Raffles Institution
H2 Mathematics (9758)
Solution for 2017 A-Level Paper 2

Section A: Pure Mathematics

Question 1

No.	Suggested Solution	Remarks for Student
(i)	Curve given by parametric equations $x=\frac{3}{t}, y=2t$ To find points on the curve that lie on the line $y = 2x$, we have for some t, $y=2x \Rightarrow 2t=2\left(\frac{3}{t}\right)$ $t^2=3 \Leftrightarrow t=\pm\sqrt{3}$ $t=\sqrt{3} \Rightarrow x=\sqrt{3},\ y=2\sqrt{3}$ $t=-\sqrt{3} \Rightarrow x=-\sqrt{3},\ y=-2\sqrt{3}$ $A\left(\sqrt{3},2\sqrt{3}\right)$ and $B\left(-\sqrt{3},-2\sqrt{3}\right)$ Alternatively, express the curve in Cartesian form, $x=\frac{3}{t} \Leftrightarrow t=\frac{3}{x}, y=2t=\frac{6}{x}$ And solve the simultaneous equations $y=\frac{6}{x}$ and $y=2x$. $2x=\frac{6}{x} \Leftrightarrow x=\pm\sqrt{3}$ $A\left(\sqrt{3},2\sqrt{3}\right)$ and $B\left(-\sqrt{3},-2\sqrt{3}\right)$ Length $AB=\sqrt{\left(2\sqrt{3}\right)^2+\left(4\sqrt{3}\right)^2}=\sqrt{60}=2\sqrt{15}$	
(ii)	$x=\frac{3}{t}, y=2t=\frac{6}{x}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{3}{t^2}, \frac{\mathrm{d}y}{\mathrm{d}t}=2 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{3}t^2$ Equation of tangent at point P: $y-2p=-\frac{2}{3}p^2\left(x-\frac{3}{p}\right)$ $y-2p=-\frac{2}{3}p^2x+2p$ $y=-\frac{2}{3}p^2x+4p$	

	When $x=0: y=4p$ When $y=0: \frac{2}{3}p^2x=4p \Rightarrow x=\frac{6}{p}$ Thus, D is $\left(\frac{6}{p},0\right)$ and E is $(0,4p)$ Mid-point F is $\left(\frac{3}{p},2p\right)$ which is the point P. $x=\frac{3}{p}, y=2p=\frac{6}{x}$. That is, $xy = 6$.	

Question 2

No.	Suggested Solution	Remarks for Student
(i)	$u_1=3,\ S_{13}=\frac{13}{2}\left[2(3)+(13-1)d\right]=156$ $\Rightarrow d=\frac{3}{2}$	
(ii)	$u_1=3,\ S_{13}=\frac{3\left(1-r^{13}\right)}{1-r}=156$ $\Rightarrow 3\left(1-r^{13}\right)=156(1-r)$ $\Rightarrow 1-r^{13}=52-52r$ $\therefore r^{13}-52r+51=0 \ldots(1)$ Note that $r = 1$ satisfies (1). But if $r = 1$, then $S_{13} = 3(13) = 39 \neq 156$. So common difference cannot be 1. Using GC, $r = 1.210024 = 1.21$ (3 s.f.) or $-1.451067 = -1.45$ (3 s.f.)	
(iii)	$3(1.2100)^{n-1}>100\left(3+\frac{3}{2}(n-1)\right)$ Use GC to get inequality or table of values, Smallest $n=42$	

Question 3

No.	Suggested Solution	Remarks for Student
(a)(i)	$y=\mathrm{f}(x) \xrightarrow{\text{Replace } x \text{ with } 2x} y=\mathrm{f}(2x)$ Scaling parallel to the x-axis by factor of $\frac{1}{2}$. $\therefore$ Curve $y=\mathrm{f}(2x)$ cuts the axes at $\left(\frac{1}{2}a,0\right),(0,b)$	
(ii)	$y=\mathrm{f}(x) \xrightarrow{\text{Replace } x \text{ with } x - 1} y=\mathrm{f}(x-1)$ Translation of 1 unit in the positive x-direction. $\therefore$ Curve $y=\mathrm{f}(x-1)$ cuts the x-axis at $(a+1,0)$	
(iii)	$y=\mathrm{f}(x) \xrightarrow{\text{Replace } x \text{ with } x - 1} y=\mathrm{f}(x-1) \xrightarrow{\text{Replace } x \text{ with } 2x} y=\mathrm{f}(2x-1)$ Translation of 1 unit in the positive x-direction followed by scaling parallel to the x-axis by factor of $\frac{1}{2}$. $\therefore$ Curve $y=\mathrm{f}(2x-1)$ cuts the x-axis at $\left(\frac{a+1}{2},0\right)$	
(iv)	We reflect the graph of $y=\mathrm{f}(x)$ about the line $y=x$ to obtain the graph of $y=\mathrm{f}^{-1}(x)$. $\therefore$ Curve $y=\mathrm{f}^{-1}(x)$ cuts the axes at $(0,a),(b,0)$	
(b)(i)	$a=1$ g is undefined when $x=1$.	
(ii)	$\begin{aligned} \mathrm{g}^2(x) &= \mathrm{g}(\mathrm{g}(x)) \\ &= \mathrm{g}\left(1-\frac{1}{1-x}\right) \\ &= \mathrm{g}\left(\frac{-x}{1-x}\right) \\ &= 1-\frac{1}{1+\frac{x}{1-x}} \\ &= 1-\frac{1}{\frac{1}{1-x}} \\ &= x \end{aligned}$	Note that if g has an inverse and $\mathrm{h}(\mathrm{g}(x))=x$ then $\mathrm{g}^{-1}=\mathrm{h}$. Here, $\mathrm{g}(\mathrm{g}(x))=x$ So immediately we actually know $\mathrm{g}^{-1}=\mathrm{g}$

	Let $y = 1 - \dfrac{1}{1-x} \Rightarrow 1 - y = \dfrac{1}{1-x}$ $\Rightarrow 1 - x = \dfrac{1}{1-y}$ $\therefore x = 1 - \dfrac{1}{1-y}$ $\mathrm{g}^{-1}(x) = 1 - \dfrac{1}{1-x} = \mathrm{g}(x)$	
(iii)	$\mathrm{g}^2(b) = \mathrm{g}^{-1}(b) \Rightarrow b = 1 - \dfrac{1}{1-b}$ $(b-1)^2 = 1$ $\therefore b = 0$ or 2	

Question 4

No.	Suggested Solution	Remarks for Student
(a)	Area $= \int_1^{5.5} \left[\frac{1}{2}(x-1) - (x^2 - 6x + 5) \right] \mathrm{d}x = 15.1875$	Note that we just use GC as question did not state "exact value" or "without using calculator", etc. The value of 15.1875 is exact.
(b)(i)	Volume	

	$=\pi\int_0^1\left(\dfrac{\sqrt{y}}{a-y^2}\right)^2\ \mathrm{d}y$ $=\pi\int_0^1 y\left(a-y^2\right)^{-2}\mathrm{d}y$ $=\dfrac{\pi}{2}\left[\dfrac{1}{a-y^2}\right]_0^1$ $=\dfrac{\pi}{2}\left[\dfrac{1}{a-1}-\dfrac{1}{a}\right]$ $=\dfrac{\pi}{2a(a-1)}$	
(ii)	$\dfrac{\pi}{2b(b-1)}=4\left(\dfrac{\pi}{2a(a-1)}\right)$ $\dfrac{1}{b(b-1)}=\dfrac{4}{a(a-1)}$ $4b^2-4b=a^2-a$ $4b^2-4b+a-a^2=0$ $b=\dfrac{4\pm\sqrt{16-4(4)\left(a-a^2\right)}}{8}=\dfrac{4\pm4\sqrt{1-\left(a-a^2\right)}}{8}$ Note that $a>1$, thus $a-a^2<0\Rightarrow 1-\left(a-a^2\right)>1$. Thus, $b=\dfrac{1+\sqrt{1-\left(a-a^2\right)}}{2}>1$ or $b=\dfrac{1-\sqrt{1-\left(a-a^2\right)}}{2}<1$ Given that the container is formed the same way, $b>1$, $b=\dfrac{1+\sqrt{1-\left(a-a^2\right)}}{2}$	

Section B: Statistics

Question 5

No.	Suggested Solution	Remarks for Student
(i)	6 Rs, 3 Ys $\mathrm{P}(T=2)=\left(\frac{6}{9}\right)\left(\frac{5}{8}\right)=\frac{5}{12}$ $\mathrm{P}(T=3)=\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)\left(\frac{5}{7}\right)2!=\frac{5}{14}$ $\mathrm{P}(T=4)=\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)\left(\frac{5}{6}\right)\frac{3!}{2!}=\frac{5}{28}$ $\mathrm{P}(T=5)=\left(\frac{6}{9}\right)\left(\frac{3}{8}\right)\left(\frac{2}{7}\right)\left(\frac{1}{6}\right)\frac{4!}{3!}=\frac{1}{21}$	
(ii)	$\mathrm{E}(T)=2\left(\frac{5}{12}\right)+3\left(\frac{5}{14}\right)+4\left(\frac{5}{28}\right)+5\left(\frac{1}{21}\right)=\frac{20}{7}$ $\mathrm{E}(T^2)=2^2\left(\frac{5}{12}\right)+3^2\left(\frac{5}{14}\right)+4^2\left(\frac{5}{28}\right)+5^2\left(\frac{1}{21}\right)=\frac{125}{14}$ $\mathrm{Var}(T)=\mathrm{E}(T^2)-(\mathrm{E}(T))^2=\frac{75}{98}$	
(iii)	Let X denote no. of games, out of 15, where Lee takes at least 4 counters out of the bag. $X\sim \mathrm{B}\left(15, \mathrm{P}(T\geq 4)=\frac{19}{84}\right)$ $\mathrm{P}(X\geq 5)=1-\mathrm{P}(X\leq 4)=0.238$ (3 s.f.)	

Question 6

No.	Suggested Solution	Remarks for Student
(i)	Each of the 5 families forms a unit. These 5 family units can be arranged in $5!$ ways. For a given family, the 4 family members can be arranged among themselves, in $4!$ ways, and as there are 5 families, in $4!\times4!\times4!\times4!\times4!$ ways. No. of required arrangements = $5!(4!)^5=955514880$	

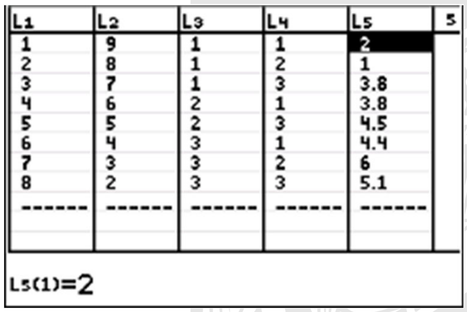
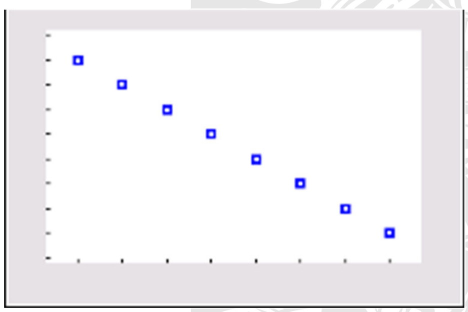
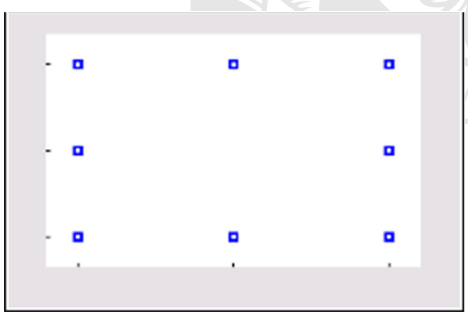
(ii)	Fathers are together with Red father (F_R) and Blue father (F_B) at the ends $F_R F_1 F_2 F_3 F_B$ OR $F_B F_1 F_2 F_3 F_R$ which can be arranged in $2(3!)$ ways. Remaining Red and Blue family (M,D,S) can be arranged in $3!\times 3!$ ways. The fathers, Red and Blue family form a unit. Together with the remaining 9 people, they can be arranged in $10!$ ways. No. of required arrangements = $10!(3!)^3(2)=1567641600$	
(iii)	Excluding the fathers, we have 15 people to arrange in a circle which can be done in $(15-1)!$ ways. There are 15 "slots" to include the fathers, which can be performed in ${}^{15}P_5$ ways. The number of ways to arrange 20 individuals in a circle is $(20-1)!$ Required probability = $\dfrac{(15-1)!\,{}^{15}P_5}{(20-1)!}=\dfrac{1001}{3876}\,(\text{or } 0.258)$	

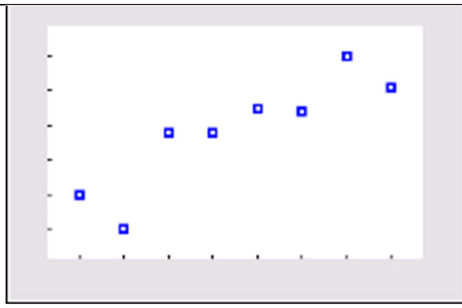
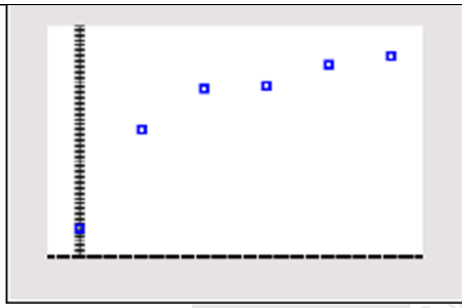
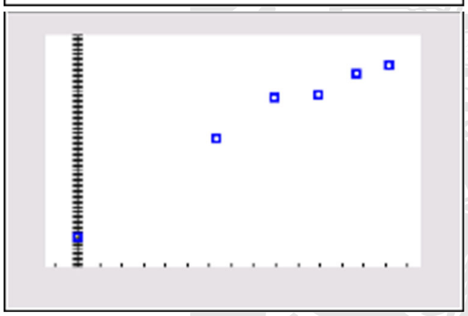
Question 7

No.	Suggested Solution	Remarks for Student
(i)	Every biscuit bar has equal chance of being selected, and chance of selection of one biscuit bar is not affected or influenced by the selection of another biscuit bar.	
(ii)	Unbiased estimate of population mean is $\bar{x}=-\dfrac{7.7}{40}+32=31.8075\approx 31.8$ Unbiased estimate of population variance is $s^2=0.2453269\approx 0.245$	
(iii)	Null hypothesis, $\mathrm{H}_0:\mu=32$ Alternative hypothesis, $\mathrm{H}_1:\mu\neq 32$ where μ is the population mean mass of biscuit bars. Perform a two-tailed test at 1% significance level. Under H_0, $\bar{X}\sim \mathrm{N}\left(32,\dfrac{0.2453269}{40}\right)$ approximately by Centre Limit Theorem since $n=40$ is large $p\text{-value}=2\mathrm{P}\left(\bar{X}<31.8075\right)=0.0139699$	Question is about claim is 32 grams.

	Since p-value $= 0.0139699 > 0.01$, we do not reject $H_0 : \mu = 32$, and conclude that there is no significant evidence at 1% level to claim that the mean mass of biscuit bars is not 32 grams.	
(iv)	Since the sample size is large and the sample is random, Central Limit Theorem can be applied such that the distribution of the **sample mean** is approximately normal.	

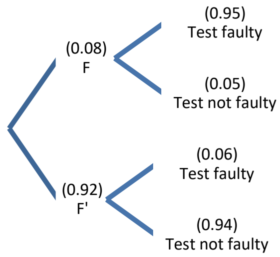
Question 8

<table>
<tr><th>No.</th><th>Suggested Solution</th><th>Remarks for Student</th></tr>
<tr><td>(a)</td><td>

<table>
<tr><th>L1</th><th>L2</th><th>L3</th><th>L4</th><th>L5</th></tr>
<tr><td>1</td><td>9</td><td>1</td><td>1</td><td>2</td></tr>
<tr><td>2</td><td>8</td><td>1</td><td>2</td><td>1</td></tr>
<tr><td>3</td><td>7</td><td>1</td><td>3</td><td>3.8</td></tr>
<tr><td>4</td><td>6</td><td>2</td><td>1</td><td>3.8</td></tr>
<tr><td>5</td><td>5</td><td>2</td><td>3</td><td>4.5</td></tr>
<tr><td>6</td><td>4</td><td>3</td><td>1</td><td>4.4</td></tr>
<tr><td>7</td><td>3</td><td>3</td><td>2</td><td>6</td></tr>
<tr><td>8</td><td>2</td><td>3</td><td>3</td><td>5.1</td></tr>
</table>
L5(1)=2
L1 and L2 for (i), L3 and L4 for (ii), L1 and L5 for (iii)</td><td></td></tr>
<tr><td>(i)</td><td></td><td>Note that perfect fit is required as question wants product moment correlation coefficient to be –1 NOT approximately –1.</td></tr>
<tr><td>(ii)</td><td></td><td>8 points that form shapes like square or circle would suffice.</td></tr>
</table>

<table>
<tr><td>(iii)</td><td></td><td></td></tr>
<tr><td>(b)(i)</td><td>
(D) $y = a\sqrt{x} + b$</td><td>Note that curvature-wise, it seems that log is also suitable. However, we noted $x = 0$ is defined for this scatter plot, so we have to reject (C)</td></tr>
<tr><td>(ii)</td><td>
<table>
<tr><th>L1</th><th>L2</th><th>L3</th><th>L4</th><th>L5</th></tr>
<tr><td>0</td><td>70</td><td>0</td><td>------</td><td>------</td></tr>
<tr><td>40</td><td>104</td><td>6.3246</td><td></td><td></td></tr>
<tr><td>80</td><td>118</td><td>8.9443</td><td></td><td></td></tr>
<tr><td>120</td><td>119</td><td>10.954</td><td></td><td></td></tr>
<tr><td>160</td><td>126</td><td>12.649</td><td></td><td></td></tr>
<tr><td>200</td><td>129</td><td>14.142</td><td></td><td></td></tr>
<tr><td>------</td><td>------</td><td>------</td><td></td><td></td></tr>
</table>
L3(1)=0

LinReg
y=ax+b
a=4.182113873
b=74.04787017
r²=.9620762849
r=.9808548745
$y = 4.18\sqrt{x} + 74.0$
Product moment correlation coefficient = 0.981</td><td></td></tr>
<tr><td>(iii)</td><td>$r = 0.981$ is near 1 which suggest good fit of the model. Besides, the value 189 is within the given range of values of x, so we did not extrapolate the information.</td><td></td></tr>
</table>

Question 9

No.	Suggested Solution	Remarks for Student
(i)	The probability that a kitchen light is faulty is a constant value 0.08. Whether or not a kitchen light is faulty is independent of each other.	Fixed number 12 is not assumption as it is given in the question. Either faulty or not faulty is not a good assumption in this context as there cannot be other possibility.
(ii)	Let X denote no. of lights, out of 12, that is faulty in the box. $X \sim \mathrm{B}(12, 0.08)$ $\mathrm{P}(X \geq 1) = 1 - \mathrm{P}(X = 0) = 0.632334 \approx 0.632$	
(iii)	Let $Y =$ no. of boxes, out of 20, that has at least 1 faulty in the box. $Y \sim \mathrm{B}(20, \mathrm{P}(X \geq 1))$ $\mathrm{P}(Y = 20) = 0.000104454 \approx 0.000104$ Alternatively There are 20 boxes in a carton, and the number faulty in each box is independent of each other. $\mathrm{P}(X_1 \geq 1, X_2 \geq 1, ..., X_{20} \geq 1)$ $= \mathrm{P}(X_1 \geq 1)\mathrm{P}(X_2 \geq 1)\cdots\mathrm{P}(X_{20} \geq 1)$ $= (\mathrm{P}(X \geq 1))^{20}$ $= (0.632334)^{20}$ $= 0.000104454 \approx 0.000104$	
(iv)	Let Y denote no. of lights, out of 240, that is faulty in the carton. $Y \sim \mathrm{B}(240, 0.08)$ $\mathrm{P}(Y \geq 20) = 1 - \mathrm{P}(Y \leq 19) = 0.458334 \approx 0.458$	
(v)	The event described in (iii) is a subset of the event described in (iv). Elaboration as follow. If each box in a carton has at least one faulty light, the carton will definitely has at least 20 faulty lights. However, is there are at least 20 faulty lights in a carton, say 21, one box could have 3 faulty lights and another 18 has one faulty light each. So, there is one box in the carton with no faulty light.	

(vi)	Let F = event that a light is faulty  P(F' | test faulty) $= \dfrac{0.92 \times 0.06}{0.08 \times 0.95 + 0.92 \times 0.06}$ $= \dfrac{69}{164}$ (or 0.421)	
(vii)	P(quick test is reliable) $= 0.08 \times 0.95 + 0.92 \times 0.94$ $= 0.9408$	
(viii)	While (vii) suggests that the quick test is 94% reliable, calculations in (vi) show that of the lights tested as faulty, 42% of them are mistakes. The quick test is not worthwhile.	

Question 10

No.	Suggested Solution	Remarks for Student
(i)	Let X denote mass of a metal sphere in grams. $X \sim \mathrm{N}\left(20, 0.5^2\right)$ $\mathrm{P}\left(X > 20.2\right) = 0.344578 \approx 0.345$	
(ii)	Let Y denote mass of a coated metal sphere in grams. $Y = 1.1X \sim \mathrm{N}\left(22, 0.55^2\right)$ $\mathrm{P}\left(21.5 < Y < 22.45\right)$ $= 0.611722$ ≈ 0.612 OR $\mathrm{P}\left(21.5 < 1.1X < 22.45\right)$ $= \mathrm{P}\left(\frac{21.5}{1.1} < X < \frac{22.45}{1.1}\right)$ $= 0.611722$ ≈ 0.612	The intent of this question seems to suggest we should define a new random variable Y.
(iii)	Let W denote mass of a metal bar in grams. $W \sim \mathrm{N}\left(\mu, \sigma^2\right)$ Given $\mathrm{P}\left(W > 12.2\right) = 0.6$ and $\mathrm{P}\left(W < 12\right) = 0.25$ $\mathrm{P}\left(\frac{W-\mu}{\sigma} > \frac{12.2-\mu}{\sigma}\right) = 0.6$ $\quad$ $\mathrm{P}\left(\frac{W-\mu}{\sigma} < \frac{12-\mu}{\sigma}\right) = 0.25$ $\frac{12.2-\mu}{\sigma} = -0.25335$ and $\frac{12-\mu}{\sigma} = -0.67449$ $\mu - 0.25335\sigma = 12.2$ $\quad$ $\mu - 0.67449\sigma = 12$ Mean, $\mu = 12.320 \approx 12.3$ Standard deviation, $\sigma = 0.47490 \approx 0.475$	
(iv)	T denote total mass of a component, in grams. $T = W + Y_1 + Y_2$ $\mathrm{E}\left(T\right) = \mathrm{E}\left(W\right) + 2\mathrm{E}\left(Y\right) = 12.320 + 44 = 56.320$ $\mathrm{Var}\left(T\right) = \mathrm{Var}\left(W\right) + 2\mathrm{Var}\left(Y\right) = 0.83053$ $T \sim \mathrm{N}(56.320, 0.83053)$ $\mathrm{P}\left(T > k\right) = 0.75$ $\therefore k = 55.7$	