XINMIN SECONDARY SCHOOL
新民中学
SEKOLAH MENENGAH XINMIN
Preliminary Examinations 2023

CANDIDATE NAME	Mark Scheme
CLASS	
INDEX NUMBER	

ADDITIONAL MATHEMATICS 4049/02

Paper 2 **29 August 2023**

Secondary 4 Express/ 5 Normal Academic **2 hour 15 minutes**

Setter : Mr Johnson Chua
Vetter : Mrs Wong Li Meng
Moderator: Ms Pang Hui Chin

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST

Write your name, register number and class in the spaces at the top of this page.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is **90.**

Errors	Qn No.	Errors	Qn No.
Accuracy		Simplification	
Brackets		Units	
Geometry		*Marks Awarded*	
Presentation		*Marks Penalised*	

For Examiner's Use
90

Parent's/Guardian's Signature:

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{r} a^{n-r} b^r + \ldots + b^n ,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \tfrac{1}{2} bc \sin A$$

1 Solve the equation $36^x - 6^{x+1} = 27$. [4]

$6^{2x} - 6^x(6) = 27$

$(6^x)^2 - 6(6^x) = 27$

let $u = 6^x$

$\therefore u^2 - 6u - 27 = 0$ — (M1)

$(u-9)(u+3) = 0$.

$[u = -3$ or $u = 9]$ — (M1)

$6^x = -3$ or $6^x = 9$

(rej.)

$\lg 6^x = \lg 9$ — (M1)

$x = \frac{\lg 9}{\lg 6}$

$= 1.2262$

≈ 1.23 — (A1)

2 In 2015, the population of a town was estimated at 15 000.

In 2020, the numbers were estimated to have increased to 17 500.

Analysts believe that the population, N, can be modelled by the formula

$N = 15000e^{kt}$, where t is the time in years after 2015.

(a) Calculate the population of the town, to the nearest person, in 2030. [4]

$N = 15000\,e^{kt}$

$17500 = 15000\,e^{k(5)}$ — (M1)

$e^{5k} = \dfrac{17500}{15000}$

$5k = \ln \dfrac{17500}{15000}$ OR $5K = \ln \dfrac{7}{6}$

$k = 0.030830$ (5sf) — (M1) OR $\dfrac{1}{5}\ln \dfrac{17500}{15000}$ or $\dfrac{1}{5}\ln \dfrac{7}{6}$

$\therefore N = 15000\,e^{0.030830t}$ OR $N = 15000\,e^{\frac{t}{5}\ln \frac{17500}{15000}}$

In 2030,

$N = 15000\,e^{0.030830(15)}$ — (M1) OR $N = 15000\,e^{3\ln \frac{17500}{15000}}$

$= 23819.3$

≈ 23819 — (A1)

$\therefore$ population: 23819

(b) The town is labelled "overcrowded" if the population exceeds 30000.

Estimate the year in which the town is first labelled "overcrowded". [2]

$15000e^{0.030830t} = 30000$ — (M1) [also accept if students use 30001 on RHS]

$0.030830t = \ln 2$

$t = 22.482$

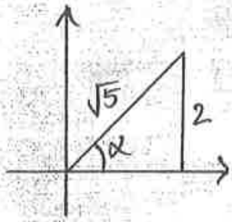
≈ 23

$2015 + 23 = 2038$

Ans: Year 2038 — (A1)

3 (a) Given that $\sin\alpha = \dfrac{2}{\sqrt{5}}$, where α is an acute angle,

show that $\sin(45° + \alpha) = \dfrac{3\sqrt{10}}{10}$. [3]



Base: $\sqrt{5-2^2}$

$= 1$

$\therefore \cos\alpha = \frac{1}{\sqrt{5}}$ — (M1)

$\sin(45° + \alpha) = \sin 45° \cos\alpha + \cos 45° \sin\alpha$

$= \underbrace{\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{5}}} + \frac{1}{\sqrt{2}} \cdot \frac{2}{\sqrt{5}}$ — (M1: $\sin 45° = \cos 45° = \frac{1}{\sqrt{2}}$)

$= \frac{1}{\sqrt{10}} + \frac{2}{\sqrt{10}}$

$= \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$ — (A1)

(b) Hence, find the value of $\cot^2(45°+\alpha)$. [3]

Recall: $1+\cot^2\theta = \operatorname{cosec}^2\theta$

$$\cot^2\theta = \operatorname{cosec}^2\theta - 1$$

$$\therefore \cot^2(45°+\alpha) = \operatorname{cosec}^2(45°+\alpha) - 1 \quad \text{— (M1)}$$

$$\operatorname{cosec}(45°+\alpha) = \frac{10}{3\sqrt{10}} \quad \text{— (M1)}$$

$$\cot^2(45°+\alpha) = \left(\frac{10}{3\sqrt{10}}\right)^2 - 1$$

$$= \frac{100}{90} - 1$$

$$= \frac{1}{9} \quad \text{— (A1)}$$

Alternative

$$\cot^2(45°+\alpha) = \frac{\cos^2(45°+\alpha)}{\sin^2(45°+\alpha)}$$

$$\cos(45°+\alpha) = \cos 45° \cos\alpha - \sin 45° \sin\alpha$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{5}}\right) - \frac{1}{\sqrt{2}}\left(\frac{2}{\sqrt{5}}\right)$$

$$= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} = -\frac{1}{\sqrt{10}} \quad \text{— (M1)}$$

OR $\cos^2(45°+\alpha)$:

$$= 1 - \sin^2(45°+\alpha)$$

$$= 1 - \left(\frac{3\sqrt{10}}{10}\right)^2$$

$$= \frac{1}{10}$$

$$\therefore \cos(45°+\alpha) = \pm\frac{1}{\sqrt{10}} \quad \text{— (M1)}$$

$$\therefore \cot^2(45°+\alpha) = \frac{\left(-\frac{1}{\sqrt{10}}\right)^2}{\left(\frac{3\sqrt{10}}{10}\right)^2} \quad \text{— (M1)}$$

$$= \frac{\left(\frac{1}{10}\right)}{\left(\frac{9}{10}\right)}$$

$$= \frac{1}{9} \quad \text{— (A1)}$$

Alternative #2:

$$\cot^2(45°+\alpha) = \operatorname{cosec}^2(45°+\alpha) - 1$$

$$= \left[\frac{1}{\sin^2(45°+\alpha)} - 1\right] \quad \text{— (M1)}$$

$$\frac{1}{\left(\frac{3\sqrt{10}}{10}\right)^2} - 1 \quad \text{— (M1)}$$

$$= \frac{10}{9} - 1$$

$$= \frac{1}{9} \quad \text{— (A1)}$$

[Turn over

4 **(a)** Given that $f(x) = ax^3 - 3x^2 - 2x + b$, where a and b are constants, find the values of a and b given that f(x) has a factor of $x - 3$ and leaves a remainder of 4 when divided by $x + 1$. [4]

$f(x) = ax^3 - 3x^2 - 2x + b$

$f(3) = 0$

$a(3)^3 - 3(3)^2 - 2(3) + b = 0$ — (M1)

$27a + b = 33$ — ①

$f(-1) = 4$

$a(-1)^3 - 3(-1)^2 - 2(-1) + b = 4$ — (M1)

$-a - 3 + 2 + b = 4$

$-a + b = 5$ — ②

① − ②: $27a - (-a) = 28$

$28a = 28$

$a = 1$ — (A1)

Sub $a = 1$ into ②: $-1 + b = 5$

$b = 6$ — (A1)

(b) **(i)** Factorise $250x^3+54y^3$. $a^3+b^3=(a+b)(a^2-ab+b^2)$ [3]

$2(125x^3+27y^3)$

$=2[(5x)^3+(3y)^3]$ — (M1)

$=2(5x+3y)((5x)^2+(3y)^2-5x(3y))$

$=2\underbrace{(5x+3y)}_{(A1)}\underbrace{(25x^2-15xy+9y^2)}_{(A1)}$

(ii) Hence, given that x and y are positive integers, explain why $250x^3+54y^3$ is an even number. [1]

Since 2, which is an even no., is a factor of $250x^3+54y^2$, this makes $250x^3+54y^3$ an even no.

OR

Since part (i) is a multiple of 2, an even no., this makes it an even no. [OR it is even when multiplied by 2]

A1 for either.

5 (a) Prove the identity $\dfrac{\sin x}{\operatorname{cosec} x-1}+\dfrac{\sin x}{\operatorname{cosec} x+1}=2\tan^2 x$. [4]

LHS : $\dfrac{\sin x}{\left(\frac{1}{\sin x}-1\right)}+\dfrac{\sin x}{\left(\frac{1}{\sin x}+1\right)}$ — (M1)

$= \dfrac{\sin x}{\left(\frac{1-\sin x}{\sin x}\right)}+\dfrac{\sin x}{\left(\frac{1+\sin x}{\sin x}\right)}$

$= \dfrac{\sin^2 x}{1-\sin x}+\dfrac{\sin^2 x}{1+\sin x}$

$= \dfrac{\sin^2 x(1+\sin x)+\sin^2 x(1-\sin x)}{1-\sin^2 x}$ — (M1 for combining)

$= \dfrac{\sin^2 x\,(1+\sin x+1-\sin x)}{\cos^2 x}$ — (M1 for denominator, $1-\sin^2 x=\cos^2 x$)

$= \dfrac{\sin^2 x(2)}{\cos^2 x}$

$= 2\tan^2 x$ (A1)

Alternative

LHS: $\dfrac{(\operatorname{cosec} x+1)\cdot\sin x+(\operatorname{cosec} x-1)\cdot\sin x}{\operatorname{cosec}^2 x-1}$ — (M1 for combining fractions)

$= \dfrac{\sin x\,(\operatorname{cosec} x+1+\operatorname{cosec} x-1)}{\cot^2 x}$ — (M1 for $\operatorname{cosec}^2 x-1=\cot^2 x$)

$= \dfrac{\sin x\,(2\operatorname{cosec} x)}{\cot^2 x}$

$= \dfrac{\sin x\left(\frac{2}{\sin x}\right)}{\cot^2 x}$ — (M1 : $\operatorname{cosec} x=\frac{1}{\sin x}$)

$= \dfrac{2}{\cot^2 x} = 2\tan^2 x$ — (A1)

(b) Hence, find the exact solution(s) to the equation

$$\frac{\sin\theta}{\operatorname{cosec}\theta-1}+\frac{\sin\theta}{\operatorname{cosec}\theta+1}=5\sec\theta-4, \text{ for } -\pi\le\theta\le\pi.$$ [5]

$2\tan^2\theta = 5\sec\theta-4$

$2\tan^2\theta - 5\sec\theta + 4 = 0$

$2(\sec^2\theta-1) - 5\sec\theta + 4 = 0$ — (M1)

$2\sec^2\theta - 2 - 5\sec\theta + 4 = 0$

$2\sec^2\theta - 5\sec\theta + 2 = 0$

$(\sec\theta-2)(2\sec\theta-1) = 0$ — (M1)

$\sec\theta = 2$ or $\sec\theta = \frac{1}{2}$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$ (rej) — (A1)

$B.A = \frac{\pi}{3}$ — (M1)

not linked

$\underbrace{\theta = -\frac{\pi}{3}, \frac{\pi}{3}}_{(A1)}$

<u>Alt.</u> $2\tan^2\theta = 5\sec\theta - 4$

$\frac{2\sin^2\theta}{\cos^2\theta} = \frac{5}{\cos\theta} - 4$

$2\sin^2\theta = 5\cos\theta - 4\cos^2\theta$

$2(1-\cos^2\theta) = 5\cos\theta - 4\cos^2\theta$ — (M1)

$2 - 2\cos^2\theta = 5\cos\theta - 4\cos^2\theta$

$2\cos^2\theta - 5\cos\theta + 2 = 0$

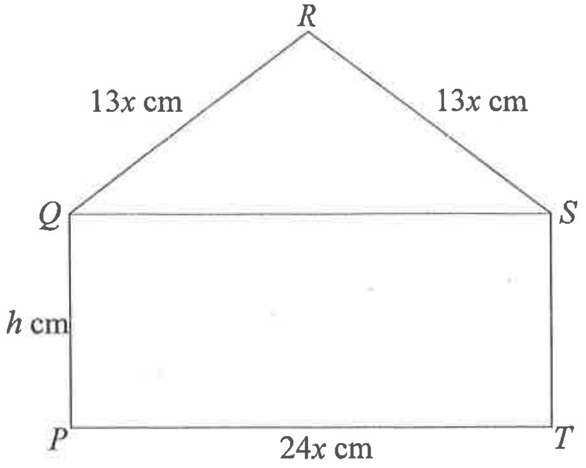
$(2\cos\theta - 1)(\cos\theta - 2) = 0$ — (M1)

$\cos\theta = \frac{1}{2}$ or $\cos\theta = 2$ (rej) — (A1)

BA: $\frac{\pi}{3}$ — (M1)

$\theta = -\frac{\pi}{3}, \frac{\pi}{3}$ — (A1)

6

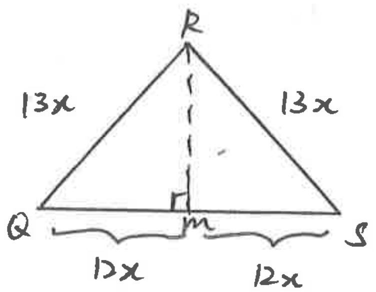


The diagram above shows a window frame, PQRST.
QRS is an isosceles triangle while PQST is a rectangle.

$QR = RS = 13x$ cm, $QP = h$ cm and $PT = 24x$ cm.

Given that the total perimeter of the window frame is 720 cm,

(a) show that the area of the window frame, A cm^2, is given by $A = 8640x - 540x^2$. [4]



$Rm = \sqrt{(13x)^2 - (12x)^2}$ — (M1)

$= 5x$

$2(13x) + 2(h) + 24x = 720$ — (M1)

$2h = 720 - 50x$

$h = 360 - 25x$

$A = \frac{1}{2}(24x)(5x) + (360 - 25x)(24x)$ — (M1)

$= 60x^2 + 8640x - 600x^2$

$= 8640 - 540x^2$ — (A1)

(b) Given that x can vary, an interior designer claimed that the value of h has to be 160 in order to obtain a maximum area for the window frame. Do you agree with his claim? Justify your answer with relevant workings. [5]

$$\frac{dA}{dx} = 8640 - 1080x$$

$$\frac{d^2A}{dx^2} = -1080 \ (<0)$$

Since $\frac{d^2A}{dx^2} < 0$, area is a maximum. } (B1)

<u>to find stationary value of x</u>

$$\frac{dA}{dx} = 0$$

$\therefore 8640 - 1080x = 0$ — (M1)

$1080x = 8640$

$x = 8$ — (A1)

When $x = 8$, $h = 360 - 25(8)$ — (M1)

$= 160$

$\therefore$ I <u>agree</u> with the interior designer's claim that when $h = 160$, area is maximum. } (A1)

<u>Alternative</u>

$A = 8640x - 540x^2$

$= -540(x^2 - 16x)$ — (M1)

$= -540(x^2 - 16x + 8^2) + 540(8^2)$

$= -540(x-8)^2 + 34\ 560$ — (M1)

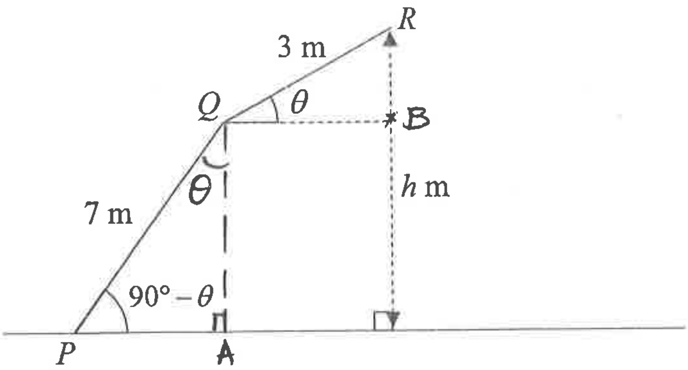
[Since coeff of x^2 is negative, A is at max. when $x = 8$] — (B1)

When $x = 8$, $h = 360 - 25(8)$ — (M1)

$= 160.$

$\therefore$ I <u>agree</u> with interior designer's claim that h has to be 160 to obtain maximum area. } (A1)

7



The diagram shows two rods, PQ and QR, of length 7 m and 3 m respectively. PQ is hinged at P while QR is hinged at Q. The rod PQ can turn about at P and is inclined at an angle of $90° - \theta$ to the horizontal ground, where $0° \le \theta \le 90°$. The rod QR can turn about at Q in such a way that its inclination to the horizontal ground is θ. The vertical distance of R from the horizontal ground is h m.

(a) Find the values of the integers a and b for which $h = a\sin\theta + b\cos\theta$. [2]

$\angle PQA = 180° - (90° - \theta) - 90°$

$= \theta$

$h = QA + RB$

$\cos\theta = \frac{QA}{7} \Rightarrow QA = 7\cos\theta$

$\sin\theta = \frac{RB}{3} \Rightarrow RB = 3\sin\theta$

(M1 for either)

$\therefore \; h = 3\sin\theta + 7\cos\theta$

$[\therefore a = 3, \; b = 7]$ — (A1)

(b) Using the values of a and b found in part **(a)**, express h in the form $R\sin(\theta + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. [3]

$R = \sqrt{3^2 + 7^2}$ — (M1)

$= \sqrt{58}$

$\alpha = \tan^{-1}\left(\frac{7}{3}\right)$ — (M1)

$= 66.801°$ (3dp)

$\therefore \; h = \sqrt{58}\sin(\theta + 66.801°)$

$= \sqrt{58}\sin(\theta + 66.8°)$ — (A1) (also accept if students left $\sqrt{58}$ in 3sf: 7.62)

Hence

(c) state the maximum value of h and find the corresponding value of θ, [3]

$h_{max} = \sqrt{58}$ — (A1)

$\sqrt{58}\sin(\theta + 66.801°) = \sqrt{58}$

$\sin(\theta + 66.801°) = 1$ — (M1)

$\theta + 66.801° = 90°$

$\theta = 90° - 66.801°$

$= 23.199°$

$\approx 23.2°$ (1dp) — (A1)

(d) find the value of h when PQ is inclined at an angle of $35°$ to the horizontal. [2]

$90° - \theta = 35°$

$\theta = 55°$

$\therefore h = \underbrace{7\cos 55° + 3\sin 55°}$ OR $\underbrace{\sqrt{58}\sin(55° + 66.801°)}$

[M1 for subst. $\theta = 55°$].

$= 6.4724$ (5sf)

≈ 6.47 (3sf) — (A1)

8 **(a)** The curve $y = kx^n$, where k and n are constants, passes through the points $(2, 64)$, $(3, 486)$ and $(a, \frac{1}{512})$. Find the values of k, n and a. [5]

Sub (2,64) and (3,486)

$$64 = k(2)^n \quad \text{— ①}$$
$$486 = k(3)^n \quad \text{— ②}$$
(M1 for either)

$\frac{①}{②}$: $\frac{64}{486} = \frac{2^n}{3^n}$ — (M1)

$$\frac{32}{243} = \left(\frac{2}{3}\right)^n$$

$$\left(\frac{2}{3}\right)^5 = \left(\frac{2}{3}\right)^n$$

$$\therefore n = 5 \quad \text{— (A1)}$$

Alternative

$$\lg\left(\frac{32}{243}\right) = \lg\left(\frac{2}{3}\right)^n$$
$$n = \lg\left(\frac{32}{243}\right) \div \lg\left(\frac{2}{3}\right)$$
$$= 5.$$

Alternative #2

from ①: $k = \frac{64}{2^n}$ — ③

Sub ③ into ②: $486 = \frac{64}{2^n} \cdot 3^n$ — (M1)

$$\frac{486}{64} = \left(\frac{3}{2}\right)^n$$

$$\frac{243}{32} = \left(\frac{3}{2}\right)^n$$

$$\left(\frac{3}{2}\right)^5 = \left(\frac{3}{2}\right)^n$$

$$\therefore n = 5 \quad \text{— (A1)}$$

Sub $n = 5$ into ①: $64 = k(2)^5$

$$k = 2 \quad \text{— (A1)}$$

$\therefore y = 2x^5 \Rightarrow \frac{1}{512} = 2a^5$

$$a^5 = \frac{1}{1024}$$

$$a = \frac{1}{4} \quad \text{— (A1)}$$

(b) Solve the equation $\log_3 y - \log_y 9 = 1$. [5]

$\log_3 y - \dfrac{\log_3 9}{\log_3 y} = 1$ — (M1)

$\log_3 y - \dfrac{2}{\log_3 y} = 1$ → $\underline{(\log_3 y)^2} - 2 = \log_3 y$

let $x = \log_3 y$

$2\log_3 y \neq (\log_3 y)^2$

$2\log_3 y = \log_3 y^2$

$\therefore x - \dfrac{2}{x} = 1$

$x^2 - 2 = x$

$x^2 - x - 2 = 0$ — (M1)

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$ — (M1)

$\therefore \log_3 y = 2$ or $\log_3 y = -1$

$y = 3^2$ or $y = 3^{-1}$

$\underbrace{= 9}_{(A1)}$ $\quad \underbrace{= \frac{1}{3}}_{(A1)}$

9 **(a)** Differentiate xe^{2x} with respect to x. [2]

$\frac{d}{dx}\left(xe^{2x}\right)$

$= e^{2x} + x(2e^{2x})$ — (M1)

$= e^{2x} + 2xe^{2x}$ — (A1)

(b) A particle moves along the curve $y = xe^{2x}$ in such a way that the y-coordinate is decreasing at a constant rate of 0.2 units per second. Find the rate of change of the x-coordinate at the point where $x = -1$. [2]

$\frac{dy}{dt} = -0.2$

at $x = -1$, $\frac{dy}{dx} = e^{2(-1)} + 2(-1)e^{2(-1)}$

$= e^{-2} - 2e^{-2}$

$= -e^{-2}$

$\therefore -0.2 = -e^{-2}\left(\frac{dx}{dt}\right)$ — (M1) [Award M1 even if Students sub $\frac{dy}{dt} = 0.2$]

$\frac{dx}{dt} = \frac{-0.2}{-e^{-2}}$

$= 1.4778$

≈ 1.48 (3sf) (also accept $0.2e^{2}$ or $\frac{e^{2}}{5}$) — (A1)

(c) Use your answer from part **(a)** to show that $\int_0^2 xe^{2x}\,dx = \frac{3e^4+1}{4}$ **[4]**

$$\int_0^2 e^{2x} + 2xe^{2x}\,dx = \left[xe^{2x}\right]_0^2 \quad \text{— (M1)}$$

$$\int_0^2 2xe^{2x}\,dx = \left[xe^{2x}\right]_0^2 - \int_0^2 e^{2x}\,dx$$

$$= 2e^4 - \left[\frac{e^{2x}}{2}\right]_0^2 \quad \text{— (M1 for } \int e^{2x}\,dx)$$

$$\int_0^2 2xe^{2x}\,dx = 2e^4 - \left(\frac{e^4}{2} - \frac{1}{2}\right) \quad \text{— (M1 for subst. of integrals)}$$

$$= 2e^4 - \frac{e^4}{2} + \frac{1}{2}$$

$$= \frac{3e^4+1}{2}$$

$$2\int_0^2 xe^{2x}\,dx = \frac{3e^4+1}{2}$$

$$\int_0^2 xe^{2x}\,dx = \frac{3e^4+1}{4} \quad \text{— (A1)}$$

Alternative

$$\int e^{2x} + 2xe^{2x}\,dx = xe^{2x} + c \quad \text{— (M1)}$$

$$\int 2xe^{2x}\,dx = xe^{2x} - \int e^{2x}\,dx + c$$

$$= xe^{2x} - \frac{e^{2x}}{2} + d \quad \text{— (M1)}$$

$$\int x e^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + e$$

$$\int_0^2 xe^{2x} = \left[\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right]_0^2$$

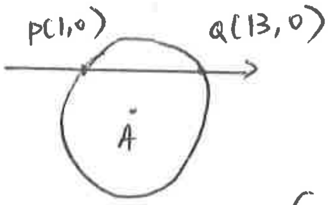
$$= \left(\frac{2e^4}{2} - \frac{e^4}{4}\right) - \left(-\frac{e^0}{4}\right) \quad \text{— (M1)}$$

$$= \frac{3e^4}{4} + \frac{1}{4}$$

$$= \frac{3e^4+1}{4} \quad \text{— (A1)}$$

10 A circle, C, of centre A intersects the x-axis at points $P(1,0)$ and $Q(13,0)$.

(a) Given that the y-coordinate of A is negative and that the radius of C_1 is $\sqrt{61}$, show that the general form of the equation of C is $x^2+y^2-14x+10y+13=0$. [4]



x-coordinate of A: $\frac{1+13}{2}=7$ — (M1)

$\therefore A(7, y)$

$(7-1)^2+(y-0)^2=61$ — (M1)

$6^2+y^2=61$

$y^2=61-36$

$y^2=25$

$y=5$ (rej) or -5

$\therefore A(7,-5)$

$\therefore (x-7)^2+(y+5)^2=(\sqrt{61})^2$ — (M1, allow ECF)

$x^2-14x+49+y^2+10y+25=61$

$x^2+y^2-14x+10y+13=0$ — (A1)

Let A be (a,b).

$\therefore AP=\sqrt{(a-1)^2+b^2}$

$AQ=\sqrt{(a-13)^2+b^2}$

$(a-1)^2+b^2=(a-13)^2+b^2$ — (M1)

$a^2-\cancel{2ab}\,2a+1=a^2-26a+169$

$24a=168$

$a=7$

$\therefore \sqrt{(7-1)^2+b^2}=\sqrt{61}$ or $\sqrt{(7-13)^2+b^2}=\sqrt{61}$ — (M1)

$b^2=61-36$

$b^2=25$

$b=5$ or -5

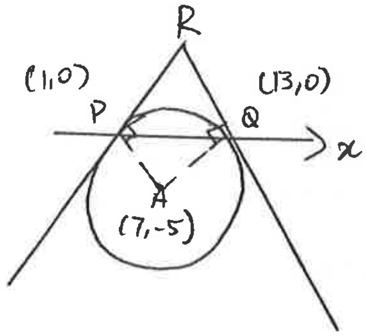
$\therefore A=(7,-5)$

$(x-7)^2+(y+5)^2=(\sqrt{61})^2$ — (M1)

$x^2-14x+49+y^2+10y+25=61$

$x^2+y^2-14x+10y+13=0$ — (A1)

(b) R is a point which lies outside the circle.
Find the coordinates of R such that triangle APR and triangle AQR are right angle triangles which are congruent to each other. [4]



$m_{AP} = \dfrac{-5-0}{7-1} = \dfrac{-5}{6}$ $\qquad$ $m_{AQ} = \dfrac{-5-0}{7-13} = \dfrac{5}{6}$

$\left[m_{PR} = \dfrac{6}{5} \qquad m_{QR} = -\dfrac{6}{5} \right]$ — M1 for either

$\left| \begin{array}{l} \text{Eq}^{n} \text{ of } PR: y - 0 = \frac{6}{5}(x-1) \\ ①: y = \frac{6}{5}x - \frac{6}{5} \end{array} \qquad \begin{array}{l} \text{Eq}^{n} \text{ of } QR: y - 0 = -\frac{6}{5}(x-13) \\ ②: y = -\frac{6}{5}x + \frac{78}{5} \end{array} \right|$ — M1 for either.

$\therefore$ Coordinates of R − ① = ② : $\dfrac{6}{5}x - \dfrac{6}{5} = -\dfrac{6}{5}x + \dfrac{78}{5}$ — (M1)

$\dfrac{12x}{5} = \dfrac{84}{5} \Rightarrow x = 7$

at $x = 7$, $y = \dfrac{36}{5}$ $\qquad \therefore R\underbrace{\left(7, \frac{36}{5}\right)}_{(A1)}$

Alternative

- M1 for findg. M_{PR} or M_{QR}
- M1 for findg Eq^{n} of PR or Eq^{n} of QR

} Refer above.

- x-coordinate of R: $x = 7$ — (B1)

$\therefore x = 7$ — ①

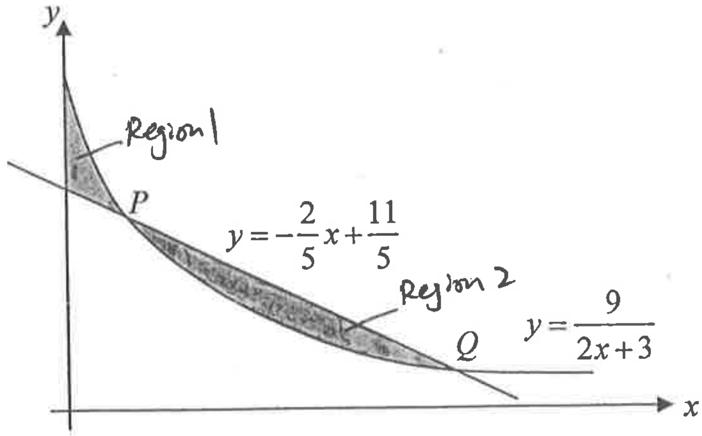
$\left[y = \frac{6}{5}x - \frac{6}{5} \;\middle|\; y = -\frac{6}{5}x + \frac{78}{5} \right]$ — ②

Sub ① into ②: $y = \dfrac{36}{5}$. $\quad \therefore R\left(7, \dfrac{36}{5}\right)$ — (A1)

(c) . State the equation of another circle that passes through P and Q and has the same radius as C. [2]

$\underbrace{(x-7)^2}_{(B1)} + \underbrace{(y-5)^2}_{(B1)} = 61$

11



The line $y=-\frac{2}{5}x+\frac{11}{5}$ intersects the curve $y=\frac{9}{2x+3}$ at points P and Q.

Find the area of the shaded region and express it in the form $a+b\ln\frac{25}{27}$, where a and b are constants. [10]

$y=-\frac{2}{5}x+\frac{11}{5}$ — ①

$y=\frac{9}{2x+3}$ — ②

$\frac{-2x+11}{5}=\frac{9}{2x+3}$ — (M1)

$(-2x+11)(2x+3)=45$

$-4x^2-6x+22x+33=45$

$-4x^2+16x-12=0$
$x^2-4x+3=0$
} M1 for either.

$(x-1)(x-3)=0$

$[\therefore x=1 \text{ or } x=3]$ — (M1)

Continuation of working space for question 10.

Area of Region 1

$$\int_0^1 \frac{9}{2x+3}\,dx - \int_0^1 -\frac{2x}{5} + \frac{11}{5}\,dx \quad \text{— (M1)}$$

$$= \left[\frac{9\ln(2x+3)}{2}\right]_0^1 - \left[\frac{-2x^2}{5(2)} + \frac{11x}{5}\right]_0^1 \quad \text{— (M1, M1 for correct integration)}$$

$$= \frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - \left(-\frac{1}{5} + \frac{11}{5}\right)$$

$$= \frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - 2$$

Area of region 2

$$\int_1^3 -\frac{2x}{5} + \frac{11}{5}\,dx - \int_1^3 \frac{9}{2x+3}\,dx \quad \text{— (M1)}$$

$$= \left[-\frac{x^2}{5} + \frac{11x}{5}\right]_1^3 - \left[\frac{9}{2}\ln(2x+3)\right]_1^3$$

$$= \left[-\frac{9}{5} + \frac{33}{5} - \left(-\frac{1}{5} + \frac{11}{5}\right)\right] - \left(\frac{9}{2}\ln 9 - \frac{9}{2}\ln 5\right)$$

$$= \frac{24}{5} - 2 - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 5$$

total area

$$\frac{9}{2}\ln 5 - \frac{9}{2}\ln 3 - 2 + \frac{24}{5} - 2 - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 5 \quad \text{— (M1) for adding area}$$

$$= \frac{4}{5} + \frac{9}{2}\left(\ln 5 - \ln 3 - \ln 9 + \ln 5\right)$$

$$= \frac{4}{5} + \frac{9}{2}\left(\ln\frac{5\times 5}{3\times 9}\right)$$

$$= \underbrace{\frac{4}{5}}_{(A1)} + \frac{9}{2}\underbrace{\ln\left(\frac{25}{27}\right)}_{(A1)}$$

--- End of Paper ---

(Q10.) Alternative (after solving $x=1$ or $x=3$ on pg 22)

when $x=0$,

$y = -\frac{2}{5}x + \frac{11}{5}$

$= \frac{11}{5}$

when $x=1$, $y = \frac{9}{2+3}$

$= \frac{9}{5}$ $\therefore P(1, \frac{9}{5})$

when $x=3$, $y = \frac{9}{6+3}$ $\therefore Q(3,1)$

$= 1$.

area of region 1

$\int_0^1 \frac{9}{2x+3}\,dx - \frac{1}{2}\left(\frac{11}{5}+\frac{9}{5}\right)(1)$ —— M1

$= \left[9\left(\frac{\ln(2x+3)}{2}\right)\right]_0^1 - 2$ —— M1 integration
—— M1 trapezium area.

$= \frac{9\ln 5}{2} - \frac{9\ln 3}{2} - 2$

$= \frac{9}{2}(\ln 5 - \ln 3) - 2$

$= \frac{9}{2}\ln\frac{5}{3} - 2$

area of region 2

$\frac{1}{2}\left(\frac{9}{5}+1\right)(3-1) - \int_1^3 \frac{9}{2x+3}\,dx$ —— M1

$= 2\frac{4}{5} - \left[9\left(\frac{\ln(2x+3)}{2}\right)\right]_1^3$

$= 2\frac{4}{5} - \frac{9}{2}\ln 9 + \frac{9}{2}\ln 5$

$= 2\frac{4}{5} - \frac{9}{2}(\ln 9 - \ln 5)$

$= 2\frac{4}{5} - \frac{9}{2}\ln\frac{9}{5}$

$\therefore$ area of shaded region

$= \left(\frac{9}{2}\ln\frac{5}{3} - 2\right) + \left(2\frac{4}{5} - \frac{9}{2}\ln\frac{9}{5}\right)$ —— M1

$= \frac{4}{5} + \frac{9}{2}\left(\ln\frac{5}{3} - \ln\frac{9}{5}\right)$

$= \frac{4}{5} + \frac{9}{2}\ln\left(\frac{5}{3} \div \frac{9}{5}\right)$

$= \underbrace{\frac{4}{5}}_{A1} + \underbrace{\frac{9}{2}\ln\frac{25}{27}}_{A1}$.