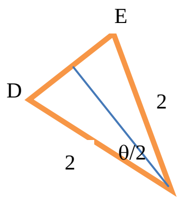
Section A: Pure Mathematics (40 marks)

1 DE = $2\left[2\sin\frac{\theta}{2}\right] = 4\sin\frac{\theta}{2}$

Alternative, use cosine rule



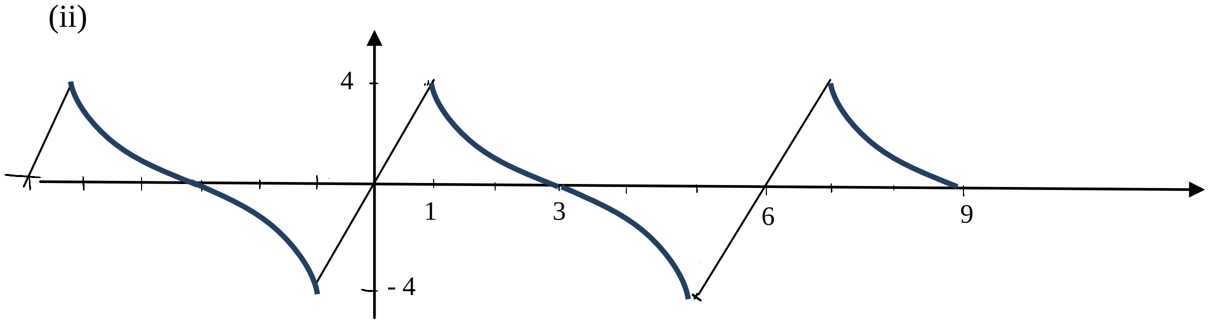
Perimeter = 4x2 + 2 [$4\sin\frac{\theta}{2}$] = $8[1+4\sin\frac{\theta}{2}]$

Unit cost = $\frac{28000}{8[1+\sin\frac{\theta}{2}]} = 3500\left(1+\sin\frac{\theta}{2}\right)^{-1}$

$$\approx 3500\left(1+\frac{\theta}{2}\right)^{-1}$$
$$= 3500\left(1-\frac{\theta}{2}\ldots\right)$$
$$\approx 3500-1750\theta$$

2 (i) f(4) = f(-2) = -f(2) = -1.

(ii)



(iii) $\int_{-5}^{7}|f(x)|\,dx = 4\left[\frac{1}{2}\times4\times1+\int_{1}^{3}(3-x)^2\,dx\right]$

$$=4\left[2+\left[\frac{(3-x)^3}{-3}\right]_1^3\right]$$

$$= 8-\frac{4}{3}[0-8] = \frac{56}{3}$$

3 line l : $2-x=\frac{y-3}{3}$, $z=2$ $\Rightarrow$ $\underset{\sim}{r}=\begin{pmatrix}2\\3\\2\end{pmatrix}+\lambda\begin{pmatrix}-1\\3\\0\end{pmatrix}$ $\lambda\in\mathbb{R}$

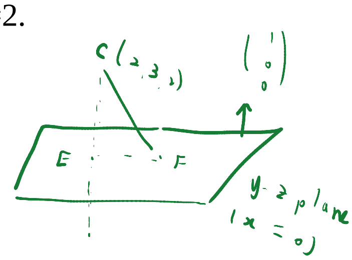
(i) $\overrightarrow{CD}=\begin{pmatrix}3\\\alpha\\\beta\end{pmatrix}-\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}1\\\alpha-3\\\beta-2\end{pmatrix}=k\begin{pmatrix}-1\\3\\0\end{pmatrix}$

Hence k = -1, α = 0 and β = 2

(ii) Equation of *y-z* plane is x = 0.

At the point of intersection F of the line *l* and the *y-z* plane, 2-λ=0 ⇒ λ=2.

Hence F (0, 9, 2)



(iii) By observation, $\overrightarrow{OE}=\begin{pmatrix}0\\3\\2\end{pmatrix}$ and hence $\overrightarrow{OC'}=\begin{pmatrix}-2\\3\\2\end{pmatrix}$

Alternative method: Let E be the foot of ⊥ from C to y-z plane $\underset{\sim}{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=0$

Line CE: $\underset{\sim}{r}=\begin{pmatrix}2\\3\\2\end{pmatrix}+\mu\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}2+\mu\\3\\2\end{pmatrix}$.

When line CE meets y-z plane: $\underset{\sim}{r}=\begin{pmatrix}2+\mu\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=0 \quad\Rightarrow\quad \mu=-2$. $\therefore \overrightarrow{OE}=\begin{pmatrix}2\\3\\2\end{pmatrix}+(-2)\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\3\\2\end{pmatrix}$

By mid-point theorem, $\overrightarrow{OE}=\dfrac{\overrightarrow{OC}+\overrightarrow{OC}'}{2} \quad\Rightarrow\quad \overrightarrow{OC}'=2\overrightarrow{OE}-\overrightarrow{OC}=2\begin{pmatrix}0\\3\\2\end{pmatrix}-\begin{pmatrix}2\\3\\2\end{pmatrix}=\begin{pmatrix}-2\\3\\2\end{pmatrix}$

4 (a) $\dfrac{dy}{dx}+4(x-y)^2\cos^2 x=\sin^2 x$

Using $u=x-y$, $\quad \dfrac{du}{dx}=1-\dfrac{dy}{dx}\Rightarrow\dfrac{dy}{dx}=1-\dfrac{du}{dx}$

$1-\dfrac{du}{dx}+4u^2\cos^2 x=\sin^2 x$

$\dfrac{du}{dx}=4u^2\cos^2 x+1-\sin^2 x$

$\dfrac{du}{dx}=\left(1+4u^2\right)\cos^2 x$

$\displaystyle\int\frac{1}{1+4u^2}\ du=\frac{1}{2}\int\left(1+\cos 2x\right)dx$

$\displaystyle\frac{1}{4}\int\frac{1}{\frac{1}{4}+u^2}\ du=\frac{1}{2}\int\left(1+\cos 2x\right)dx$

$\dfrac{1}{4}\left[2\tan^{-1}2u\right]=\dfrac{1}{2}\left(x+\dfrac{1}{2}\sin 2x\right)+c$

$\tan^{-1}\left[2\left(x-y\right)\right]=x+\dfrac{1}{2}\sin 2x+c$

(b)(i) $\dfrac{dx}{dt}=R-kx^2$, k is a positive constant

At $x=2R$, $\dfrac{dx}{dt}=0$

$0=R-k(2R)^2\Rightarrow 0=R(1-4kR)\Rightarrow k=\dfrac{1}{4R}$

$\dfrac{dx}{dt}=R-\dfrac{x^2}{4R}$

$\displaystyle\int\frac{4R}{4R^2-x^2}\ dx=\int 1dt$

$\ln\left|\dfrac{2R+x}{2R-x}\right|=t+C$

$\dfrac{2R+x}{2R-x}=\pm e^C e^t \quad\Rightarrow\quad \dfrac{2R+x}{2R-x}=Ae^t$, $A=\pm e^C$

At $t=0, x=0 \Rightarrow A=1$ Hence $\frac{2R+x}{2R-x}=e^t \Rightarrow x=2R\left[\frac{e^t-1}{1+e^t}\right]$

(b)(ii) $x=2R\left[\frac{1-\frac{1}{e^t}}{\frac{1}{e^t}+1}\right]$, As $t\to\infty$, $\frac{1}{e^t}\to 0 \Rightarrow x \to 2R$ [or use graphical method]

The amount of drug in the body will not exceed 2R mg regardless of the period of treatment.

5 $z^3=\frac{1}{\sqrt{2}}\left(1+\frac{2}{2a-1}\mathrm{i}\right)$

$\arg(z^3)=-\frac{\pi}{4} \Rightarrow \frac{2}{2a-1}=-1 \Rightarrow a=-\frac{1}{2}$

(i) $z^3=\frac{1}{\sqrt{2}}(1-\mathrm{i}) \Rightarrow z^3=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{4}+2k\pi\right)}$, k = 0, +1, -1

$z=\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{12}+\frac{2k\pi}{3}\right)}, k=0,\pm 1$

$z=\cos\left(\frac{-\pi}{12}\right)+i\sin\left(\frac{-\pi}{12}\right)$, $z=\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(\frac{-3\pi}{4}\right)$ and $z=\cos\left(\frac{7\pi}{12}\right)+i\sin\left(\frac{7\pi}{12}\right)$

$\sqrt{2}\left(\frac{w}{1+i}\right)^3=1-\mathrm{i} \Rightarrow \left(\frac{w}{\sqrt{2}e^{i\frac{\pi}{4}}}\right)^3=\frac{1-\mathrm{i}}{\sqrt{2}}$

$\frac{w}{\sqrt{2}e^{i\frac{\pi}{4}}}=z \Rightarrow w=\sqrt{2}e^{i\frac{\pi}{4}}z$

The points representing *w* is an anti-clockwise rotation of the points representing *z* about the origin by $\frac{\pi}{4}$, followed by an enlargement by factor $\sqrt{2}$ about the origin.

Section B: Statistics (60 marks)

6 PP EEE C R T I V

i) number of ways = $\frac{6!\times 7\times 6\times 5\times 4}{3!\times 2!}=50400$

ii) P _ _ P _ _ _ _ _ _

Case 1: 2 E $7!=5040$

Case 2: 1 E $\left({}^5C_1\times 2!\right)\times\frac{7!}{2!}=25200$ Total = 30240

7. (a) (i) $P(A\cup B)\le 1$ and $P(A\cap B)=P(A)+P(B)-P(A\cup B)$

$\ge 0.7+0.8-1$

= 0.5

(i) $P(A'\cup B)+P(A)-P(A\cap B)=1$ (Use Venn Diagram)

$P(A\cap B)=0.85+0.7-1=0.55$

$P(A|B)=\frac{P(A\cap B)}{P(B)}=\frac{0.55}{0.8}=\frac{11}{16}=0.6875$

A & B are not independent as P(A|B) ≠ P(A) . or $P(A\cap B)\ne P(A).P(B)$

(ii) If $P(A\cap B)=0.5$, then $P(A\cup B)=1$, event *A* & event *B* are exhaustive.

(b) Total no. of points = $1+2+3+...+9+10=55$ <u>OR</u> ${}^{11}C_2 = 55$

Prob. = ${}^{11}C_1 \times \frac{10}{55} \times \frac{9}{54} \times \frac{8}{53} = \frac{8}{159} \approx 0.0503$ <u>OR</u> ${}^{11}C_1 \times \frac{{}^{10}C_3}{{}^{55}C_3} = \frac{8}{159}$

8 **i)** Let X = number of hot drinks sold in five minutes.

Let Y = number of cold drinks sold in five minutes

$X \sim Po(2)$ $\quad\quad Y \sim Po(2.5)$

$X+Y \sim Po(4.5)$

$P(X+Y>4) = 1-P(X+Y \le 4)$

= 1- 0.532103…

= 0.467896….= 0.468

ii) Required probability = $P(X=0 / X+Y \le 4)$

$$= \frac{P(X=0 \cap X+Y \le 4)}{P(X+Y \le 4)}$$

$$= \frac{P(X=0)P(Y \le 4)}{P(X+Y \le 4)} = \frac{0.13534 \times 0.89118}{0.53210} = 0.227$$

(iii) Let W= number of periods with more than 4 drinks sold out of the twenty periods.

$W \sim B(20, 0.46790)$

$P(W \ge n) < 0.4$

$1-P(W \le n-1) < 0.4 \quad \Rightarrow P(W \le n-1) > 0.6$

From GC: $P(W \le 9) = 0.527 < 0.6$

$P(W \le 10) = 0.696 > 0.6$

Least *n*-1 =10 $\quad\quad \Rightarrow$ least *n* = 11

(iv) Number of hot drinks sold in 30 minutes = $H \sim Po(12)$

Number of cold drinks sold in 30 minutes = $C \sim Po(15)$

Since both λ >10, $H \sim N(12,12)$ and $C \sim N(15,15)$

$H-C \sim N(-3, 27)$

$P(H-C>0) = P(H-C>0.5) = 0.25029... = 0.250$

9 (a) $n=16$ $\bar{x} = \frac{2848}{16} = 178$,

$$s^2 = \frac{1}{15}\left[509884 - \frac{2848^2}{16}\right] = 196, \quad s = 14$$

Hypothesis: $H_0 : \mu = \mu_0$ against $H_1 : \mu \ne \mu_0$

As σ^2 is unknown, *n* is small, conduct a 2-tailed t –test,

assuming $X \sim$ *Normal Distribution* $N(\mu, \sigma^2)$, test statistics: $T = \frac{\bar{X}-\mu_0}{\frac{S}{\sqrt{16}}} \sim t(15)$

At 5% significance level, do not reject H_0 if $-2.131 < t < 2.131$.

$$-2.131 < \frac{178-\mu_0}{\frac{14}{\sqrt{16}}} < 2.131$$

$$-7.4585 < 178-\mu_0 < 7.4585$$

$$170.5415 < \mu_0 < 185.4585 \quad \Rightarrow \quad 171 < \mu_0 < 185$$

(b) Combine sample: $n = 16+36 = 52$, $120 = \frac{1}{36}\left[\sum c^2 - \frac{6660^2}{36}\right] \Rightarrow \sum c^2 = 1236420$

$$\sum y = 2848 + 185(36) = 9508$$

and $\sum y^2 = 509884 + 1236420 = 1746304$.

Unbiased estimate for μ is = $\bar{y} = \frac{9508}{52} = 182.8461538$

Unbiased estimate for σ^2 is = $s^2 = \frac{1}{51}\left[1746304 - \frac{9508^2}{52}\right] = 152.9954751$

Hypothesis: $H_0 : \mu = 180$ against $H_1 : \mu > 180$

As σ^2 is unknown, n is large, conduct a 1-tailed z –test,

test statistics: $Z = \frac{\bar{X} - 180}{\frac{s}{\sqrt{52}}} \sim N(0,1)$ [by CLT]

From GC, $z = 1.65928407$, $p = 0.04852928$

$p - value = P(Z > 1.65928407)$ when μ = 180)

= prob that the sample mean cholesterol attains a value of more than 182.8461538 if the population mean cholesterol content in eggs is 180

Or The *p*-value is the lowest level of significance for which the null hypothesis of mean cholesterol level being 180, will be rejected.

Or The *p*-value is the probability of obtaining a test statistic more than 1.659, assuming that H_0 is true.

10 Let L = waiting time of a patient at Lee's Clinic and H = waiting time of a patient at Hope Clinic.

$L \sim N(25, 8^2) \quad H \sim N(37, 4^2)$

i) Let $X = \frac{L_1 + L_2 + L_3 + L_4 + L_5}{5} \sim N\left(25, \frac{8^2}{5}\right)$

$$X - H \sim N\left(25-37, \frac{8^2}{5} + 4^2\right) = N(-12, 28.8)$$

Required probability = $P(|X-H| \geq 5)$

$= P(X-H \geq 5) + P(X-H \leq -5)$

= 0.000768 + 0.90395

= 0.905 (to 3 sf)

Assumption: the waiting times of all patients are independent.

ii) Let Y = number of patients with waiting time more than 25 minutes

$Y \sim B(n, 0.5)$

Since n > 40, $\therefore$ np > 5 and n(1 – p) > 5

$\Rightarrow \quad Y \sim N(0.5n, 0.25n)$ approximately.

$P(Y \le 40) < 0.95$

$\Rightarrow P(Y \le 40.5) < 0.95$

$\Rightarrow P\left(Z \le \dfrac{40.5 - 0.5n}{0.5\sqrt{n}}\right) < 0.95$

$\Rightarrow \dfrac{40.5 - 0.5n}{0.5\sqrt{n}} < 1.64485$

$\Rightarrow 0.5n + 1.64485\left(0.5\sqrt{n}\right) - 40.5 > 0$

By GC, $n > 67.487$. Least n = 68

iii) Let W = number of patients treated for influenza in a sample of 20.

$W \sim B(20, 0.2)$

$E(W) = 20(0.2) = 4$ and $Var(W) = 20(0.2)(0.8) = 3.2$

Since n = 60, by CLT, $\overline{W} \sim N\left(4, \dfrac{3.2}{60}\right)$

$P\left(\overline{W} > 3.5\right) = 0.9848... = 0.985$

iv) Normal model is not likely to be appropriate as the waiting times of the combined group of patients may follow a bi-modal distribution.

11. (i) $\overline{\theta} = 45$, $\overline{T} = \dfrac{80+t}{7}$ lies on the regression line.

$$\frac{80+t}{7} = 35.857 - 0.51429(45) \Rightarrow t = 8.998 \approx 9 \text{ (nearest integer)}$$

(ii) When $\theta = 70$, $T = -0.143$.

Reason 1 : this estimate is not realistic since the time taken cannot be negative.

Reason 2: scatter diagram suggests a curved relationship rather than a straight line.

(iii) $\theta(T - a) = b \Rightarrow T = a + \dfrac{b}{\theta}$

Regression line of T on $\dfrac{1}{\theta}$: $T = -8.1556 + \dfrac{804.86}{\theta}$

$30 = -8.1556 + \dfrac{804.86}{\theta} \Rightarrow \theta = 21.1\,°C$

[Note: the wrong regression line gives 21.2 $°C$]