic fields.</u> In the diagrams below, the compasses give the direction of the magnetic field.

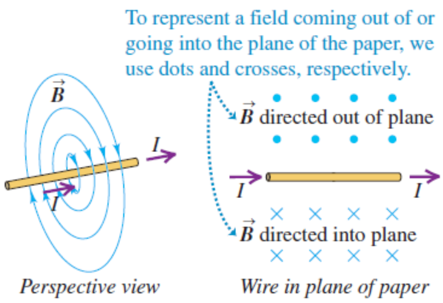
FIGURE 33.2 Response of compass needles to a current in a straight wire.



The magnetic field lines around a current-carrying wire are circular, with direction is given by the right-hand grip rule. It gives the direction that a magnetic force would act on a north pole at that point (which is why the compasses align like that). The magnetic field weakens as you get further away.

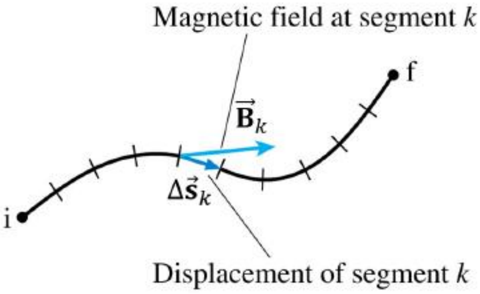


In fact, since there are no magnetic monopoles (all magnetic sources are dipoles or combinations of dipoles) magnetic field lines always curl around the source that produces them, like that of a dipole. If you follow a magnetic field line all the way, it will form a closed loop, and inside that loop is the magnetic source (in this case, a current).



[7] Strictly speaking, $\vec{B}$ is *not* the magnetic field strength, but you can think of it as such.
[8] Magnetic monopoles (North or South poles by themselves) are hypothesized to exist, but we have yet to isolate them: https://en.wikipedia.org/wiki/Magnetic_monopole

At this point we need to talk a little about **line integrals.**[9] When discussing Gauss' Law, we saw that Gauss' Law is essentially a surface integral that sums up the electric flux over the surface – to be precise, going over each little area dA and taking the dot product of $\vec{E}$ with the area vector $d\vec{A}$ normal to the area. As we've seen above, taking a similar approach with $\vec{B}$ doesn't give a useful or interesting result (Gauss' Law for magnetism).

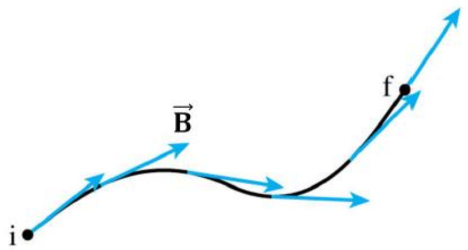


Instead, we can try adding fluxes along a line (or path). Draw a path of length ℓ and cut it into small lengths $d\ell$. The vector $d\vec{\ell}$ points in the direction of the path. For each little length $d\ell$, take the dot product of $\vec{B}$ with $d\vec{\ell}$, and sum this for the entire path. This is called a line or path integral.

$$\int \vec{B}\cdot d\vec{\ell} = \int B\, d\ell \cos\theta$$

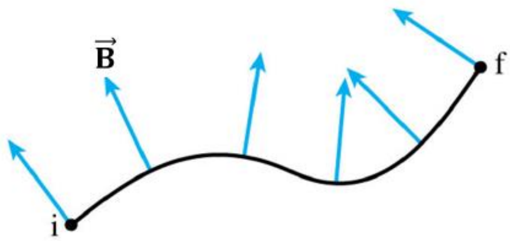
If the magnetic field is *everywhere* tangent to a line of length ℓ and has the same magnitude B at every point,

$$\int \vec{B}\cdot d\vec{\ell} = \int B\, d\ell \cos 0 = B\int d\ell = B\ell$$



If the magnetic field is *everywhere* perpendicular to the line,

$$\int \vec{B}\cdot d\vec{\ell} = \int B\, d\ell \cos 90° = 0$$



Now, since the magnetic field curls around its source, if you draw a loop and add the fluxes along it, you should get something proportional to the enclosed current:

$$\int \vec{B}\cdot d\vec{\ell} \propto I_{enc}$$

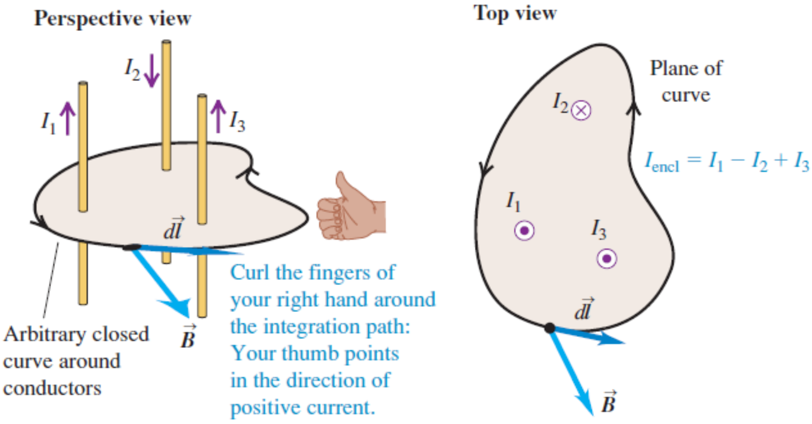
Since ℓ is a loop, to make that explicit we add a circle to the integral sign. This gives Ampere's Law:

Ampere's Law states that the net magnetic flux along any closed loop is equal to the net current enclosed inside the loop multiplied by μ_0. The closed loop is called an Amperian loop.

$$\oint \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$$

- μ_0 is the magnetic permeability of free space, given in the formula sheet as $4\pi \times 10^{-7}\ \mathrm{H\ m^{-1}}$
- Like Gauss' Law, Ampere's Law is always true, but it may not always be useful – it is only useful when there is symmetry that can be exploited. (Since in the H3 syllabus this is the only way to calculate the magnetic flux density, all questions will have symmetry that can be exploited using Ampere's Law)

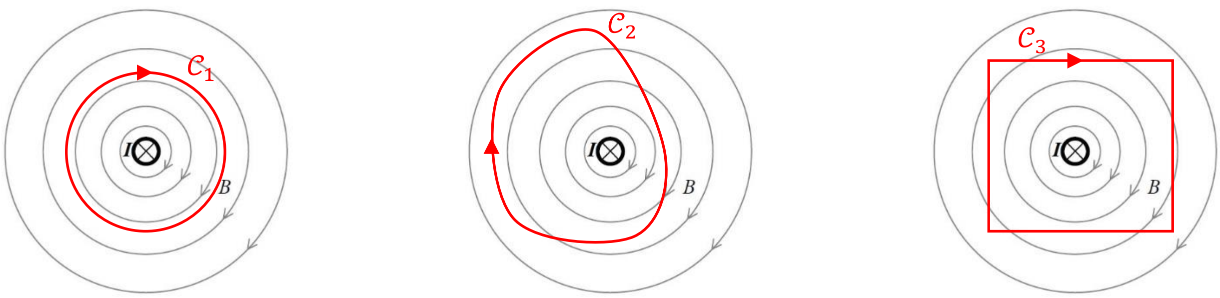
[9] Also known as a path integral or contour integral.



(A circled dot ⊙ means the current is going out of the plane of the page, a circled cross ⊗ means the current is going into the plane of the page)

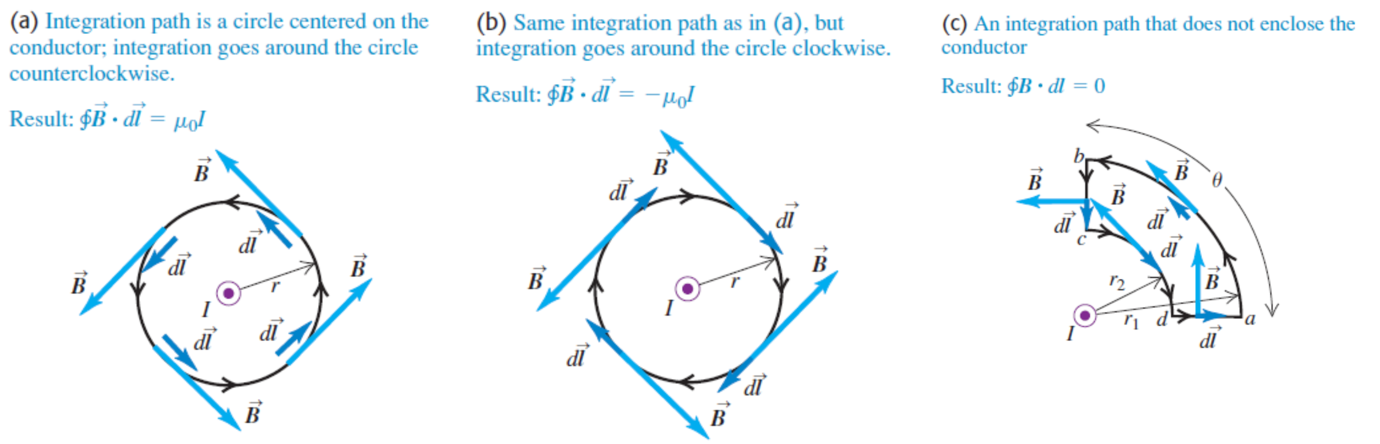
One can imagine the enclosed current (I_{enc}) as the current that "pokes through" the area bounded by the loop, regardless of the angle in which it does so – just like how you can poke a pencil through a sheet of paper from a variety of angles, but the pencil always pokes from above the paper to below the paper, or vice versa (corresponding to current in the other direction)

The loop need not be a circle – Ampere's law holds for any shape, so long as it is a loop. All of these Amperian loops will give the same result:



So, just as we did for Gauss' Law, a careful choice of Amperian loop will let $\vec{B}$ be constant *and* parallel or perpendicular to $d\vec{\ell}$ along the loop, so that we don't actually need to do any integration!

The direction that you traverse the loop matters though, since if $\vec{B}$ is in the opposite direction to $d\vec{\ell}$ the dot product will be negative. This also means that currents outside the loop will not contribute to the line integral:



5.3 Applying Ampere's Law

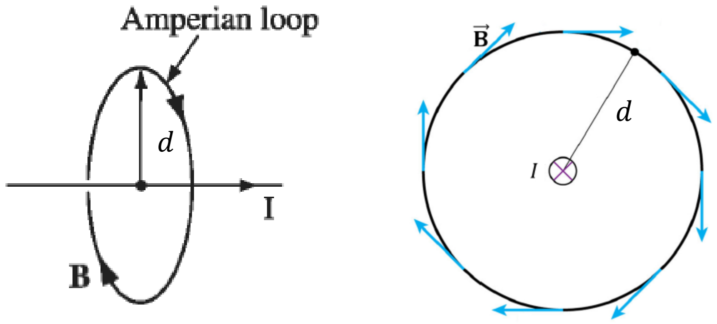
Let's verify that Ampere's Law works, by using it to calculate the magnetic field due to a current in a wire.

Lecture Example 9

A thin long straight wire carries a steady current I. Find the magnetic flux density B at a distance d from the wire.

Solution

Draw a circular Amperian loop C with radius d centred on the wire.



By symmetry, the magnitude of B is constant along the loop.

$$\oint_C \vec{B} \cdot d\vec{\ell} = \oint_C B\, d\ell = B \oint_C d\ell = B(2\pi d)$$

Therefore, Ampere's Law becomes:

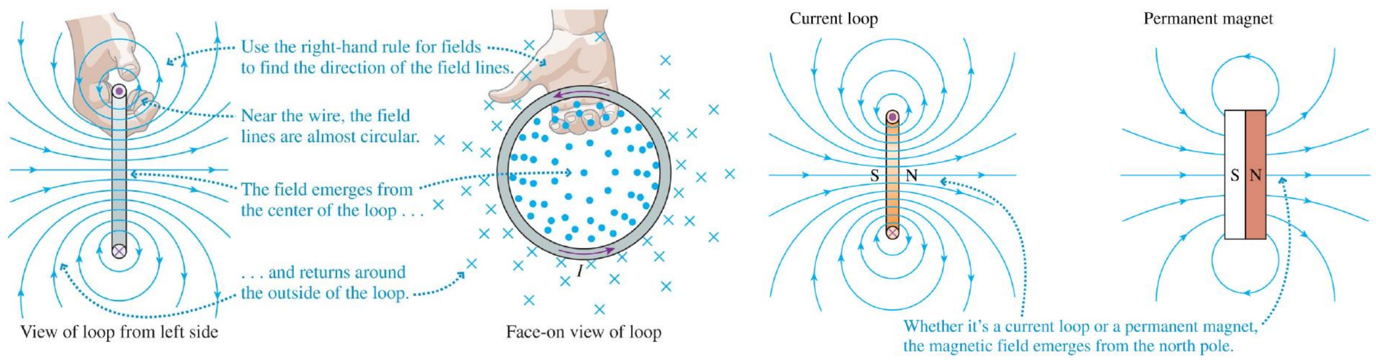
$$B(2\pi d) = \mu_0 I$$

$$\boxed{B = \frac{\mu_0 I}{2\pi d}}$$

(This result is given in the formula sheet of the A-level exam, and can be used in solving related problems)

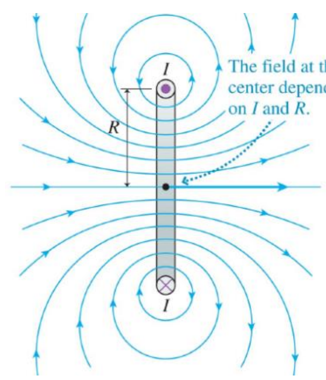
This is what we expected! (We learnt this result in H2 Physics.)

Let's look at another example. The magnetic field due to a circular loop of wire looks a bit like that of a short bar magnet:

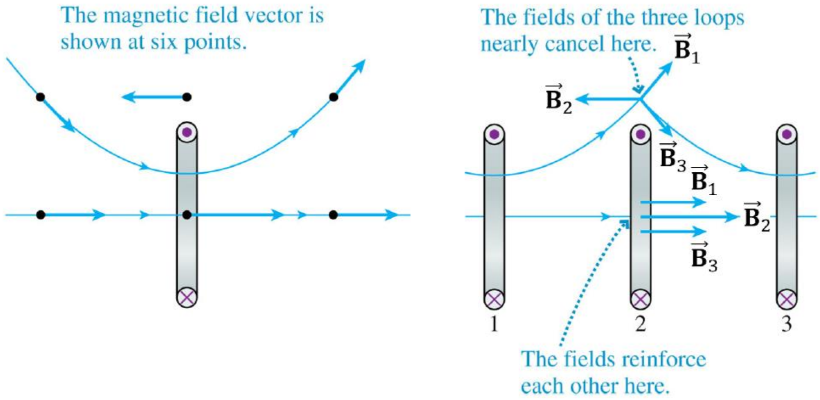


The magnetic flux density at the centre of a circular loop of radius R is:

$$B = \frac{\mu_0 I}{2R}$$



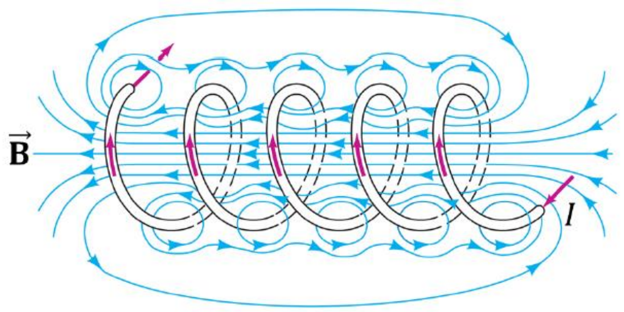
When multiple circular loops of wire are brought near each other, the magnetic field in the centre of the loops is strengthened:



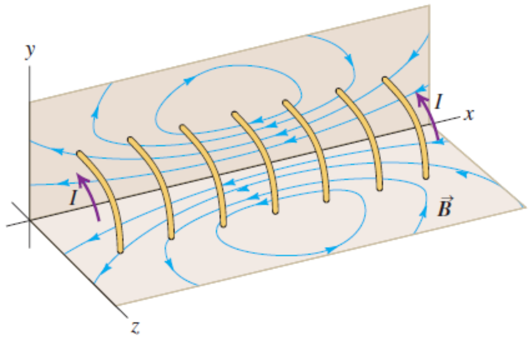
If there are N loops adjacent to each other, the magnetic flux density in the centre increases N times, so the magnetic flux density of a flat circular coil is:

$$\boxed{B = \frac{\mu_0 N I}{2R}}$$

While we can't prove this easily using Ampere's Law (because B is not conveniently constant or zero along any path we can draw), we can use if the coil is long enough. A **solenoid** is a helical coil of wire with the same current I passing through each loop of the coil. Solenoids may have hundreds or thousands of turns, so instead of counting all the turns, we characterise them by number of turns per unit length, n. Since the turns are tightly wound, we can treat them as an infinite series of circular loops, each carrying current I.



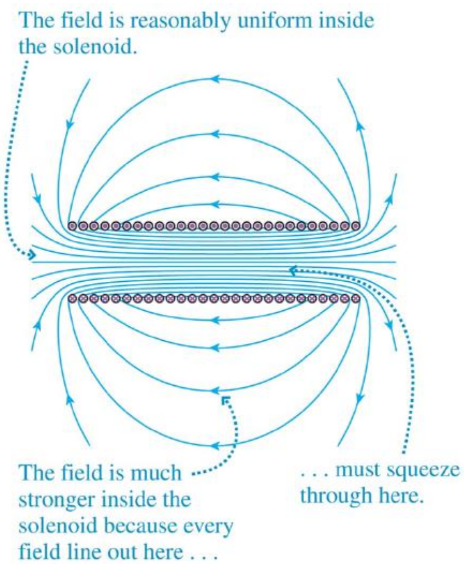
Magnetic field of a loosely-wound coil/solenoid



Magnetic field of a series of circular loops

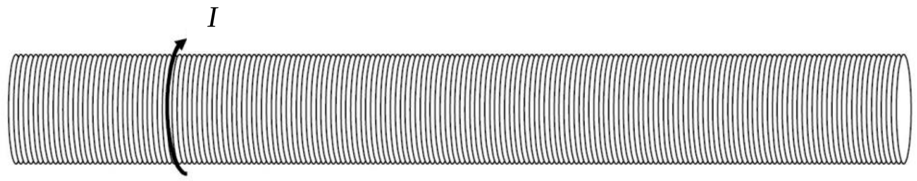
Notice that in the middle, $\vec{B}$ is approximately uniform inside the solenoid, and is approximately zero outside the solenoid. (This is only true if the solenoid is very long such that you can't "see" the ends from the middle, or if you zoom in very close to the solenoid so that you can't see the divergence at the ends). If you aren't convinced, tutorial question C7 will explore the fields around a solenoid in more detail.

Now, how can we find the magnetic field at the centre of the solenoid?



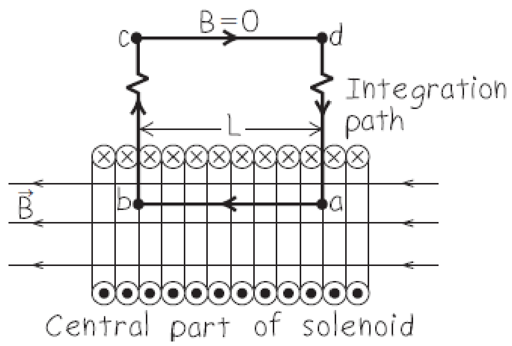
Lecture Example 10

A long solenoid with n loops of wire per unit length carries current I. Find the magnetic flux density B at the centre of the solenoid.



Solution:

Draw a square Amperian loop C of side L, where side *ab* is inside the solenoid and side *cd* is somewhere outside the solenoid.



Ampere's Law states that:

$$\oint_C \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$\int_{ab} \vec{B} \cdot d\vec{\ell} + \int_{bc} \vec{B} \cdot d\vec{\ell} + \int_{cd} \vec{B} \cdot d\vec{\ell} + \int_{da} \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$$

The magnetic flux density B along *cd* is zero, so $\int_{cd} \vec{B} \cdot d\vec{\ell} = 0$.
The magnetic flux density is perpendicular to *bc* and *da*, so $\vec{B} \cdot d\vec{\ell} = 0$ there, meaning $\int_{bc} \vec{B} \cdot d\vec{\ell} = \int_{da} \vec{B} \cdot d\vec{\ell} = 0$.
The magnetic flux density along *ab* is constant and parallel to $d\vec{\ell}$, so $\vec{B} \cdot d\vec{\ell} = B\, d\ell$.
Thus:

$$\int_{ab} B\, d\ell + 0 + 0 + 0 = \mu_0 I_{enc}$$

$$\therefore BL = \mu_0 I_{enc}$$

If there are N turns of wire in the loop, and each wire carries current I, $I_{enc} = NI$

$$B = \frac{\mu_0 N I}{L}$$

Since $n = \frac{N}{L}$ is the number of turns per unit length, we get:

$$\boxed{B = \mu_0 n I}$$

(This result is given in the formula sheet of the A-level exam, and can be used in solving related problems)

The above derivation assumed that the solenoid is *long*, and therefore edge effects can be ignored (so that the field inside the coil is uniform, and the field outside is zero). This is not the case for a *flat* (i.e. *short*) circular coil where the radius r is close to or greater than the length of the coil – then, $B = \frac{\mu_0 N I}{2R}$ at the centre, as discussed earlier.

While we could also have solved these problems by trying to sum up the magnetic field due to current elements, just as we summed up the electric field due to charges, this turns out to be much more complicated,[10] so this is not in the H3 syllabus.

	Electric fields	Magnetic fields
Summing up the charges/currents	Superposition of charges	(not in syllabus)
Summing up the flux	Gauss' Law	Ampere's Law

At this point, it is worth reminding you of the following results from H2 Physics that could be relevant to certain problems:

- Force on a moving charge: $F = q\vec{v} \times \vec{B} = qvB \sin\phi$
- Force on a current-carrying conductor: $F = I\vec{\ell} \times \vec{B} = BIL \sin\phi$

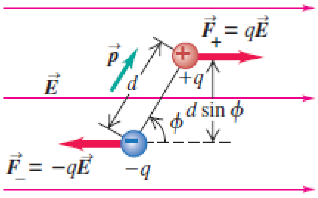
Now, you are ready to try part C of the tutorial.

[10] It is called the Biot-Savart Law, and involves integrating over a cross-product, which is beyond the mathematical requirements of H3 Physics.

6 Dipoles in Fields

We first looked at dipoles at the end of Section 4. Dipoles by themselves may not be very interesting to study, but they do exhibit interesting behaviour in fields.

21.31 The net force on this electric dipole is zero, but there is a torque directed into the page that tends to rotate the dipole clockwise.



Consider an electric dipole of charge q that makes an angle ϕ with a uniform electric field E. The net force on the dipole is zero, but the couple gives a net torque τ:

$$\vec{\tau} = \vec{d} \times \vec{F} = \vec{d} \times q\vec{E}, \qquad \tau = (d)(qE\sin\phi)$$

If we want to express the torque purely in terms of the electric field E, we can rearrange it:

$$\tau = (qd)(E\sin\phi)$$

This looks like a cross product between qd and E! So let's define a new quantity:

The electric dipole moment p is a vector that points from the negative end to the positive end of an electric dipole and its magnitude is the product of the charge and the distance of separation.

$$\vec{p} = q \cdot \vec{d}$$

Now our expression looks like this:

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}, \qquad \tau = pE\sin\phi}$$

By rotating the dipole from some initial angle ϕ_1 to ϕ_2, we do work on the system, storing potential energy:

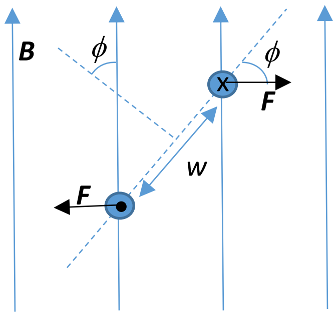
$$W = \int_{\phi_1}^{\phi_2} \tau \, d\phi' = \int_{\phi_1}^{\phi_2} pE\sin\phi \; d\phi = -pE\cos\phi_2 - (-pE\cos\phi_1)$$

Since the work done on the system is equal to the change in potential energy (i.e. $W = \Delta U = U_f - U_i$), the two terms must represent the initial and final potential energies of the system, thus:

$$\boxed{U = -pE\cos\phi}$$

When $\phi = 0$ the system is in equilibrium with minimum potential energy (most negative). When $\phi = 180°$ the system is also in equilibrium but has maximum potential energy (most positive). We'll explore this in the tutorial.

The same thing happens for magnetism when we consider a rectangular current loop of width w and length L in a uniform magnetic field (the figure shows the side view):



$$\vec{\tau} = \vec{w} \times \vec{F} = \vec{w} \times \vec{B}IL, \qquad \tau = BILw\sin\phi$$

But Lw is the area A of the loop:

$$\tau = BIA\sin\phi$$

Once again, we can define a magnetic dipole moment:

The magnetic dipole moment μ is a vector that points from the south pole to the north pole of a magnetic dipole and its magnitude is the product of the current and the area of the loop.

$$\vec{\mu} = I \cdot \vec{A}$$

Then the torque is:

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}, \qquad \tau = \mu B\sin\phi}$$

Don't confuse this with μ_0, the magnetic permeability of free space!

And once again, by rotating it, we do work, changing the potential energy:

$$\Delta U = W = \int_{\phi_1}^{\phi_2} \tau \, d\phi = \int_{\phi_1}^{\phi_2} \mu B \sin\phi \; d\phi = -\mu B \cos\phi_2 - (-\mu B \cos\phi_1)$$

So the potential energy of the magnetic dipole is:

$$\boxed{U = -\mu B \cos\phi}$$

Lecture Example 11

An electric dipole is in a uniform electric field of magnitude 5.00×10^5 N C^{-1}, oriented as shown below. The charges are $\pm 1.60 \times 10^{-19}$ C and are separated by a distance of 1.25×10^{-10} m.

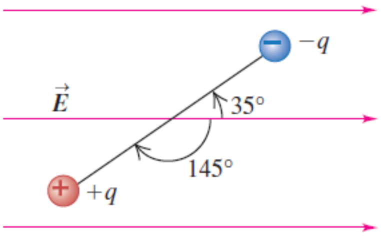
Find the:

(a) magnitude of the force on the positive and negative charge, and the net force on the dipole

(b) magnitude of the electric dipole moment

(c) magnitude and direction of the torque on the dipole

(d) potential energy of the system.



Solution:

(a) The force on each charge is given by $F = qE$, and is the same on both charges as they have the same magnitude of charge and the electric field is uniform.

$$F = qE = (5.00 \times 10^5)(1.60 \times 10^{-19}) = 8.00 \times 10^{-15} \text{ N}$$

As the force on the positive charge acts to the right and the force on the negative charge acts to the left, there is no net force on the dipole.

(b)

$$\begin{aligned} |\vec{p}| &= q|\vec{d}| \\ &= (1.60 \times 10^{-19})(1.25 \times 10^{-10}) \\ &= 2.00 \times 10^{-29} \text{ C m} \end{aligned}$$

(c)

$$\begin{aligned} |\vec{\tau}| &= |\vec{p} \times \vec{E}| \\ &= pE \sin 145° \\ &= (2.00 \times 10^{-29})(5.00 \times 10^5) \sin 145° \\ &= 5.74 \times 10^{-24} \text{ N m} \quad \text{(anticlockwise)} \end{aligned}$$

(d).

$$\begin{aligned} U &= -pE\cos\phi \\ &= -(2.00 \times 10^{-29})(5.00 \times 10^5) \cos 145° \\ &= 8.2 \times 10^{-24} \text{ J} \end{aligned}$$

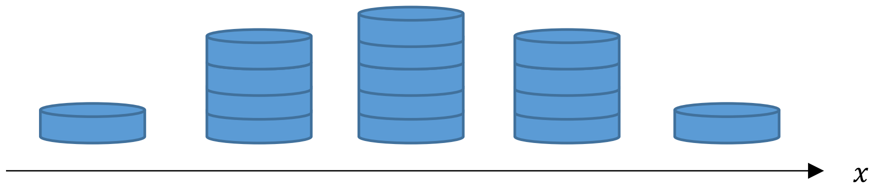
Now, you are ready to try part D of the tutorial.

Appendix 1: Multiple Integrals

You will not be expected to solve multiple integrals in H3 Physics, but a working knowledge of them will be helpful in understanding the equations. This appendix gives you an overview of what a multiple integral means.

Double integrals in Cartesian coordinates

Imagine you have some $1 coins laid out in a line:



How might you find the total amount of money you have? First you'd count the value of the coins in each stack:

and then add up the total value of all the stacks:

$$\$1 + \$4 + \$5 + \$4 + \$1 = \$15$$

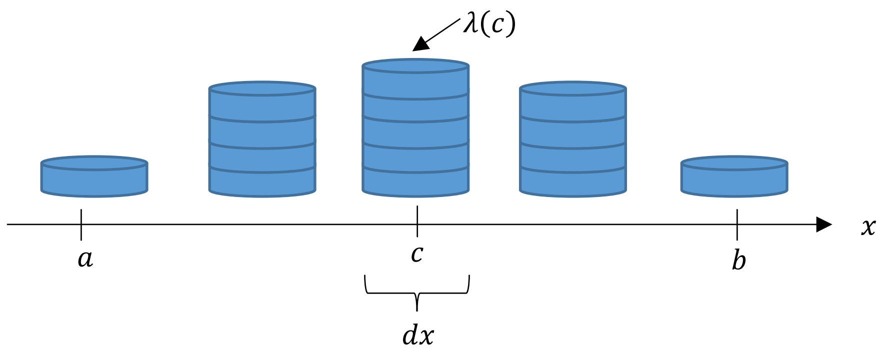
Integrating to add up the total amount of something (e.g. charge, mass, etc) is much like that. Denote the position of the stack by x. The number of coins in each stack is given by a function, $\lambda(x)$. Let's call this a "**linear density function**" (see Section 2.3). In this case, $\lambda(x) = 5 - (x-2)^2$. Then the total amount of money T is:

$$T = \sum_{x=0}^{4} 5 - (x-2)^2 = 15$$

This is a discrete sum; but if we had a continuous sum (as explained in Section 2.1) then we'd have:

$$T = \int_0^4 5 - (x-2)^2 \, dx$$

In general the amount of money in a line would be:

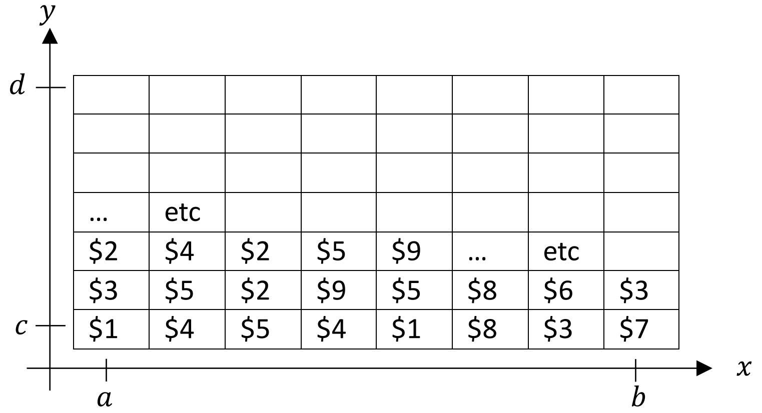


$$T = \int_a^b \lambda(x) \, dx$$

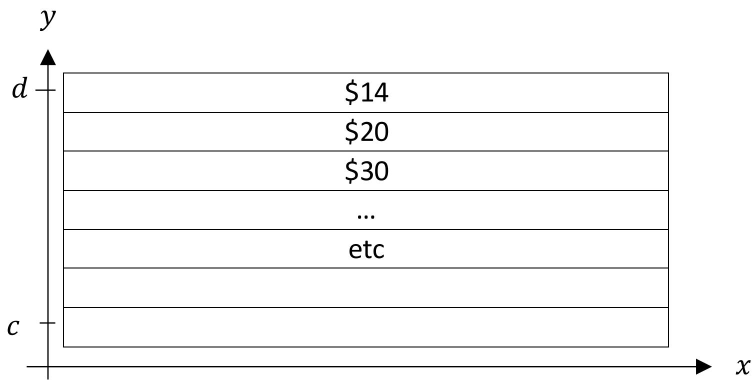
When considering a line of charge, λ is the *linear charge density,* where $\lambda = Q/L$.

Finally consider one more special case: **when $\lambda = 1$, we get the length of the line**. In other words, $\int dx = x$ which is a length.

Now, let's say your stacks of coins were spread out over an area like this.



You might first add up everything in each row:



Then add up all the rows (i.e. add up the one column that remains)

$$\$14 + \$20 + \$30 + \cdots$$

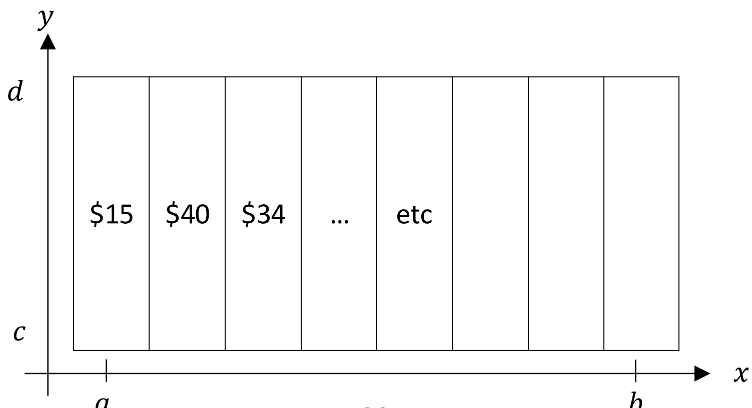
In other words:

$$T = \sum_{y=c}^{d} \left(\sum_{x=a}^{b} \sigma(x, y) \right)$$

Where $\sigma(x, y)$ is a function that tells you how much money is in each little box. σ must depend on x and y because the coordinates of each box are (x, y) Let's call σ a "**surface density function**" (See Section 2.3). If this was a continuous sum:

$$T = \int_{y=c}^{y=d} \left(\int_{x=a}^{x=b} \sigma(x, y)\, dx \right) dy$$

Note that instead of summing the rows first, we could have summed up the columns instead:



Then the sum would look like this:

$$T = \sum_{x=a}^{b} \left(\sum_{y=c}^{d} \sigma(x,y) \right), \qquad T = \int_{x=a}^{x=b} \left(\int_{y=c}^{y=d} \sigma(x,y)\, dy \right) dx$$

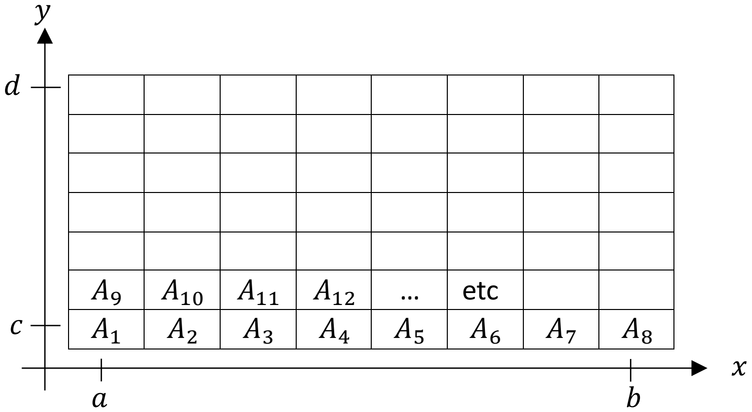
So, the order of integration doesn't matter! We are free to perform whichever one we want first.

When considering a surface of charge, σ is the *surface charge density,* and $\sigma = Q/A$ (see Section 2.3).

Finally, consider one more special case: **when $\sigma = 1$, we get the area of the surface**, i.e. $\int\int dx\, dy = xy$ which is length × breadth.

Generalising Double Integrals to other coordinates

Another way to look at this integral is to label each small piece of the grid A_1, A_2, A_3 ... etc:



The ith area is denoted by A_i. Then σ gives the amount of money in each small area, and is now a function of A_i, i.e. $\sigma(A_i)$. Summing over all the areas,

$$T = \sum_{A_i} \sigma(A_i)$$

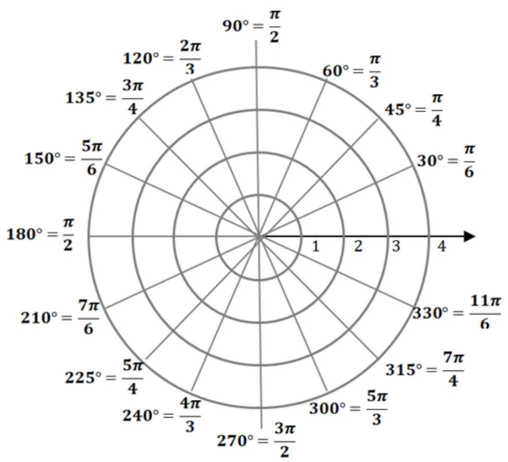
In integral form,

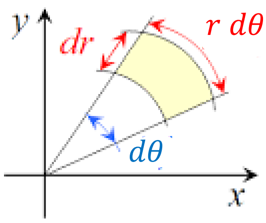
$$T = \int \sigma\, dA$$

Notice that the area of one square is $dA = (dx)(dy)$. So:

$$T = \int \sigma\, dA = \iint \sigma\, dx\, dy = \iint \sigma\, dy\, dx$$

dA is called the **area element.** Working in terms of dA is more "general", because we can "cut up" the area to be integrated in different ways (thus getting a different amount of money in each little area) but still get the same result after adding everything up. For example, consider the polar coordinate system, which has coordinates (r, θ):





When $d\theta$ and dr are really small, the curved lines are approximately straight, and you get a rectangle with sides dr and $r\, d\theta$. So the area is $dA = (dr)(r\, d\theta)$.

As you can see, the area of each little piece of the grid is not constant! It increases as you get further away from the centre. In fact, the area of each little piece of grid, dA, is given by $dA = (dr)(r\, d\theta)$.

So in this new coordinate system, σ is a function of r and θ, so $\sigma(r, \theta)$. And thus:

$$T = \int \sigma\, dA = \iint \sigma\, (dr)(r\, d\theta) = \iint \sigma\, (r\, d\theta)(dr) = \iint \sigma\, r\, dr\, d\theta$$

Therefore, when integrating over an area, we're free to choose any coordinate system we want, and we will get the same answer. (Don't forget to choose appropriate limits for your integrals!).

Note that **you are not required to solve double (or triple) integrals at H3 Physics. Questions will guide you to simplify the problem into a single integral.** However, an understanding of what double integrals are will help in your understanding of the equations (e.g. Gauss' Law).

Example: Area of shape

Let's say we want to find the area of a circle of radius R, centred on the polar coordinate grid. We can do this by summing up the areas of all the little area elements dA inside the circle. (Set $\sigma = 1$)

$$A = \int dA = \int_{r=0}^{r=R} \int_{\theta=0}^{\theta=2\pi} r\, dr\, d\theta = \int_{\theta=0}^{\theta=2\pi} \frac{1}{2} R^2\, d\theta = (2\pi)\left(\frac{1}{2} R^2\right) = \pi R^2$$

Note that the order of integration doesn't matter: we could have integrated over θ first, then r, and we would get the same result.

(Try it in cartesian coordinates – you'll find that it's significantly more tedious!)

Double integrals over curved surfaces

The examples we looked at earlier dealt with plane (flat) areas. What if the area is a curved surface, like part of a sphere? In that case, the area element changes correspondingly. See Section 2.2 for the area elements in cylindrical and spherical coordinates in 3D.

Triple integrals

The same idea applies to volumes as well. This time, we cut up the volume into little pieces of volume dV (i.e. the **volume element**), and add up the amount of money/charge/whatever in all the little volume elements. It's a bit like finding the total price of a Lego set – each Lego brick has a certain value, and so you add up the total cost of all the Lego bricks.

Once again, let's cut up the volume into little volume elements $V_1, V_2, V_3, \ldots$ etc, each with volume dV. Let the function that tells us the cost of each brick be ρ. We can specify ρ as a function of each volume element, $\rho(V_i)$. Then:

$$T = \sum_{V_i} \rho(V_i)$$

In integral form (for a continuous sum):

$$T = \int \rho\, dV$$

If we are working in Cartesian coordinates, each V_i is specified by its coordinates (x, y, z). So ρ is also a function of x, y, z, i.e. $\rho(x, y, z)$. Then:

$$T = \sum_{x=a}^{b} \sum_{y=c}^{d} \sum_{z=e}^{f} \rho(x, y, z), \qquad T = \int_{x=a}^{x=b} \int_{y=c}^{y=d} \int_{z=e}^{z=f} \rho(x, y, z)\, dz\, dy\, dx$$

Just as for double integrals, the order of integration doesn't matter, and we are free to interchange dx, dy, dz.

The volume element in different coordinate systems (see Section 2.2):

- Cartesian: $dV = dx\, dy\, dz$
- Cylindrical: $dV = r\, d\theta\, dr\, dz$
- Spherical: $dV = r^2 \sin\theta\ dr\, d\theta\, d\phi$

For charges in a volume, we have the *volume charge density*, $\rho = Q/V$ (see Section 2.3).

Finally, consider one more special case: **when $\rho = 1$, we get the volume of the shape**, i.e. $\int\int\int dx\, dy\, dz = xyz$ which is length × breadth × height.

Again, note that **you are not required to solve triple integrals at H3 Physics. Questions will guide you to simplify the problem into a single integral.** However, an understanding of what triple integrals are will help in your understanding of the equations.

Video examples of triple integrals:

- Volume of a trapezoid in Cartesian coordinates: https://www.youtube.com/watch?v=BY-2zoSmsRU
- Volume of a cylinder in Cartesian coordinates: https://www.youtube.com/watch?v=MSe29iESv2k
- Volume of a cylinder in Cylindrical coordinates: https://www.youtube.com/watch?v=akuenZwWO-w (notice how much easier it is!)

Example: Triple integrals to find moment of inertia

(Note: this example is to demonstrate how triple integrals can be used. **It is not required knowledge at H3 Physics.**)

A spherical shell has inner radius r_1 and outer radius r_2. Its density is given by $\rho(x, y, z) = kz$ where k is a constant. Find its moment of inertia.

Solution

The moment of inertia of a point mass at distance r from the axis is $I = mr^2$. Thus, the moment of inertia of the entire sphere is:

$$I = \int r^2\, dm = \int r^2 \rho\, dV$$

In spherical coordinates, the volume element is $dV = (r\, d\theta)(r \sin\theta\ d\phi)(dr) = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

$$I = \int_{r=r_1}^{r=r_2} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} r^2 \rho\, (r^2 \sin\theta\, dr\, d\theta\, d\phi)$$

ρ is given to us in Cartesian coordinates, but since $z = r\cos\theta$, we convert to spherical coordinates: $\rho = k(r\cos\theta)$:

$$I = \int_{r=r_1}^{r=r_2} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} r^2 (k\, r\cos\theta)\, (r^2 \sin\theta\, dr\, d\theta\, d\phi) = \int_{r=r_1}^{r=r_2} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} k\, r^5 \cos\theta \sin\theta\, dr\, d\theta\, d\phi$$

Triple integrals can be integrated one at a time (order does not matter), treating the other variables as constants. Evaluating the integral in ϕ, we get a factor of 2π:

$$I = \int_{r=r_1}^{r=r_2} \int_{\theta=0}^{\theta=\pi} (2\pi)\, k\, r^5 \cos\theta \sin\theta\, dr\, d\theta$$

Evaluating the integral in r next, factoring out constants, and applying a trigonometric identity:

$$I = \int_{\theta=0}^{\theta=\pi} (2\pi)\, k \left(\frac{r_2^4 - r_1^4}{4}\right) \cos\theta \sin\theta\, d\theta = (2\pi)\, k \left(\frac{r_2^4 - r_1^4}{4}\right) \int_{\theta=0}^{\theta=\pi} \cos\theta \sin\theta\, d\theta$$

$$= (2\pi)\, k \left(\frac{r_2^4 - r_1^4}{4}\right) \int_{\theta=0}^{\theta=\pi} \frac{1}{2} \sin 2\theta\, d\theta$$

Integrating, $\int_0^{\pi} \frac{1}{2} \sin 2\theta\, d\theta = 0$

$$\therefore I = 0$$

(Intuitively, this must be correct because spherically symmetric objects can be treated as point masses, and so have zero moment of inertia about the centre.)

Summary

So, in summary, for some function f:

Single integrals integrate over one variable. If it is a displacement, the result is the sum of f over a length (e.g. line integral) $$\int f\,dx$$ If the path is a closed loop, we can denote this with a circle: $$\oint f\,dx$$	Double integrals integrate over two variables. If they are displacements, the result is the sum of f over an area (e.g. surface integral) $$\iint f\,dx\,dy$$ If the surface is a closed surface, we can denote this with a circle: $$\oiint f\,dA$$	Triple integrals integrate over three variables. If they are displacements, the result is the sum of f over a volume $$\iiint f\,dx\,dy\,dz$$

The left hand side of Gauss' Law (Section 4.1) is the *net* electric flux Φ through a *closed surface* $\mathcal{S}$(obtained by cutting $\mathcal{S}$ into area elements dA, calculating the fluxes through each surface element $\vec{E}\cdot d\vec{A}$, and adding it up):

$$\Phi_E = \oiint_{\mathcal{S}} \vec{E}\cdot d\vec{A}$$

The left hand side of Ampere's Law (Section 5.2) is the *net* magnetic flux along a *closed loop* $\mathcal{C}$ (obtained by cutting $\mathcal{C}$ into line elements $d\ell$, calculating the fluxes along each line element $\vec{B}\cdot d\vec{\ell}$, and adding it up):

$$\Phi_B = \oint_{\mathcal{C}} \vec{B}\cdot d\vec{\ell}$$

Appendix 2: Common results for common distributions

You can't use these results directly (unless they are given in the formula sheet), but after solving enough tutorial problems you should be able to spot the pattern below, which can help you check your answers in the exam.

For electric fields (Gauss' Law):

Outside this distribution…	… the field strength is …	… the potential[11] is …
Point charge Spherically symmetric charge[12] [draw a **spherical** Gaussian surface]	$E \propto \frac{1}{r^2}$ In fact: $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$	$V \propto \frac{1}{r}$ In fact, from H2 physics: $V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$
Line of charge Cylindrically symmetric charge[13] [draw a **cylindrical** Gaussian surface]	$E \propto \frac{1}{r}$ In fact: $E = \frac{\lambda}{2\pi\varepsilon_0 r}$	$V \propto -\ln r$ Or equivalently: $V \propto \ln\frac{1}{r}$
Plane of charge[14] [draw a **cuboidal** Gaussian surface]	$E = \text{constant}$ In fact: $E = \frac{\sigma}{2\varepsilon_0}$	$V \propto r$ In fact, from H2 physics: $V = Ed$

For magnetic fields (Ampere's Law):

Outside this distribution…	… the flux density is …
Current Cylindrically symmetric current[15] [draw a **circular** Amperian loop]	$B \propto \frac{1}{r}$ In fact: $B = \frac{\mu_0 I}{2\pi d}$
Plane of current[16] [draw a **rectangular** Amperian loop]	$B = \text{constant}$ In fact, since a solenoid is effectively two planes of current: $B = \frac{1}{2}B_{inside\ solenoid} = \frac{1}{2}\mu_0 nI$

Note the similarity with Gauss' Law's results for a cylinder of charge and plane of charge!

Anything encountered in the H3 exams are pretty much guaranteed to be some combination of these.

[11] This assumes the charges are positive. If the charges are negative, there will be an overall a negative sign.
[12] i.e. $\rho =$ constant, or ρ only depends on r. (Intuitively, if you stand in the centre of the sphere and look in any direction, you should see the same arrangement of charge).
[13] i.e. $\rho =$ constant, or ρ only depends on r. This ignores edge effects – must be a *long* cylinder or close to the middle of the line/cylinder
[14] i.e. $\sigma =$ constant. This ignores edge effects – must be a *large* plane or close to the surface
[15] i.e. $J =$ constant, or J only depends on r. Again, this ignores edge effects.
[16] i.e. $K =$ constant. Again, this ignores edge effects.

H3 : Electricity and Magnetism

Capacitors and Inductors	**Learning Outcomes** Students should be able to:
Capacitance and inductance	(a) define capacitance and the farad. (b) define mutual inductance, self-inductance and the henry. (c) show an understanding that the self-inductance (inductance) of a circuit can result in a self-induced e.m.f.
Dielectrics and ferromagnetic materials	(d) show a qualitative understanding that dielectric materials enhance capacitance, and that dielectric breakdown can occur when the electric field is sufficiently strong (knowledge of the quantitative modification of electric fields in matter through the permittivity is not required). (e) show a qualitative understanding that ferromagnetic materials enhance inductance and that this enhancement is non-linear especially near saturation (knowledge of the quantitative modification of magnetic fields in matter through the permeability is not required).
Energy in a capacitor and in an inductor	(f) derive, from the definition of work done by a force, that the potential energy stored in a capacitor is $U = \frac{1}{2}CV^2$, and apply this to solve related problems. (g) derive, from the definition of work done by a force, that the potential energy stored in an inductor is $U = \frac{1}{2}LI^2$, and apply this to solve related problems.
Circuits with capacitors and inductors	(h) solve problems using the formulae for the combined capacitance of two or more capacitors in series and/or parallel. (i) solve problems using the formulae for the combined inductance of two or more inductors in series and/or parallel. (j) solve problems involving circuits with resistors, capacitors, and sources of constant e.m.f. (includes solving first-order differential equations). *[RC series circuits with constant e.m.f. source]* (k) solve problems involving circuits with resistors, inductors, and sources of constant e.m.f. (includes solving first-order differential equations). *[RL series circuits with constant e.m.f. source]* (l) solve problems involving circuits with inductors and capacitors only (includes solving second-order differential equations). *[LC series circuits without e.m.f. source]* (m) solve problems involving circuits with resistors, inductors and capacitors only (students are not expected to solve the general second-order differential equations, though they can be asked to show that particular solutions work). *[RLC series circuits without e.m.f. source]*

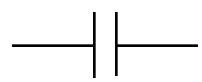
H3 : Electricity and Magnetism

B2: Capacitors and Inductors

B2.1 Capacitors and capacitance

Any two conductors separated by an insulator form a capacitor. Although the capacitor is initially neutral, by transferring electric charge from one of the conductors to another, one of the conductors can be made to hold a positive charge $+Q$ while the other has a negative charge $-Q$. In such a state, we say that a charge Q has been stored on the capacitor (note that the net charge on the capacitor is still zero).

In a circuit diagram, a capacitor is represented by the symbol



When a capacitor has charge Q, a potential difference V exist between the two conductors.

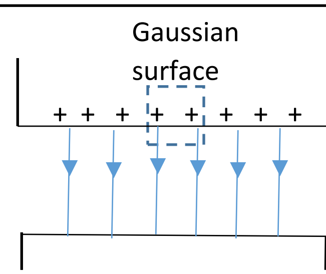
The ratio of the charge on the capacitor to the potential difference between the two conductors V is called the capacitance C. The capacitance is a measure of the capacitor's ability to store energy. It depends only on the shapes and sizes of the conductors and the nature of the insulating material between them.

$$C = \frac{Q}{V}$$

The S.I. unit of capacitance is called farad (F), one farad is a very large capacitance. Capacitors found in most applications are usually in the range of picofarads to microfarads.

$$1F = 1\ CV^{-1}$$

Case study: The parallel plate capacitor



From Gauss' Law, $E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$

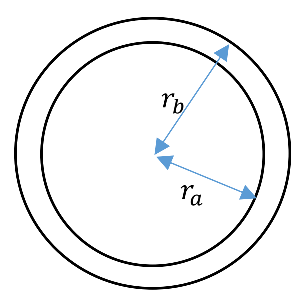
Where σ is surface charge density, A is the surface area

Also, for parallel plates, $E = \frac{V}{d}$

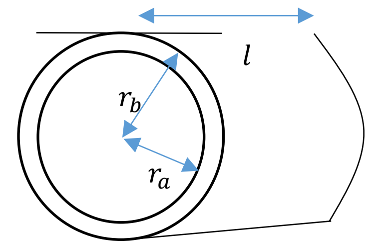
Hence $V = \frac{d}{\epsilon_0 A} Q$

The capacitance of a parallel plate capacitor is thus $C = \frac{\epsilon_0 A}{d}$

Similarly, by considering the electric field between the two conductors of a capacitor, we can also find that the capacitance of a spherical capacitor comprising two concentric spherical conducting shells separated by a vacuum with inner radius r_a (positive charge) and outer radius r_b (negative charge) as $C = 4\pi\epsilon_0 \frac{r_a r_b}{r_b - r_a}$.



Also, the capacitance of a cylindrical capacitor of inner radius r_a (positive charge) and outer radius r_b (negative charge) and length l is $C = \frac{2\pi\epsilon_0 l}{ln(r_b/r_a)}$



B2.2 Capacitors in Series/Parallel Connection

When capacitors are connected in series, the magnitude of charge is the same on all plates of all the capacitors. The potential difference of the individual capacitors are not necessarily the same however (unless the capacitors have the same capacitance).

The total potential difference across the capacitors connected in series is the sum of the potential difference of the individual capacitors.

$$V = V_1 + V_2 + \cdots$$

By definition, we can rewrite this bearing in mind that the magnitude of charge is the same on all plates of all the capacitors

$$\frac{Q}{C} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots$$

Finally, the effective capacitance of all the capacitors connected in series is calculated by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots \text{ (capacitors in series)}$$

When capacitors are connected in parallel, the total charge on the equivalent capacitor is the sum of the charges on each of the capacitor.

$$Q = Q_1 + Q_2 + \cdots$$

By definition, we can rewrite this bearing in mind that the potential difference across each capacitor is the same since they are connected in parallel,

$$CV = VC_1 + VC_2 + \cdots$$

Finally, the effective capacitance of all the capacitors connected in series is calculated by

$$C = C_1 + C_2 + \cdots \text{ (capacitors in parallel)}$$

B2.3 Energy stored in a capacitor

Electrical potential energy can be stored in a capacitor due to work done in charging it. The total work that has been done to charge a capacitor (i.e. to move a charge Q from one plate to the other plate of the capacitor) is given by

$$U = \int_0^Q v dq = \int_0^Q \frac{q}{C} dq = \frac{1}{2}\frac{Q^2}{C}$$

where v is the potential difference across the plates at an instant in time and q is the amount of charge on the capacitor at an instant in time.

Example:

A capacitor with capacitance $C_1 = 8.0\ \mu F$ is initially charged by connecting it to a source of 120 V. The capacitor will then have a charge of $Q = CV = 960\ \mu C$ and the energy stored in the capacitor is $U = \frac{Q^2}{2C} = 0.0576\, J$.

When this capacitor is connected to another capacitor that is initially uncharged and has capacitance $C_2 = 4.0\ \mu F$, charges will redistribute and eventually, both capacitors will have the same potential difference across them. Hence,

$$V_1 = V_2$$

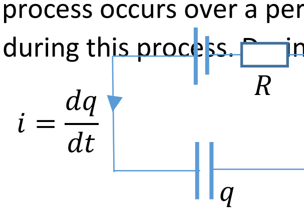
$$\frac{Q_1}{C_1} = \frac{Q - Q_1}{C_2}$$

Hence the charge on the first capacitor is now $Q_1 = 640\ \mu C$ and the second capacitor will have $Q_2 = 320\ \mu C$. The first capacitor will have stored energy $U_1 = \frac{{Q_1}^2}{2C_1} = 0.0256\, J$ and the second capacitor will have stored energy $U_2 = \frac{{Q_2}^2}{2C_2} = 0.0128\, J$.

Note that the total energy stored in the two capacitors is less than the energy that was initially stored in the first capacitor due to energy loss in the form of heat and electromagnetic radiation.

Example: **RC circuit.**

A capacitor can be charged by connecting it to a battery. The charging process occurs over a period of time. Electrical energy from the battery is transferred to the capacitor during this process. During this process, we have the equation $\xi = \frac{dq}{dt}R + \frac{q}{C}$



Solving the differential equation yields $\int_0^q \frac{dq'}{q' - \xi C} = -\int_0^t \frac{dt'}{RC}$

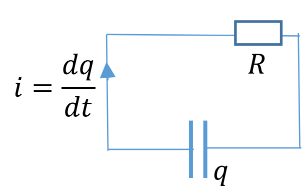
At any instant in time, the charge on the capacitor, $q = C\xi\left(1 - e^{-t/RC}\right)$

The instantaneous current is then $\frac{dq}{dt} = i = \frac{\xi}{R}e^{-t/RC} = I_0 e^{-t/RC}$

Further, if the battery is now removed, stored energy in the capacitor is now released back into the circuit and dissipated at the resistor. During this process, we have the equation $\frac{q}{C} = \left(-\frac{dq}{dt}\right)R$

Note that a minus sign is needed as the charge on the capacitor is decreasing during this process and hence $\frac{dq}{dt}$ is a negative number.

Solving the first order differential equation yields, $\int_Q^q \frac{dq'}{q'} = -\frac{1}{RC}\int_0^t dt'$



At any instant in time, the charge on the capacitor is $q = Qe^{-t/RC}$

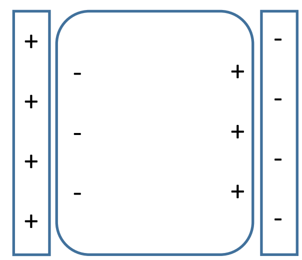
The instantaneous current is $i = \frac{dq}{dt} = -\frac{Q}{RC}e^{-\frac{t}{RC}} = I_0 e^{-t/RC}$

B2.4 Dielectrics in a Capacitor

Between the two plates of a capacitor we can insert a non-conducting material, known as a dielectric.

The dielectric serves three functions.

i) It allows the two plates to be in close proximity without actual contact.
ii) The maximum potential difference that can be achieved between the two plates before discharge occurs can be increased with the use of a dielectric. When the potential difference is high enough for the material to cause partial ionization, a conduction through the material is possible and it is known as dielectric breakdown.
iii) The capacitance of the capacitor is increased with the use of a dielectric. The ratio of the capacitance when there is a dielectric compared to when there isn't, is called the dielectric constant of the material, $K = \frac{C}{C_0} = \frac{V_0}{V}$ (when Q remains constant).



Induced bound surface charges reduces E field between plates

As the dielectric is non-conducting, inserting it between the plates of a capacitor polarizes the dielectric generating an internal E field that opposes the E field E_0 due to the plates of the capacitor. Hence the electric field as well as the potential difference reduces by a factor of K. The resultant E field is thus $E = E_0/K$.

From the expression for energy stored in a capacitor, $U = \frac{Q^2}{2C}$, we can also see that inserting a dielectric with no change in the charge stored on the capacitor will reduce the energy stored by a factor of *K*. The energy lost occurs during the process of inserting the dielectric. Similarly, one will have to expend energy when removing the dielectric to replace the energy stored in the capacitor.

Further, from $U = \frac{Q^2}{2C} = \frac{Q^2}{2KC_0} = \frac{Q^2}{2K\epsilon_0 A/d} = \frac{Ad}{2K}\epsilon_0 E_0^2$

where *A* is the area of the plate and *d* is the distance between the two plates,

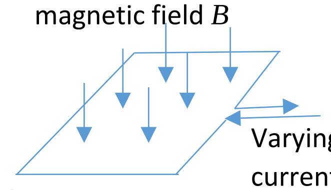
we have energy density $u = \frac{U}{Ad} = \frac{1}{2K}\epsilon_0 E_0^2 = \frac{1}{2}K\epsilon_0 E^2 = \frac{1}{2}\epsilon E^2$

where ϵ is known as the permittivity of the dielectric, defined as $\epsilon = K\epsilon_0$

The expression for **energy density** $u = \frac{1}{2}\epsilon E^2$ is actually valid for any electric field configuration even though we only derived it for the case of parallel plate capacitor.

B2.5 Inductance

A coil of wire of *N* turns in a circuit that carries a varying current is an inductor. As the current is not constant, the magnetic field it produces resulted in a changing magnetic flux linkage at the coil produced by the current is also varying. This results in an induced emf in the coil of wire. According to Lenz's Law, the induced emf results in an induced current that tends to oppose the change in the current through the loop.



$$\xi_{self} = -N\frac{d\Phi}{dt}$$

where Φ is the magnetic flux of a single turn of wire in the coil.

If we define a quantity called inductance (also known as self-inductance as the magnetic field is produced by the current in the coil itself) as a ratio of the magnetic flux linkage of the coil to the current that generates the magnetic field, $L = \frac{N\Phi}{i}$, then $\frac{d(N\Phi)}{dt} = L\frac{di}{dt}$ and the induced emf is

$$\xi_{self} = -L\frac{di}{dt}$$

Example: A toroidal solenoid with cross sectional area of $A = 5.0\text{ cm}^2$ and mean radius $r = 0.10$ m is closely wound with 200 turns of wire. The self inductance of the toroidal solenoid can be calculated by

$$L = \frac{N\Phi}{i} = \frac{N}{i}A\left(\frac{\mu_0 Ni}{2\pi r}\right) = \frac{\mu_0 N^2 A}{2\pi r}$$

$$L = \frac{(4\pi \times 10^{-7})(200)^2(5.0 \times 10^{-4})}{2\pi(0.10)} = 4.0 \times 10^{-5}H$$

Further, if there is a second coil of wire nearby, carrying a varying current i_2, the first coil will also experience an induced emf due to the magnetic field generated by the second coil, and we can write

$$\xi_{mutual} = -N\frac{d\Phi_2}{dt}$$

If we define a quantity called mutual inductance *M* as the ratio of the magnetic flux linkage (in the first coil due to current flowing in the second coil that generates the magnetic field) to the current in the second coil, $M = \frac{N\Phi_2}{i_2}$, and $\frac{d(N\Phi_2)}{dt} = M\frac{di_2}{dt}$ and the induced emf is

$$\xi_{mutual} = -M\frac{di_2}{dt}$$

The mutual inductance is a constant that depends only on the geometry of the two coils (i.e. the size, shape, number of turns, orientation of the coils and the separation of the coils etc).

The S.I. unit of inductance is called henry (H), in honor of the American physicist Joseph Henry.

$$1\,H = \frac{1Wb}{A} = 1\,V.\frac{s}{A} = 1\,\Omega.s = 1\,JA^{-2}$$

Example: A solenoid has 1000 turns and is 0.50 m long with cross sectional area of 10 cm^2. A second coil of wire wound around the solenoid has 10 turns. Calculate the mutual inductance.

Consider the second coil, the mutual inductance is given by

$$M = \frac{N_2 \Phi}{i_1} = \frac{N_2}{i_1} A \left(\frac{\mu_0 N_1 i_1}{l} \right) = \frac{\mu_0 A N_1 N_2}{l}$$

$$M = \frac{(4\pi \times 10^{-7})(10 \times 10^{-4})(1000)(10)}{0.50} = 2.5 \times 10^{-5} H$$

Inductors in a circuit has two important effects.

(i) Firstly, because the induced emf in inductors opposes changes in current, it acts as a magnetic ballast, causing the current to change slower.

(ii) Secondly, inductors serve as an alternative energy storage. When current through inductors are increasing, the inductors extract electrical energy and stores it as magnetic potential energy. When current through inductors are decreasing, the stored magnetic potential energy is then released back into the circuit as electrical energy.

B2.6 Energy stored in an inductor

Consider an ideal inductor (with zero resistance) and let's try to calculate the amount of magnetic potential energy that is stored in the process of increasing the current in the inductor. The potential difference across the terminals of the inductor is simply the induced emf and the electrical power that is extracted from the circuit (to be stored as magnetic potential energy) is given by

$$P = \xi_{induced} i = \left(L \frac{di}{dt} \right) (i)$$

Hence the total magnetic potential energy stored is given by

$$\Delta U = \int dU = \int P dt = \int_0^I L i di = \frac{1}{2} L I^2$$

Unlike a resistor where the power is dissipated, in an inductor, the power consumed when the current is increasing is not dissipated, but it is temporarily stored as magnetic potential energy. This amount of energy is returned to the circuit when the current in the inductor decreases.

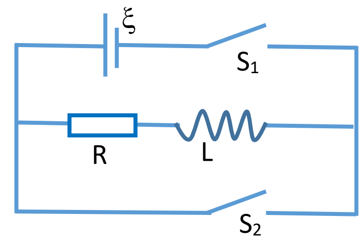
It can also be shown, by substituting L for an expression in terms of μ for a particular configuration, that the **magnetic energy density** u in a material for any magnetic field configuration in a material with constant permeability μ, is given by

$$u = \frac{B^2}{2\mu}$$

Example: R-L Circuit.

The inductor helps to prevent rapid changes in current. This effect is observed in a simple R-L circuit.

Suppose both switches are initially open. When switch S_1 is closed, in the period before a steady current is obtained, we have,



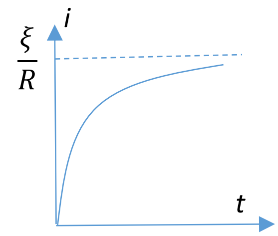
$$\xi = iR + L\frac{di}{dt} \text{ ---- } (*)$$

Solving the first order differential equation,

$$-\int_0^t \frac{R}{L} dt' = \int_0^i \frac{di'}{i' - \frac{\xi}{R}}$$

$$-\frac{Rt}{L} = ln\left(\frac{i - \frac{\xi}{R}}{-\frac{\xi}{R}}\right)$$

$$i = \frac{\xi}{R}\left(1 - e^{-(R/L)t}\right)$$



For convenience, it is common for people to get a sense of the rapidity of current buildup by referring to the time constant τ, i.e. the time needed for the current to reach 63% of its final steady value. This is given by $\tau = L/R$.

When S_1 is off and S_2 is on, the inductor again slows down the rate at which the steady current falls to zero. Energy that was stored in the inductor during the phase when current was increasing is now released.

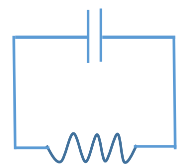
$-L\frac{di}{dt} = iR$ … … …(**) (note that $\frac{di}{dt}$ is a negative number hence a minus sign is needed)

Solving yields $\int_{\frac{\xi}{R}}^{i} \frac{di}{i} = -\int_0^t \frac{R}{L} dt$, and $i = \frac{\xi}{R} e^{-\frac{R}{L}t}$

Example: The L-C circuit.

A circuit comprising an inductor and capacitor will result in an oscillatory movement of charges between the two components, and together with it, the to-and-fro flow of energy that is alternately stored as electrical potential energy (in the capacitor) and magnetic potential energy (in the inductor).

Consider as the start of the cycle, the state where a fully charged capacitor is connected to an inductor. During this phase,



$$\frac{q}{C} = -L\frac{di}{dt}$$

From which we obtain a second order differential equation for the charge that remains on the capacitor plates as

$$\frac{d^2q}{dt^2} = -\frac{q}{LC}$$

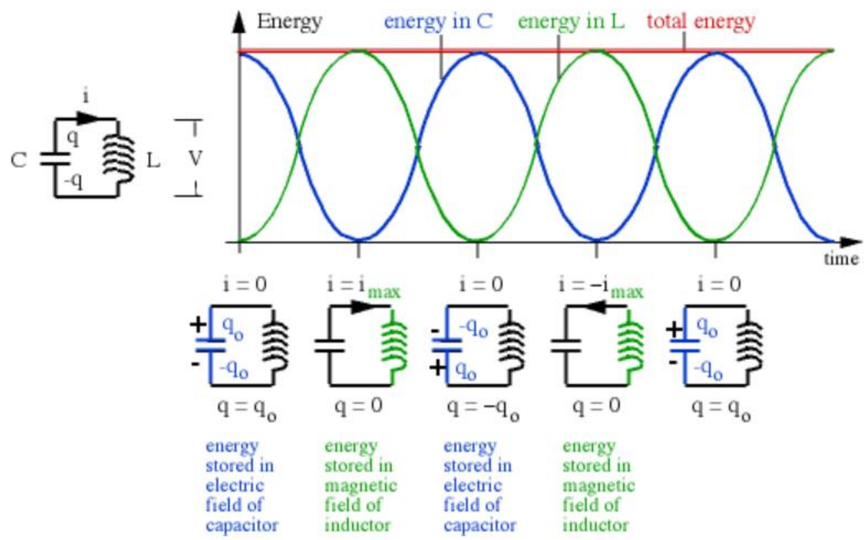
This is the characteristic equation of simple harmonic oscillations of frequency given by $f = \frac{1}{2\pi}\omega = \frac{1}{2\pi}\sqrt{\frac{1}{LC}}$ and $q = Qcos(\omega t)$

In considering the energy distribution of the circuit, note that the energy stored in the capacitor is given by $U_{capacitor} = \frac{1}{2}\frac{q^2}{C}$ and that the energy transferred to the inductor is stored as $U_{inductor} = \frac{1}{2}Li^2$. Since no energy is lost, we can write or show $\frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{Q^2}{C}$

Solving yields an equation for the instantaneous current,

$$i = \pm\frac{1}{\sqrt{LC}}\sqrt{Q^2 - q^2}$$

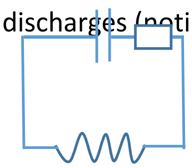
The following shows the transformation of energies between capacitor and inductor.



In reality, there is always some resistance in a circuit and hence we consider the L-R-C circuit comprising an inductor, a capacitor and a resistor. Charge distribution still oscillates between the capacitor and inductor but the oscillations are now damped.

Example: L-R-C circuit.

Consider the start of the cycle when the capacitor is fully charged. As it discharges (noting that $i = \frac{dq}{dt} < 0$), we have the equation,



$$\frac{q}{C} = -L\frac{di}{dt} - iR$$

Rearranging yields $\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = 0$

This is a second-order differential equation in *q*. While you are generally not expected to solve second-order differential equations, you can be asked to verify possible solutions for these equations. You should also be able to recognize the general solutions for the various cases discussed below.

The solutions can be verified by differentiating it twice and substituting the expressions for $\frac{dq}{dt}$ and $\frac{d^2q}{dt^2}$ back into the differential equation.

Solution of the Differential Equation:

The auxiliary equation is given by $k^2 + \left(\frac{R}{L}\right)k + \left(\frac{1}{LC}\right) = 0$

With roots given by $k_1 = -\frac{1}{2}\left(\frac{R}{L}\right) + \sqrt{\frac{\left(\frac{R}{L}\right)^2}{4} - \left(\frac{1}{LC}\right)}$ $\quad k_2 = -\frac{1}{2}\left(\frac{R}{L}\right) - \sqrt{\frac{\left(\frac{R}{L}\right)^2}{4} - \left(\frac{1}{LC}\right)}$

Case One : If determinant of auxiliary equation is positive (2 real roots)

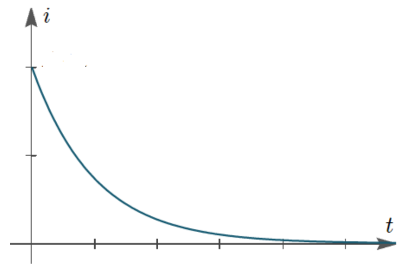
General Solution is of the form

$$q = C_1 e^{k_1 t} + C_2 e^{k_2 t}$$

where k_1 and k_2 are negative.

Resistance is so large that there is no oscillation.

Also referred to as over-damping.

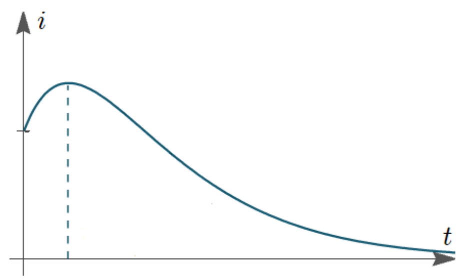


Case Two : If determinant of auxiliary equation is zero (1 real root)

General Solution is of the form

$$q = (C_1 + C_2 t)e^{-Rt/2L}.$$

Resistance is so large that there is no oscillation.
Also referred to as critical damping.



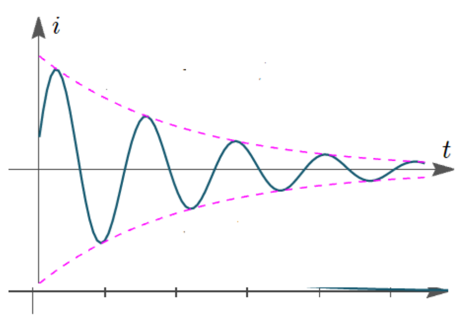
Case Three : if determinant of auxiliary equation is negative (2 complex roots)

$$k_1 = -\frac{1}{2}\left(\frac{R}{L}\right) + i\sqrt{\left(\frac{1}{LC}\right) - \frac{\left(\frac{R}{L}\right)^2}{4}}$$

$$k_2 = -\frac{1}{2}\left(\frac{R}{L}\right) - i\sqrt{\left(\frac{1}{LC}\right) - \frac{\left(\frac{R}{L}\right)^2}{4}}$$

The general solution is given by

$$q = Q_0 e^{-(R/2L)t} cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t + \emptyset\right).$$



This scenario is for lightly damped oscillations.

The frequency of the oscillation is given in the coefficient of t in the cosine function,

$$\omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

The exponential term accounts for the loss of energy from the system over time due to damping effect caused by resistance in the circuit. Note that the maximum amount of charge that is present on the capacitor (when the cosine term equals 1) decreases over time.

The cosine term accounts for the oscillatory shift of energy stored between the capacitor and the inductor.

We can further calculate the half-life time $t_{1/2}$ required for the maximum charge on the capacitor to be reduced to half its initial value by setting the cosine term to one and results in $t_{1/2} = \frac{2L}{R} ln2$

We can also calculate the half-life time $T_{1/2}$ required for energy stored in the system to be reduced to half its initial value. This calculation can be simplified if we consider the instance when all the energy in the system is stored in the capacitor, then $E = \frac{q^2}{2C} = \frac{1}{2}\frac{Q^2}{2C}$. This occurs when $= \frac{Q_0}{\sqrt{2}}$.

Since at this instant $q = Q_0 e^{-(R/2L)t}$, we can obtain the expression

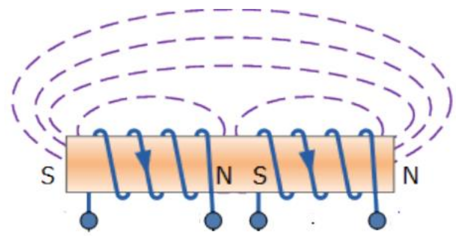
$T_{1/2} = \frac{L}{R} ln2 = 2t_{1/2}$.

B2.7 Inductors in Series/Parallel

Consider two inductors of self-inductance L_1 and L_2 that can be connected in series or in parallel such that they are not too close to each other so that mutual inductance is negligible.

When two inductors are connected in series, there are two possible configurations, the first is where the coils are connected such that the current flowing through the coils are both clockwise when seen from the same end. The other configuration is when the current flowing through the two coils are opposite in direction when seen from the same end.

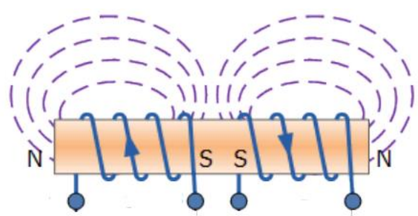
If the two coils are connected such that the magnetic field of the second coil aids or strengthens the magnetic field produced by the first coil, this will result in a stronger induced emf in the first coil



$$-L_{eff}\frac{di}{dt} = -L_1\frac{di}{dt} - M_1\frac{di}{dt} - L_2\frac{di}{dt} - M_2\frac{di}{dt}$$

$$L_{eff} = L_1 + L_2 + 2M$$

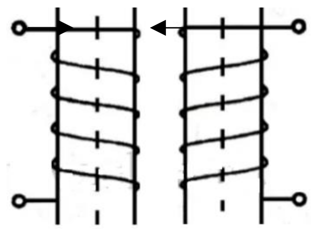
If the two coils are connected such that the magnetic field of the second coil opposes the magnetic field of the first coil, then the induced emf in the first coil is weakened.



$$-L_{eff}\frac{di}{dt} = -L_1\frac{di}{dt} + M_1\frac{di}{dt} - L_2\frac{di}{dt} + M_2\frac{di}{dt}$$

$$L_{eff} = L_1 + L_2 - 2M$$

When the two inductors are connected in parallel, there are again two possible configurations. For the first configuration where the magnetic field of the second coil helps to strengthen the magnetic field in the first coil, the induced emf in the first coil is now stronger. Further, the induced emf ξ for each branch is the same and given by



$$\xi = -L_1\frac{di_1}{dt} - M\frac{di_2}{dt} = -M\frac{di_1}{dt} - L_2\frac{di_2}{dt}$$

From which we can write down,

$$\frac{dI_1}{dt} = -\frac{\xi(L_2 - M)}{L_1L_2 - M^2}$$

$$\frac{dI_2}{dt} = -\frac{\xi(L_1 - M)}{L_1L_2 - M^2}$$

Since $I = I_1 + I_2$, and $\xi_{eff} = -L_{eff}\frac{dI}{dt} = \xi$

$$L_{eff} = \frac{-\xi}{\frac{dI_1}{dt} + \frac{dI_2}{dt}} = \frac{L_1L_2 - M^2}{L_1 + L_2 - 2M}$$

For the other configuration,

$$L_{eff} = \frac{L_1L_2 - M^2}{L_1 + L_2 + 2M}$$

We can also consider the energy stored in the inductors when they are connected in series or parallel. For example, if we consider two inductors connected in series such that they aid each other,

$$U_{total} = U_1 + U_2 = \left(\frac{1}{2}L_1i_1^2 + \int i_1\left(M\frac{di_2}{dt}\right)dt\right) + \left(\frac{1}{2}L_2i_2^2 + \int i_2\left(M\frac{di_1}{dt}\right)dt\right)$$

$$U_{total} = \frac{1}{2}L_1i_1^2 + \frac{1}{2}L_2i_2^2 + Mi_1i_2 = \frac{1}{2}\left(\sqrt{L_1}i_1 - \sqrt{L_2}i_2\right)^2 + \left(\sqrt{L_1L_2} + M\right)i_1i_2$$

Similarly, when inductors are connected such that they oppose each other, we will have less energy stored and given by

$$U_{total} = \frac{1}{2}\left(\sqrt{L_1}i_1 - \sqrt{L_2}i_2\right)^2 + \left(\sqrt{L_1L_2} - M\right)i_1i_2$$

Since stored energy cannot be negative, we require in all situations,

$$\left(\sqrt{L_1L_2} - M\right) \geq 0$$

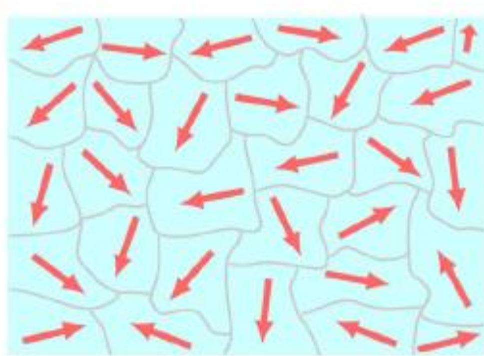
The ratio $\kappa = \frac{M}{L_1L_2} \leq 1$ is also known as the coupling coefficient where $\kappa = 0$ refer to the case where the two inductors are not coupled.

B2.8 Magnetism in Materials

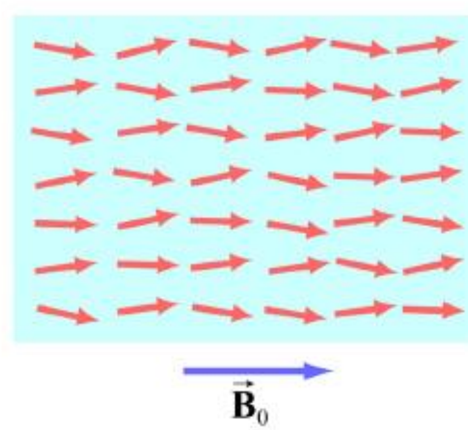
Moving electrons in atoms generate magnetic fields. Due to the random nature of the electrons' motion, the magnetic field generated by the electron tends to cancel out such that the net magnetic field generated by the material is insignificant. However, when there is an external magnetic field present, the motion of the electrons can be influenced. In such a situation, the magnetic field generated by electrons in the material is significant and may either enhance or weaken the external field.

There are three types of magnetization observed in materials. The difference in behavior is largely due to the extent of influence each atom has on others around it. When this influence is weak, the magnetic field generated by the atoms are pretty randomized in direction. In the presence of an external magnetic field, materials where the magnetic field of the material enhances the external field are known as ***paramagnetic*** materials and where the magnetic field of the material weakens the external field, they are known as ***diamagnetic*** materials.

When the influence of atoms amongst themselves are strong (much stronger than the influence of thermal agitation), we have the formation of ***magnetic domains***, large regions where every atom in the region are aligned with each other even in the absence of external field. When an external field is present, domains that are in alignment with the external field grow in size while those that are not aligned diminishes. The external field is enhanced significantly in this way and the extent is many times larger than that of paramagnetic materials. We refer to these materials as ***ferromagnetic*** materials.



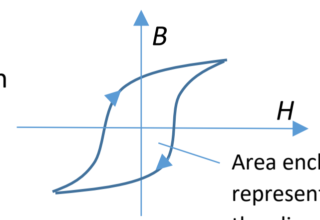
(a) Ferromagnetic domains



(b) Alignment of magnetic moments in the direction of the external field B

An important feature of ferromagnetic materials is that the strong atom-atom interaction being much stronger than thermal agitation results in the magnetic domains persisting even in the absence of external magnetic field. This allows permanent magnets to be made.

The persistence of magnetic domains also means that the material's behavior in the presence of external magnetic field H depends on the existing state of the magnetic domains when the field is being applied. This is known as magnetic hysteresis and we can see from the graph how an applied external magnetic field influences the net magnetic field within the material B.



The tendency of magnetic domains to align themselves to the external field also further differentiates ferromagnetic materials into hard and soft materials. Hard ferromagnetic materials have large hysteresis loops, they are difficult to magnetize and are suited for making permanent magnets. Soft ferromagnetic materials lose their magnetization easily once the external field is removed and also generates less heat.

The effect of inductors can be greatly enhanced if ferromagnetic materials are inserted into the coil of wire, in which case, we should use $\mu = K_m \mu_0$ where K_m is the relative permeability of the material. For ferromagnetic materials near saturation level, K_m can reach several thousand.

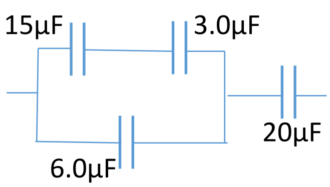
Self-Review Questions

Capacitors

1. An isolated charged conducting sphere of radius 12.0 cm creates an electric field of 4.90×10^4 N/C at a distance 21.0 cm from its center. Calculate its surface charge density and capacitance.

2. When a potential difference of 150 V is applied to the plates of a parallel capacitor, the plates carry a surface charge density of 30.0 nC/cm^2. What is the spacing between the plates?

3. Four capacitors are connected as shown in the diagram below. Find the equivalent capacitance of the arrangement and the charge on each capacitor if 15.0 V is applied across the arrangement.



4. A $450\ \mu F$ capacitor is charged to 295 V. A wire is then connected between the plates. How much thermal energy is produced if all the energy stored in the capacitor goes into heating the wire?

5. Two parallel plates have equal and opposite charges. When there is vacuum between the plates, the electric field is $E = 3.20 \times 10^5$ V/m. When the space is filled with dielectric, the electric field is $E' = 2.50 \times 10^5$ V/m. What is the charge density on each surface of the dielectric? Determine the dielectric constant.

Inductors

6. A 12.0 V battery is connected in series with a 10.0 Ω resistor and a 2.00 H inductor. How long will it take the current to reach 50.0% of its steady value?

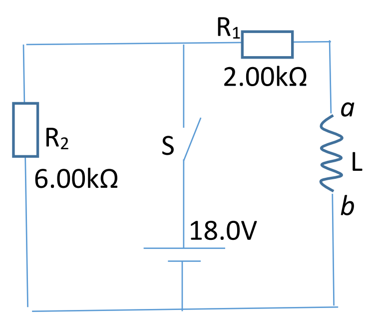
7. Two toroidal solenoids are wound around the same form so that the magnetic field of one passes through the turns of the other. Solenoid 1 has 700 turns, and solenoid 2 has 400 turns. When the current in solenoid 1 is 6.52 A, the average flux through each turn of solenoid 2 is 0.0320 Wb. a) What is the mutual inductance of the pair of solenoids? b) When the current in solenoid 2 is 2.54 A, what is the average flux through each turn of solenoid 1?

8. An air-core solenoid with 68 turns is 8.00 cm long and has a diameter of 1.20 cm. How much energy is stored in its magnetic field when it carries a current of 0.770 A?

Circuits involving capacitors and inductors

9. Calculate the resonant frequency in an LC circuit with inductance of 500 mH and capacitance of 0.100 μF.

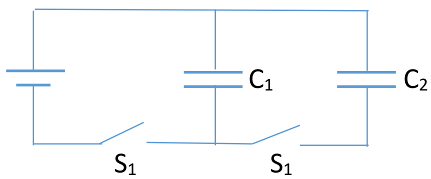
10. In the circuit shown, the switch is initially closed and steady state conditions reached. The switch is opened at $t = 0$. (a) Find the initial potential difference across *L* and state which end is at a higher potential. (b) sketch graphs of the current in each of the two resistors, treating steady state directions as positive.



Tutorial Questions

1. A 50.0 m length of coaxial cable has an inner conductor of diameter 2.58 mm and carries a charge of 8.10 μC. The surrounding conductor has an inner diameter of 7.27 mm and a charge of -8.10 μC. Calculate the capacitance between the two conductors.

2. Consider the circuit shown below where $C_1 = 6.00\ \mu F$, $C_2 = 3.00\ \mu F$ and the cell has e.m.f. of 20.0 V. Capacitor C_1 is first charged by closing switch S_1. Switch S_1 is now open and S_2 now closed. Calculate the initial charge acquired by C_1 and the final charge on each capacitor.



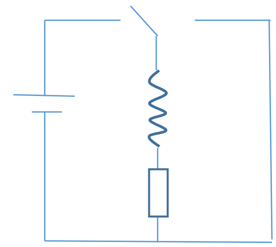
3. Determine the change in stored energy of a parallel plate capacitor of plate area *A*, separation *x* and charge *Q* when the plate separation is increased by *dx*. Find an expression for the force that attracts the plates together.

4. A parallel plate capacitor with only air between the plates is charged by connecting it to a battery. The capacitor is then disconnected from the battery, without any of the charges leaving the plates. a) A voltmeter reads 45.0 V when placed across the capacitor. When a dielectric is inserted between the plates, completely filling the space, the voltmeter reads 11.5V. What is the dielectric constant of this material? (b) What will the voltmeter read if the dielectric is pulled partway out so it fills only one-third of the space between the plates?

5. A parallel plate capacitor of area *A* has the space between the plates filled with two slabs of dielectric, one with constant K_1 and one with constant K_2. Each slab has thickness $d/2$, where *d* is the plate separation. Find the capacitance in terms of *A*, *d*, K_1, K_2.

6. A solenoid 25.0cm long and with a cross-sectional area of 0.500 cm^2 contains 400 turns of wire and carries a current of 80.0 A. Calculate a) the magnetic field in the solenoid, b) energy density in the magnetic field if the solenoid is filled with air, c) the total energy contained in the coil's magnetic field (assume the field is uniform) d) the inductance of the solenoid.

7. A 140 mH inductor and a 4.90 Ω resistor are connected with a switch to a 6.00 V battery as shown. (a) If the switch is thrown to the left, how much time elapses before the current reaches 220 mA? (b) What is the current in the inductor 10.0 s after the switch is closed? (c) Now the switch is quickly thrown to the right, how much time elapses before the current falls to 160 mA?



8. A fixed inductor $L = 1.05\ \mu H$ is used in series with a variable capacitor in the tuning section of a radiotelephone on a ship. What capacitance tunes the circuit to the signal from a transmitter broadcasting at 6.30 MHz?

9. Consider an *LC* circuit in which $L = 500\ mH$ and $C = 0.100\ \mu F$. a) What is the resonant frequency ω_0? b) If a resistance of $1.00\ k\Omega$ is introduced into this circuit, what is the frequency of the damped oscillations?