Raffles Institution
H2 Mathematics
Solution for 2015 A-Level Paper 2

Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
(i)	32 m	Note that we need $16-\frac{1}{2}h \geq 0$
(ii)	$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\sqrt{16-\frac{1}{2}h}}{10} \Rightarrow \int 10\left(16-\frac{1}{2}h\right)^{-\frac{1}{2}}\mathrm{d}h=\int \mathrm{d}t$ $\Rightarrow \frac{20}{\left(-\frac{1}{2}\right)}\sqrt{16-\frac{1}{2}h}=t+c$ $\therefore t=b-40\sqrt{16-\frac{1}{2}h}$ where $b=-c$ $h=0$ when $t=0 \Rightarrow b=160$ Thus, $t=160-40\sqrt{16-\frac{1}{2}h}$ Half its maximum height, that is, $h=16$ we have $t=160-40\sqrt{16-\frac{1}{2}(16)}=160-40\sqrt{8}\approx 46.9$ years	

Question 2

No.	Suggested Solution	Remarks for Student
	$L: \underset{\sim}{r} = \begin{pmatrix}1\\-2\\-4\end{pmatrix} + \lambda\begin{pmatrix}2\\3\\-6\end{pmatrix},\ \lambda \in \mathbb{R}$	
(i)	Acute angle = $\cos^{-1}\left\lvert\dfrac{\begin{pmatrix}2\\3\\-6\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}}{\left\lvert\begin{pmatrix}2\\3\\-6\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}1\\0\\0\end{pmatrix}\right\rvert}\right\rvert = \cos^{-1}\dfrac{2}{7} = 73.4^\circ$	
(ii)	$\overrightarrow{OP} = \begin{pmatrix}2\\5\\-6\end{pmatrix}$ Any point, R, on L has position vector, $\overrightarrow{OR} = \begin{pmatrix}1+2\lambda\\-2+3\lambda\\-4-6\lambda\end{pmatrix}$ for some $\lambda \in \mathbb{R}$. $\left\lvert\overrightarrow{PR}\right\rvert = \sqrt{33} \quad \Rightarrow \left\lvert\begin{pmatrix}1+2\lambda-2\\-2+3\lambda-5\\-4-6\lambda+6\end{pmatrix}\right\rvert = \sqrt{33}$ $\Rightarrow \sqrt{(2\lambda-1)^2+(3\lambda-7)^2+(2-6\lambda)^2} = \sqrt{33}$ $\Rightarrow 4\lambda^2-4\lambda+1+9\lambda^2-42\lambda+49+4-24\lambda+36\lambda^2 = 33$ $\Rightarrow 49\lambda^2-70\lambda+21=0$ $\Rightarrow 7\lambda^2-10\lambda+3=0$ $\Rightarrow (7\lambda-3)(\lambda-1)=0$ $\Rightarrow \lambda = \dfrac{3}{7} \quad \text{or} \quad \lambda = 1$	

	$\overrightarrow{OR}=\begin{pmatrix}1+2\left(\frac{3}{7}\right)\\-2+3\left(\frac{3}{7}\right)\\-4-6\left(\frac{3}{7}\right)\end{pmatrix}$ or $\overrightarrow{OR}=\begin{pmatrix}1+2\\-2+3\\-4-6\end{pmatrix}=\begin{pmatrix}3\\1\\-10\end{pmatrix}$ $=\begin{pmatrix}\frac{13}{7}\\-\frac{5}{7}\\-\frac{46}{7}\end{pmatrix}$ $\overrightarrow{OR'}=\frac{1}{2}\left[\begin{pmatrix}3\\1\\-10\end{pmatrix}+\begin{pmatrix}\frac{13}{7}\\-\frac{5}{7}\\-\frac{46}{7}\end{pmatrix}\right]=\begin{pmatrix}\frac{17}{7}\\\frac{1}{7}\\-\frac{58}{7}\end{pmatrix}$	
(iii)	$\begin{pmatrix}2-1\\5+2\\-6+4\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}1\\7\\-2\end{pmatrix}\times\begin{pmatrix}2\\3\\-6\end{pmatrix}=\begin{pmatrix}-36\\2\\-11\end{pmatrix}$ Required plane is $-36x+2y-11z=-36(2)+2(5)-11(-6)=4$.	

Question 3

No.	Suggested Solution	Remarks for Student
	$\mathrm{f}: x \mapsto \dfrac{1}{1-x^2},\ x \in \mathbb{R},\ x>1$	
(a)(i)	Method 1: $\mathrm{f}'(x)=\dfrac{2x}{\left(1-x^2\right)^2}>0$ for $x>1$ Thus, f is an increasing function for $x > 1$. Thus f has an inverse. Method 2: Sketch graph of f and explain using horizontal line.	
(ii)	Let $y=\dfrac{1}{1-x^2}$ $\Rightarrow \dfrac{1}{y}=1-x^2$ $\Rightarrow x^2=1-\dfrac{1}{y}$ $\Rightarrow x=\sqrt{1-\dfrac{1}{y}}$ $\because x>1$ $\mathrm{f}^{-1}(x)=\sqrt{1-\dfrac{1}{x}}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=(-\infty,0)$	
(b)	$\mathrm{g}: x \mapsto \dfrac{2+x}{1-x^2},\ x \in \mathbb{R},\ x \neq \pm 1$ Let $y=\dfrac{2+x}{1-x^2}$ $\Rightarrow y\left(1-x^2\right)=2+x$ $\Rightarrow yx^2+x+(2-y)=0$ For the equation to be defined, discriminate ≥ 0, thus, $1-4y(2-y)\geq 0$ $4y^2-8y+1\geq 0$ $4\left(y^2-2y+1\right)-3\geq 0$ $4(y-1)^2-3\geq 0$ $\left(2y-2-\sqrt{3}\right)\left(2y-2+\sqrt{3}\right)\geq 0$ $y\geq\dfrac{2+\sqrt{3}}{2}$ or $y\leq\dfrac{2-\sqrt{3}}{2}$	

Question 4

No.	Suggested Solution	Remarks for Student
(a)	Let P_n be the statement $\sum_{r=1}^{n} r(r+2)(r+5) = \frac{1}{12}n(n+1)(3n^2+31n+74)$ for $n \in \mathbb{Z}^+$. For $n=1$, $LHS = \sum_{r=1}^{1} r(r+2)(r+5) = 1\times3\times6 = 18$ $RHS = \frac{1}{12}(1)(1+1)(3+31+74) = \frac{216}{12} = 18 = LHS$ Therefore, P_1 is true. Assume P_k is true for some $k \in \mathbb{Z}^+$, i.e. $\sum_{r=1}^{k} r(r+2)(r+5) = \frac{1}{12}k(k+1)(3k^2+31k+74)$ We need to show P_{k+1} is true, i.e. $\sum_{r=1}^{k+1} r(r+2)(r+5) = \frac{1}{12}(k+1)\left[(k+1)+1\right]\left(3(k+1)^2+31(k+1)+74\right)$ For $n=k+1$,	

	$\sum_{r=1}^{k+1} r(r+2)(r+5)$ $=\sum_{r=1}^{k} r(r+2)(r+5)+(k+1)(k+1+2)(k+1+5)$ $=\frac{1}{12}k(k+1)\left(3k^2+31k+74\right)+(k+1)(k+3)(k+6)$ $=\frac{1}{12}(k+1)\left(k\left(3k^2+31k+74\right)+12(k+3)(k+6)\right)$ $=\frac{1}{12}(k+1)\left(3k^3+31k^2+74k+12k^2+108k+216\right)$ $=\frac{1}{12}(k+1)\left(3k^3+43k^2+182k+216\right)$ $=\frac{1}{12}(k+1)(k+2)\left(3k^2+37k+108\right)$ $=\frac{1}{12}(k+1)(k+2)\left(3\left(k^2+2k+1\right)+31(k+1)+74\right)$ $=\frac{1}{12}(k+1)\left[(k+1)+1\right]\left(3(k+1)^2+31(k+1)+74\right)$ Therefore $P_k \Rightarrow P_{k+1}$ is true. Since P_1 is also true, P_n is true for $n \in \mathbb{Z}^+$ by Mathematical Induction.	
(b)(i)	$\frac{2}{4r^2+8r+3}=\frac{2}{(2r+1)(2r+3)}=\frac{1}{2r+1}-\frac{1}{2r+3}$	
(ii)	$\sum_{r=1}^{n}\frac{2}{4r^2+8r+3}=\sum_{r=1}^{n}\left(\frac{1}{2r+1}-\frac{1}{2r+3}\right)$ $=\frac{1}{3}-\frac{1}{5}$ $+\frac{1}{5}-\frac{1}{7}$ $+\frac{1}{7}-\frac{1}{9}$ ... $+\frac{1}{2n-1}-\frac{1}{2n+1}$ $+\frac{1}{2n+1}-\frac{1}{2n+3}$ $=\frac{1}{3}-\frac{1}{2n+3}$	

(iii)	$\left\|\frac{1}{3}-\frac{1}{2n+3}-\frac{1}{3}\right\|<10^{-3} \quad \Rightarrow \frac{1}{2n+3}<10^{-3}$ $\Rightarrow n>498.5$ smallest $n=499$	

Section B: Statistics

Question 5

No.	Suggested Solution	Remarks for Student
(i)	Not possible to obtain the sampling frame, that is, not possible to get the list of all customers, with their age, who patronize the supermarket.	
(ii)	Assume manager wants to get opinions of 10 customers from each of the following groups (by ages) Children: Below 12 years old, Teenagers: 12 to 18 years old, Young Adult: 18 to 35 years old, Mature Adult: 35 to 50 years old, Above 50 years old. Approach the first 10 customers who are willing to provide their opinions using a questionnaire.	
(iii)	As mentioned in (ii), we are approaching the first 10 customers who are willing to provide their opinions. We will not obtain opinions of those who are not as vocal.	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	Let X be the number of red sweets in a small packet of 10 sweets. $X \sim \mathrm{B}(10, 0.25)$ $\mathrm{P}(X \geq 4) = 1 - \mathrm{P}(X \leq 3) = 1 - 0.775875 \approx 0.224$	
(ii)	Let X be the number of red sweets in a large packet of 100 sweets. $Y \sim \mathrm{B}(100, 0.25)$ Since $n = 100$ is large, and $np = 25 > 5$ and $n(1-p) = 75 > 5$, $Y \sim \mathrm{N}(25, 18.75)$ approximately. $\mathrm{P}(Y \geq 30) = \mathrm{P}(Y \geq 29.5)$ by continuity corrections $= 0.14935 \approx 0.149$	Be sure to check the conditions and state the approximate distribution used.
(iii)	For 9740 syllabus: Let W be the number large packets, out of 15, containing at least 30 red sweets. $W \sim \mathrm{B}(15, 0.14935)$ $\mathrm{P}(W \leq 3) = 0.824655 \approx 0.825$ For 9758 syllabus: Without using approximation for $\mathrm{P}(Y \geq 30)$, $W \sim \mathrm{B}(15, \mathrm{P}(Y \geq 30))$, that is, $W \sim \mathrm{B}(15, 0.14954)$ $\mathrm{P}(W \leq 3) = 0.82407 \approx 0.824$	For 9740 syllabus: Exam report suggests that students need to use answer from (ii), as the only answer given is 0.825.

Question 7

No.	Suggested Solution	Remarks for Student
(i)	Number of errors per page are assumed to remain constant uniformly. Errors occure independently between pages of the newspaper.	
	Let X be the number of errors on one page. $X \sim \text{Po}(1.3)$	
(ii)	$Y = X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \sim \text{Po}(7.8)$ $\text{P}(Y > 10) = 1 - \text{P}(Y \le 10) = 1 - 0.83523 \approx 0.165$	
(iii)	$W = X_1 + X_2 + \ldots + X_n \sim \text{Po}(1.3n)$ $\text{P}(W < 2) < 0.05$ $\text{P}(W = 0) + \text{P}(W = 1) < 0.05$ $e^{-1.3n} + 1.3ne^{-1.3n} < 0.05$ $e^{-1.3n}(1 + 1.3n) - 0.05 < 0$ ...(1) Solving (1) using GC graphically or table of values, least $n = 4$	To avoid careless mistake, we should also use the original inequality, $\text{P}(W < 2) < 0.05$, to check the answer.

Question 8

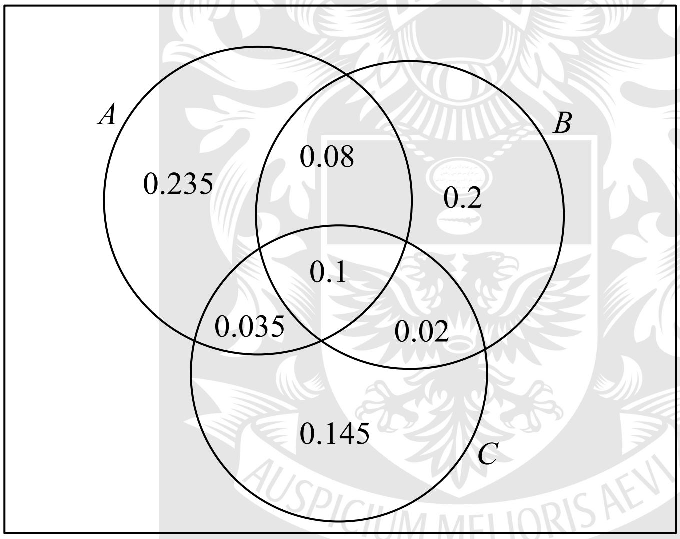
No.	Suggested Solution	Remarks for Student
	Let X be the mass of a pineapple in kg. $X \sim \mathrm{N}\left(\mu, \sigma^2\right)$ unbiased estimate of μ, $\bar{x} = 0.8825$ unbiased estimate of σ^2, $s^2 = 0.074785^2 \approx 0.00559$ Null hypothesis, $\mathrm{H}_0 : \mu = 0.9$ Alternative hypothesis, $\mathrm{H}_1 : \mu < 0.9$ Perform an one-tailed test at 10% significance level. Under H_0, $\bar{X} \sim N\left(0.9, \frac{0.074785^2}{8}\right)$ approximately. p-value $= \mathrm{P}\left(\bar{X} < 0.8825\right) = 0.265 > 0.1$. Since p-value = 0.265 > 0.1, we do not reject H_0 and thus we do not have sufficient evidence at 10% level of significance to doubt the claim of the stall owner that the mean mass of the pineapples is at least 0.9kg.	

Question 9

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td></td><td>$\mathrm{P}(A)=0.45,\ \mathrm{P}(B)=0.4,\ \mathrm{P}(C)=0.3,\ \mathrm{P}(A\cap B\cap C)=0.1.$
A and B are independent.
A and C are independent.</td><td></td></tr>
<tr><td>(i)</td><td>$\mathrm{P}(B\mid A)=\mathrm{P}(B)=0.4$ since A and B are independent.</td><td></td></tr>
<tr><td>(ii)</td><td>$\mathrm{P}(A)=0.45,\ \mathrm{P}(B)=0.4,\ \mathrm{P}(C)=0.3,\ \mathrm{P}(A\cap B\cap C)=0.1.$

A and B are independent, so $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)=0.18.$
A and C are independent, so $\mathrm{P}(A\cap C)=\mathrm{P}(A)\mathrm{P}(C)=0.135.$
B and C are independent, so $\mathrm{P}(B\cap C)=\mathrm{P}(B)\mathrm{P}(C)=0.12.$

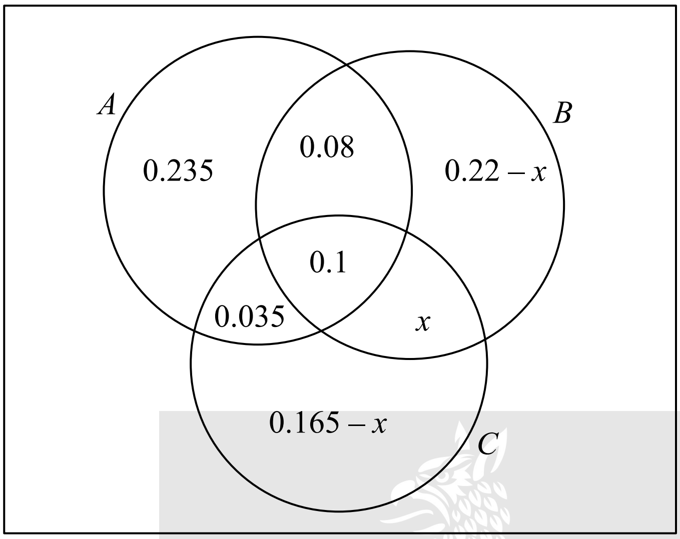
Using Venn diagram,
$\mathrm{P}(A'\cap B'\cap C')=1-\mathrm{P}(A\cup B\cup C)=1-0.815=0.185$


</td><td></td></tr>
<tr><td>(iii)</td><td>$\mathrm{P}(A)=0.45,\ \mathrm{P}(B)=0.4,\ \mathrm{P}(C)=0.3,\ \mathrm{P}(A\cap B\cap C)=0.1.$

A and B are independent, so $\mathrm{P}(A\cap B)=\mathrm{P}(A)\mathrm{P}(B)=0.18.$
A and C are independent, so $\mathrm{P}(A\cap C)=\mathrm{P}(A)\mathrm{P}(C)=0.135.$

$\mathrm{P}(A'\cap B'\cap C')=1-\mathrm{P}(A\cup B\cup C)$</td><td>Use venn diagram to visualize $\mathrm{P}(A'\cap B'\cap C')$ $=1-\mathrm{P}(A\cup B\cup C)$</td></tr>
</table>

Note that



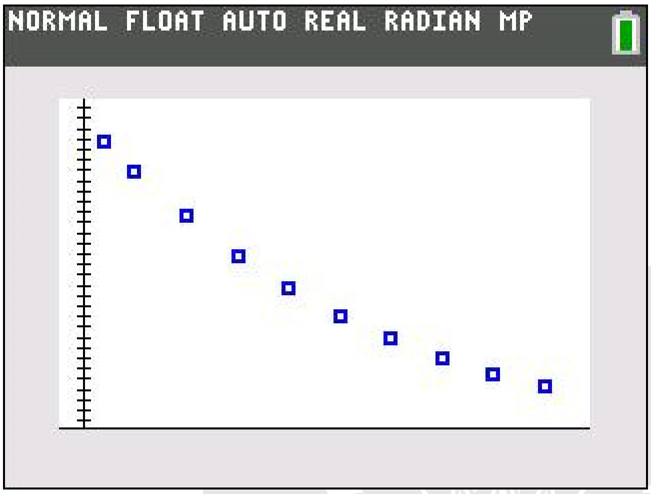
$x \neq 0.02$ ($\because B$ and C not independent)

$$\begin{aligned} \mathrm{P}(A' \cap B' \cap C') &= 1 - \mathrm{P}(A \cup B \cup C) \\ &= 0.165 + x \end{aligned}$$

$$\begin{aligned} \max \mathrm{P}(A' \cap B' \cap C') &= 0.165 + \text{max value of } x \\ &= 0.165 + 0.165 \\ &= 0.33 \end{aligned}$$

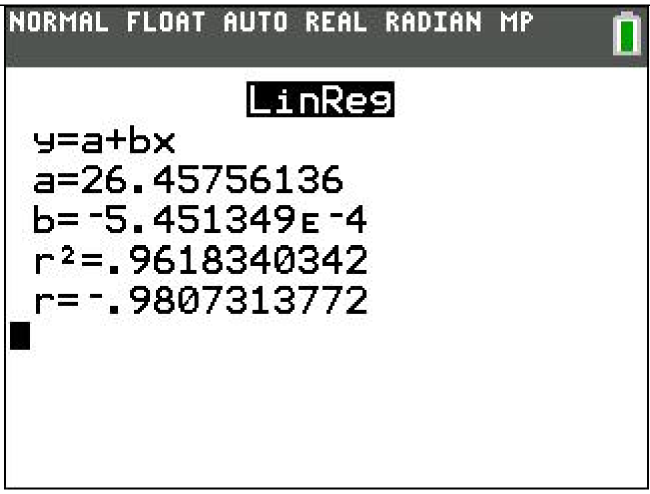
$$\begin{aligned} \min \mathrm{P}(A' \cap B' \cap C') &= 0.165 + \text{min value of } x \\ &= 0.165 + 0 \\ &= 0.165 \end{aligned}$$

Question 10

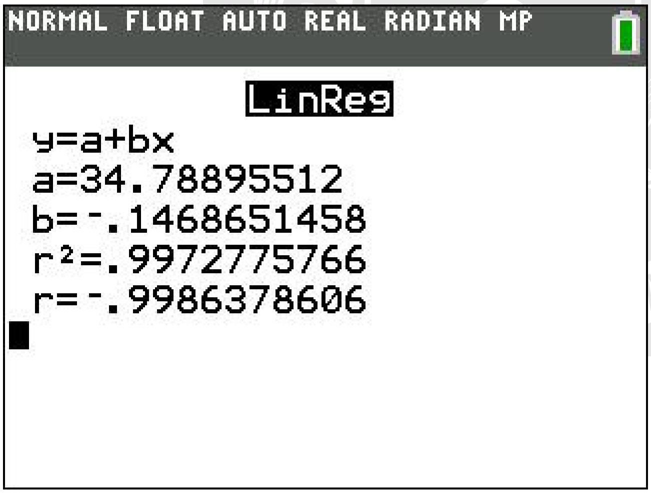
<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)(i)</td><td>


</td><td>Remember to label x-axis as h(independent variable) and y-axis as P(dependent variable).</td></tr>
<tr><td>(ii)</td><td>
NORMAL FLOAT AUTO REAL RADIAN MP
<table>
<tr><th>L1</th><th>L2</th><th>L3</th><th>L4</th><th>L5</th><th>4</th></tr>
<tr><td>27.8</td><td>2000</td><td>7.6009</td><td>44.721</td><td>------</td><td></td></tr>
<tr><td>24.9</td><td>5000</td><td>8.5172</td><td>70.711</td><td></td><td></td></tr>
<tr><td>20.6</td><td>10000</td><td>9.2103</td><td>100</td><td></td><td></td></tr>
<tr><td>16.9</td><td>15000</td><td>9.6158</td><td>122.47</td><td></td><td></td></tr>
<tr><td>13.8</td><td>20000</td><td>9.9035</td><td>141.42</td><td></td><td></td></tr>
<tr><td>11.1</td><td>25000</td><td>10.127</td><td>158.11</td><td></td><td></td></tr>
<tr><td>8.89</td><td>30000</td><td>10.309</td><td>173.21</td><td></td><td></td></tr>
<tr><td>7.04</td><td>35000</td><td>10.463</td><td>187.08</td><td></td><td></td></tr>
<tr><td>5.52</td><td>40000</td><td>10.597</td><td>200</td><td></td><td></td></tr>
<tr><td>4.28</td><td>45000</td><td>10.714</td><td>212.13</td><td></td><td></td></tr>
</table>
$L_4=\sqrt{L_2}$

(a)
</td><td></td></tr>
</table>

<table>
<tr><td></td><td>


(b)


(c)


</td><td></td></tr>
<tr><td>(iii)</td><td>(c) has product moment correlation coefficient nearest to –1.
Equation of best model:
$P = 34.789 - 0.14687\sqrt{h} \approx 34.8 - 0.147\sqrt{h}$</td><td></td></tr>
<tr><td>(iv)</td><td>$P = 34.789 - 0.14687\sqrt{3.28h}$ (where h is in metres)
$\approx 34.8 - 0.266\sqrt{h}$</td><td></td></tr>
</table>

Question 11

No.	Suggested Solution	Remarks for Student
	Available letters 1C, 2A, 2B, 1G, 1E, 1S Total: 8 letters.	
(i)	Total number of arrangements $= \frac{8!}{2!2!} = 10080$	
(ii)	$10080 - 1 = 10079$	
(iii)	$6! = 720$	
(iv)	By Complement Principle, number of different arrangements that can be made with no two adjacent letters the same is = (i) – (iii) – Number of ways with only "AA" adjacent and "BB" not adjacent – Number of ways with only "BB" adjacent and "AA" not adjacent $= 10080 - 720 - 2[5!\,{}^6C_2] = 5760$	

Question 12

No.	Suggested Solution	Remarks for Student
	Let X be the mass of an apple in grams. $X \sim \mathrm{N}\left(300, 20^2\right)$ Let Y be the mass of a pear in grams. $Y \sim \mathrm{N}\left(200, 15^2\right)$	
(i)	$X_1 + X_2 + \ldots X_5 \sim \mathrm{N}(1500, 2000)$ $\mathrm{P}(X_1 + X_2 + \ldots X_5 > 1600) = 0.0127$	
(ii)	$Y_1 + Y_2 + \ldots Y_8 \sim \mathrm{N}(1600, 1800)$ $(X_1 + X_2 + \ldots X_5) - (Y_1 + Y_2 + \ldots Y_8) \sim \mathrm{N}(-100, 3800)$ $\mathrm{P}(X_1 + X_2 + \ldots X_5 > Y_1 + Y_2 + \ldots Y)$ $= \mathrm{P}((X_1 + X_2 + \ldots X_5) - (Y_1 + Y_2 + \ldots Y_8) > 0)$ $= 0.0524$	
(iii)	$0.85(X_1 + X_2 + \ldots X_5) \sim \mathrm{N}(1275, 1445)$ $0.9(Y_1 + Y_2 + \ldots Y_8) \sim \mathrm{N}(1440, 1458)$ $0.85(X_1 + X_2 + \ldots X_5) + 0.9(Y_1 + Y_2 + \ldots Y_8) \sim \mathrm{N}(2715, 2903)$ $\mathrm{P}(0 < 0.85(X_1 + X_2 + \ldots X_5) + 0.9(Y_1 + Y_2 + \ldots Y_8) < 2750)$ $= 0.742$	