Chapter 0

Review of Basic Concepts (Self-learning)

Objectives:

Chapter 0 is a review of basic concepts of some selected topics you have learnt at 'O' level which are important prerequisites for the 'A' level course.

0.1 Equations and Inequalities

- **Conditions for a quadratic equation to have:**
  - **i)** **two real roots**
  - **ii)** **two equal roots**
  - **iii)** **no real roots**
- **Conditions for $ax^2 + bx + c$ to be always positive (or negative)**

0.1.1 Review of Quadratic Functions & Equations

The expression $f(x) = ax^2 + bx + c$ where x is a real variable, a, b and c are real constants and $a \neq 0$, is called a **quadratic function**. It is a **polynomial of degree 2** as the highest power of x is 2. We can always rewrite $f(x)$ in the form $a(x+p)^2 + q$, where $p, q \in \mathbb{R}$, by **completing the square**.

Completing the Square:

Complete the square for $2x^2 - x + 3$. Hence find the value of x for which the given quadratic expression takes the minimum value and write down this minimum value.

Solution:

Make coefficient of x^2 to be 1   *add square of half the coefficient of x*

$$2x^2 - x + 3 = 2\left(x^2 - \frac{1}{2}x\right) + 3 = 2\left[\underbrace{x^2 - \frac{1}{2}x + \left(-\frac{1}{4}\right)^2}_{\text{A perfect square}} - \left(-\frac{1}{4}\right)^2\right] + 3 = 2\left[\left(x - \frac{1}{4}\right)^2 - \frac{1}{16}\right] + 3$$

$$= 2\left(x - \frac{1}{4}\right)^2 + \frac{23}{8}.$$

The quadratic expression takes the minimum value when $x = \frac{1}{4}$ and this minimum value is $2\left(\frac{1}{4} - \frac{1}{4}\right)^2 + \frac{23}{8} = \frac{23}{8}$.

The equation $ax^2 + bx + c = 0$ $(a \neq 0)$ is called a **quadratic equation**. The quadratic equation can be solved by completing the square, where the roots are given by $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Example 1: Complete the square for (i) $3x^2+5x+4$ (ii) $3-4x-x^2$

Solution:

i) $3x^2+5x+4=3\left(x^2+\frac{5}{3}x+\frac{4}{3}\right)$

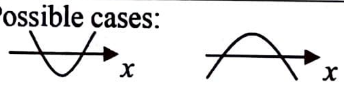
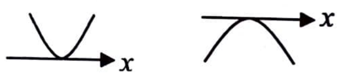
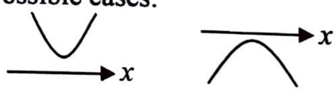
$$=3\left[\left(x+\frac{5}{6}\right)^2-\left(\frac{5}{6}\right)^2+\frac{4}{3}\right]=3\left(x+\frac{5}{6}\right)^2+\frac{23}{12}$$

ii) $3-4x-x^2=-(x^2+4x-3)$

$$=-\left[(x+2)^2-(2)^2-3\right]=-(x+2)^2+7$$

0.1.2 Nature of Roots

For any quadratic equation of the form $ax^2+bx+c=0$, where $a\neq 0$, there are three possible cases for the roots of the equation as summarized below:

	Nature of roots	Discriminant	Graphical representation of $y=ax^2+bx+c$
Case 1	2 real and distinct roots	$b^2-4ac>0$	Possible cases:  $a>0$ $a<0$
Case 2	Equal real roots	$b^2-4ac=0$	Possible cases:  $a>0$ $a<0$
Case 3	No real roots	$b^2-4ac<0$	Possible cases:  $a>0$ $a<0$

Cases 1 and 2: Roots are real

Case 3: If $a>0$, then ax^2+bx+c is always +ve. If $a<0$, then ax^2+bx+c is always –ve.

Example 2: The quadratic equation $x^2-4x-1=2k(x-5)$, where k is a constant, has two equal roots. Find the possible values of k.

Solution:

Rearranging, we have

$x^2-4x-1=2k(x-5)$

$x^2-(4+2k)x+(10k-1)=0$

For equal roots, we have $b^2-4ac=0$

$[-(4+2k)]^2-4(1)(10k-1)=0$

$4k^2-24k+20=0$

$k^2-6k+5=0$

$(k-1)(k-5)=0$

$\therefore\quad k=1 \quad \text{or} \quad k=5$

Example 3: Show that the equation $x^2 + kx = 4 - 2k$ has real roots for all real values of k.

Solution:
Rearranging, we have
$x^2 + kx - 4 + 2k = 0$

For real roots $b^2 - 4ac \geq 0$,

$$(k)^2 - 4(1)(2k-4) = k^2 - 8k + 16$$
$$= (k-4)^2$$

Since k is real, $(k-4)^2 \geq 0$ for all values of k. Hence the roots of the equation are real for all real values of k.

0.1.3 Relation of Roots and Coefficients

Suppose the quadratic equation $ax^2 + bx + c = 0$ has roots α and β. Then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

Comparing coefficients, we have

$$\text{Sum of roots, } \alpha + \beta = -\frac{b}{a}; \text{ product of roots, } \alpha\beta = \frac{c}{a}.$$

0.2 Polynomials

- **Multiplication and division of polynomials**
- **Use of remainder and factor theorems**

0.2.1 Review of Multiplication of Polynomials

Example 4: Find the products of each of the following:

i) $(3x+5)(2x+7)$
ii) $(5x^3 + 2x - 3)(x^2 - 3x + 6)$

Solution:

i) $(3x+5)(2x+7) = 3x(2x+7) + 5(2x+7) = 6x^2 + 31x + 35$

ii) $(5x^3 + 2x - 3)(x^2 - 3x + 6) = 5x^3(x^2 - 3x + 6) + 2x(x^2 - 3x + 6) - 3(x^2 - 3x + 6)$
$= 5x^5 - 15x^4 + 32x^3 - 9x^2 + 21x - 18$

0.2.2 Review of Long Division of Polynomials

Video resource: http://www.youtube.com/watch?v=o_1_eCscjSw

Example: $y = \dfrac{x^3 - 2x^2 - 13x + 6}{x+3}$

Step 1: Set up the long division.

The divisor (what you are dividing by) goes on the outside of the division box. The dividend (what you are dividing into) goes on the inside of the division box.

When you write out the dividend, make sure that you write it in **descending order** of the degree of the variables, (insert 0's for any missing terms).

$$x+3\overline{)x^3-2x^2-13x+6}$$

Step 2: Divide 1st term of dividend by first term of divisor to get first term of the quotient.

The quotient (answer) is written above the division box.

Make sure that you line up the first term of the quotient with the term of the dividend that has the same degree.

$$\begin{array}{r} x^2 \\ x+3\overline{)x^3-2x^2-13x+6} \end{array}$$

Step 3: Take the term found in step 2 and multiply it by the divisor.

Make sure that you line up all terms of this step with the term of the dividend that has the same degree.

$$\begin{array}{r} x^2 \\ x+3\overline{)x^3-2x^2-13x+6} \\ x^3+3x^2 \end{array}$$

Step 4: Subtract this from the line above.

Make sure that you subtract EVERY term found in step 3, not just the first one.

$$\begin{array}{r} x^2 \\ x+3\overline{)x^3-2x^2-13x+6} \\ \underline{x^3+3x^2} \\ -5x^2-13x+6 \end{array}$$

Step 5: Repeat until done.
You keep going until the degree of the remainder is less than the degree of the divisor.

$$\begin{array}{rrl} & x^2-5x+2 & \longleftarrow \text{quotient} \\ \text{divisor} \longrightarrow & x+3\overline{)x^3-2x^2-13x+6} & \longleftarrow \text{dividend} \\ & \underline{x^3+3x^2} & \\ & -5x^2-13x+6 & \\ & \underline{-5x^2-15x} & \\ & 2x+6 & \\ & \underline{2x+6} & \\ & 0 & \longleftarrow \text{remainder} \end{array}$$

Step 6: Write out the answer.

Your answer is the quotient that you ended up with on the top of the division box. If you have a remainder, write it over the divisor and include it in your final answer.

$$y = \frac{x^3 - 2x^2 - 13x + 6}{x + 3} = x^2 - 5x + 2$$

Example 5: Find the quotient and remainder for $(x^3 - 2x^2 - 13x + 6) \div (-x + 2)$.

$$\begin{array}{r|l} & -x^2 + 13 \\ -x+2 & x^3 - 2x^2 - 13x + 6 \\ & x^3 - 2x^2 \\ \hline & -13x + 6 \\ & -13x + 26 \\ \hline & -20 \end{array}$$

$$\frac{x^3 - 2x^2 - 13x + 6}{-x + 2} = -x^2 + 13 - \frac{20}{-x + 2}$$

0.2.3 Review of Remainder Theorem

When a polynomial $f(x)$ is divided by a linear divisor $ax - b$, $(a \neq 0)$, the remainder is $f\left(\frac{b}{a}\right)$.

Example 6: Find the remainder when $4x^3 - 2x^2 + 7x - 4$ is divided by $x - 1$.

Solution:
Remainder $R = f(1)$

$$= 4(1)^3 - 2(1) + 7(1) - 4 = 5$$

0.2.4 Review of Factor Theorem

The factor theorem states that if $f\left(\frac{b}{a}\right) = 0$, then $ax - b$ is a factor of the polynomial $f(x)$.

Conversely, if $ax - b$ is a factor of $f(x)$, then $f\left(\frac{b}{a}\right) = 0$ and $f(x)$ is divisible completely by $ax - b$.

Example 7: Find the value of k for which $x - 2$ is a factor of $f(x) = 3x^3 - 2x^2 + 5x + k$. Hence find the remainder $f(x)$ when it is divided by $x + 3$.

Solution:
$f(x) = 3x^3 - 2x^2 + 5x + k$

$x - 2$ is a factor of $f(x)$ $\Rightarrow$ $f(2) = 0$,

$3(2)^3 - 2(2)^2 + 5(2) + k = 0$

$k = -26$

$\therefore \quad f(x) = 3x^3 - 2x^2 + 5x - 26$

When $f(x)$ is divided by $(x + 3)$, remainder $R = f(-3)$

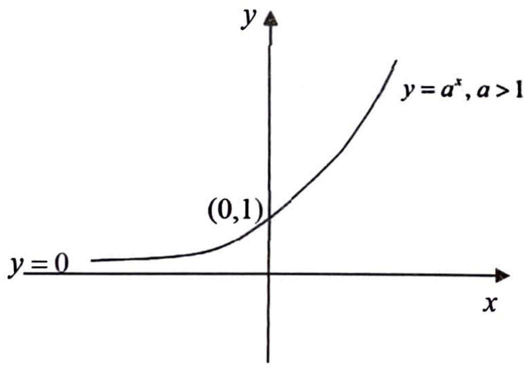
$\therefore \quad R = 3(-3)^3 - 2(-3)^2 + 5(-3) - 26 = -140$

0.3 Exponential and logarithmic functions

- **Functions a^x, e^x, $\log_a x$, $\ln x$ and their graphs**
- **Laws of logarithms**
- **Equivalence of $y = a^x$ and $x = \log_a y$**
- **Change of base of logarithms**
- **Function $|x|$ and graphs of $|f(x)|$, where $f(x)$ is linear, quadratic or trigonometric**
- **Solving simple equations involving exponential and logarithmic functions**

0.3.1 Functions a^x, $\log_a x$ and their Graphs

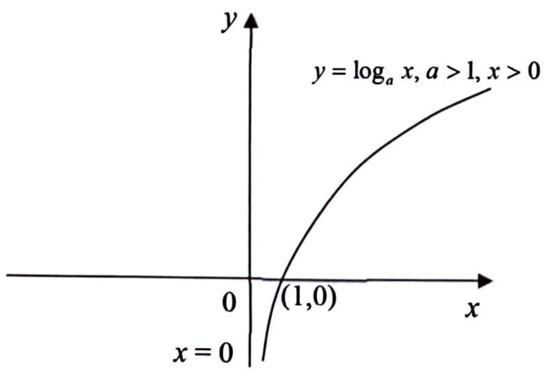
Exponential Functions $y = a^x$, $a > 0$



Important properties of an exponential function
- The y-intercept is 1.
- The line $y = 0$ is the horizontal asymptote.
- The function is increasing for $a > 1$.
- All the y values are strictly positive for all real values of x as the graph lies entirely above the x-axis.

Questions to ponder: What is the shape of the graph when $0 < a < 1$?

Logarithmic Functions $y = \log_a x$, $a > 0$, $x > 0$



Important properties of a logarithm function
- The x-intercept is 1.
- The line $x = 0$ is the vertical asymptote.
- All the x values are strictly positive for all real values of x as the graph lies entirely on the right of the y-axis.

0.3.2 Review of Logarithms

If a number y is expressed as the exponent of another number a, i.e. $y = a^x$ $(a > 0,\ a \neq 1)$, we say that x is the logarithm of y to the base a. We write this as $x = \log_a y$.

In general:	$y = a^x \iff x = \log_a y,\quad a > 0, a \neq 1$

Example 8: Convert the following to logarithmic form:

i) $3^{-2} = \frac{1}{9}$

ii) $2^x = 8$

Solution:

i) $3^{-2} = \frac{1}{9} \iff \log_3 \frac{1}{9} = -2$

ii) $2^x = 8 \iff \log_2 8 = x$

0.3.3 Law of Indices and Logarithms

Important rules for indices:	Laws of logarithms
If a and b are positive and m and n are positive integers then, 1) $a^m \times a^n = a^{m+n}$ 2) $a^m \div a^n = a^{m-n}$ 3) $(a^m)^n = a^{mn}$ 4) $a^m \times b^m = (a \times b)^m$ 5) $a^m \div b^m = \left(\frac{a}{b}\right)^m$ 6) $a^0 = 1$ 7) $a^{-n} = \frac{1}{a^n}$ 8) $a^{\frac{1}{n}} = \sqrt[n]{a}$ 9) $a^{\frac{m}{n}} = \sqrt[n]{a^m}$	If a, m and n are positive, and $a > 0, a \neq 1$ 1) Product law $\log_a mn = \log_a m + \log_a n$ 2) Quotient law $\log_a \frac{m}{n} = \log_a m - \log_a n$ 3) Power law $\log_a m^n = n \log_a m$

Example 9a: Evaluate $\log_2 \frac{\sqrt{2}}{64}$ without the use of a calculator.

Solution

(a) $\log_2 \frac{\sqrt{2}}{64} = \log_2 \sqrt{2} - \log_2 64$

$$= \log_2 2^{\frac{1}{2}} - \log_2 2^6$$

$$= \frac{1}{2}\log_2 2 - 6\log_2 2$$

$$= \frac{1}{2} - 6 = -5\frac{1}{2}$$

Example 9b: Simplify and express $2\lg 3 + 5\lg 2$ as a single logarithm.

Solution

(b) $2\lg 3 + 5\lg 2 = \lg 3^2 + \lg 2^5 = \lg(9\times 32) = \lg 288$

Example 9c: Simplify $\ln a + 3\ln b - 2\ln ab$ into a single logarithm in terms of a and b.

Solution

(c) $\ln a + 3\ln b - 2\ln ab = \ln a + \ln b^3 - \ln(ab)^2 = \ln\frac{ab^3}{(ab)^2} = \ln\frac{b}{a}$

0.3.4 Change of base of logarithms

In general, to convert $\log_a b$ to logarithms of base c, we have

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Example 10: Solve the following logarithmic equation $\log_3 N + \log_9 N = 6$.

Solution:

$\log_3 N + \log_9 N = 6$

$\log_3 N + \frac{\log_3 N}{\log_3 9} = 6$

$\log_3 N + \frac{\log_3 N}{2} = 6$

$3\log_3 N = 12$

$\log_3 N = 4$

$N = 3^4 = 81$

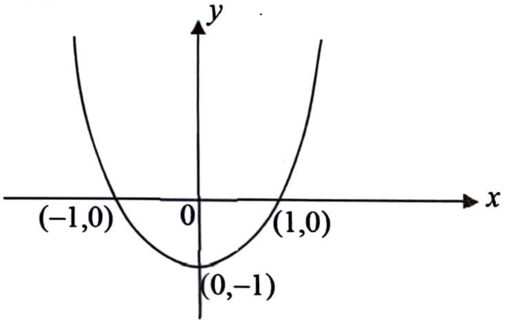
0.3.5 Review of Function $|x|$ and Graph of $|f(x)|$, where $f(x)$ is Linear, Quadratic or Trigonometric

The absolute value or the modulus of x, denoted by $|x|$, is defined by

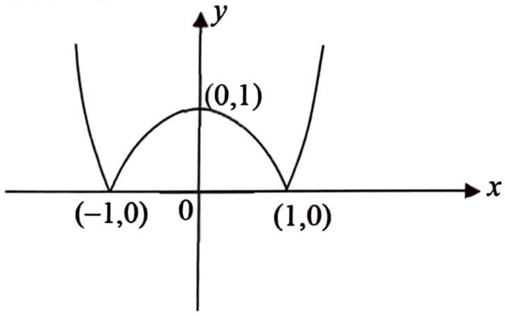
$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Example: Graph the following functions

(i) $y = x^2 - 1$,



(ii) $y = |x^2 - 1|$.



$y = \lvert f(x)\rvert$ is obtained by reflecting in the x-axis, the negative portion of the graph of $y = f(x)$ (i.e. $y < 0$), while preserving the positive portion (i.e. $y > 0$), of the graph.

0.3.6 Solving Simple Exponential and Logarithmic Equations

Example 11: Solve the following equations:

i) $e^{2x} - 5e^x + 6 = 0$

ii) $\log_{10}(3x+2) - 2\log_{10} x = 1 - \log_{10}(5x-3)$

Solution:

i) $e^{2x} - 5e^x + 6 = 0$

$(e^x)^2 - 5e^x + 6 = 0$

Let $y = e^x$,

$y^2 - 5y + 6 = 0$

$(y-2)(y-3) = 0$

$\therefore y = 2$ or $y = 3$

i.e. $e^x = 2$ or $e^x = 3$

$x = \ln 2$ or $x = \ln 3$

ii)

$$\log_{10}(3x+2)-2\log_{10}x=1-\log_{10}(5x-3)$$
$$\log_{10}(3x+2)-2\log_{10}x+\log_{10}(5x-3)=1$$
$$\log_{10}\frac{(3x+2)(5x-3)}{x^2}=1$$
$$\frac{(3x+2)(5x-3)}{x^2}=10^1$$
$$15x^2-9x+10x-6=10x^2$$
$$5x^2+x-6=0$$
$$(5x+6)(x-1)=0$$
$$\therefore\ x=-1\frac{1}{5}\quad\text{or}\quad x=1$$

Substituting $x=-1\frac{1}{5}$ into the original equation, we find $2\log_{10}\left(-1\frac{1}{5}\right)$ is not defined, so $x=-1\frac{1}{5}$ is not a solution. Substituting $x = 1$ into the original equation, we find that it satisfies the equation. Therefore, the solution is $x = 1$

0.4 Review of Partial Fractions

Include cases where the denominator is of the form

- $(ax+b)(cx+d)$
- $(ax+b)(cx+d)^2$
- $(ax+b)(x^2+c^2)$

You should carry out the following steps when finding partial fractions of a proper fraction:

<u>**STEP 1:**</u>
Check whether the algebraic fraction is **proper** or **improper**. If it is an **improper** fraction, convert it to a sum of polynomial and a **proper** fraction using long division.

<u>**STEP 2:**</u>
Factorise the denominator **completely**.

<u>**STEP 3:**</u>
Find the unknown constants $A, B, C, \ldots$ of the partial fractions. This is done by
(a) substituting specific values for x or
(b) equating coefficients of like terms.

Recall the following **partial fractions** decomposition:

(I)	**Non-repeated linear factors:**	$\frac{px+q}{(ax+b)(cx+d)}=\frac{A}{ax+b}+\frac{B}{cx+d}$

Example 12: Express $\dfrac{2x^2+3x+4}{(1-2x)(1-x^2)}$ as a sum of partial fractions.

Solution:

Write $\dfrac{2x^2+3x+4}{(1-2x)(1-x^2)}=\dfrac{2x^2+3x+4}{(1-2x)(1-x)(1+x)}$

$$=\frac{A}{1-2x}+\frac{B}{1-x}+\frac{C}{1+x}$$

$$=\frac{A(1-x)(1+x)+B(1-2x)(1+x)+C(1-2x)(1-x)}{(1-2x)(1-x)(1+x)}$$

1. Check for proper or improper fraction.
2. Denominator must be completely factorised

Equating the numerator: $2x^2+3x+4 = A(1-x)(1+x)+B(1-2x)(1+x)+C(1-2x)(1-x)$

Put $x=1$: $2(1)^2+3(1)+4=A(1-1)(1+1)+B(1-2)(1+1)+C(1-2)(1-1)$

$\Rightarrow \quad 9=-2B \quad \Rightarrow \quad B=-\dfrac{9}{2}$

$x=-1$: $2(-1)^2+3(-1)+4=A(1+1)(1-1)+B(1+2)(1-1)+C(1+2)(1+1)$

$\Rightarrow \quad 3= 6C \quad \Rightarrow \quad C=\dfrac{1}{2}$

$x=\dfrac{1}{2}$: $2\left(\dfrac{1}{2}\right)^2+3\left(\dfrac{1}{2}\right)+4=A\left(1-\dfrac{1}{2}\right)\left(1+\dfrac{1}{2}\right)+B(1-1)\left(1+\dfrac{1}{2}\right)+C(1-1)\left(1-\dfrac{1}{2}\right)$

$\Rightarrow \quad 6=\dfrac{3}{4}A \quad \Rightarrow \quad A=8$

$$\therefore \quad \frac{2x^2+3x+4}{(1-2x)(1-x^2)}=\frac{8}{1-2x}-\frac{9}{2(1-x)}+\frac{1}{2(1+x)}$$

Example 13: Express $\dfrac{x^3-1}{(x+1)(x+2)}$ as a sum of partial fractions.

Solution:
Using long division,

$$\frac{x^3-1}{(x+1)(x+2)} = (x-3)+\frac{7x+5}{(x+1)(x+2)}$$

1. Check for proper or improper fraction.
2. Denominator must be completely factorised

Write $\dfrac{7x+5}{(x+1)(x+2)}=\dfrac{A}{x+1}+\dfrac{B}{x+2}$

$$=\frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

Equating the numerator: $7x+5=A(x+2)+B(x+1)$

Put $x=-1$: $7(-1)+5=A(-1+2)+B(-1+1) \quad \Rightarrow \quad A=-2$

$x=-2$: $7(-2)+5=A(-2+2)+B(-2+1) \quad \Rightarrow \quad -9=-B \quad \Rightarrow \quad B=9$

$$\therefore \quad \frac{x^3-1}{(x+1)(x+2)}=(x-3)-\frac{2}{x+1}+\frac{9}{x+2}$$

(II) Repeated linear factors: $\dfrac{px^2+qx+r}{(ax+b)(cx+d)^2}=\dfrac{A}{ax+b}+\dfrac{B}{cx+d}+\dfrac{C}{(cx+d)^2}$

Example 14: Express $\dfrac{2x^2-7}{(x+1)(x-3)^2}$ as a sum of partial fractions.

Solution:

Write $\dfrac{2x^2-7}{(x+1)(x-3)^2}=\dfrac{A}{x+1}+\dfrac{B}{x-3}+\dfrac{C}{(x-3)^2}$

$$=\frac{A(x-3)^2+B(x+1)(x-3)+C(x+1)}{(x+1)(x-3)^2}$$

> 1. Check for proper or improper fraction.
> 2. Denominator must be completely factorised

Equate the numerator: $2x^2-7 = A(x-3)^2+B(x+1)(x-3)+C(x+1)$

Put $x=-1$: $2(-1)^2-7 = A(-1-3)^2+B(-1+1)(-1-3)+C(-1+1)$

$$\Rightarrow \quad -5=16A \quad\Rightarrow\quad A=-\frac{5}{16}$$

$x=3$: $2(3)^2-7 = A(3-3)^2+B(3+1)(3-3)+C(3+1)$

$$\Rightarrow \quad 11=4C \quad\Rightarrow\quad C=\frac{11}{4}$$

Equate coefficient of x^2: $2=A+B \quad\Rightarrow\quad 2=-\dfrac{5}{16}+B \quad\Rightarrow\quad B=\dfrac{37}{16}$

$$\therefore \quad \frac{2x^2-7}{(x+1)(x-3)^2}=-\frac{5}{16(x+1)}+\frac{37}{16(x-3)}+\frac{11}{4(x-3)^2}$$

(III) Non-repeated quadratic factor: $\dfrac{px^2+qx+r}{(ax+b)(x^2+c^2)}=\dfrac{A}{ax+b}+\dfrac{Bx+C}{x^2+c^2}$

Example 15: Express $\dfrac{5x^2-x+2}{(2x+1)(x^2+1)}$ as a sum of partial fractions.

Solution:

Write $\dfrac{5x^2-x+2}{(2x+1)(x^2+1)}=\dfrac{A}{2x+1}+\dfrac{Bx+C}{x^2+1}$

$$=\frac{A(x^2+1)+(Bx+C)(2x+1)}{(2x+1)(x^2+1)}$$

> 1. Check for proper or improper fraction.
> 2. Denominator must be completely factorised

Equating the numerator: $5x^2-x+2=A(x^2+1)+(Bx+C)(2x+1)$

Put $x=-\dfrac{1}{2}$: $5\left(-\dfrac{1}{2}\right)^2-\left(-\dfrac{1}{2}\right)+2=A\left[\left(-\dfrac{1}{2}\right)^2+1\right]+\left[B\left(-\dfrac{1}{2}\right)+C\right]\left[2\left(-\dfrac{1}{2}\right)+1\right]$

$$\Rightarrow \quad \frac{15}{4}=\frac{5}{4}A \quad\Rightarrow\quad A=3.$$

Equate coefficients of x^2: $5=A+2B \quad\Rightarrow\quad B=1.$

Equate constants: $2=A+C \quad\Rightarrow\quad C=-1.$

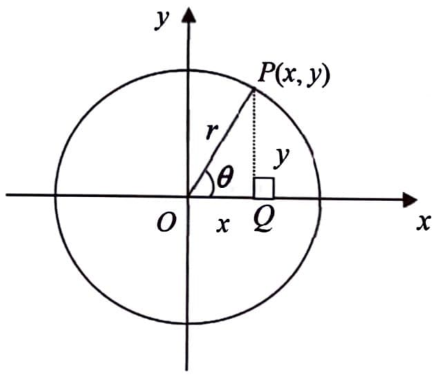
$$\therefore \quad \frac{5x^2-x+2}{(2x+1)(x^2+1)}=\frac{3}{2x+1}+\frac{x-1}{x^2+1}.$$

0.5 Review of Basic Trigonometry

0.5.1 Definition of Trigonometric Functions and Basic Concepts

Suppose the line OP can rotate about O in an anticlockwise direction and makes an angle θ with the positive x-axis. We divide the complete revolution into 4 **quadrants** and take the positive y-axis at 90°. Let (x, y) be the coordinates of P. x and y will be positive or negative dependent on which quadrant P lies in.

Note: θ is positive if OP rotates in an anti-clockwise direction and negative if OP rotates in a clockwise direction.



The diagram shows that P lies in the first quadrant and θ is an acute angle. From the right-angled triangle OPQ, we define

$\sin\theta = \dfrac{y}{r}$ $\quad$ $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$, $\sin\theta \neq 0$

$\cos\theta = \dfrac{x}{r}$ $\quad$ $\sec\theta = \dfrac{1}{\cos\theta}$, $\cos\theta \neq 0$

$\tan\theta = \dfrac{y}{x}$, $x \neq 0$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$, $\tan\theta \neq 0$

Note:
(1) θ is in radians (or degree)
(2) The above trigonometric functions are true even if θ is not acute.

Trigonometric Ratios for a General Angle
Since r is positive, the signs of the ratios depend on the quadrant that $P(x,y)$ is in.

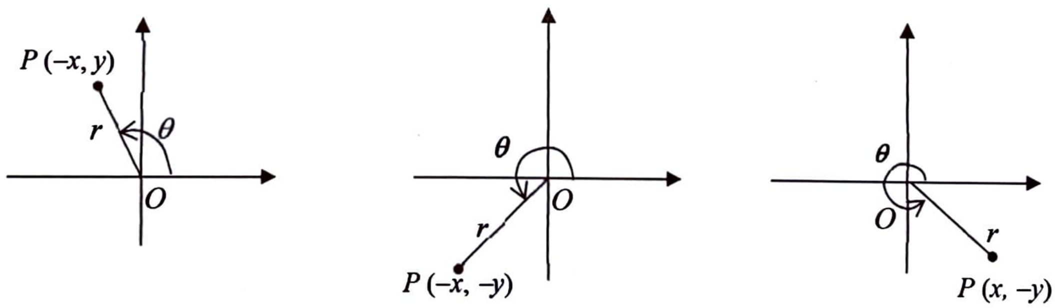


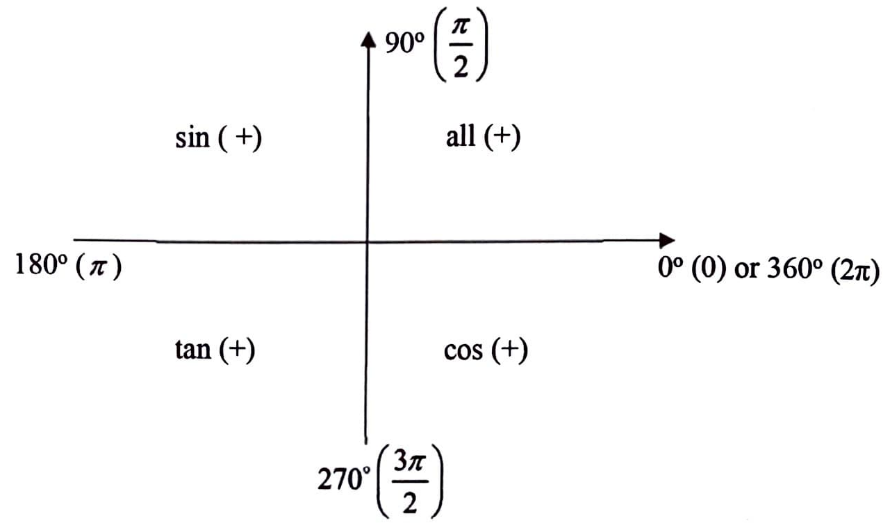
diagram 1 $\quad$ diagram 2 $\quad$ diagram 3

Consider diagram 1 (second quadrant) where θ is obtuse
Let $OP = r$ (r is always positive)

$\sin\theta = \dfrac{+y}{+r}$ is positive $\quad$ $\cos\theta = \dfrac{-x}{+r}$ is negative $\quad$ $\tan\theta = \dfrac{+y}{-x}$ is negative

Similarly, we can check for the sign of the trigonometric ratios for θ between 180° and 270° and θ between 270° and 360°.

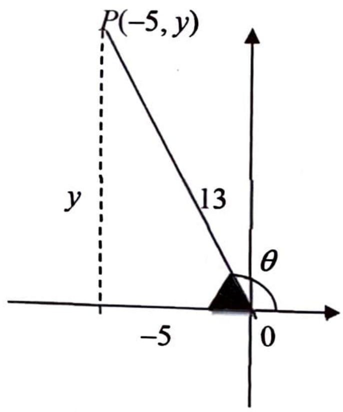
We can summarise the signs of the trigonometric ratios by the following diagram.



Example 16: Given that $\cos\theta = -\frac{5}{13}$ and $\tan\theta < 0$, evaluate $\sin\theta$ and $\tan\theta$ without the use of a calculator.

Solution:

$\cos\theta$ is negative $\Rightarrow$ θ lies in the 2nd or 3rd quadrants
$\tan\theta$ is negative $\Rightarrow$ θ lies in the 2nd or 4th quadrants
$\therefore$ θ lies in the 2nd quadrant



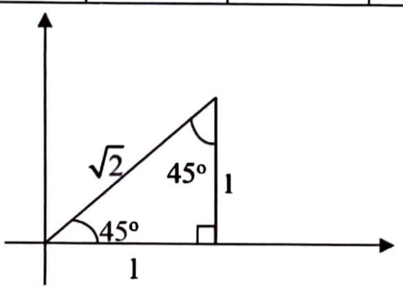
$$y^2 + (-5)^2 = 13^2 \Rightarrow y^2 = 169 - 25 = 144$$

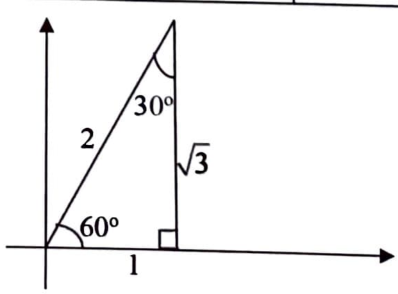
Since $y > 0$, $y = \sqrt{144} = 12$

$$\therefore \sin\theta = \frac{y}{r} = \frac{12}{13}; \quad \tan\theta = \frac{y}{x} = \frac{12}{-5} = -\frac{12}{5}$$

Table of values of functions of some special angles

θ	0° (0)	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$	180° (π)	270° $\left(\frac{3\pi}{2}\right)$	360° (2π)
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	−1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	−1	0	1
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	0	undefined	0





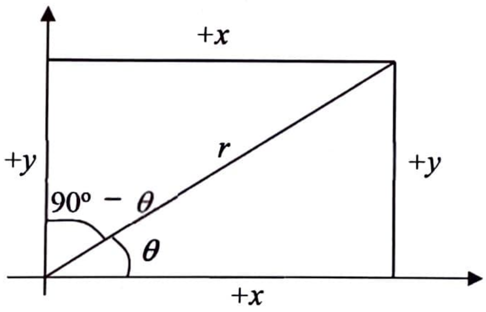
Complementary Angles

$$\sin\theta = \frac{+y}{r} = \cos\left(90^o - \theta\right)$$

$$\cos\theta = \frac{+x}{r} = \sin\left(90^o - \theta\right)$$

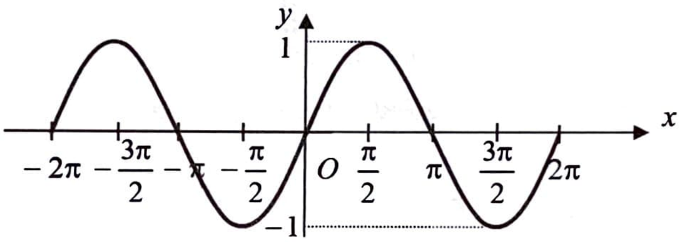
$$\tan\theta = \frac{+y}{+x} = \cot\left(90^o - \theta\right)$$

$$\cot\theta = \frac{+x}{+y} = \tan\left(90^o - \theta\right)$$



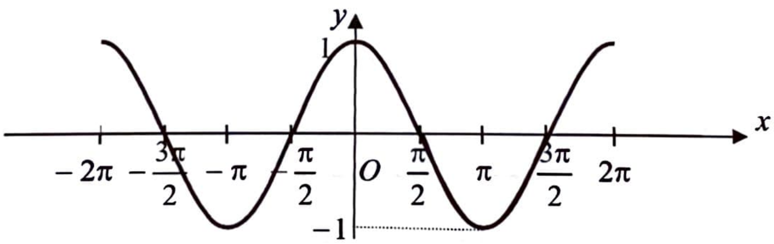
0.5.2 Graphs of Trigonometric Functions (for $-2\pi \le x \le 2\pi$)

The Graph of $y = \sin x$



The graph of $y = \sin x$ is **periodic** with **period** 2π i.e. $\sin x = \sin(x + 2\pi)$.

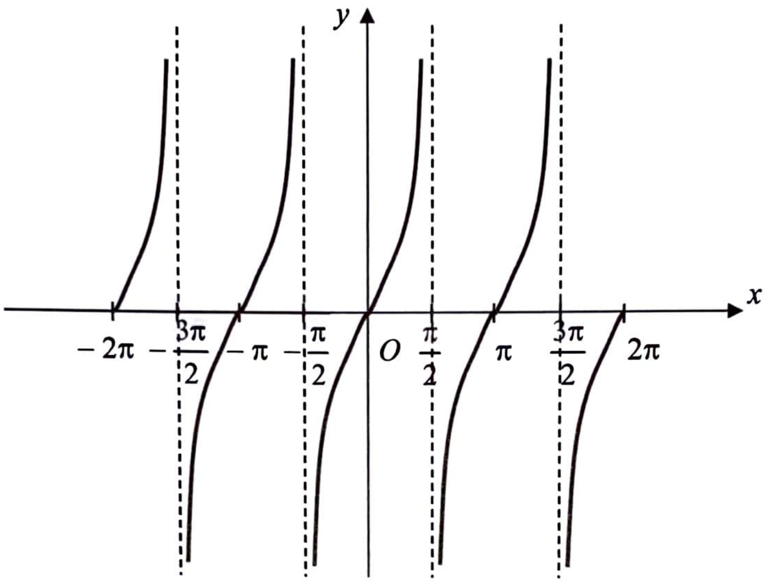
The Graph of $y = \cos x$



The graph of $y = \cos x$ is **periodic** with **period** 2π i.e. $\cos x = \cos(x + 2\pi)$.

Note that the cosine graph can be obtained from the sine graph by shifting it to the left along the x-axis through $\frac{\pi}{2}$ radians i.e. $\cos x = \sin\left(x + \frac{\pi}{2}\right)$.

The Graph of $y = \tan x$



The graph of $y = \tan x$ is **periodic** with **period** π i.e. $\tan x = \tan(x+\pi)$.

0.5.3 Trigonometric Identities

0.5.3.1 Basic Identities

(i) $\sin(-x) = -\sin x$, $\cos(-x) = \cos x$, $\tan(-x) = -\tan x$

(ii) $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, $\cos x = \sin\left(\frac{\pi}{2} - x\right)$, $\tan x = \cot\left(\frac{\pi}{2} - x\right)$

(iii) $\sin^2 x + \cos^2 x = 1$

(iv) $1 + \tan^2 x = \sec^2 x$

(v) $1 + \cot^2 x = \operatorname{cosec}^2 x$

0.5.3.2 The Addition Formulae (in MF26 – formulae booklet in 'A' level)

(i) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

(ii) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

Example 16: Without the use of the calculator, evaluate
(Hint: Use Addition formulae)

(i) $\cos\frac{5\pi}{12}$,

(ii) $\tan 105^\circ$ exactly.

Solution:

(i) $$\cos\left(\frac{5\pi}{12}\right)=\cos\left(\frac{\pi}{4}+\frac{\pi}{6}\right)$$
$$=\cos\frac{\pi}{4}\cos\frac{\pi}{6}-\sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)-\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$=\frac{1}{4}\left(\sqrt{6}-\sqrt{2}\right)$$

(ii) $$\tan 105°=\tan(60°+45°)$$
$$=\frac{\tan 60°+\tan 45°}{1-\tan 60°\tan 45°}$$
$$=\left(\frac{\sqrt{3}+1}{1-\sqrt{3}(1)}\right)=\frac{\sqrt{3}+1}{1-\sqrt{3}}$$
$$=\frac{\sqrt{3}+1}{1-\sqrt{3}}\left(\frac{1+\sqrt{3}}{1+\sqrt{3}}\right)=\frac{4+2\sqrt{3}}{-2}$$
$$=-2-\sqrt{3}$$

0.5.3.3 The Double Angle Formulae (in MF26)

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\cos 2A = \cos^2 A - \sin^2 A = 1 - 2\sin^2 A = 2\cos^2 A - 1$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Example 16: Prove that $\dfrac{\sin\theta}{1+\cos\theta}\equiv\tan\dfrac{\theta}{2}$.

Solution:

$$\frac{\sin\theta}{1+\cos\theta}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$
$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$
$$=\tan\frac{\theta}{2}$$

0.5.3.4 The Factor Formulae (in MF26 and will be covered in Integration)

(i) $\sin P+\sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$

(ii) $\sin P-\sin Q = 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$

(iii) $\cos P+\cos Q = 2\cos\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$

(iv) $\cos P-\cos Q = -2\sin\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$

Example 17: Prove that $\sin x + \sin 3x + \sin 5x + \sin 7x = 16\sin x\cos^2 x\cos^2 2x$.

Solution:

$$\sin x+\sin 3x+\sin 5x+\sin 7x = 2\sin\left(\frac{3x+x}{2}\right)\cos\left(\frac{3x-x}{2}\right)+2\sin\left(\frac{7x+5x}{2}\right)\cos\left(\frac{7x-5x}{2}\right)$$

$$= 2\sin 2x\cos x + 2\sin 6x\cos x = 2\cos x(\sin 2x+\sin 6x)$$

$$= 2\cos x\left[2\sin\left(\frac{6x+2x}{2}\right)\cos\left(\frac{6x-2x}{2}\right)\right] = 4\cos x\ \sin 4x\ \cos 2x$$

$$= 4\cos x(\cos 2x)(2\sin 2x\cos 2x) \quad = 8\cos x\cos^2 2x(2\sin x\cos x)$$

$$= 16\sin x\cos^2 x\cos^2 2x \qquad \text{(proved)}$$

0.5.3.5 The *R*-Formulae

For any $a, b > 0$,

(i) $a\sin x \pm b\cos x = R\sin(x\pm\alpha)$ where $R=\sqrt{a^2+b^2}$, $\tan\alpha = \frac{b}{a}$ where $0\le\alpha\le\frac{\pi}{2}$

(ii) $a\cos x \pm b\sin x = R\cos(x\mp\alpha)$

Example 18: Express 5 sin θ + 12 cos θ in the form R sin ($\theta + \alpha$), where R is positive and α is acute, giving the value of α to the nearest 0.1°.

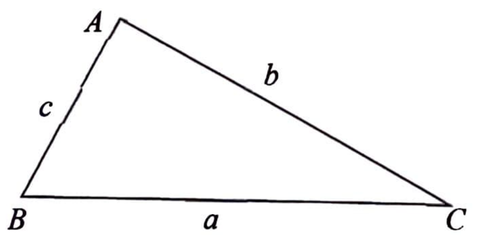
Solution:

$$5\sin\theta + 12\cos\theta = \sqrt{5^2+12^2}\sin\left(\theta+\tan^{-1}\frac{12}{5}\right)$$

$$= 13\sin(\theta + 67.4°)$$

0.5.4 The Sine and Cosine Rules

Consider triangle ABC,



Sine rule

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

Cosine rule

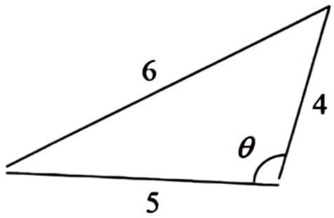
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note: the largest angle is opposite the longest side!

Example 19: The lengths of the sides of a triangle are 4 cm, 5 cm and 6 cm. The size of the largest angle of the triangle is θ. Calculate the value of cos θ and hence show that sin θ = $\frac{a}{b}\sqrt{7}$ where a and b are integers to be determined.

Solution:

The largest angle is opposite the longest side. So the angle θ is opposite the side of length 6 cm.



By cosine rule, $\cos\theta = \dfrac{4^2+5^2-6^2}{2(4)(5)} = \dfrac{1}{8}$.

Using the basic identity $\sin^2\theta + \cos^2\theta = 1$, we obtain

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\sin\theta = \pm\sqrt{\left(1-\cos^2\theta\right)} = \pm\sqrt{\left(1-\left(\frac{1}{8}\right)^2\right)} = \pm\frac{3}{8}\sqrt{7}$$

Since $\cos\theta = \dfrac{1}{8} > 0$, θ is acute.

Hence $\sin\theta = \dfrac{3}{8}\sqrt{7}$, i.e., $a = 3$ and $b = 8$.

Chapter 1

Basic Graphing Techniques (Self-learning)

At the end of this chapter, students should be able to
- use a graphing calculator to graph a given function;
- find characteristics of graphs such as symmetry, intersections with the axes, turning points and asymptotes;
- determine the equations of asymptotes, axes of symmetry and restrictions on the possible values of x and (or) y and show these features on a graph.

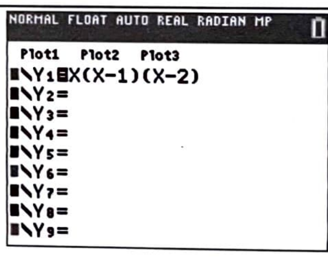
1.1 Characteristics of graphs

When sketching graphs, it is important to show the key characteristics of the graphs - axial intercepts, stationary points and asymptotes. These characteristics allow us to visualize the behaviour of a function by looking at its graph. In the 'A' level syllabus, **you are required to label clearly the coordinates of the axial intercepts, stationary points and the equations of the asymptotes** on your sketch.

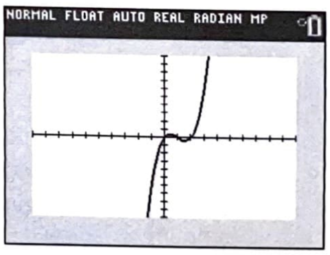
1.2 Graphing using Graphing Calculator (GC)

Example 1: Graph the curve $y = x(x-1)(x-2)$ using the GC.

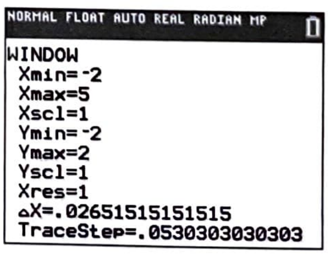
1. Press [y=] and key in the function into Y_1. Press [X,T,θ,n] to get the variable x.



2. Press [graph] to display the graph.

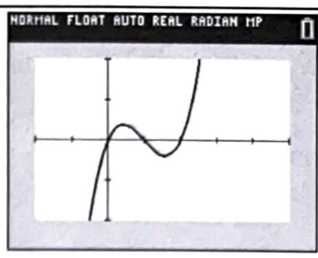


3. Press [window] to display the WINDOW editor. You may change the individual settings in [window] to see different parts of the graph.



4. Press [graph]

Alternative method: Press [zoom][6] (for **ZStandard**) to display the graph using a preset viewing. The standard viewing shows the graph for $-10 \le x \le 10$ and $-10 \le y \le 10$.



1.3 Axial intercepts

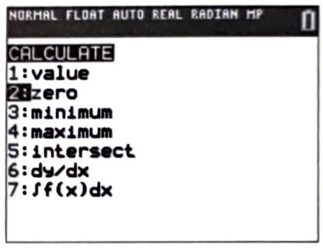
These are points where the curve intersects the x- and y-axes.

Example 2: Find the coordinates of the points of intersection of $y = -x^3 - 3x + 1$, using the GC.

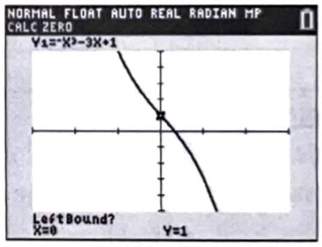
To find the x-intercepts.

First sketch the curve on your GC

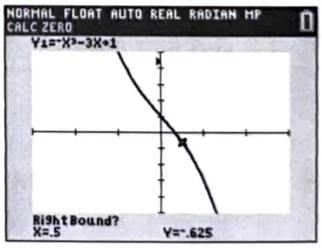
1. Press [2nd] [trace] (for **CALC**) then select **2: zero**. Press [enter].



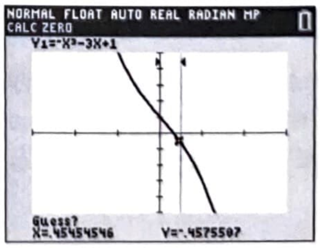
2. Respond to **Left Bound?** by moving the cursor [◄] to just left of the axial intercept point and pressing [enter].



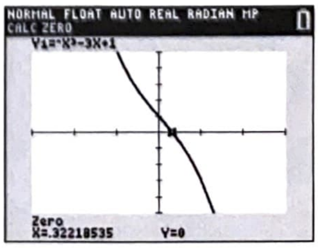
3. Respond to **Right Bound?** by moving the cursor [►] to just right of the axial intercept point and pressing [enter].



4. Respond to **Guess?** by pressing [enter].

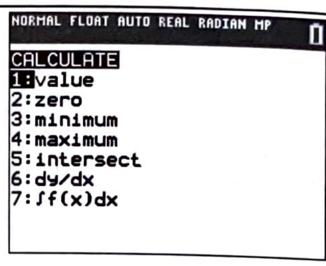


$\therefore$ the x-intercept is at $x = 0.322$.

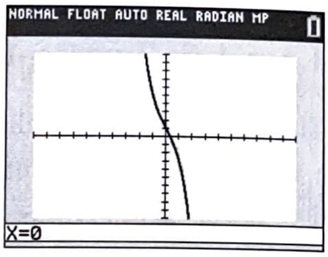


To find the *y*-intercepts.

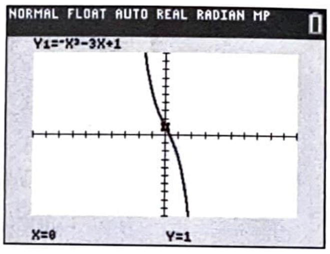
1. Press [2nd] [trace] (for **CALC**) then select **1: value**. Press [enter].



2. Key in 0 for the *x*-value and press [enter].



$\therefore$ the *y*-intercept is at $y = 1$.



Note:
One of the limitations of the graphing calculator is that **it does not give exact answers**. If exact answers are required by the question, the algebraic or analytical approach should be used.

In addition, you may need to adjust the window settings appropriately to see the complete graph. Hence it is important to know the general characteristics of the different types of graphs.

1.4 Stationary points and their nature

Stationary points of curves are points where $\frac{dy}{dx} = 0$ (i.e. gradient = 0). Suppose $\frac{dy}{dx} = 0$ at $x = k$, then we can say that a stationary point occurs at $x = k$. From here, we can proceed to determine the nature of this stationary point using either the first derivative test or the second derivative test. We can also find stationary points of a curve using a graphing calculator.

Example 3: Find the coordinates of the turning points on the curve $y = x(x-1)(x-2)$ and state the nature of each turning point.

Analytical Method

$$y = x(x-1)(x-2) = x^3 - 3x^2 + 2x \quad \Rightarrow \quad \frac{dy}{dx} = 3x^2 - 6x + 2$$

To find turning points, set $\frac{dy}{dx} = 0$

$$3x^2 - 6x + 2 = 0$$

$$\therefore \quad x = \frac{6 \pm \sqrt{36-24}}{6} = \frac{3 \pm \sqrt{3}}{3} = 0.423 \text{ or } 1.58 \text{ (to 3 significant figure)}$$

At $x = 0.423$, $\quad y = 0.385$
At $x = 1.58$, $\quad y = -0.385$
The coordinates of the turning points are (0.423, 0.385) and (1.58, –0.385).

Using 2nd derivative test,

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=0.423} = 6(0.423) - 6 = -3.462 < 0$$

$\therefore$ (0.423, 0.385) is a maximum point

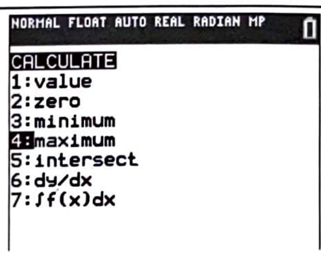
$$\left.\frac{d^2y}{dx^2}\right|_{x=1.58} = 6(1.58) - 6 = 3.48 > 0$$

$\therefore$ (1.58, –0.385) is a minimum point.

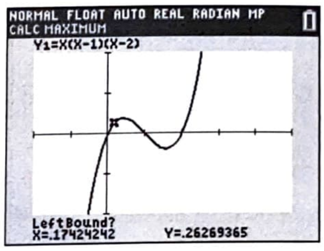
Using GC

Sketch the curve on your GC first.

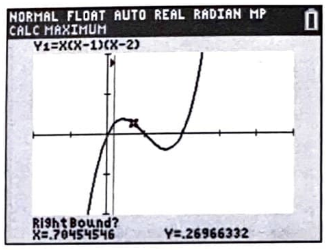
1. Press [2nd] [trace](for **CALC**) then select **4:maximum**. Press [enter]



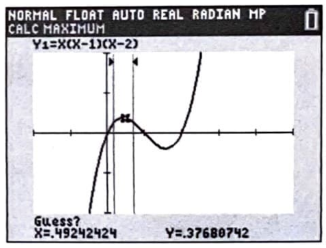
2. Respond to **Left Bound?** by moving the cursor [◄] to the left of the maximum point and pressing [enter].



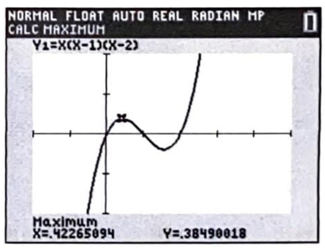
3. Respond to **Right Bound?** by moving the cursor [►] to the right of the maximum point and pressing [enter].



4. Respond to **Guess?** by pressing [enter].



∴ the maximum point is (0.423, 0.385)



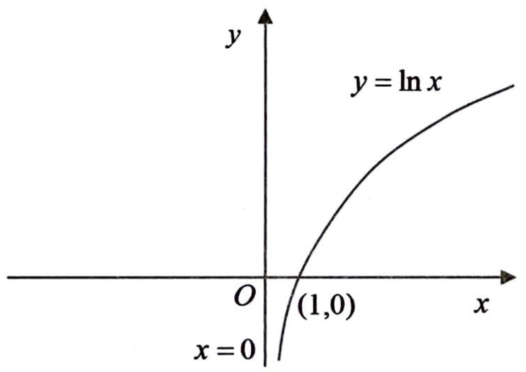
Similarly use the '**minimum**' command to get the minimum point (1.58, –0.385).

1.5 Asymptotes

An asymptote is a line or curve that a graph approaches as it heads towards infinity. Recall that the line $x = 0$ is the asymptote of the graph $y = \ln x$, and that the line $y = 0$ is the asymptote of the graph $y = e^x$. We usually use **dotted lines** to represent asymptotes in graphs. One of the limitations of the GC when sketching the graph is that, the asymptotes are not shown. Hence we need to find the asymptotes manually.

Vertical asymptotes are vertical lines which often arises due to the restrictions in the domain (set of all values of x that is allowed) of the graphs, i.e. we say $x = a$ is a vertical asymptote if y **approaches** infinity or negative infinity, when x **approaches** a. Mathematically, this is written as "$y \to \infty$ or $y \to -\infty$ when $x \to a$."

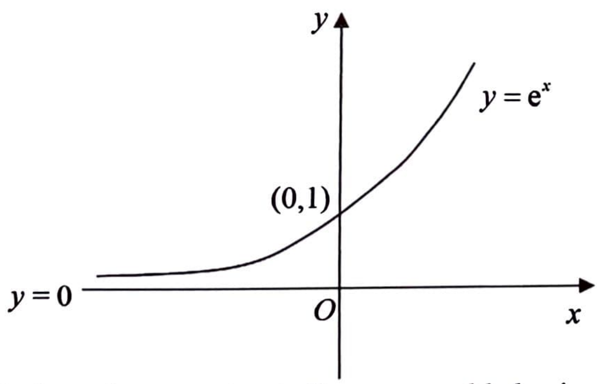
Example 4: Sketch $y = \ln x$, showing the equation of the asymptote and the coordinates of the point of intersection of the graph with the axes.



Important properties of $y = \ln x$
- The x-intercept is (1, 0).
- The line $x = 0$ is the vertical asymptote.
- The function is *strictly increasing* for $x > 0$ (i.e. $x_1 < x_2 \Leftrightarrow \ln(x_1) < \ln(x_2)$).
- $y = \ln x$ is a special case of $y = \log_a x$, where $a = \mathrm{e}$ (Euler's number).

Horizontal asymptotes are lines that show how the graph behaves as $x \to \pm\infty$.
i.e. we say $y = a$ is a horizontal asymptote if $x \to \infty$ or $x \to -\infty$ when $y \to a$.

Example 5: Sketch $y = \mathrm{e}^x$, showing the equation of the asymptote and the coordinates of the point of intersection of the graph with the axes.



Important properties of $y = \mathrm{e}^x$
- The y-intercept is (0, 1).
- The line $y = 0$ is the horizontal asymptote.
- The function is *strictly increasing*.
- $y > 0$ for all real values of x.

Horizontal asymptotes indicate general behaviour of the graphs at the extreme values of x ($x \to \pm\infty$). Thus, it is possible for the graph to intersect the horizontal asymptotes when the values of x are not at the extremes.

<u>Note</u>: It is important to label the equations of all asymptotes which the graph approaches on the sketch even if they coincide with the *x*-axis or *y*-axis.

Example 6: Find the asymptotes for the curve $y = 2 + \dfrac{3}{x-1}$.

We see that as x tends to positive or negative infinity, y will tend to 2.

So as $x \to \infty,\ \dfrac{3}{x-1} \to 0$ so $y \to 2$

$x \to -\infty,\ \dfrac{3}{x-1} \to 0$ so $y \to 2$

So $y = 2$ is an asymptote. We also refer to this as a horizontal asymptote.

To find the vertical asymptote, we see that for y to tend to positive infinity, x must tend to a value slightly more than 1.
For y to tend to negative infinity, x must tend to a value slightly less than 1.
So we say as $x \to 1^+,\ y \to +\infty$

$x \to 1^-,\ y \to -\infty$

Therefore, y tends towards infinity as x tends to 1. $x = 1$ is an asymptote and we refer it as a vertical asymptote.

We will deal with this problem again in **Example 9**. We will learn the short cuts as well.

Example 7: Find the asymptotes for the curve $y = x + \dfrac{1}{x}$.

As $x \to \infty,\ \dfrac{1}{x} \to 0$ so $y \to x$

$x \to -\infty,\ \dfrac{1}{x} \to 0$ so $y \to x$

So $y = x$ is an asymptote

As $x \to \infty,\ y \to \infty$

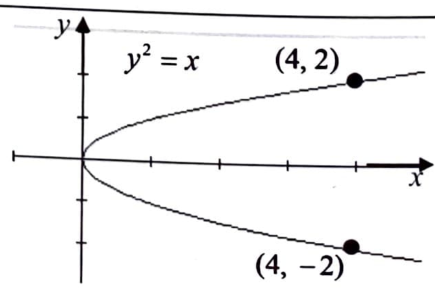
$x \to -\infty,\ y \to -\infty$

This means that the curve does not have a ~~vertical~~ horizontal asymptote.

1.6 Symmetry

(A) Symmetry about x-axis

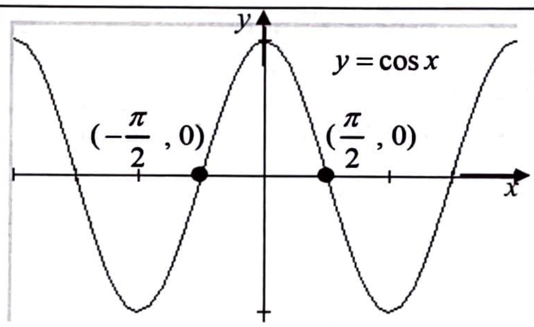
A curve is symmetric about the x-axis, if (x, y) is a point on the curve then $(x, -y)$ is also a point on the curve.
In other words, when y is replaced by $-y$, the equation of the curve remains unchanged.
Example: $y^2 = x$



(B) Symmetry about y-axis
(Even Function, $f(x) = f(-x)$**)**

A curve is symmetric about the y-axis, if (x, y) is a point on the curve then $(-x, y)$ is also a point on the curve.
In other words, the equation of the curve remains unchanged when x is replaced by $-x$. This is a property of **even functions.**

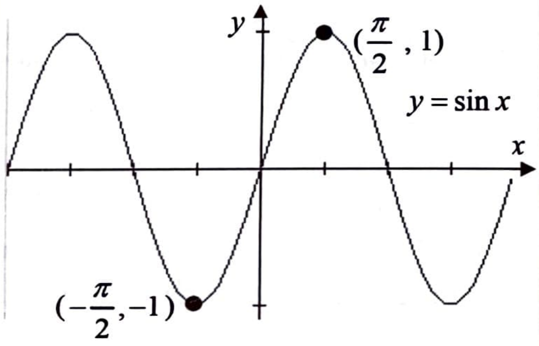
Examples: $y = x^2$, $y = \cos x$, $y = |x|$



(C) Symmetry about the Origin
(Odd Function, $f(-x) = -f(x)$**)**

A curve is symmetric about the origin, if (x, y) is a point on the curve then $(-x, -y)$ is also a point on the curve. In other words, the equation of the curve remains unchanged when x and y are replaced by $-x$ and $-y$ simultaneously. This is a property of **odd functions.**

Examples: $y = x^3$, $y = \sin x$, $y = \tan x$



Example 8: [N2014/P1/Q7 (modified)]
It is given that $f(x) = x^6 - 3x^4 - 7$. Show that $f(x) = f(-x)$. What can be said about the real roots?

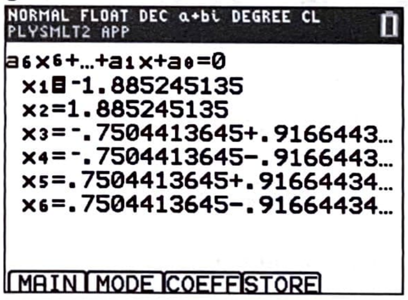
Solutions

$$f(-x) = (-x)^6 - 3(-x)^4 - 7$$
$$= x^6 - 3x^4 - 7$$
$$= f(x)$$

If α is a real root, then $-\alpha$ is also a real root.

Using GC

Students can make use of the PolySmlt2 App to get a hint of what the roots are like.



Note: x_3, x_4, x_5, x_6 are complex roots. Don't worry about it first. We will learn about complex numbers in JC2

1.7 Power Functions $y = ax^n$, where a is positive

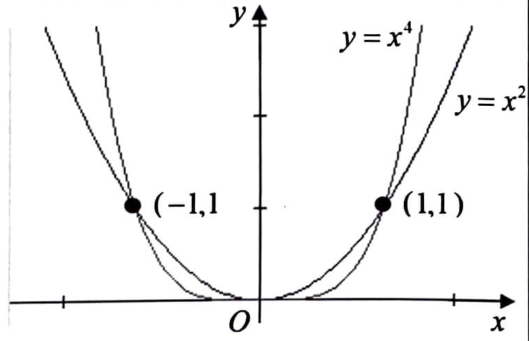
<table>
<tr><th>Cases</th><th>Examples where $a = 1$</th></tr>
<tr><td>

(A) *n* is an even positive integer,

When n is an even positive integer,

- the graph of $y = ax^n$ is symmetrical about the y-axis (even function);
- all these curves have three common points of intersection, namely $(0,0)$, $(1, 1)$ and $(-1, 1)$.

</td><td>



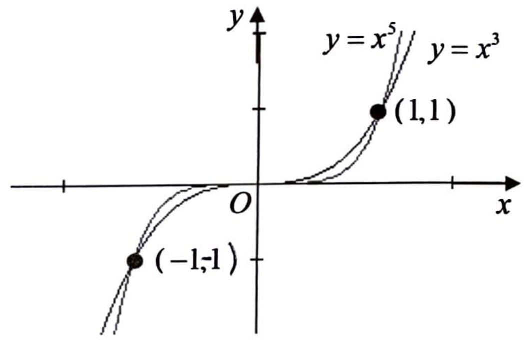
</td></tr>
<tr><td>

(B) *n* is an odd positive integer where $n \geq 3$

When n is an odd positive integer,

- the graph of $y = ax^n$ is symmetrical about the origin (odd function);
- all these curves have three common points of intersection, namely $(0,0)$, $(1, 1)$ and $(-1, -1)$.

</td><td>



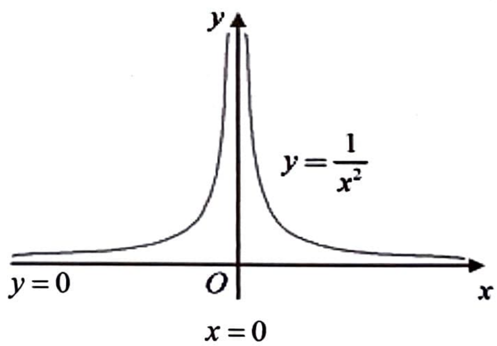
</td></tr>
<tr><td>

(C) *n* is an even negative integer

When n is an even negative integer,

- the graph of $y = ax^n$ is symmetrical about the y-axis (even function);
- all these curves have two common points of intersection, namely $(1, 1)$ and $(-1, 1)$;
- the line $x = 0$ is a vertical asymptote;
- the line $y = 0$ is a horizontal asymptote.

</td><td>



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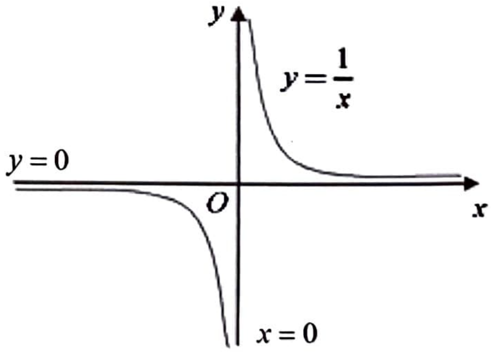
<tr><td>

(D) *n* is an odd negative integer

When n is an odd negative integer,

- the graph of $y = ax^n$ is symmetrical about the origin (odd function);
- all these curves have two common points of intersection, namely $(1, 1)$ and $(-1, -1)$;
- the line $x = 0$ is a vertical asymptote;
- the line $y = 0$ is a horizontal asymptote.

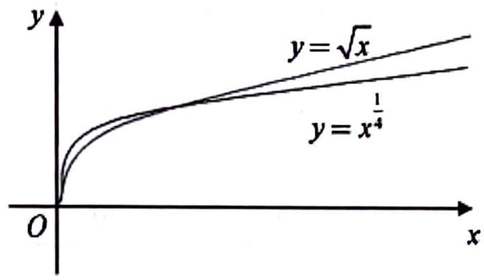
</td><td>



</td></tr>
</table>

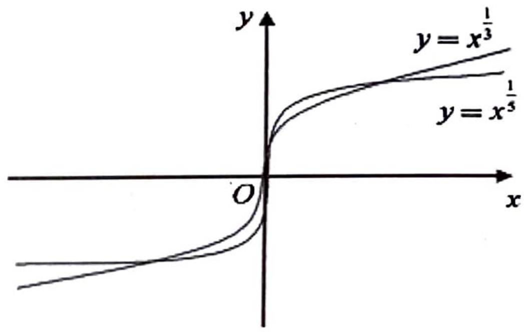
(E) ***n* is a rational number of the form $\frac{1}{k}$ where *k* is an even positive integer (e.g. $\frac{1}{2}, \frac{1}{4}$)**

When k is even, the set of values of x that satisfy $y = ax^n$ is the set of non-negative real numbers. Do you know why?



(F) ***n* is a rational number of the form $\frac{1}{k}$ where *k* is an odd positive integer where $k \geq 3$ (e.g. $\frac{1}{3}, \frac{1}{5}$)**

If k is odd, the set of values of x that satisfy $y = ax^n$ is all real numbers.
The graph is symmetrical about the origin (odd function).

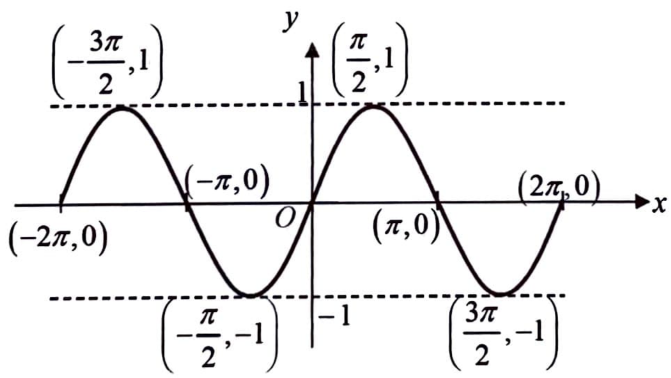


Question: What will happen to the graphs if $a < 0$?

1.8 Trigonometric Functions

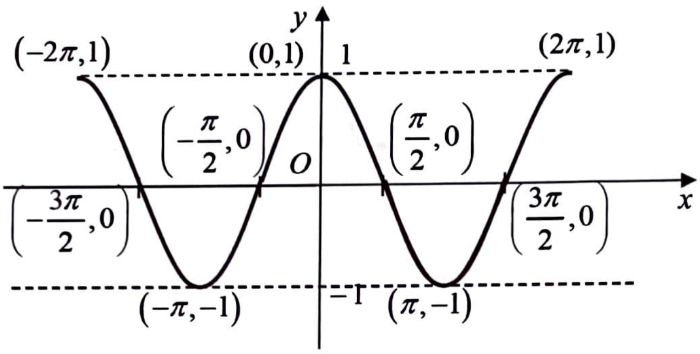
Remember to use radian mode whenever we deal with calculus (differentiation or integration). To change or check the mode in your GC, press [mode] and in the fourth line you can select either RADIAN or DEGREE.

(A) Graph of $y = \sin x$, $-2\pi \leq x \leq 2\pi$.



Important features of $y = \sin x$
- The amplitude is 1.
- The period is 2π.
- Symmetrical about origin.

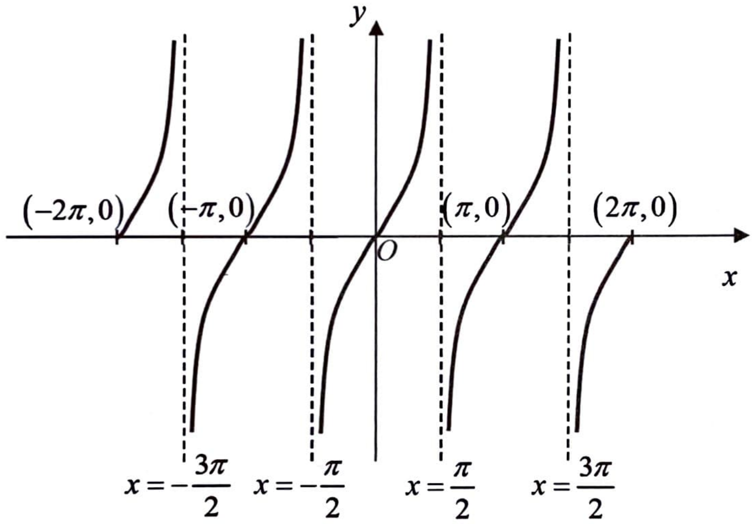
(B) Graph of $y = \cos x$, $-2\pi \le x \le 2\pi$



Important features of $y = \cos x$

- The amplitude is 1.
- The period is 2π.
- Symmetrical about y-axis.

(C) Graph of $y = \tan x$, $-2\pi \le x \le 2\pi$



Important features of $y = \tan x$

- The period is π.
- The asymptotes are $y = \frac{\pi}{2} + n\pi$, where n is an integer.
- Symmetrical about origin

1.9 Periodic Functions

In the previous section, we have discussed about trigonometric functions. These functions are examples of **periodic functions**. Notice that $\sin(x + 2\pi) = \sin(x)$ (period 2π) and $\tan(x + \pi) = \tan(x)$ (period π). In general, a function is periodic if

$$f(x + d) = f(x),$$

where d is the **period** of the function. We will see examples of periodic functions in Example 12.

1.10 Rectangular hyperbola: $y = \frac{ax+b}{cx+d}$, where $a, b, c, d \in \mathbb{R}$, $c \neq 0$, $x \neq -\frac{d}{c}$

The rectangular hyperbola $y = \frac{ax+b}{cx+d}$, where $c \neq 0, x \neq -\frac{d}{c}$ looks like either figure 1(a) or 1(b) below.

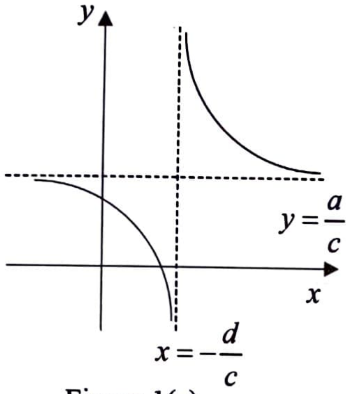


Figure 1(a)

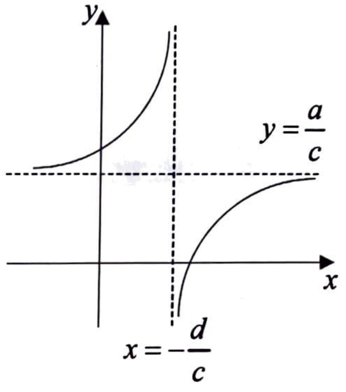


Figure 1(b)

The asymptotes divide the paper into four sectors, and the graph must lie in two opposite sectors. In fact, the graph is symmetrical about the point of intersection of the asymptotes. To draw the graph:

Step 1: Do long division on $y = \frac{ax+b}{cx+d}$ to find the vertical and horizontal asymptotes.

Step 2: Find the axial intercepts.

Once the asymptotes are drawn, we can use the axial intercepts to determine the shape of the graph (i.e. either figure 1(a) or 1(b)).

Example 9: Sketch the graph of $y=\dfrac{2x+1}{x-1}$, $x\neq 1$, showing clearly the coordinates of all axial intercepts and equations of asymptotes.

Solutions

Step 1:

Find the equations of vertical and horizontal asymptotes

Apply long division : $y=\dfrac{2x+1}{x-1}=2+\dfrac{3}{x-1}$

Vertical asymptote (set denominator = 0): $x-1=0 \Rightarrow x=1$

Horizontal asymptote (let $x\to\pm\infty$): As $x\to\pm\infty$, $\dfrac{3}{x-1}\to 0$,

$y\to 2$

In summary,

$y=\dfrac{2x+1}{x-1}=2+\dfrac{3}{x-1}$

Vertical asymptote: $x=1$

Horizontal asymptote: $y=2$

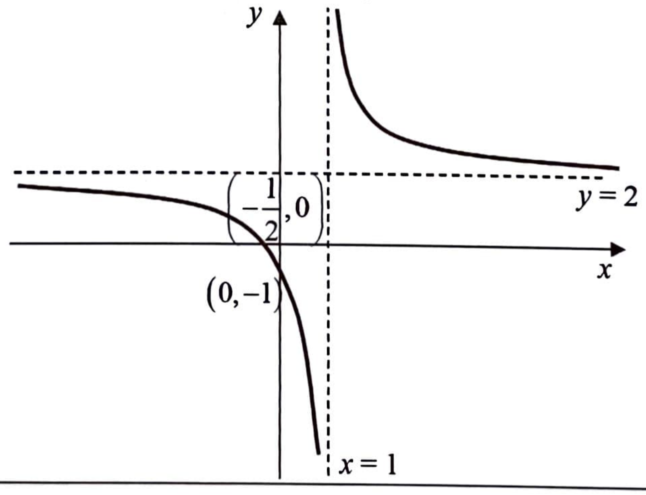
$\therefore$ Asymptotes: $x=1$ & $y=2$

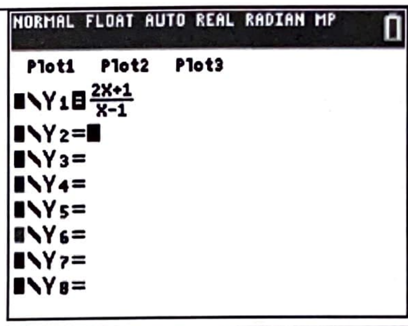
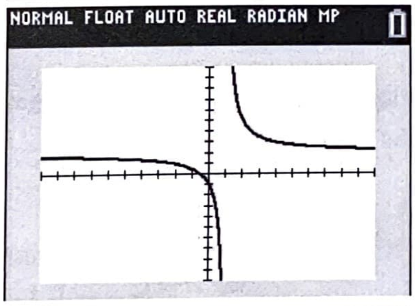
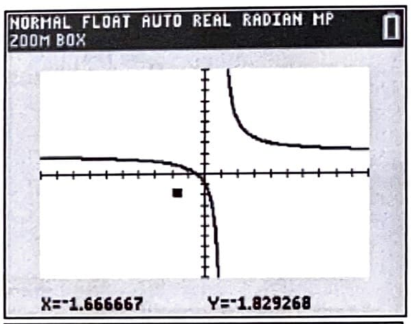
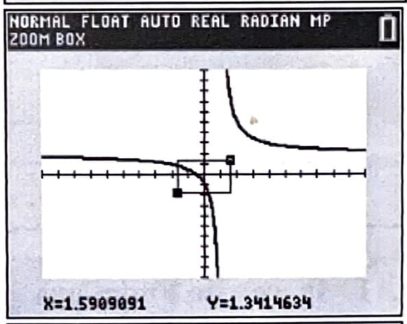
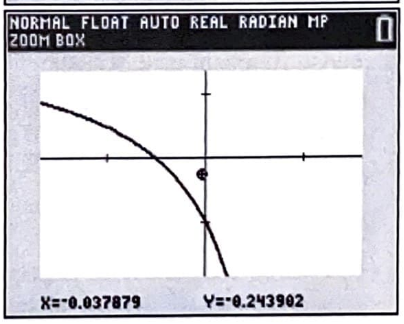
Step 2: Find the coordinates of axial intercepts

When $x=0$, $y=-1$

When $y=0$, $x=-\dfrac{1}{2}$

Step 3: Sketch the graph with clear labelling for the key features



Optional Step: Key in the expression into GC to visualise the shape of the graph. 1. Key y= followed by alpha y= to get the fraction function. Select 1: n/d and key in the equation.	
2. Press graph	
3. To see the location of the axial intercepts clearly, you have to use the zoom feature. You can use zoom 1:ZBox. A cursor will appear and you can define the start point of the box by pressing enter.	
4. Move the cursor to box out the area you want and the screen will show an enlarged part of the graph when you press enter.	 

Can you think of a simpler way to identify the horizontal asymptote for $y=\dfrac{ax+b}{cx+d}$? (Hint: by observation)

Example 10: Sketch the graph of $y = \frac{1}{2-x}$, $x \neq 2$, showing clearly the coordinates of the points of intersection of the graph with the coordinate axes and the equations of the asymptotes.

Solutions

Long division: Since this is a proper fraction, $y = 0 + \frac{1}{2-x}$.

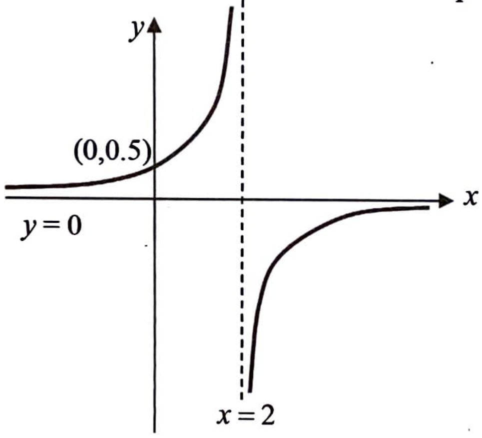
Vertical asymptote (set denominator = 0):
$2 - x = 0 \Rightarrow x = 2$

Horizontal asymptote (let $x \to \pm\infty$):
As $x \to \pm\infty$, $y \to 0$

Asymptotes: $x = 2$ and $y = 0$
(If the asymptotes are the coordinate axes, you do not need to represent them using dotted lines.)

Axial intercepts: $x = 0 \Rightarrow y = \frac{1}{2}$

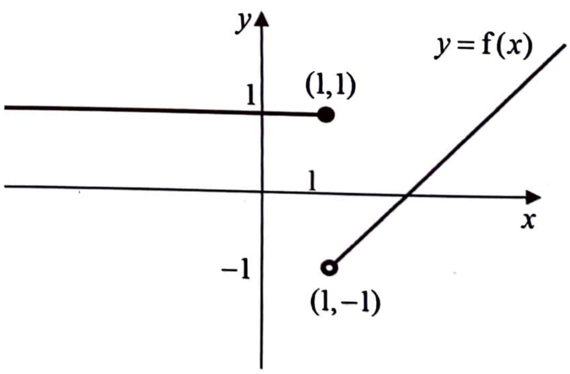
$y = 0 \Rightarrow$ no values of x exist $\therefore$ there is no x-intercept



1.11 Piecewise Functions

A piecewise function is one that has different definitions on different intervals. An example of a piecewise function is

$$f(x) = \begin{cases} 1, & x \leq 1 \\ x-2, & x > 1 \end{cases}$$

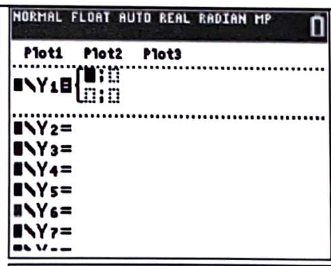


Remark: We use a **solid dot** to indicate that the point is included in the graph of $y = f(x)$; an **empty circle** is used to indicate that the point is not included in the graph of $y = f(x)$. The graph may not be continuous (disjoint).

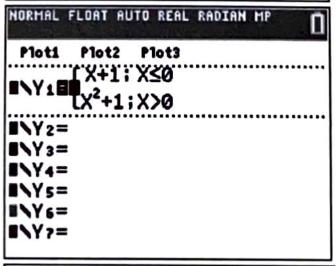
Example 11: Sketch the graph of $f(x)=\begin{cases} x+1, & x \le 0 \\ x^2+1, & x>0 \end{cases}$ for $x \in \mathbb{R}$

Sketch the curve on your GC

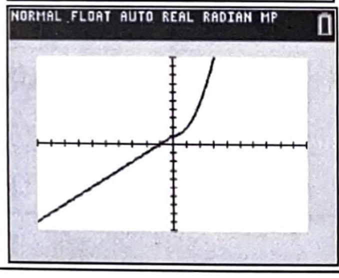
1. Press [y=] [math] and scroll down to select B:Piecewise(
   For Pieces, move the cursor [◀] [▶] to increase or decrease the number of pieces. For this case, choose 2 pieces.



2. Enter the equation. Press [2nd] [math] to select the inequality signs.

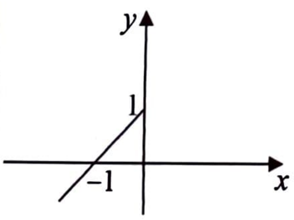


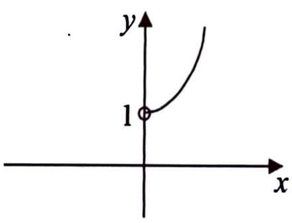
3. Press [graph] to see the piecewise graph.



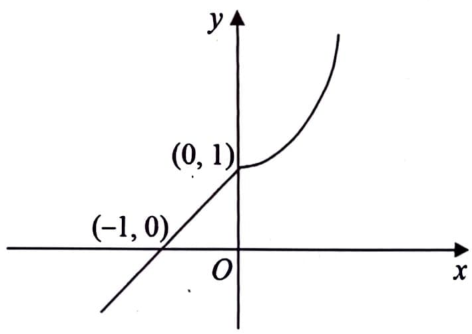
Alternative Solution

Sketch the graphs in their respective intervals:





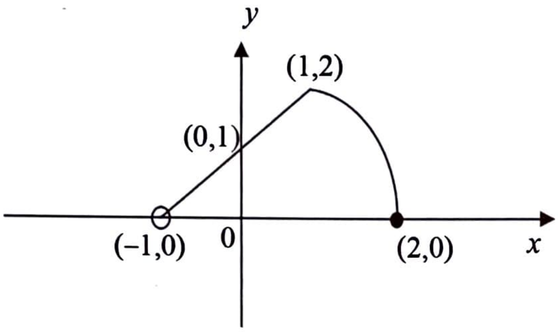
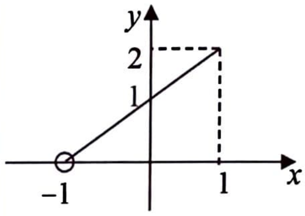
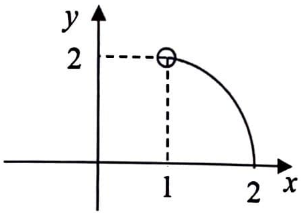
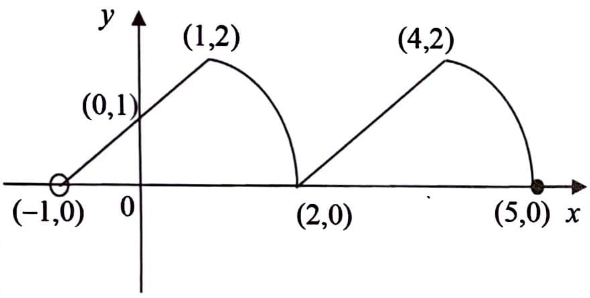
Piece them together to obtain the final graph.



Why is there no empty circle for the point (0, 1) on the final graph?

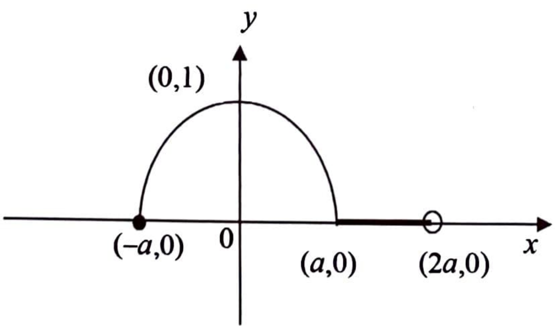
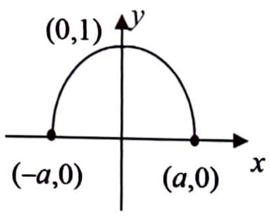
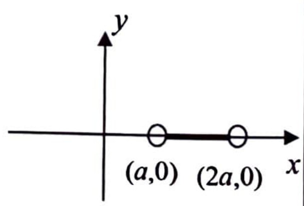
Example 12: Sketch the graph of $f(x) = \begin{cases} x+1, & -1 < x \le 1 \\ 2x(2-x), & 1 < x \le 2 \end{cases}$ for $-1 < x \le 2$.

Given that $f(x+3) = f(x)$ for all real values of x, sketch the graph of $y = f(x)$ for $-1 \le x \le 5$.

Solutions	Think Zone
	Sketch the graphs in their respective intervals:   Piece them together to obtain the final graph.
	$f(x+3) = f(x)$ implies that the period of the function is 3. The graph is repeated after every interval of 3 units. Note: This repeatition makes the graph periodic.

Example 13: [N2013/P1/Q5(i)(modified)]

Sketch the graph of $f(x) = \begin{cases} \sqrt{1-\dfrac{x^2}{a^2}}, & -a \le x \le a \\ 0, & a < x < 2a \end{cases}$ for $-a \le x < 2a$, where a is a real constant.

Solutions	Think Zone
	Sketch the graphs in their respective intervals:   Piece them together to obtain the final graph.