H2 Topic 9

Oscillations

Do you know why that most grandfather clocks, invented in 1656, will have a pendulum of length ≈1m while smaller versions have a pendulum of length ≈25cm? Do you know that these clocks will 'slow down' on top of Mount Everest?

These clocks were the most accurate time keepers until quartz was invented in 1927!

What is the period of these clocks (1.0m and 0.25m)?

A ***0.5 s*** ***B*** ***1.0 s*** ***C*** ***2.0 s*** ***D*** ***1 minute***

Content

- Simple harmonic motion
- Energy in simple harmonic motion
- Damped and forced oscillations: Resonance

Learning Outcomes

Candidates should be able to:

(a) describe simple examples of free oscillations.
(b) investigate the motion of an oscillator using experimental and graphical methods.
(c) understand and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.
(d) recognise and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.
(e) recall and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.
(f) recognise and use

$$v = v_0 \cos \omega t$$

$$v = \pm \omega \sqrt{(x_0{}^2 - x^2)}$$

(g) describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.
(h) describe the interchange between kinetic and potential energy during simple harmonic motion.
(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in cases such as a car suspension system.
(j) describe practical examples of forced oscillations and resonance.
(k) describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.
(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

References:

1 Advanced Level Physics by Loo Kwok Wai
2 College Physics by Young and Geller
3 Physics for Scientist and Engineers (5th Edition) by Serway

1 Introduction

(a) describe simple examples of free oscillations.

An **oscillation** is a periodic motion of an object about a certain mean equilibrium position with a continuous interchange of kinetic energy and potential energy.
Periodic (or harmonic) motion refers to any motion that repeats itself in equal intervals of time.

Examples: vibration of tuning fork, boat bobbing at anchor, a child playing on a swing, bells, pendulum of a grandfather clock, diaphragms in telephones and speaker systems and surging pistons in engines of cars.

When an object is displaced from its fixed equilibrium position and is free to oscillate, it will oscillate about this position with the *natural frequency* of the system. (We will go into more details of this natural frequency with a simple pendulum case study later.)

The object will oscillate forever under **free oscillations** as energy is conserved and quantities such as amplitude and period remain constant.

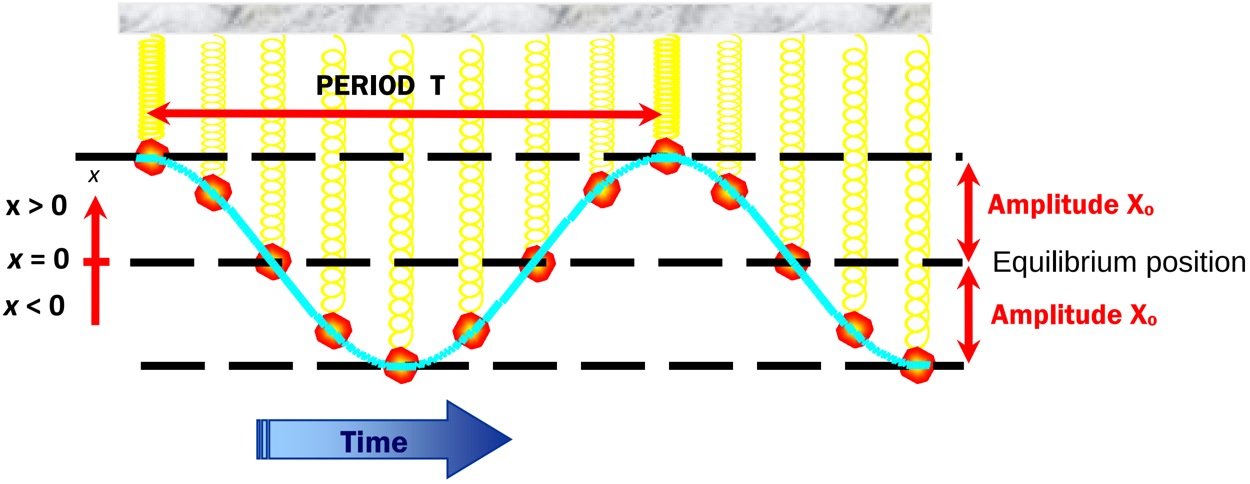
In real life, dissipative forces like air resistance and friction will cause the oscillations to fade with time, as we observed in our many practical exercises on oscillations. These oscillations are known as damped oscillations. **Damped oscillations** occur when there is a continuous transfer of energy to the surroundings such that the energy in the system decreases with time, hence the amplitude of the motion progressively decreases with time.

Forced oscillations are caused by continual input of energy by external source to an oscillating system to compensate the loss due to damping in order to maintain the amplitude of the oscillation.

1.1 Describing Oscillations

(c)	**understand and use the terms amplitude, period, frequency, angular frequency and phase difference and express the period in terms of both frequency and angular frequency.**

Consider displacement with time of a spring-mass oscillating in frictionless environment:



a. ***Equilibrium position (or neutral position)*** is the position at which no net force acts on the oscillating mass.

b. ***Displacement*** **(*x*)** is the distance of the oscillating mass from its equilibrium position at any instant in a stated direction.

c. ***Amplitude*** **(x_o)** is the maximum displacement of the oscillating mass from the equilibrium position in either direction.

d. ***Period*** **(*T*)** is the time taken for one complete oscillation. Unit: second.

e. ***Frequency*** **(*f*)** is the number of complete to-and-fro cycles per unit time made by the oscillating object. Unit: Hertz. 1 Hz is equal to one cycle per second.

f. ***Angular frequency*** **(ω)** of an oscillation is frequency $\times$ 2π. It is the angle in radians by which the phase of the motion changes per unit time.
Unit: radian per second (**rad s^{-1}**).

g. ***Phase*** is an angle in either degrees or radians which gives a measure of the fraction of a cycle that has been completed by an oscillating particle or by a wave.

h. **Phase difference** is a measure of <u>how much</u> one wave is <u>out of step</u> with another. It is measured in either degrees or radians.

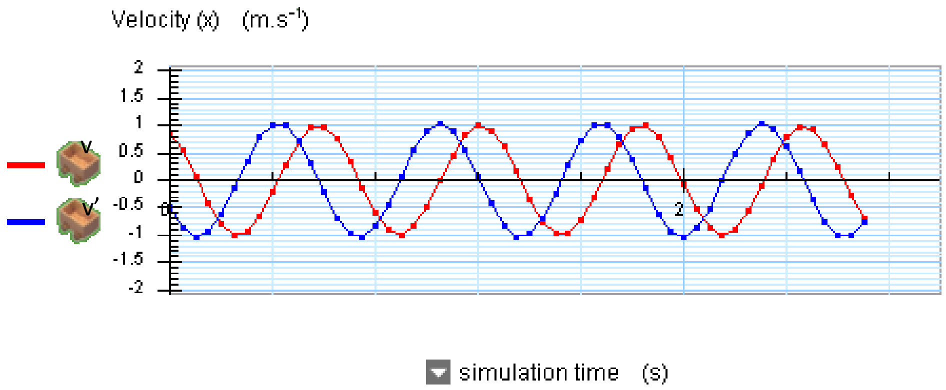
i. **In-phrase, Out of *Phase, Antiphase***

In phase: phase difference is zero.

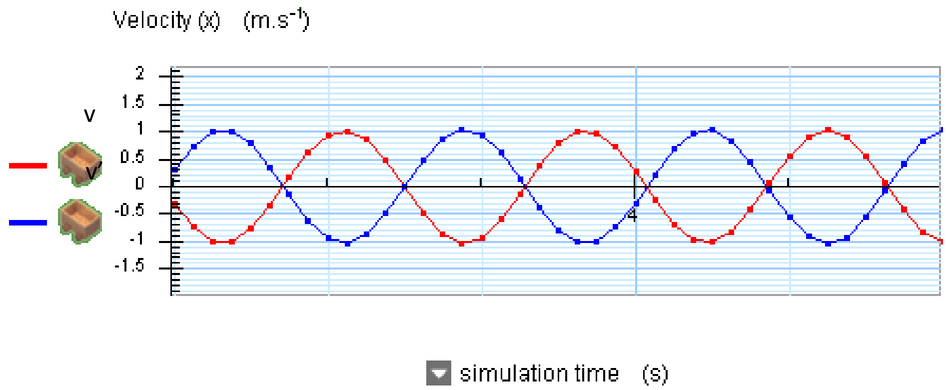
Out of phase: phase difference is not zero.

Antiphase: phase difference is 180 degrees or **π** radians.

Examples of Phase Difference



Phase difference of $\pi/2$, ($\Delta t = \frac{1}{4}T$)
If $v = v_0 \cos kt$, then $v' = v_0' \cos (kt - \pi/2) = -v_0' \sin kt$



Phase difference of π, completely out of phase, ($\Delta t = \frac{1}{2}T$)
If $v = v_0 \sin kt$, then $v' = v_0' \sin (kt - \pi) = - v_0' \sin kt$

Example 1

A man standing at the dock was observing the bobbing motion of a speedboat on the seawater just below the dock. He estimated that the boat could reach two extreme positions which were 3.0 m and 4.0 m below the dock surface respectively. He also counted that the boat would bob on average 30 times in 15 seconds. Assume that the speedboat was executing free oscillation, determine its
(a) amplitude, (b) period, (c) frequency, (d) vertical distance between the equilibrium position and the dock surface and (e) the angular frequency.

Solution

(a) $x_0 = \frac{1}{2}(4.0 - 3.0) = \frac{1}{2}(1.0) =$ **0.50 m**

(b) $T = \dfrac{15}{30} =$ **0.50 s**

(c) $f = \dfrac{1}{T} = \dfrac{1}{0.50} =$ **2.0 Hz**

(d) vertical distance = $\frac{1}{2}(4.0 + 3.0) =$ **3.5 m**

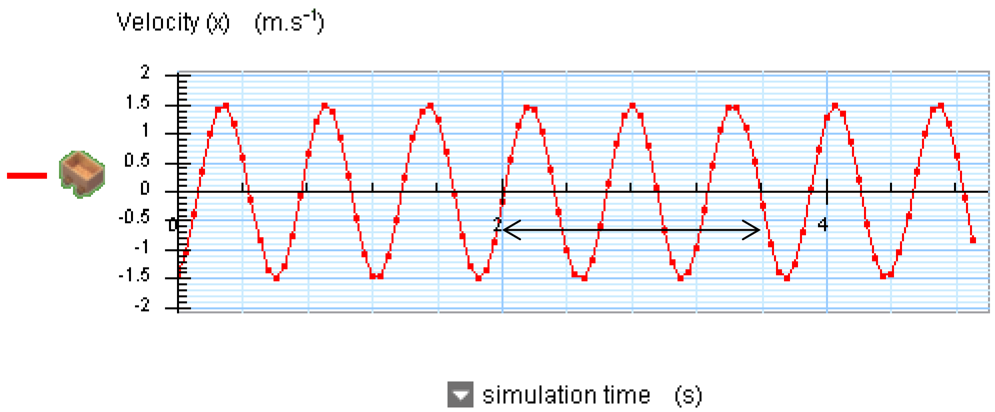
(e) $\omega = 2\pi f = 2\pi f = 2\pi(2.00) = 4.00\pi =$ **12.6 rads^{-1}**

<u>Example 2</u>
A pendulum takes 31.7 seconds to complete 25 oscillations, if another similar pendulum is 1.70 seconds behind the first, what is the phase difference between the pendulums?

Solution
31.7 ÷ 25 = 1.268 = 1.27 s
The period is 1.27 s

Since the period of the pendulum is 1.27s, the actual time difference in the oscillation is Δt= 1.70 – 1.268 = 0.432 s
$\phi = 2\pi\Delta t/T = 2.14$ rad

<u>Example 3</u>
The velocity time graph of the oscillating cart is shown below. Determine the
(a) Period (b) Frequency (c) Angular Frequency



Solution
(a) 2.5 waves in 1.6 seconds, therefore period is 0.64 s
(b) f = 1/T = 1.56 Hz
(c) $\omega = 2\pi f = 9.82$ rad s^{-1}

Food for thought
Is circular motion a type of oscillatory motion? Why? Can ω be negative? Why?
Is Angular frequency = Angular velocity?

2 Dynamics of Simple Harmonic Motion (SHM)

(d)	**recognise and use the equation $a = -\omega^2 x$ as the defining equation of simple harmonic motion.**
(e)	**recall and use $x = x_0 \sin \omega t$ as a solution to the equation $a = -\omega^2 x$.**

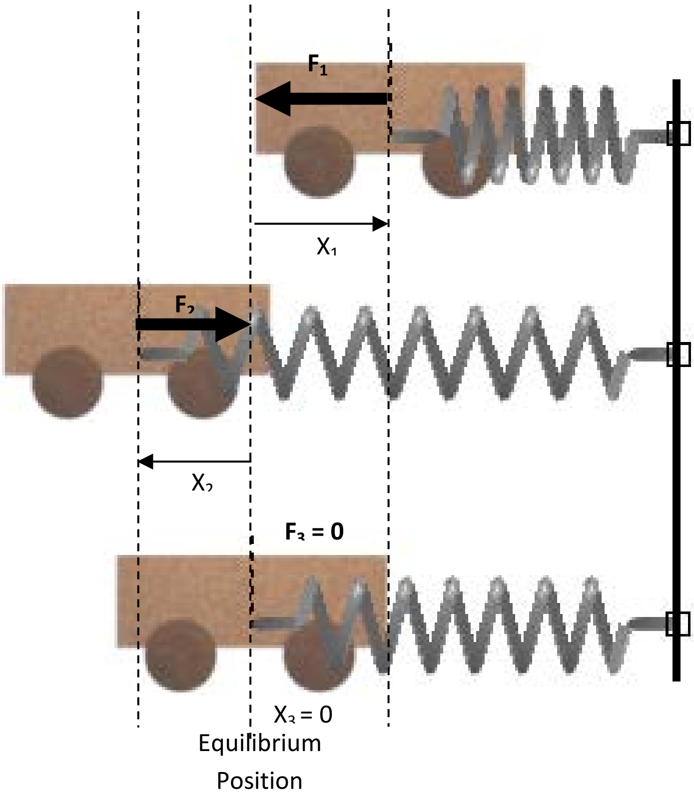
The example of the cart oscillating due to the spring is a classic case of simple harmonic motion. SHM is the simplest form of oscillatory motion. For SHM to take place, we assume that the objects are in free oscillation (no energy lost through resistive forces) and that there is a restoring force (i.e. Hooke's law) to bring the object back to the fixed (equilibrium) position O.

<ins>Case 1</ins>
The spring is compressed, the displacement is to the right while the spring exerts a pushing force to the left

<ins>Case 2</ins>
The spring is stretched, the displacement is to the left while the spring exerts a pulling force to the right

<ins>Case 3</ins>
The spring is in equilibrium position, there is no net force (but the cart can be moving)

As mentioned earlier, the direction of the force (or acceleration) is in opposite direction to the displacement, hence $a = \frac{d^2x}{dt^2} = -\omega^2 x$.



<ins>Enrichment (Not required to solve ODE for Exams)</ins>

Note: *Equations of the form* $\frac{d^2x}{dt^2} = -\omega^2 x$ *are known as differential equations. Since we differentiate x with respect to time twice, this is a second order ordinary differential equation (ODE). Mathematicians devised many methods in solving ODE which we will not be going through in details here. But the sine/cosine like behaviour of the harmonic oscillation gives us a hint that the solution to x may be related to sine/cosine functions.*

In order to find the equation of motion for simple harmonic motion, we will need to look for a function x(t) that satisfy the differentiate equation of the form $\frac{d^2x}{dt^2} = -\omega^2 x$. *Just like solving for the equation of motions for kinematics, we look at the displacement-time and velocity-time graph for inspirations. We could use the sine wave like displacement-time and velocity-time graph of the SHM to help us understand the problem better.*

As smart physics students, let's assume that the function x has the general form

$$x = A \sin Bt$$

where A and B are constants to be solved.

Enrichment (Not required to solve ODE for Exams)

Given that the angular frequency of the system is $\omega = 2\pi/T$ or $2\pi = T\omega = BT$, we can deduce that constant B = ω since the sine wave need to make one complete wave for every time interval T. We get

$$x = A\sin\omega t$$

If we let the maximum displacement of the oscillation be x_0, then A = x_0 since $-1 \leq \sin\omega t \leq 1$.
We get

$$x = x_0 \sin\omega t$$

This is the general solution for simple harmonic oscillation. The displacement of the object in SHM is given by the expression $x = x_0 \sin\omega t$ so we will know where our object is at any given point of time.

Let's verify that the expression $x = x_0 \sin\omega t$ satisfies the equation a= $\frac{d^2x}{dt^2} = -\omega^2 x$

$$\frac{dx}{dt} = x_0\omega\cos\omega t$$

$$\frac{d^2x}{dt^2} = -x_0\omega^2 \sin\omega t = -\omega^2(x_0 \sin\omega t) = -\omega^2 x \text{ (Verified)}$$

IMPORTANT!

Therefore, the general equation **defining** simple harmonic motion is given by

$$a = -\omega^2 x$$

Food for Thought
If a body's motion is given by a = $\omega^2 x$, is the body undergoing SHM? Why?

DEFINITION!

Simple Harmonic motion (SHM) is defined as oscillatory motion of a particle whose acceleration is directly proportional to its displacement from the equilibrium position and this acceleration is always in opposite direction to its displacement.

Solution to SHM

$x = x_0 \sin\omega t$ is a solution to $a = -\omega^2 x$

This equation is in the formula list

Food for Thought
If we start off with $x = x_0 \cos\omega t$ instead, we could obtain an alternative solution to the general equation of SHM, $a = -\omega^2 x$. Why? What is the difference between $x = x_0 \cos\omega t$ and $x = x_0 \sin\omega t$? (Hint: Think of the starting position x of the object in SHM, if it starts at equilibrium position versus starting at maximum displacement)

The general solution for $a = -\omega^2 x$ is $x = x_0 \sin(\omega t + \phi)$, what is ϕ denoted here?

2.1 Velocity of SHM

(f)	**recognise and use** $v = v_0 \cos\omega t, v = \pm\omega\sqrt{(x_0{}^2 - x^2)}$

Since we have shown that $x = x_0 \sin\omega t$, the velocity of the object in simple harmonic motion is simply given by the expression

Therefore the velocity of the object at a given time *t* is

$$v = \frac{dx}{dt} = x_0\omega\cos\omega t = v_0\cos\omega t \text{ , where } v_0 = x_0\omega$$

$$v = v_0\cos\omega t$$

This equation is in the formula list

Food for Thought
In circular motion, the linear velocity is given by $v = r\,\omega$, can you draw any similarities to the expression of v_0 above?

To get rid of the trigonometric functions, we can try to square the functions of *x* and *v*,

$v^2 = x_0^2\omega^2\cos^2\omega t$ --- Eqn (1)

$x^2 = x_0^2\sin^2\omega t$,

$\omega^2 x^2 = x_0^2\omega^2\sin^2\omega t$ --- Eqn (2)

Adding Eqn (1) and Eqn (2) together,

$$v^2 + \omega^2 x^2 = x_0^2\omega^2\sin^2\omega t + x_0^2\omega^2\cos^2\omega t$$

$$v^2 + \omega^2 x^2 = x_0^2\omega^2$$

$$v^2 = \omega^2(x_0^2 - x^2)$$

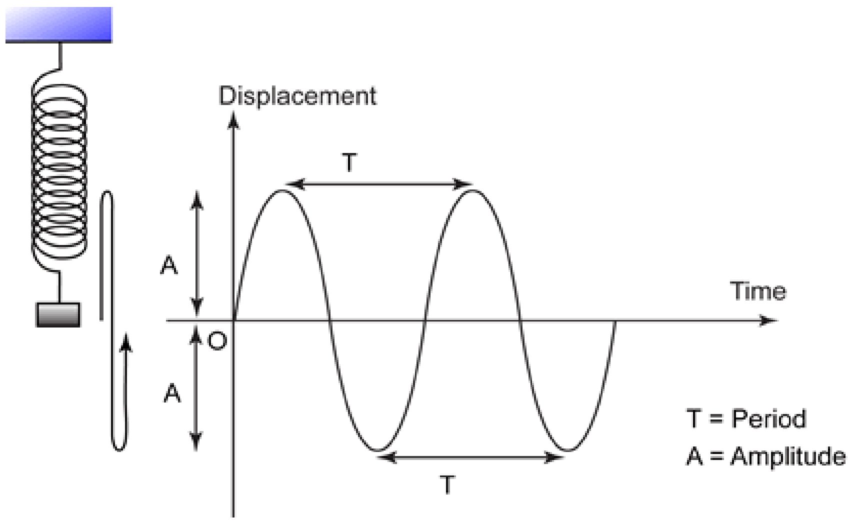
$$v = \pm\omega\sqrt{(x_0^2 - x^2)}$$

Therefore the velocity of the object at position *x* is given by $v = \pm\omega\sqrt{(x_0^2 - x^2)}$

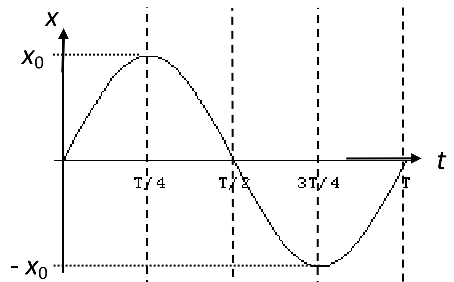
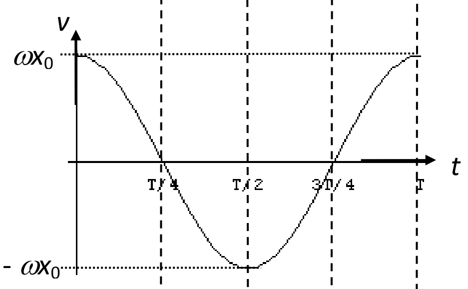
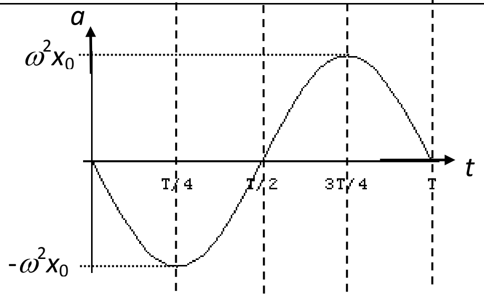
This equation is in the formula list

2.2 Graphical Representation

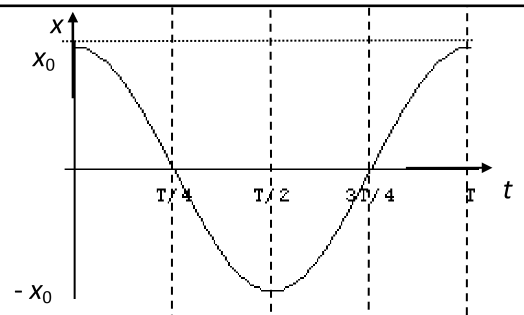
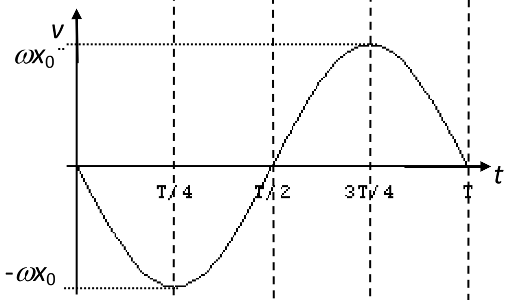
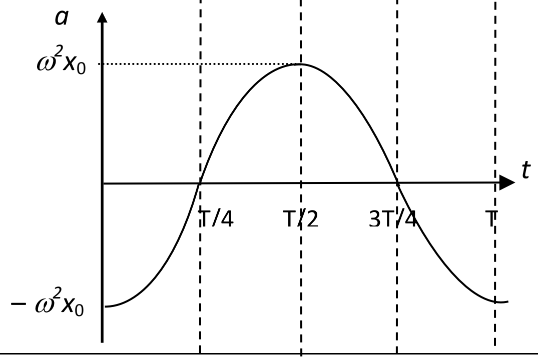
(b)	**investigate the motion of an oscillator using experimental and graphical methods.**
(g)	**describe, with graphical illustrations, the changes in displacement, velocity and acceleration during simple harmonic motion.**



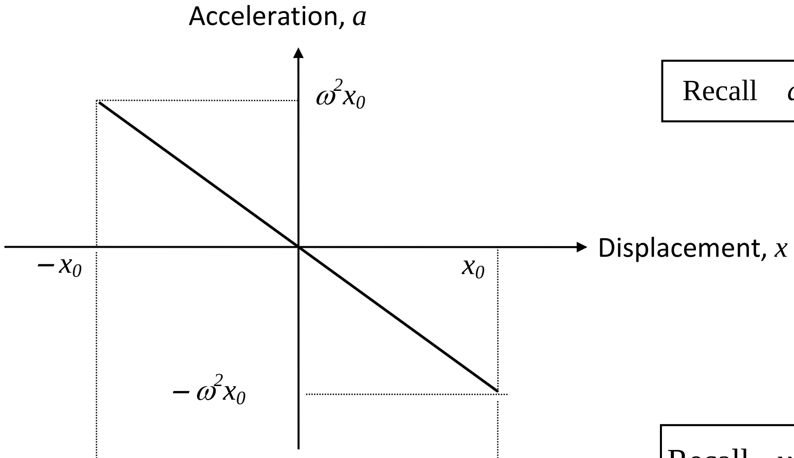
<u>Case 1: Mass initially at equilibrium position, i.e. when $t = 0$, $x = 0$</u>
The corresponding **<u>displacement-time</u>** graph, **<u>velocity-time</u>** graph and **<u>acceleration-time</u>** graph are as shown.

<u>1) Displacement-Time Graph</u> $x = x_0 \sin \omega t$	
<u>2) Velocity-Time Graph</u> $v = \omega x_0 \cos \omega t$ $(v = \frac{dx}{dt})$	
<u>3) Acceleration-Time Graph</u> $a = -\omega^2 x_0 \sin \omega t$ $(a = \frac{dv}{dt} = \frac{d^2x}{dt^2})$	

Case 2: Mass initially at maximum displacement, i.e. When $t = 0$, $x = x_0$

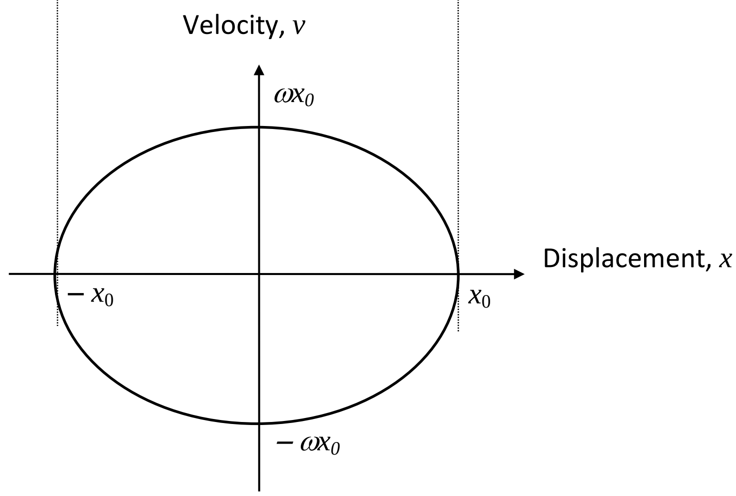
1) **Displacement-Time Graph** $\boldsymbol{x = x_0 \cos \omega t}$	
2) **Velocity-Time Graph** $\boldsymbol{v = \underline{-\omega x_0 \sin \omega t}}$ ($v = \dfrac{dx}{dt}$)	
3) **Acceleration-Time Graph** $\boldsymbol{a = \underline{-\omega^2 x_0 \cos \omega t}}$ ($a = \dfrac{dv}{dt} = \dfrac{d^2x}{dt^2}$)	

The graphical representations for *v* and *a* against *x* are as follows:



Recall $a = -\omega^2 x$

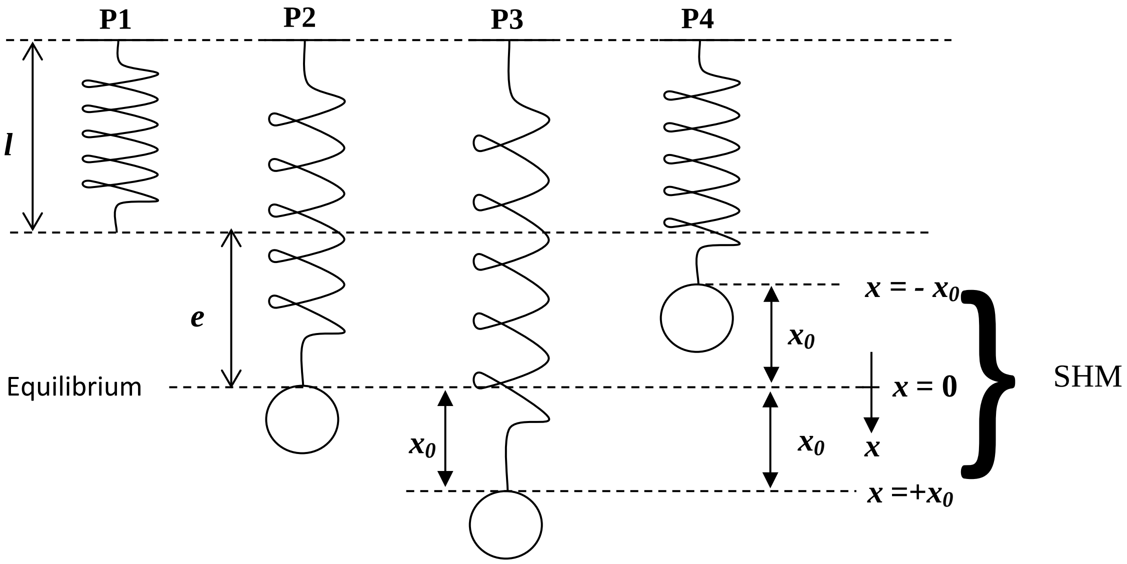
Recall $v = \pm\omega\sqrt{x_o^2 - x^2}$

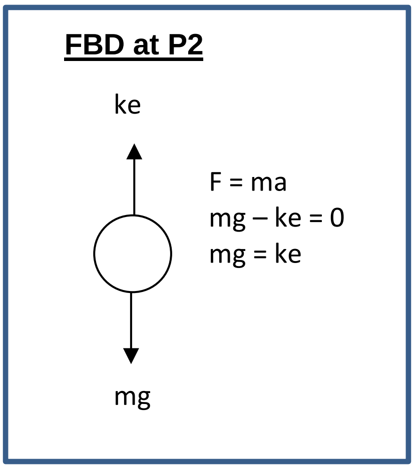


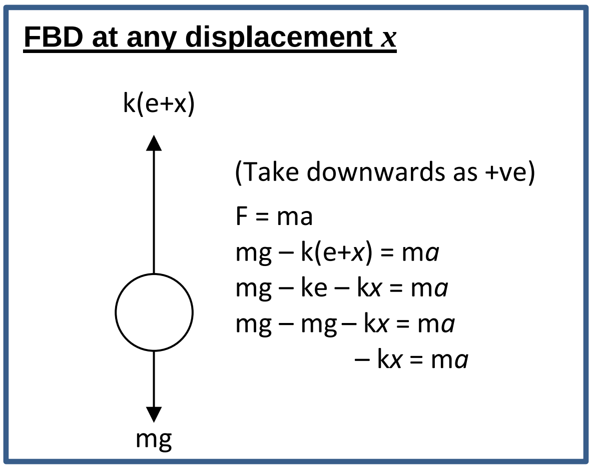
2.3 Case Studies

A Spring- Mass System

In this case, we load a spring of natural length *l* with a mass *m*. The spring extends by *e* as a result of the mass to counter the weight of the mass. To achieve SHM, we pull the mass to extend the spring by a further distance x_0 and release it. The mass oscillates freely about the equilibrium position between $x = x_0$ and $x = -x_0$.







Note the acceleration *a* of the mass at a displacement *x* from the equilibrium point is given by

$$a = -\frac{k}{m}x$$

$$a = -\omega^2 x$$

$$\therefore \ \omega = \sqrt{\frac{k}{m}}$$

For SHM, $\omega = 2\pi f$ and $f = \frac{1}{T}$, we obtained $\omega = \frac{2\pi}{T}$

Period $T = 2\pi\sqrt{\frac{m}{k}}$

Derivation not required

We also know that $mg = ke$, hence $\dfrac{m}{k} = \dfrac{e}{g}$,

Therefore, Period $T = 2\pi\sqrt{\dfrac{e}{g}}$

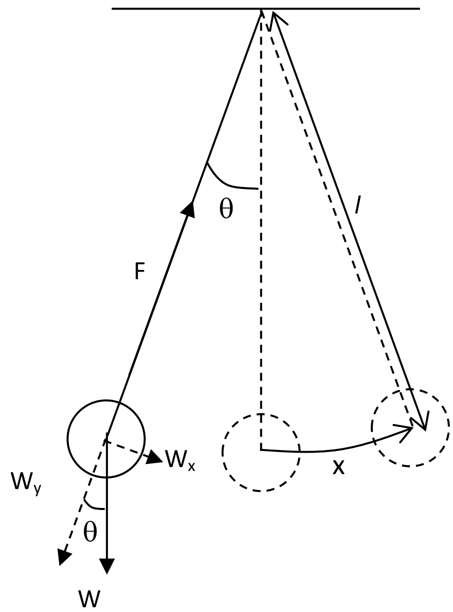
Derivation not required

This is a commonly used expression when we conduct our oscillation experiment. Eventually, for a fixed mass, the period of oscillation for this experiment depends only on the amount of extension which in turn depends on the stiffness of the spring, as long as Hooke's law applies.

Challenge
Can you derive the same expressions above for a horizontal spring mass system?

A Simple Pendulum

Another case study that we can look into in details would be our simple pendulum. In this case, we hang a mass *m* on a non extensible light string of length *l* and set it to swing back and forth as shown in the diagram on the right. The restoring force in this case comes from gravity. To be exact, the restoring force is the W_x component of the weight as indicated.



$$W_x = W\sin\theta$$

For small angle θ, $\sin\theta \approx \theta$ (In radians),
And arc length $x = l \times \theta$
Combining, we get

$$W_x = -\frac{xW}{l} = ma$$

Applying $a = -\omega^2 x$,

$$-\frac{xW}{ml} = a = -\omega^2 x$$

Rearranging and substituting $W = mg$

$$\omega = \sqrt{\frac{g}{l}}$$

Hence period T is

$$T = 2\pi\sqrt{\frac{l}{g}}$$

We include a negative sign is because the displacement is in the opposite direction of the restoring force.

Derivation not required

This is another equation that we see in oscillation practical quite often. The only assumption in this case is that the angle of oscillation is small so that the small angle approximation holds. Therefore it is important to keep the amplitude of oscillation small for pendulum experiment so that the curved arc *x* is approximately straight. In this case, the period of oscillation depends on the length of the pendulum and the acceleration due to gravity. It is independent of mass!

Example 4

A mass oscillates with an amplitude of 0.0030 m and a period of 0.020 s.
(a) Find the frequency and angular frequency
(b) State equation of displacement, given that the mass is initially at equilibrium position
(c) Find the equation of velocity and equation of acceleration.

Given: x_0 = 0.0030 m, T = 0.020 s

(a) $f = \dfrac{1}{T} = \dfrac{1}{0.020} = \underline{\textbf{50 Hz}}$

ω = 2 π f = 2 π f = 2 π (50) = 100 π = **<u>314 rads⁻¹</u>**

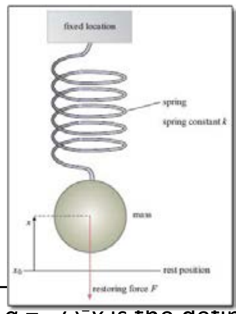
(b) $x = x_0 \sin \omega t$

$x = \underline{\textbf{0.0030 sin 314 } t}$

(c) $x = x_0 \sin \omega t$

$$v = \frac{dx}{dt} = \frac{d(x_o \sin \omega t)}{dt} = \omega x_o \cos \omega t$$

$$a = \frac{dv}{dt} = \frac{d(\omega x_o \cos \omega t)}{dt} = -\omega^2 x_o \sin \omega t = -\omega^2 x$$



Note: $a = -\omega^2 x$ is the defining equation for SHM

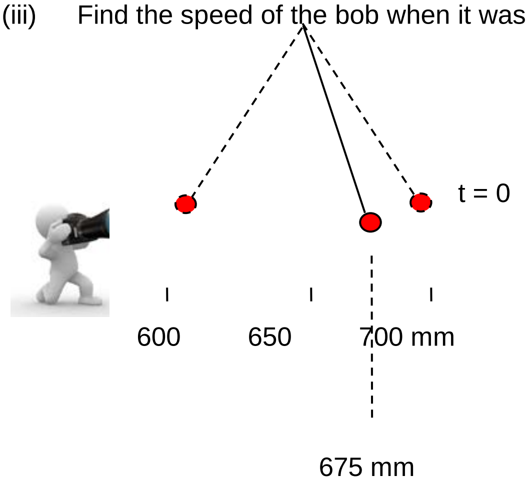
∴ $v = 314\,(0.0030) \cos 314\,t = \underline{\textbf{0.942 cos 314 } t}$
∴ $a = -\,314^2\,(0.0030) \sin 314\,t = \underline{\textbf{- 296 sin 314 } t}$

Example 5

In order to check the speed of a camera shutter, the camera was used to photograph the bob of a simple pendulum moving in front of a horizontal scale.

The extreme positions of the bob were at the 600 mm and 700 mm marks. The photograph showed that while the shutter was open the bob moved from 700 mm to 675 mm mark.

Given that the period of the pendulum was 2.0 s.
(i) How long did the shutter remain open?
(ii) Without calculation, comment on the time taken for the bob to travel from 675 mm to 650 mm.
(iii) Find the speed of the bob when it was at the 675 mm mark.



(i) Given T = 2.0 s

$$x_o = \frac{700-600}{2} = 50\,mm$$

$$x = 675 - 650 = 25\,mm$$

At $t = 0$, $x = x_0$

Hence $x = x_0 \cos \omega t$

$$25 = 50 \cos\left(\frac{2\pi}{2.0} t\right)$$

t = **<u>0.33 s</u>**

(ii) Time taken should be <u>shorter</u> because the speed of the bob increases towards equilibrium.

(iii) At 675 mm mark, from (i), t = 0.33 s

$v = -\,\omega\, x_0 \sin \omega t$

$$v = -\left(\frac{2\pi}{2.0}\right)(50)\sin\left(\frac{2\pi}{2.0}(0.33)\right)$$

= **<u>-135 mm s⁻¹</u>**

Note: Negative since it is moving to the left

2.4 Energy of Simple Harmonic Motion (SHM)

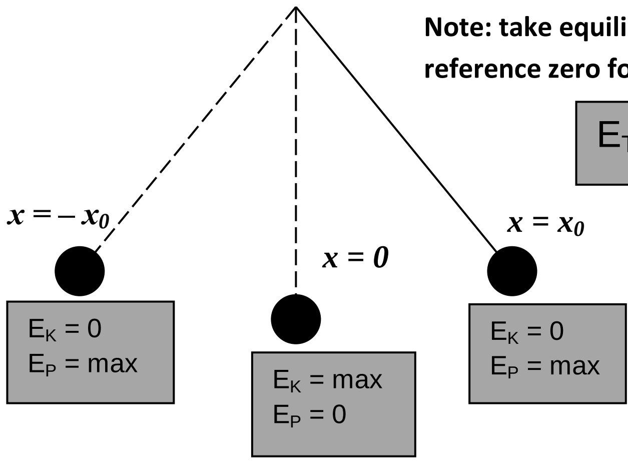
(h)	**describe the interchange between kinetic and potential energy during simple harmonic motion.**

A Simple Pendulum

An object executing SHM interchanges its **kinetic and potential energies**. For the simple pendulum system, the bob loses kinetic energy after passing through the middle of the swing, and then stores the energy as potential energy as it rises to the top of the swing.

Note: take equilibrium position as reference zero for Potential Energy

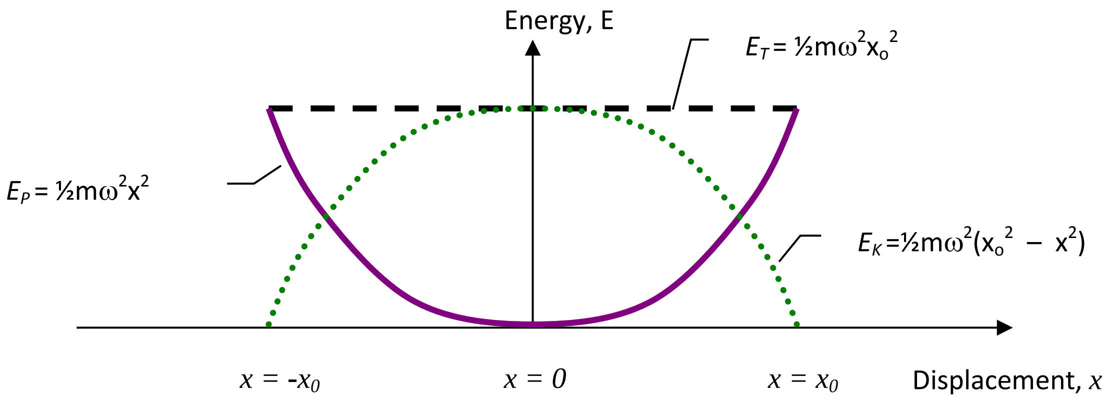
$E_T = E_K + E_P$



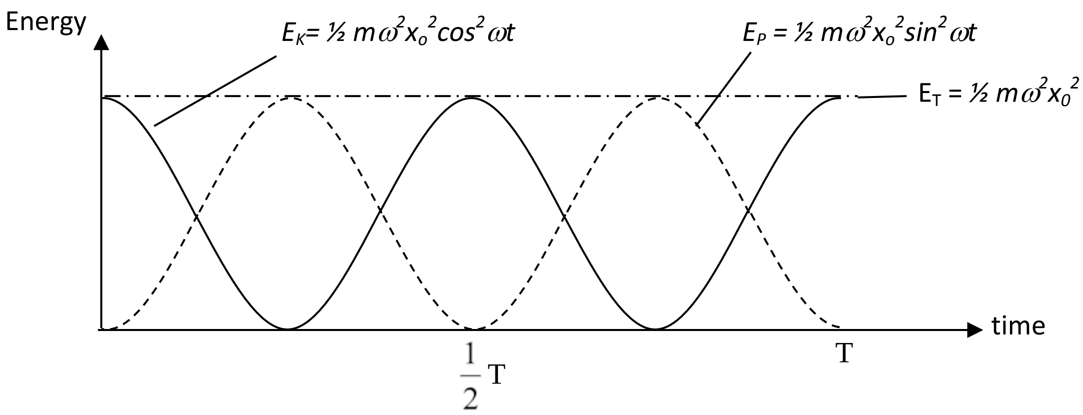
Assume air resistance to be negligible,

	Function of displacement (x)	**Function of time (t)**
Kinetic Energy E_K	Sub. $v = \pm\ \omega\sqrt{x_o^2 - x^2}$ into $E_K = ½mv^2$ To get $\mathbf{E_K = ½m\omega^2(x_o^2 - x^2)}$	If $x = x_o \sin \omega t$, Then differentiating x w.r.t time, $v = \omega x_o \cos \omega t$ Which will be sub. into $E_K = ½mv^2$ To get $\mathbf{E_K = ½m\omega^2 x_o^2 \cos^2 \omega t}$
Potential Energy E_P	$E_P = E_T - E_K$ $= ½m\omega^2x_o^2 - ½m\omega^2(x_o^2 - x^2)$ Hence, $\mathbf{E_P = ½m\omega^2x^2}$	Sub. $x = x_o \sin \omega t$, into $E_P = ½m\omega^2 x^2$ To get $\mathbf{E_P = ½m\omega^2 x_o^2 \sin^2 \omega t}$
Total Energy E_T	Kinetic energy E_K is maximum when $x = 0$, as velocity is maximum at the centre of oscillation. Max. $E_K = \frac{1}{2}m\omega^2x_0^2$, $E_P = 0$ (vice versa) $\mathbf{E_T} = \frac{1}{2}m\omega^2x_0^2$ (at all positions) Hence E_T is shown to be independent of its position.	$E_T = E_K + E_P$ $= \frac{1}{2}m\omega^2x_0^2\cos^2\omega t + \frac{1}{2}m\omega^2x_0^2\sin^2\omega t$ $\mathbf{E_T} = \frac{1}{2}m\omega^2x_0^2$ Hence E_T is shown to be independent of time.

The graphical representations of various energies with respect to displacement are as follows.



The graphical representation of various energies with respect to time are as shown below.



A Spring-Mass System

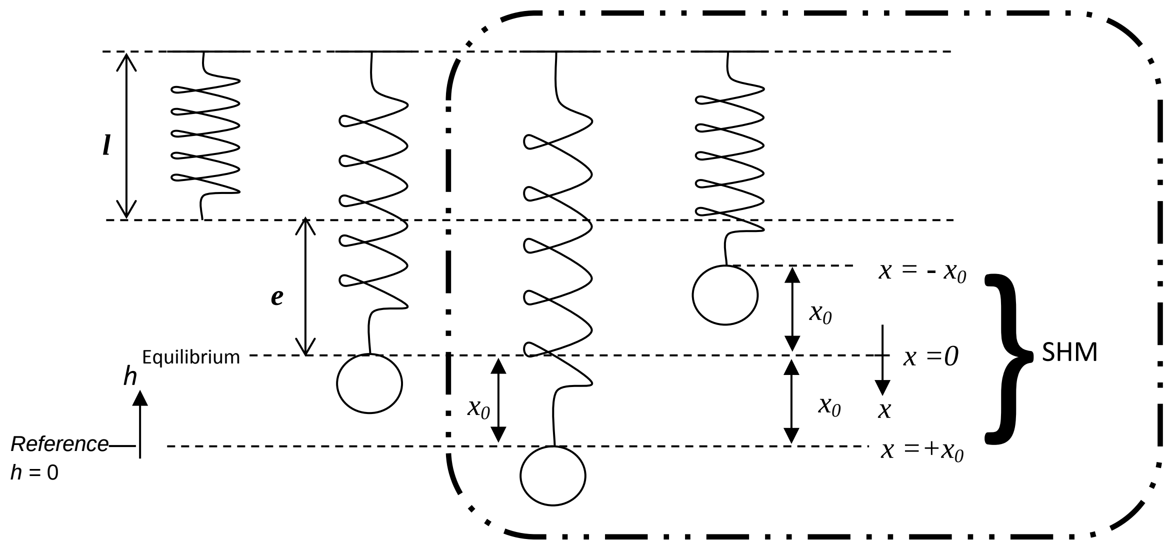
An object executing SHM in a spring-mass system interchanges three types of energies, namely **kinetic energy, gravitational potential energy and elastic potential energy**.

Kinetic energy (KE) depends on the speed of the object, which in turn depends on its displacement from the equilibrium position.

Gravitational potential energy (GPE) depends on the height of the object from an *arbitrary chosen point*, usually taken at the lowest point of the oscillation.

Elastic potential energy (EPE) depends on the extension of the spring from its original length and can be calculated by assuming Hooke's Law is valid.

Total energy (TE) is sum of the three energies and is constant throughout the SHM.



Guiding Equations:

KE = ½ mv^2 where $v = \pm\ \omega\sqrt{x_o^2 - x^2}$ and x is displacement from equilibrium position

GPE = mg*h* where h = 0 at the lowest point of the oscillation (*arbitrary*) and $h = x_0 - x$

EPE = ½ k (*e* + *x*)^2 where ***e*** is the extension of the spring at equilibrium position

TE = KE + GPE + EPE

	KE	GPE	EPE	TE
Displacement $x = x$	$\frac{1}{2} m\omega^2(x_0^2 - x^2)$	$mg(x_0 - x)$	$\frac{1}{2} k (e + x)^2$	**KE + GPE + EPE**
Highest Point $x = -x_0$	0	$2mgx_0$	$\frac{1}{2} k (e - x_0)^2$	
Equilibrium Position $x = 0$	$\frac{1}{2} m\omega^2 x_0^2$	mgx_0	$\frac{1}{2} k e^2$	
Lowest Point $x = +x_0$	0	0	$\frac{1}{2} k (e + x_0)^2$	

Note: **PE = ½ m $\omega^2 x^2$ cannot be applied directly** here because **PE** = **GPE + EPE** and we did not set GPE + EPE = 0 at x = 0.

Example 6

Hui Yi did 1.5 J of work to set a pendulum of mass 0.50 kg in oscillation. If the pendulum takes 30 seconds to complete 20 oscillations, find the maximum displacement of the pendulum.

Solution

Period of the pendulum T = 30/20 = 1.5 s
Angular frequency $\omega = 2\pi/T = 4.2$ rad s^{-1}
Using PE = $\frac{1}{2}m\omega^2x^2$,
$1.5 = \frac{1}{2} \times 0.50 \times 4.2^2\, x_0^2$
$x_0 = 0.58$ m
The displacement of the pendulum is 0.58 m

Note: If Hui Yi did 1.5 J just lifting the pendulum, she would have lifted it by 1.5 ÷ (0.5 × 9.81) = 0.31 m only. Therefore the displacement here is different from the height.

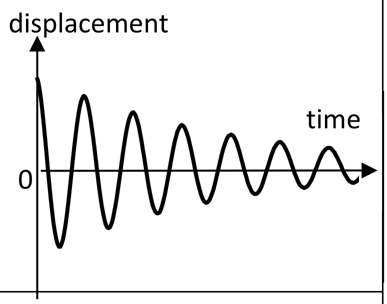
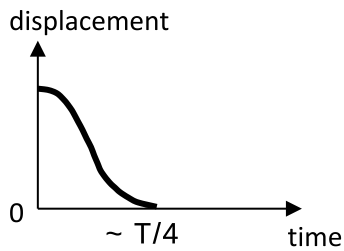
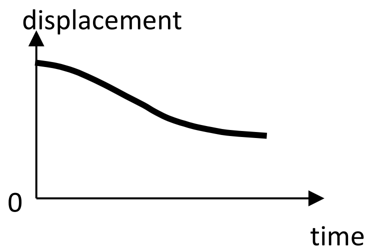
$\therefore PE = \frac{1}{2}m\omega^2 x_0^2 \sin^2 \omega t = mgh$ is valid. In fact, the displacement measured here is the arc length of the pendulum swing

3 Damped Oscillations, Forced Oscillations and Resonance

(i) describe practical examples of damped oscillations with particular reference to the effects of the degree of damping and the importance of critical damping in cases such as a car suspension system.

The assumption that there is no resistance to motion is not valid in everyday situations. Some energy will always be lost by the system to overcome dissipative forces, e.g. friction, air resistance. As energy is used to overcome resistive forces, the amplitude of the oscillation decreases since total energy is directly proportional to the square of amplitude. Such oscillations are called damped oscillation.

Damping is a process where energy is taken from an oscillating system as a result of dissipative forces.

Extent of Damping	Characteristics	Practical applications	Graphical representation
Light damping	Oscillations are maintained about the equilibrium position after the system has been displaced. The amplitude of oscillations decreases over a long time.	Meters which need to show rapid fluctuations such as “level meters’ on tape recorders are lightly damped.	
Critical damping	No oscillations occur. The motion is brought to rest in the shortest possible time.	Meters such as ammeter and voltmeter which required the needle not to oscillate at the final reading. Car suspension system.	
Heavy damping	No oscillations occur about the equilibrium position when the damping force increases beyond the point of critical damping. The system takes a long time to return to the equilibrium position relative to critically damped system.	Meters which are required to ignore transient changes (such as car fuel gauges) are overdamped. Door is heavily damped for safety reasons.	

j)	**describe practical examples of forced oscillations and resonance.**

Forced Oscillations and Resonance

- As damping causes energy to be lost, an external energy source is required to maintain oscillations at constant amplitude.

- The external agent providing the energy is called the driving force.

- The oscillator is the driven system, which undergoes forced oscillations due to the driving force.

- **Forced oscillations** are caused by continual input of energy by external source to an oscillating system to compensate the loss due to damping in order to maintain the amplitude of the oscillation.

- Examples:
  1. Engine vibrations cause bus windows to vibrate.
  2. Spinning drum causes washing machine to vibrate.

- Any mechanical system that is free to move has a natural frequency f_0 which depends on its dimensions and nature of material.

- A system undergoes forced vibrations when it is subjected to a driving force of a certain driving frequency f, there is a transfer of energy.

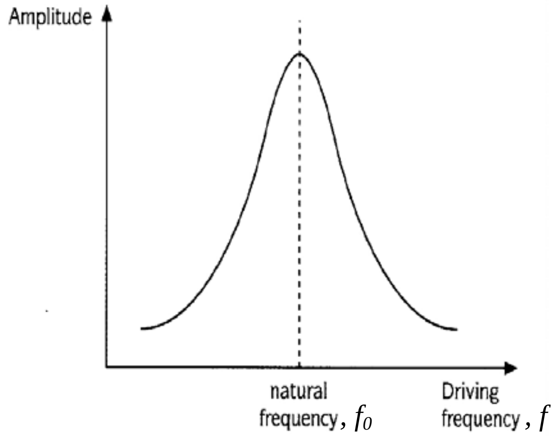
- When the driving frequency f = natural frequency f_0, the amplitude of vibration of the system becomes a maximum, and the system is set in resonance.

- **Resonance** occurs when the resulting amplitude of the system becomes a maximum when the driving frequency of external driving force equals to natural frequency of the system.
  At resonance, there is a maximum transfer of energy from the driving system to the driven system.

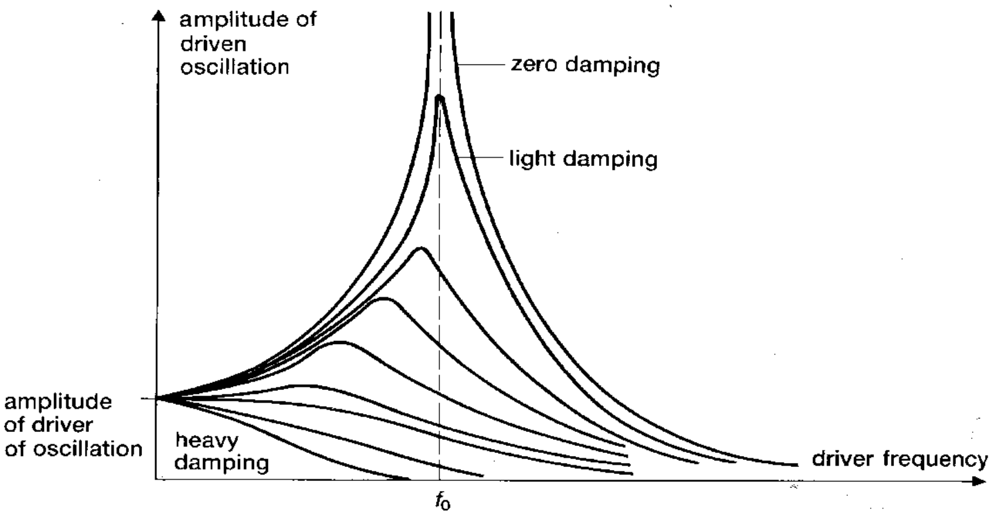
(k)	**describe graphically how the amplitude of a forced oscillation changes with frequency near to the natural frequency of the system, and understand qualitatively the factors which determine the frequency response and sharpness of the resonance.**

<u>Frequency Response (amplitude vs driving frequency graph)</u>

If the amplitude of oscillation is plotted against driving (forcing) frequency, the resulting graph is the *frequency response* of the system.



Consider an oscillating system of natural frequency f_o. When the <u>external driving frequency</u>, f is slowly increased from a small value, the amplitude of oscillation of the <u>driven oscillating system</u> will increase to a maximum value at the <u>resonant frequency</u> (where the driving frequency equals to the natural frequency of the system). Thereafter, as the driving frequency is increased further, the amplitude of oscillation of the driven system decreases.



The <u>sharpness</u> of the response depends on the damping forces involved. Theoretically, with <u>no damping</u>, the amplitude of oscillation of the driven system should tend to an infinitely large magnitude at the resonant frequency. This is because there is a continuous input of energy.

In practice, <u>damping</u> always exists and the amplitude and energy will increase until the rate of energy transfer is equal to rate of energy dissipation.

Notice that as the degree of damping is increased,

- the <u>*amplitude*</u> of the peak oscillation <u>decreases</u>.
- the <u>peak becomes broader</u> as it spreads over a wider range of frequencies (the response is <u>less sharp</u>).
- resonance occurs at a frequency <u>smaller</u> than the natural frequency.

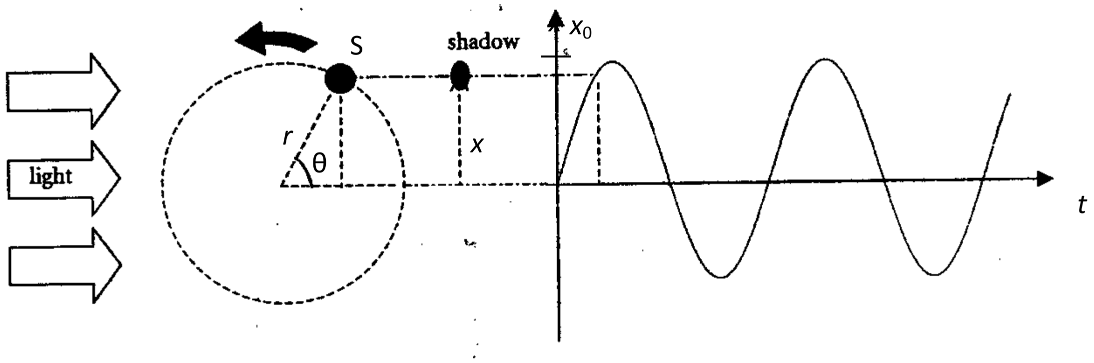
(l) show an appreciation that there are some circumstances in which resonance is useful and other circumstances in which resonance should be avoided.

Useful Resonance	Destructive Resonance
Musical Instruments Resonance is responsible for the **production of sound** in many musical instruments especially the wind instruments.	*Shattering of Glass* It has been known for **high-pitched** sound waves to shatter fragile objects. For example, an opera singer hitting a top note may shatter a wine glass at resonance.
Radio Receptions A **radio receiver** works on the principle of resonance. Our air is filled with **radio waves** of many different frequencies which the aerial picks up. The tuner can be adjusted so that the *frequency of the electrical oscillations* in the circuits is the same as that of the *radio waves* transmitted from the particular station we desired. The effect of resonance **amplifies the signals** contained in this wave while the radio waves of other frequencies are diminished.	*Earthquakes* During earthquakes, buildings are forced to oscillate in resonance with the **seismic waves**, resulting in serious damages. In regions of the world where earthquakes happen regularly, buildings may be built on *foundations that absorb the energy* of the shock waves. In this way, the vibrations are damped and the amplitude of the oscillations cannot reach dangerous levels.
Microwave Cooking In a microwave oven, microwaves with a frequency similar to the natural frequency of **vibration of water molecules** are used. When food containing water molecules is placed in the oven, the water molecules resonate, absorbing energy from the microwaves and consequently get heated up. This absorbed energy then *spreads through the food* and cooks it. The plastic or glass containers do not heat up since they do not contain water molecules.	*Human Internal Organs* In human beings, internal organs can be made to resonate in response to external frequencies, usually **below 10 Hz**. High levels of vibration can cause serious, or even fatal lung, heart, intestinal, and brain **damage**.
Magnetic Resonance Imaging Strong, varying **radio frequency** electromagnetic fields are used to cause **oscillations in atomic nuclei**. When resonance occurs, energy is absorbed by the molecules. By analyzing the **pattern of energy absorption**, a computer-generated image can be produced.	

Additional Example 1 (Comparing SHM with uniform circular motion)

Consider a turntable carrying a metallic knob S, rotating at constant angular frequency ω. Light from a source casts a shadow of the knob on a screen.

Notice how the shadow of S in uniform circular motion undergoes to-and-fro sinusoidal motion. As S moves through one revolution, the displacement of its shadow undergoes one complete sinusoidal variation with time.



(i) Show that the equation for displacement of the shadow is in the form $x = x_o \sin \omega t$.

Let r be the radius of circular motion and θ be the angle moved by the knob in time t.

From the triangle on the diagram: $\sin\theta = \dfrac{x}{r} \quad \Rightarrow \quad x = r\sin\theta$

Since angular frequency $\omega = \dfrac{\theta}{t}$ and amplitude $x_o = r$, $\Rightarrow \quad x = x_o \sin \omega t$ (shown)

(ii) A student placed a ruler against the shadow of the knob produced. She noticed that the shadow moved from one extreme position of 50 cm to the other extreme position of 90 cm in an average time of 2.5 seconds.

What is the time taken for the shadow to move from 70 cm to 60 cm?

The equilibrium position is at l = 70 cm. $\therefore$ Amplitude x_o = 20 cm.

Suppose time starts when shadow is positioned at l = 70 cm i.e. x = 0 cm,

We use $x = x_o \sin \omega t$ (since $t = 0$, $x = 0$)

Thus, $x = x_o \sin \dfrac{2\pi}{T} t$ where T = 2 x 2.5 = 5.0 s

When the shadow displaces from 70 cm to 60 cm, x = 10 cm

From $x = x_o \sin \dfrac{2\pi}{T} t$

$\therefore \quad 10 = 20 \sin \dfrac{2\pi}{5.0} t$

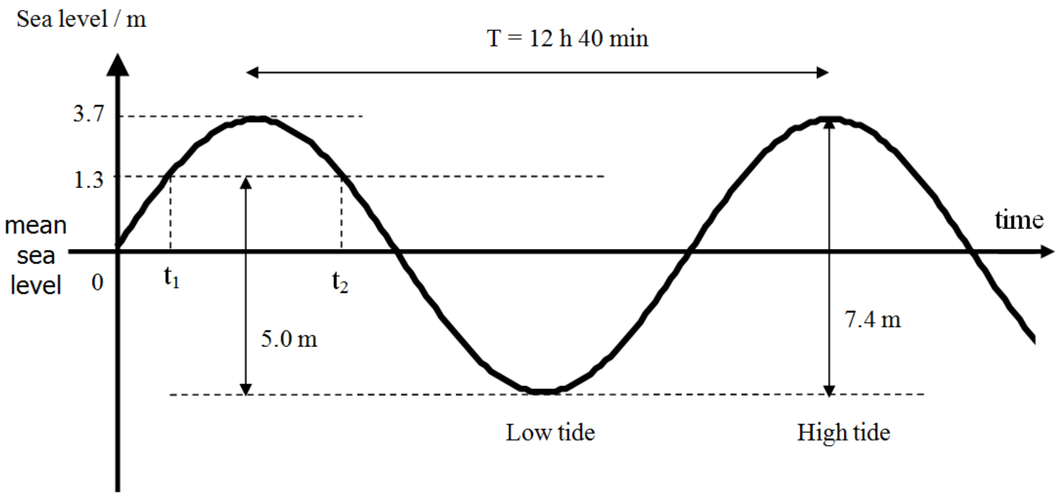
Hence, t = **0.42 s**

Additional Example 2

At Liverpool the tide is nearly sinusoidal and has a period of 12 hours 40 minutes. The difference in height between high and low tides is 7.4 m. Find

(a) the length of time within a cycle during which the water is more than 5 m above low tide.

(b) the rate at which the tide is rising or falling when the water is 5 m above low tide.



Period T = $(12 \times 60 + 40) \times 60 = 45600$ s

Angular frequency $\omega = 2\pi/T = 1.38 \times 10^{-4}$ rad s^{-1}

(a) Let t_1 and t_2 be the times when the tide is 5.0 m above the low tide i.e. 1.3 m above the mean sea level.

$x = x_0 \sin \omega t$

$1.3 = 3.7 \sin (2\pi/45600)t$

$t_1 = 2610$ s

By symmetry, $t_2 = T/2 - t_1 = 22800 - 2610 = 20200$ s

Time when sea level is > 5 m above low tide

$= t_2 - t_1 = 20200 - 2610 = 17600$ s = **4 hr 53 min**

(b) Rate of rise of tide,

$v = dx/dt = d/dt\,(x_0 \sin \omega t)$

$= \omega x_0 \cos \omega t_1$

$= (1.38 \times 10^{-4})(3.7) \cos (1.38 \times 10^{-4} \times 2610)$

$= 4.8 \times 10^{-4}$ m s^{-1} = **0.48 mm s^{-1}**

Alternatively, $v = \omega \sqrt{(x_o^2 - x^2)}$ (considering magnitude)

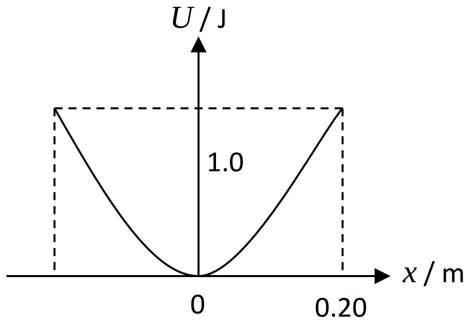
$= (1.38 \times 10^{-4})\sqrt{(3.7^2 - 1.3^2)} = 4.8 \times 10^{-4}$ m s^{-1} = **0.48 mm s^{-1}**

Additional Example 3

A particle of mass 4.0 kg moves with simple harmonic motion and its potential energy U varies with position x as shown.

(a) What is the total energy of the system?

(b) What is the period of oscillation of the mass?



(a) From the graph,

$$U_{max} = 1.0 \text{ J}$$

Total energy = U_{max} = **1.0 J**

(b) $E_T = ½m\omega^2 x_o^2$

$1.0 = ½(4.0)\,\omega^2(0.20)^2$

ω = **3.5 rad s⁻¹**

Oscillation Definition List

Oscillation	It is a periodic motion of an object about a certain mean equilibrium position with a continuous interchange of kinetic energy and potential energies.
<u>Forced</u> oscillations	Forced oscillations are caused by continual input of energy by external source to an oscillating system to compensate the loss due to damping in order to maintain the amplitude of the oscillation.
<u>Damped</u> simple harmonic oscillations	Oscillation in which there is a continuous transfer of energy to the surroundings due to dissipative force such that the energy in the system decreases with time, hence the amplitude of the motion progressively decreases with time.
Equilibrium position	***Equilibrium position (or neutral position)*** is the position at which no net force acts on the oscillating mass.
Angular frequency (ω) (2008)	***Angular frequency*** **(ω)** of an oscillation refers to the constant which characterises the particular simple harmonic oscillator and is related to its natural frequency given by $\omega = 2\pi f$. Or The angle in radians by which the phase of the motion changes per unit time. Unit: radian per second (**rad s^{-1}**).
Frequency (f) (2008)	Frequency is the number of complete to-and-fro cycles per unit time made by the oscillating object. Unit: Hertz. (1 Hz is equal to one cycle per second.)
Period (T)	***Period*** **(T)** is the time taken for one complete oscillation. Unit: second.
Phase	Phase is an angle in either degrees or radians which gives a measure of the fraction of a cycle that has been completed by an oscillating particle or by a wave.
Phase difference (2010)	Phase difference is a measure of <u>how much</u> one wave is <u>out of step</u> with another. It is measured in either degrees or radians.
In-phase	phase difference is zero.
Out of Phase	phase difference is not zero.
Anti-phase	phase difference is 180 degrees or π radians.
Displacement	Displacement is the distance of the oscillating mass from its equilibrium position at any instant in a stated direction.
Amplitude	Amplitude is the maximum displacement of the oscillating mass from the equilibrium position in either direction.
Simple Harmonic motion (SHM) (2003, 2006, 2007, 2008, 2009)	Defined as oscillatory motion of a particle whose acceleration is directly proportional to its displacement from the equilibrium position and this acceleration is always in opposite direction to its displacement.
Damping	Damping is a process where energy is taken from an oscillating system as a result of dissipative forces.
Light damping	Oscillations are maintained about the equilibrium position after the system has been displaced. The amplitude of oscillations decreases over a long time.
Critical damping	No oscillations occur. The motion is brought to rest in the shortest possible time.
Heavy damping	No oscillations occur about the equilibrium position when the damping force increases beyond the point of critical damping. The system takes a long time to return to the equilibrium position compared to critically damped system.
Natural Frequency	Any mechanical system that is free to move has a <u>natural frequency f_0</u> which depends on its dimensions and nature of material.
Resonance (2005)	Resonance occurs when the resulting <u>amplitude</u> of the system becomes a <u>maximum</u> when the <u>driving frequency</u> of external driving force equals to <u>natural frequency</u> of the system. At resonance, there is a maximum transfer of energy from the driving system to the driven system.

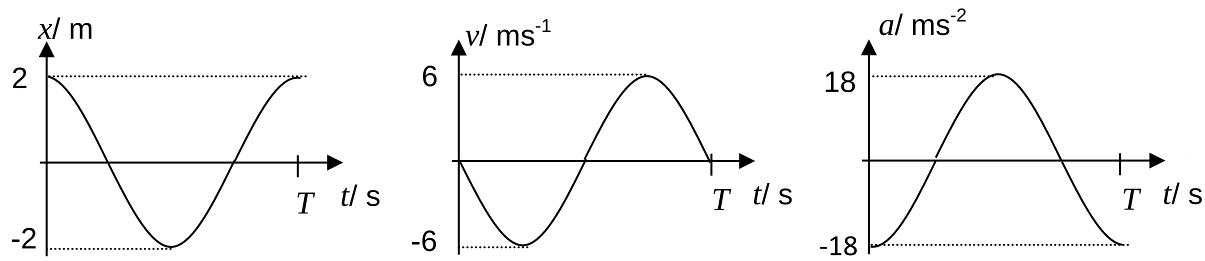
Tutorial Questions

Simple harmonic motion

Self-Attempt Questions

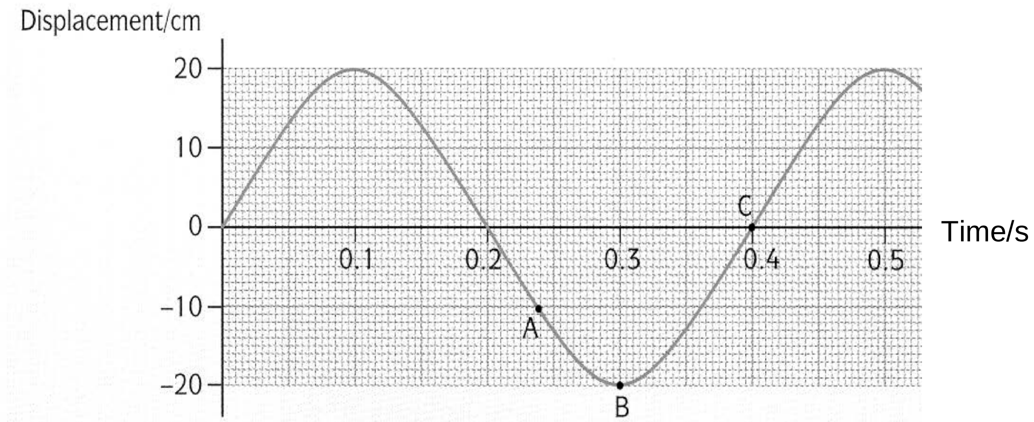
1. (a) Give two examples of free oscillations.
   (b) Define simple harmonic motion (s.h.m.). State the defining equation of s.h.m., and explain all the symbols used.
   (c) Is a bouncing ball an example of s.h.m.? Give a reason for your answer.

2. The graphs below show how the displacement x, velocity v and acceleration a of a body vary with time t when it is oscillating with simple harmonic motion.



What is the value of T? [2.1 s]

3. The figure shows the displacement-time graph of a simple harmonic oscillator.



(a) Deduce the following quantities:
   (i) amplitude, [20 cm]
   (ii) period, [0.4 s]
   (iii) frequency, [2.5 Hz]
   (iv) angular frequency, [*15.7* rads^{-1}]
   (v) displacement at A, [-10.0 cm]
   (vi) Using your answers (a)(i) and (iv), write down an equation that describes the variation of displacement, x, with time, t, as shown in the graph.
   (vii) Substitute (v) the displacement at A into the equation to find the corresponding time. Check that it agrees with the value from the graph.

(viii) At which point (A,B,C) is the velocity maximum? State whether the direction is positive or negative. Explain your answers.

(ix) At which point (A,B,C) is the acceleration maximum? State whether the direction is positive or negative. Explain your answers.

(b) Deduce the following quantities:

(i) velocity at B [*0* ms^{-1}]

(ii) velocity at C [3.14 ms^{-1}]

(iii) acceleration at B [49.3 ms^{-2}]

(iv) acceleration at C. [0 ms^{-2}]

(c) Sketch for this oscillator,

(i) the acceleration-time graph

(ii) the velocity-time graph

(iii) the velocity-displacement graph

(iv) the acceleration-displacement graph.

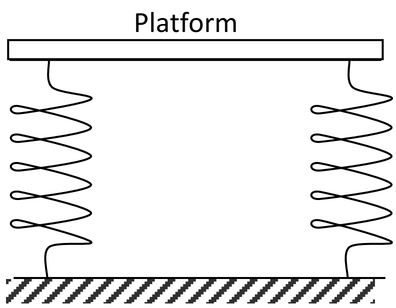
4. A 0.5 kg body performs simple harmonic motion with a frequency of 2.0 Hz and amplitude of 8 mm.

Determine the maximum velocity and the maximum acceleration and the corresponding positions of the body. [0.10 ms^{-1}, 1.26 ms^{-2}]

Discussion Questions

5. TYS (8)9 Q2 (2004 P2 Q1b)

6. A light platform is supported by two identical springs, each having spring constant 20 N m^{-1} as shown in the figure on the right



(a) Calculate the weight which must be placed on the centre of the platform in order to produce a displacement of 3.0 cm.

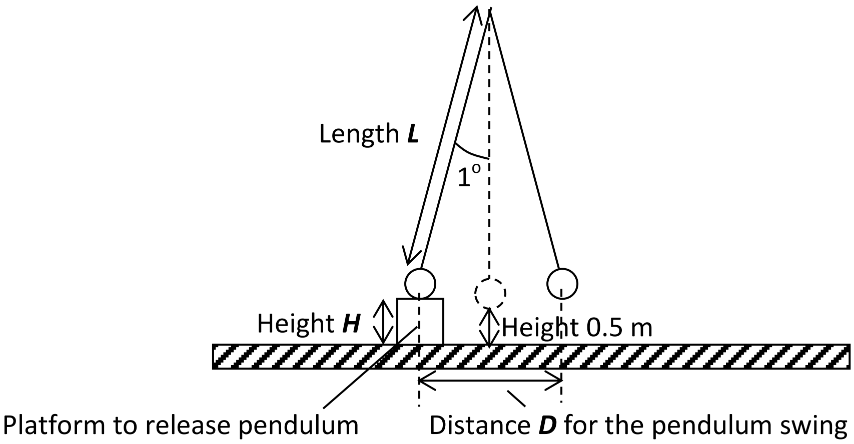
(b) The weight remains on the platform and the platform is depressed a further 1.0 cm and then released.

(i) What is the frequency of the oscillation of the platform?

(ii) Sketch the displacement-time graph of the oscillation

(iii) Mark on your sketch the times at which the acceleration of the platform is maximum and calculate this acceleration

7. Mediacorp wants to add a new twist to its new variety show game. They want to design a pendulum that has a period of 10 seconds. Contestants are to answer questions within 10 seconds and the pendulum will serve the purpose of a time keeper. Help Mediacorp by calculating the unknown dimensions (**D**,**H** & **L**) required for such a pendulum shown in the figure below. Neglect size of pendulum bob.



Energy in simple harmonic motion

Self-Attempt Question

8 Refer to the displacement-time graph in Q3. Given that the mass of the oscillator is 1 kg.

(a) At which point (A,B,C) is
(1) the potential energy maximum? (2) the kinetic energy maximum?

(b) Find the kinetic energy and the potential energy of the oscillator at C and B. [0 J, 4.93 J]

(c) Sketch for the oscillator, labelled graphs on the same horizontal time axis,
(i) the variation of its kinetic energy,
(ii) the variation of its potential energy,
(iii) the variation of its total mechanical energy.

(d) Sketch for the oscillator, labeled graphs on the same horizontal displacement axis,
(i) the variation of its kinetic energy,
(ii) the variation of its potential energy,
(iii) the variation of its total mechanical energy.

Discussion Question

9 TYS (8)21 Q3 (2010 P3 Q6)

10 TYS (8)17 Q1 (2008 P3 Q6)

Damped oscillations and forced oscillations: resonance

Self-Attempt Question

11 Explain what is meant by damped oscillation, and suggest how damping can be increased in a spring-mass oscillation.

12 (a) Sketch a set of displacement-time graphs using the same axes, to show oscillatory motion that is
(i) underdamped,
(ii) critically damped,
(iii) overdamped.

(b) Give an example of an application of critical damping.

13 Explain what is meant by forced oscillation and resonance. State a situation in which resonance is used to advantage.

Discussion Question

14 In the military, there is a long standing tradition that you do not march soldiers over a bridge in step. Every soldier is allowed to march at his own step. What could be the possible reason for this practice?

15. Why is it that when a vehicle is stationary, it vibrates violently but when the vehicle is in motion, the vibration is reduced?

16. If there is little to no damping on the receiver of the microphone, what could happen if it picks up sound signals near the natural frequency of the receiver? Sketch the amplitude – driving frequency graph and explain how a good microphone would work with the concepts of resonance.

17. The suspension of a car may be considered to be a spring under compression combined with a shock absorber which damps the vertical oscillations of the car. Sketch graphs, one in each case, to illustrate how the vertical height of the car above the road will vary with time after the car has just passed over a bump if the shock absorber is
(a) not functioning (i.e. slides without resistance)
(b) operating normally

18. TYS (8)24 Q4 (2011 P3 Q7) part (a) & (b) only

Challenge Accepted (See P14)

19. A block of mass is attached to the end of a spring, with the block free to move on a horizontal frictionless surface. When the spring is neither stretched nor compressed, the block is at the equilibrium position, where x = 0, as shown in Fig. 1.1. When the block is pulled by a distance from the equilibrium position as shown in Fig. 1.2 and subsequently released, it oscillates back and forth about its equilibrium position.

(a) Show, *from first principles*, that the motion of the object is simple harmonic motion. You may make use of Fig. 1.2 and Fig. 1.3 as part of your working.

(b) Hence, derive the expression for the period of the simple harmonic motion in term of the spring constant k and mass of the block m.

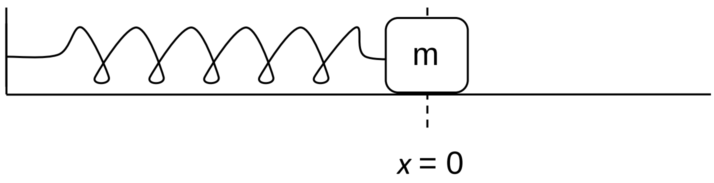


Fig. 1.1

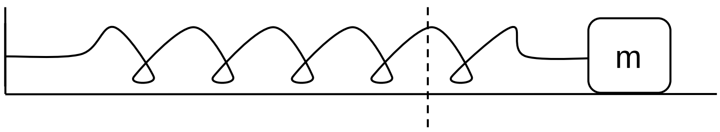


Fig. 1.2

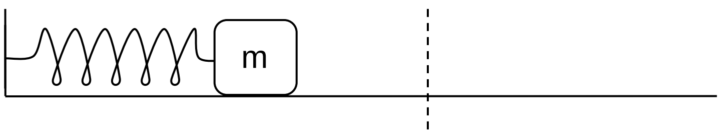


Fig. 1.3

Challenging Question (Not in Syllabus, requires a bit of Calculus and Trigonometry)

20. The defining equation for SHM can be modified for forced oscillations with damping by considering Newton's 2nd Law.

$$m\frac{d^2x}{dt^2} = -m{\omega_0}^2x$$

where ω_0 is the frequency of a free oscillator.

We can take the driving force $F = F_0\cos\omega t$, where F_0 is a constant and ω is the angular frequency of the force and the damping force $f = -kv$, we get

$$m\frac{d^2x}{dt^2} = -m{\omega_0}^2x - k\frac{dx}{dt} + F_0 cos\omega t$$

Given that the steady state solution of the damped forced oscillator is

$$x = Acos(\omega t + \phi)$$

(a) Find A and ϕ

(b) Explain how your solution in (a) explains the resonance graphs we learned in this topic.