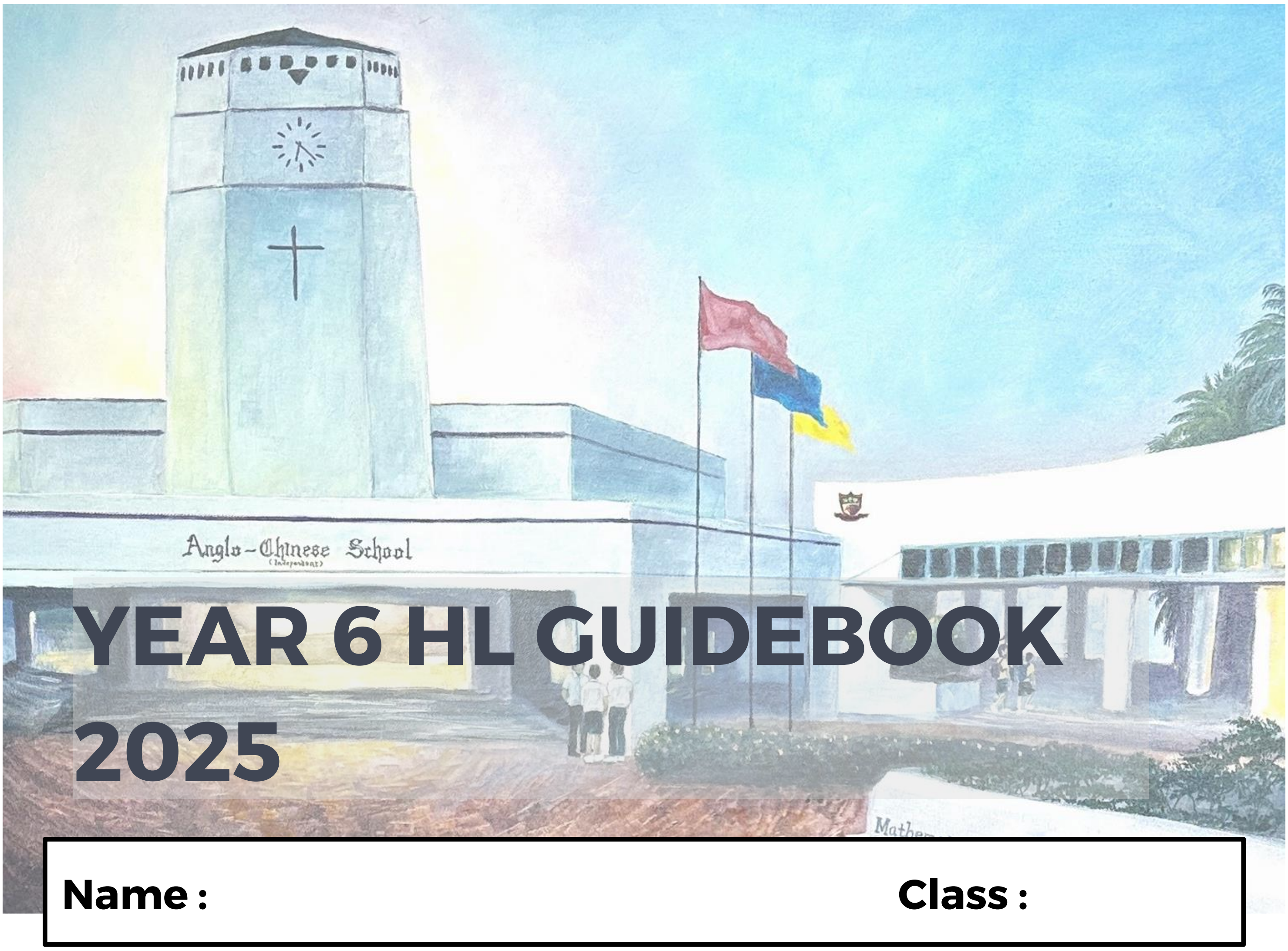
Anglo-Chinese School
(Independent)
A Methodist Institution
(Founded 1886)

MATHEMATICS:
ANALYSIS AND APPROACHES



YEAR 6 HL GUIDEBOOK 2025

Name : **Class :**

"Do nothing from selfish ambition or conceit, but in humility consider others more significant than yourselves" – Philippians 2:3 (ESV)

For Internal Circulation Only

CONTENTS

This guidebook is compiled for use by the students and teachers of ACS (Independent) only. It is meant for internal use only.

IB learner profile

The aim of all IB programmes is to develop internationally minded people who, recognizing their common humanity and shared guardianship of the planet, help to create a better and more peaceful world.

As IB learners we strive to be:

INQUIRERS
We nurture our curiosity, developing skills for inquiry and research. We know how to learn independently and with others. We learn with enthusiasm and sustain our love of learning throughout life.

KNOWLEDGEABLE
We develop and use conceptual understanding, exploring knowledge across a range of disciplines. We engage with issues and ideas that have local and global significance.

THINKERS
We use critical and creative thinking skills to analyse and take responsible action on complex problems. We exercise initiative in making reasoned, ethical decisions.

COMMUNICATORS
We express ourselves confidently and creatively in more than one language and in many ways. We collaborate effectively, listening carefully to the perspectives of other individuals and groups.

PRINCIPLED
We act with integrity and honesty, with a strong sense of fairness and justice, and with respect for the dignity and rights of people everywhere. We take responsibility for our actions and their consequences.

OPEN-MINDED
We critically appreciate our own cultures and personal histories, as well as the values and traditions of others. We seek and evaluate a range of points of view, and we are willing to grow from the experience.

CARING
We show empathy, compassion and respect. We have a commitment to service, and we act to make a positive difference in the lives of others and in the world around us.

RISK-TAKERS
We approach uncertainty with forethought and determination; we work independently and cooperatively to explore new ideas and innovative strategies. We are resourceful and resilient in the face of challenges and change.

BALANCED
We understand the importance of balancing different aspects of our lives—intellectual, physical, and emotional—to achieve well-being for ourselves and others. We recognize our interdependence with other people and with the world in which we live.

REFLECTIVE
We thoughtfully consider the world and our own ideas and experience. We work to understand our strengths and weaknesses in order to support our learning and personal development.

The IB learner profile represents 10 attributes valued by IB World Schools. We believe these attributes, and others like them, can help individuals and groups become responsible members of local, national and global communities.

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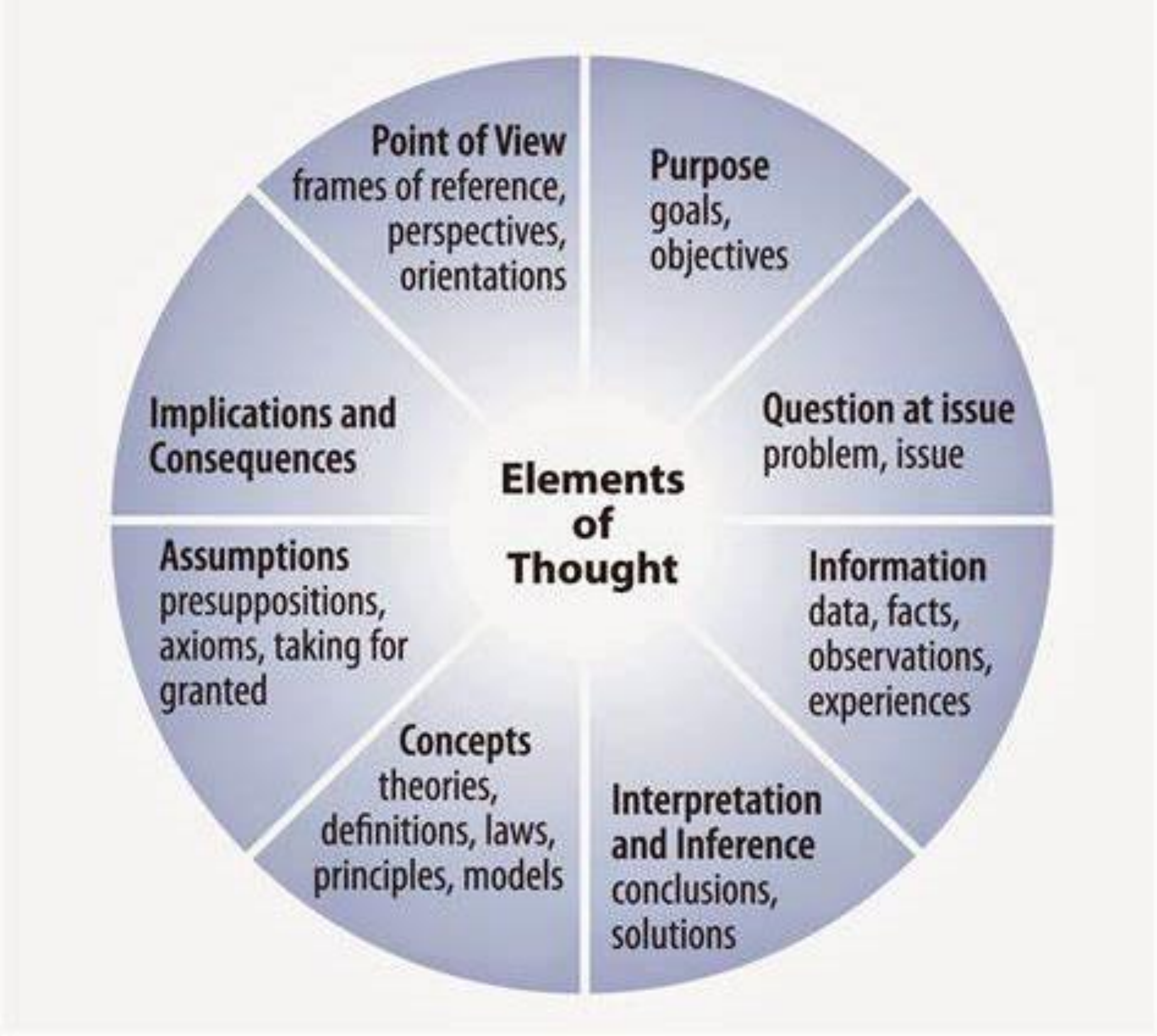
PAUL'S WHEEL OF REASONING



<u>Using Paul's Wheel of Reasoning in Mathematics</u>

- Purpose (What do you think is the purpose of giving you that fact?)
- Question at Issue (What is the question asking for in your own words?)
- Information (Please read and list all the facts and information)
- Assumptions (What do you think is the assumption behind the given information?)
- Concepts (What are the key concepts you need in order to resolve this question?)
- Point of View (Is there any other way of seeing this problem?)
- Interpretation and Inference (Given the question and the information, what do you think are the key moves to solve this problem?)
- Implication and Consequences (Based on your point of view, what would be the steps you take to solve this maths problem?)

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Diploma Programme

Mathematics: analysis and approaches HL formula booklet

For use during the course and in the examinations
First examinations 2021

Version 1.0

HIGHER LEVEL

Anglo-Chinese School (Independent)

Mathematics Department

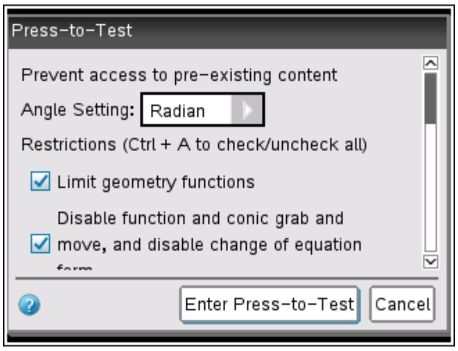
Setting Press-To-Test for TI-Nspire CX II

1) Make sure your GDC is switched **OFF**

2) Press **ESC** esc and **HOME** on buttons simultaneously.

3) A pop up box will appear.

4) Change Angle Setting to **Radians**


Press-to-Test

Prevent access to pre-existing content

Angle Setting: Radian

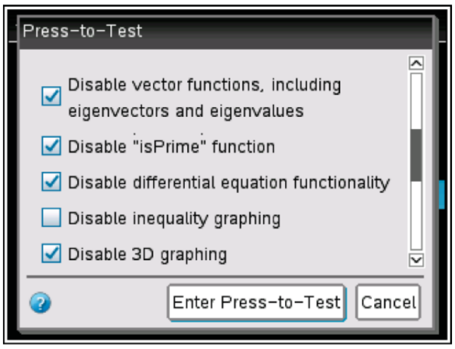
Restrictions (Ctrl + A to check/uncheck all)

☑ Limit geometry functions

☑ Disable function and conic grab and move, and disable change of equation form

5) Unselecting Functions

- **DO NOT** press ENTER enter while selecting the functions.
- Use only the TAB tab button to scroll down the list.
- Uncheck by pressing the centre of the scroll button when the function you want is highlighted.
- Applications that are approved (**MUST NOT BE TICKED**)
    - Disable inequality graphing
    - Limit trigonometric functions
    - Disable $\log_b x$ template and summation functions
    - Disable Polynomial Root Finder and Simultaneous Equation Solver
    - Disable Numerical Solver
    - Disable Sliders
- Applications that are **NOT** approved (**MUST BE TICKED**)
    - Limit geometry functions
    - Disable function and conic grab and move, and disable change of equation form
    - Disable vector functions, including eigenvectors and eigenvalues
    - Disable "isPrime" function
    - Disable differential equation graphing
    - Disable 3D graphing
    - Disable implicit graphing, conic templates, conic analysis and geometric conics.


Press-to-Test

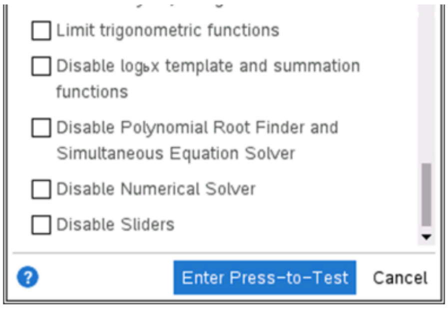
☑ Disable vector functions, including eigenvectors and eigenvalues

☑ Disable "isPrime" function

☑ Disable differential equation functionality

☐ Disable inequality graphing

☑ Disable 3D graphing


☐ Limit trigonometric functions

☐ Disable $\log_b x$ template and summation functions

☐ Disable Polynomial Root Finder and Simultaneous Equation Solver

☐ Disable Numerical Solver

☐ Disable Sliders

6) Once done, press ENTER enter

7) Your GDC will enter press-to-test mode and the LED will blink Orange

Contents

Topic 1: Number and algebra – HL

1.2	The *n*th term of an arithmetic sequence The sum of *n* terms of an arithmetic sequence	$u_n = u_1 + (n-1)d$ $S_n = \frac{n}{2}\left(2u_1 + (n-1)d\right)$; $S_n = \frac{n}{2}(u_1 + u_n)$
1.3	The *n*th term of a geometric sequence The sum of *n* terms of a finite geometric sequence	$u_n = u_1 r^{n-1}$ $S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{u_1(1 - r^n)}{1 - r}$, $r \neq 1$
1.8	The sum of an infinite geometric sequence	$S_\infty = \frac{u_1}{1-r}$, $\lvert r \rvert < 1$
1.4	Compound interest	$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$, where FV is the future value, PV is the present value, n is the number of years, k is the number of compounding periods per year, r% is the nominal annual rate of interest
1.5	Exponents and logarithms	$a^x = b \Leftrightarrow x = \log_a b$, where $a > 0, b > 0, a \neq 1$
1.7	Exponents and logarithms Exponential and logarithmic functions	$\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$ $\log_a x^m = m \log_a x$ $\log_a x = \frac{\log_b x}{\log_b a}$ $a^x = \mathrm{e}^{x \ln a}$; $\log_a a^x = x = a^{\log_a x}$ where $a, x > 0, a \neq 1$
1.9	Binomial theorem $n \in \mathbb{N}$	$(a+b)^n = a^n + {}^n\mathrm{C}_1\, a^{n-1}b + \ldots + {}^n\mathrm{C}_r\, a^{n-r}b^r + \ldots + b^n$ ${}^n\mathrm{C}_r = \frac{n!}{r!(n-r)!}$

1.10	Combinations	${}^{n}\mathrm{C}_{r} = \frac{n!}{r!(n-r)!}$
	Permutations	${}^{n}\mathrm{P}_{r} = \frac{n!}{(n-r)!}$
	Extension of binomial theorem, $n \in \mathbb{Q}$	$(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+...\right)$
1.12	Complex numbers	$z = a + b\mathrm{i}$
1.13	Modulus-argument (polar) and exponential (Euler) form	$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{e}^{\mathrm{i}\theta} = r\,\mathrm{cis}\,\theta$
1.14	De Moivre's theorem	$\left[r(\cos\theta + \mathrm{i}\sin\theta)\right]^n = r^n(\cos n\theta + \mathrm{i}\sin n\theta) = r^n\mathrm{e}^{\mathrm{i}n\theta} = r^n\,\mathrm{cis}\,n\theta$

Topic 2: Functions – HL

2.1	Equations of a straight line	$y = mx + c$; $ax + by + d = 0$; $y - y_1 = m(x - x_1)$
	Gradient formula	$m = \frac{y_2 - y_1}{x_2 - x_1}$
2.6	Axis of symmetry of the graph of a quadratic function	$f(x) = ax^2 + bx + c \Rightarrow$ axis of symmetry is $x = -\frac{b}{2a}$
2.7	Solutions of a quadratic equation	$ax^2 + bx + c = 0 \quad \Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\ a \neq 0$
	Discriminant	$\Delta = b^2 - 4ac$
2.12	Sum and product of the roots of polynomial equations of the form $\sum_{r=0}^{n} a_r x^r = 0$	Sum is $\frac{-a_{n-1}}{a_n}$; product is $\frac{(-1)^n a_0}{a_n}$

Topic 3: Geometry and trigonometry – HL

Prior learning – HL	
Area of a parallelogram	$A = bh$, where b is the base, h is the height
Area of a triangle	$A = \frac{1}{2}(bh)$, where b is the base, h is the height
Area of a trapezoid	$A = \frac{1}{2}(a+b)\,h$, where a and b are the parallel sides, h is the height
Area of a circle	$A = \pi r^2$, where r is the radius
Circumference of a circle	$C = 2\pi r$, where r is the radius
Volume of a cuboid	$V = lwh$, where l is the length, w is the width, h is the height
Volume of a cylinder	$V = \pi r^2 h$, where r is the radius, h is the height
Volume of a prism	$V = Ah$, where A is the area of cross-section, h is the height
Area of the curved surface of a cylinder	$A = 2\pi rh$, where r is the radius, h is the height
Distance between two points (x_1, y_1) and (x_2, y_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
Coordinates of the midpoint of a line segment with endpoints (x_1, y_1) and (x_2, y_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

3.1	Distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2)	$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$
	Coordinates of the midpoint of a line segment with endpoints (x_1, y_1, z_1) and (x_2, y_2, z_2)	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
	Volume of a right-pyramid	$V = \frac{1}{3}Ah$, where A is the area of the base, h is the height

	Volume of a right cone	$V=\frac{1}{3}\pi r^2 h$, where r is the radius, h is the height
	Area of the curved surface of a cone	$A=\pi rl$, where r is the radius, l is the slant height
	Volume of a sphere	$V=\frac{4}{3}\pi r^3$, where r is the radius
	Surface area of a sphere	$A=4\pi r^2$, where r is the radius
3.2	Sine rule	$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$
	Cosine rule	$c^2=a^2+b^2-2ab\cos C$; $\cos C=\frac{a^2+b^2-c^2}{2ab}$
	Area of a triangle	$A=\frac{1}{2}ab\sin C$
3.4	Length of an arc	$l=r\theta$, where r is the radius, θ is the angle measured in radians
	Area of a sector	$A=\frac{1}{2}r^2\theta$, where r is the radius, θ is the angle measured in radians
3.5	Identity for $\tan\theta$	$\tan\theta=\frac{\sin\theta}{\cos\theta}$
3.6	Pythagorean identity	$\cos^2\theta+\sin^2\theta=1$
	Double angle identities	$\sin 2\theta=2\sin\theta\cos\theta$ $\cos 2\theta = \cos^2\theta-\sin^2\theta = 2\cos^2\theta-1 = 1-2\sin^2\theta$
3.9	Reciprocal trigonometric identities	$\sec\theta=\frac{1}{\cos\theta}$ $\mathrm{cosec}\,\theta=\frac{1}{\sin\theta}$
	Pythagorean identities	$1+\tan^2\theta=\sec^2\theta$ $1+\cot^2\theta=\mathrm{cosec}^2\theta$

3.10	Compound angle identities	$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$
	Double angle identity for tan	$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$
3.12	Magnitude of a vector	$\lvert \boldsymbol{v} \rvert = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$
3.13	Scalar product	$\boldsymbol{v} \cdot \boldsymbol{w} = v_1w_1 + v_2w_2 + v_3w_3$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\boldsymbol{v} \cdot \boldsymbol{w} = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \cos\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Angle between two vectors	$\cos\theta = \dfrac{v_1w_1 + v_2w_2 + v_3w_3}{\lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert}$
3.14	Vector equation of a line	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}$
	Parametric form of the equation of a line	$x = x_0 + \lambda l,\ y = y_0 + \lambda m,\ z = z_0 + \lambda n$
	Cartesian equations of a line	$\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$
3.16	Vector product	$\boldsymbol{v} \times \boldsymbol{w} = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$, where $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert = \lvert \boldsymbol{v} \rvert \lvert \boldsymbol{w} \rvert \sin\theta$, where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$
	Area of a parallelogram	$A = \lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$ where $\boldsymbol{v}$ and $\boldsymbol{w}$ form two adjacent sides of a parallelogram
3.17	Vector equation of a plane	$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$
	Equation of a plane (using the normal vector)	$\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$
	Cartesian equation of a plane	$ax + by + cz = d$

Topic 4: Statistics and probability – HL

4.2	Interquartile range	$\text{IQR} = Q_3 - Q_1$
4.3	Mean, $\bar{x}$, of a set of data	$\bar{x} = \dfrac{\sum_{i=1}^{k} f_i x_i}{n}$, where $n = \sum_{i=1}^{k} f_i$
4.5	Probability of an event A Complementary events	$\text{P}(A) = \dfrac{n(A)}{n(U)}$ $\text{P}(A) + \text{P}(A') = 1$
4.6	Combined events Mutually exclusive events Conditional probability Independent events	$\text{P}(A \cup B) = \text{P}(A) + \text{P}(B) - \text{P}(A \cap B)$ $\text{P}(A \cup B) = \text{P}(A) + \text{P}(B)$ $\text{P}(A \mid B) = \dfrac{\text{P}(A \cap B)}{\text{P}(B)}$ $\text{P}(A \cap B) = \text{P}(A)\,\text{P}(B)$
4.7	Expected value of a discrete random variable X	$\text{E}(X) = \sum_{i=1}^{k} x_i\ \text{P}(X = x_i)$
4.8	Binomial distribution $X \sim \text{B}(n, p)$ Mean Variance	 $\text{E}(X) = np$ $\text{Var}(X) = np(1-p)$
4.12	Standardized normal variable	$z = \dfrac{x - \mu}{\sigma}$
4.13	Bayes' theorem	$\text{P}(B \mid A) = \dfrac{\text{P}(B)\ \text{P}(A \mid B)}{\text{P}(B)\ \text{P}(A \mid B) + \text{P}(B')\ \text{P}(A \mid B')}$ $\text{P}(B_i \mid A) = \dfrac{\text{P}(B_i)\text{P}(A \mid B_i)}{\text{P}(B_1)\text{P}(A \mid B_1) + \text{P}(B_2)\text{P}(A \mid B_2) + \text{P}(B_3)\text{P}(A \mid B_3)}$

4.14	Variance σ^2	$\sigma^2 = \frac{\sum_{i=1}^{k} f_i\left(x_i - \mu\right)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2$
	Standard deviation σ	$\sigma = \sqrt{\frac{\sum_{i=1}^{k} f_i\left(x_i - \mu\right)^2}{n}}$
	Linear transformation of a single random variable	$\mathrm{E}\left(aX + b\right) = a\mathrm{E}(X) + b$ $\mathrm{Var}\left(aX + b\right) = a^2\,\mathrm{Var}(X)$
	Expected value of a continuous random variable X	$\mathrm{E}(X) = \mu = \int_{-\infty}^{\infty} x\,f(x)\,\mathrm{d}x$
	Variance	$\mathrm{Var}(X) = \mathrm{E}\left[(X - \mu)^2\right] = \mathrm{E}(X^2) - \left[\mathrm{E}(X)\right]^2$
	Variance of a discrete random variable X	$\mathrm{Var}(X) = \sum (x - \mu)^2\ \mathrm{P}(X = x) = \sum x^2\ \mathrm{P}(X = x) - \mu^2$
	Variance of a continuous random variable X	$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} x^2 f(x)\,\mathrm{d}x - \mu^2$

Topic 5: Calculus – HL

5.12	Derivative of $f(x)$ from first principles	$y=f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=f'(x)=\lim\limits_{h\to 0}\left(\dfrac{f(x+h)-f(x)}{h}\right)$
5.3	Derivative of x^n	$f(x)=x^n \Rightarrow f'(x)=nx^{n-1}$
5.6	Derivative of $\sin x$	$f(x)=\sin x \Rightarrow f'(x)=\cos x$
	Derivative of $\cos x$	$f(x)=\cos x \Rightarrow f'(x)=-\sin x$
	Derivative of e^x	$f(x)=\mathrm{e}^x \Rightarrow f'(x)=\mathrm{e}^x$
	Derivative of $\ln x$	$f(x)=\ln x \Rightarrow f'(x)=\dfrac{1}{x}$
	Chain rule	$y=g(u)$, where $u=f(x) \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Product rule	$y=uv \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=u\dfrac{\mathrm{d}v}{\mathrm{d}x}+v\dfrac{\mathrm{d}u}{\mathrm{d}x}$
	Quotient rule	$y=\dfrac{u}{v} \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{v\dfrac{\mathrm{d}u}{\mathrm{d}x}-u\dfrac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$
5.15	Standard derivatives $\tan x$	$f(x)=\tan x \Rightarrow f'(x)=\sec^2 x$
	$\sec x$	$f(x)=\sec x \Rightarrow f'(x)=\sec x\tan x$
	$\operatorname{cosec} x$	$f(x)=\operatorname{cosec} x \Rightarrow f'(x)=-\operatorname{cosec} x\cot x$
	$\cot x$	$f(x)=\cot x \Rightarrow f'(x)=-\operatorname{cosec}^2 x$
	a^x	$f(x)=a^x \Rightarrow f'(x)=a^x(\ln a)$
	$\log_a x$	$f(x)=\log_a x \Rightarrow f'(x)=\dfrac{1}{x\ln a}$
	$\arcsin x$	$f(x)=\arcsin x \Rightarrow f'(x)=\dfrac{1}{\sqrt{1-x^2}}$
	$\arccos x$	$f(x)=\arccos x \Rightarrow f'(x)=-\dfrac{1}{\sqrt{1-x^2}}$
	$\arctan x$	$f(x)=\arctan x \Rightarrow f'(x)=\dfrac{1}{1+x^2}$

5.9	Acceleration Distance travelled from t_1 to t_2 Displacement from t_1 to t_2	$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$ distance $= \int_{t_1}^{t_2} \lvert v(t) \rvert \mathrm{d}t$ displacement $= \int_{t_1}^{t_2} v(t) \mathrm{d}t$
5.5	Integral of x^n Area between a curve $y = f(x)$ and the x-axis, where $f(x) > 0$	$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + C, \; n \neq -1$ $A = \int_a^b y \, \mathrm{d}x$
5.10	Standard integrals	$\int \frac{1}{x} \, \mathrm{d}x = \ln \lvert x \rvert + C$ $\int \sin x \, \mathrm{d}x = -\cos x + C$ $\int \cos x \, \mathrm{d}x = \sin x + C$ $\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + C$
5.15	Standard integrals	$\int a^x \, \mathrm{d}x = \frac{1}{\ln a} a^x + C$ $\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$ $\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin\left(\frac{x}{a}\right) + C, \quad \lvert x \rvert < a$
5.16	Integration by parts	$\int u \frac{\mathrm{d}v}{\mathrm{d}x} \mathrm{d}x = uv - \int v \frac{\mathrm{d}u}{\mathrm{d}x} \mathrm{d}x$ or $\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$

5.11	Area of region enclosed by a curve and x-axis	$A = \int_a^b \lvert y \rvert \, \mathrm{d}x$
5.17	Area of region enclosed by a curve and y-axis Volume of revolution about the x or y-axes	$A = \int_a^b \lvert x \rvert \, \mathrm{d}y$ $V = \int_a^b \pi y^2 \, \mathrm{d}x$ or $V = \int_a^b \pi x^2 \, \mathrm{d}y$
5.18	Euler's method Integrating factor for $y' + P(x)y = Q(x)$	$y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant (step length) $\mathrm{e}^{\int P(x)\mathrm{d}x}$
5.19	Maclaurin series Maclaurin series for special functions	$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$ $\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \ldots$ $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$ $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$ $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots$ $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots$

Aims

The aims of all DP mathematics courses are to enable students to:

1. develop a curiosity and enjoyment of mathematics, and appreciate its elegance and power
2. develop an understanding of the concepts, principles and nature of mathematics
3. communicate mathematics clearly, concisely and confidently in a variety of contexts
4. develop logical and creative thinking, and patience and persistence in problem solving to instil confidence in using mathematics
5. employ and refine their powers of abstraction and generalization
6. take action to apply and transfer skills to alternative situations, to other areas of knowledge and to future developments in their local and global communities
7. appreciate how developments in technology and mathematics influence each other
8. appreciate the moral, social and ethical questions arising from the work of mathematicians and the applications of mathematics
9. appreciate the universality of mathematics and its multicultural, international and historical perspectives
10. appreciate the contribution of mathematics to other disciplines, and as a particular “area of knowledge” in the TOK course
11. develop the ability to reflect critically upon their own work and the work of others
12. independently and collaboratively extend their understanding of mathematics.

Assessment objectives

Problem solving is central to learning mathematics and involves the acquisition of mathematical skills and concepts in a wide range of situations, including non-routine, open-ended and real-world problems. Having followed a DP mathematics course, students will be expected to demonstrate the following:

1. **Knowledge and understanding**: Recall, select and use their knowledge of mathematical facts, concepts and techniques in a variety of familiar and unfamiliar contexts.
2. **Problem solving:** Recall, select and use their knowledge of mathematical skills, results and models in both abstract and real-world contexts to solve problems.
3. **Communication and interpretation**: Transform common realistic contexts into mathematics; comment on the context; sketch or draw mathematical diagrams, graphs or constructions both on paper and using technology; record methods, solutions and conclusions using standardized notation; use appropriate notation and terminology.
4. **Technology**: Use technology accurately, appropriately and efficiently both to explore new ideas and to solve problems.
5. **Reasoning:** Construct mathematical arguments through use of precise statements, logical deduction and inference and by the manipulation of mathematical expressions.
6. **Inquiry approaches:** Investigate unfamiliar situations, both abstract and from the real world, involving organizing and analyzing information, making conjectures, drawing conclusions, and testing their validity.

Syllabus outline

Syllabus component	Suggested teaching hours	
	SL	HL
Topic 1—Number and algebra	19	39
Topic 2—Functions	21	32
Topic 3— Geometry and trigonometry	25	51
Topic 4—Statistics and probability	27	33
Topic 5 —Calculus	28	55
The toolkit and the mathematical exploration Investigative, problem-solving and modelling skills development leading to an individual exploration. The exploration is a piece of written work that involves investigating an area of mathematics.	30	30
Total teaching hours	**150**	**240**

All topics are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as prior learning.

Assessment outline—HL

First assessment 2021

Assessment component	Weighting
External assessment (5 hours)	**80%**
Paper 1 (120 minutes) No technology allowed. (110 marks) *Section A* Compulsory short-response questions based on the syllabus. *Section B* Compulsory extended-response questions based on the syllabus.	**30%**
Paper 2 (120 minutes) Technology required. (110 marks) *Section A* Compulsory short-response questions based on the syllabus. *Section B* Compulsory extended-response questions based on the syllabus.	**30%**
Paper 3 (60 minutes) Technology required. (55 marks) Two compulsory extended response problem-solving questions.	**20%**
Internal assessment This component is internally assessed by the teacher and externally moderated by the IB at the end of the course. **Mathematical exploration** Internal assessment in mathematics is an individual exploration. This is a piece of written work that involves investigating an area of mathematics. (20 marks)	**20%**

Syllabus content

Topic 1: Number and algebra

Concepts

Essential understandings:

Number and algebra allow us to represent patterns, show equivalencies and make generalizations which enable us to model real-world situations. Algebra is an abstraction of numerical concepts and employs variables which allow us to solve mathematical problems.

Suggested concepts embedded in this topic:

Generalization, representation, modelling, equivalence, patterns, quantity

AHL: Validity, systems.

Content-specific conceptual understandings:

- Modelling real-life situations with the structure of arithmetic and geometric sequences and series allows for prediction, analysis and interpretation.
- Different representations of numbers enable equivalent quantities to be compared and used in calculations with ease to an appropriate degree of accuracy.
- Numbers and formulae can appear in different, but equivalent, forms, or representations, which can help us to establish identities.
- Formulae are a generalization made on the basis of specific examples, which can then be extended to new examples.
- Logarithm laws provide the means to find inverses of exponential functions which model real-life situations.
- Patterns in numbers inform the development of algebraic tools that can be applied to find unknowns.
- The binomial theorem is a generalization which provides an efficient method for expanding binomial expressions.

AHL

- Proof serves to validate mathematical formulae and the equivalence of identities.
- Representing partial fractions and complex numbers in different forms allows us to easily carry out seemingly difficult calculations.
- The solution for systems of equations can be carried out by a variety of equivalent algebraic and graphical methods.

SL content

Recommended teaching hours: 19

The aim of the SL content of the number and algebra topic is to introduce students to numerical concepts and techniques which, combined with an introduction to arithmetic and geometric sequences and series, can be used for financial and other applications. Students will also be introduced to the formal concept of proof.

Sections SL1.1 to SL1.5 are content common to Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 1.1

Content	Guidance, clarification and syllabus links
Operations with numbers in the form $a \times 10^k$ where $1 \leq a < 10$ and k is an integer.	Calculator or computer notation is **not** acceptable. For example, 5.2E30 is **not** acceptable and should be written as 5.2×10^{30}.

Connections

Other contexts: Very large and very small numbers, for example astronomical distances, sub-atomic particles in physics, global financial figures

Links to other subjects: Chemistry (Avogadro's number); physics (order of magnitude); biology (microscopic measurements); sciences group subjects (uncertainty and precision of measurement)

International-mindedness: The history of number from Sumerians and its development to the present Arabic system

TOK: Do the names that we give things impact how we understand them? For instance, what is the impact of the fact that some large numbers are named, such as the googol and the googolplex, while others are represented in this form?

Download connections template

SL 1.2

Content	Guidance, clarification and syllabus links
Arithmetic sequences and series. Use of the formulae for the n^{th} term and the sum of the first n terms of the sequence. Use of sigma notation for sums of arithmetic sequences.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways. If technology is used in examinations, students will be expected to identify the first term and the common difference.
Applications.	Examples include simple interest over a number of years.
Analysis, interpretation and prediction where a model is not perfectly arithmetic in real life.	Students will need to approximate common differences.

Connections

International-mindedness: The chess legend (Sissa ibn Dahir); Aryabhatta is sometimes considered the "father of algebra"–compare with alKhawarizmi; the use of several alphabets in mathematical notation (for example the use of capital sigma for the sum).

TOK: Is all knowledge concerned with identification and use of patterns? Consider Fibonacci numbers and connections with the golden ratio.

Download connections template

SL 1.3

Content	Guidance, clarification and syllabus links
Geometric sequences and series. Use of the formulae for the n th term and the sum of the first n terms of the sequence.	Spreadsheets, GDCs and graphing software may be used to generate and display sequences in several ways.

Content	Guidance, clarification and syllabus links
Use of sigma notation for the sums of geometric sequences.	If technology is used in examinations, students will be expected to identify the first term and the ratio. **Link to:** models/functions in topic 2 and regression in topic 4.
Applications.	Examples include the spread of disease, salary increase and decrease and population growth.

Connections
Links to other subjects: Radioactive decay, nuclear physics, charging and discharging capacitors (physics).

TOK: How do mathematicians reconcile the fact that some conclusions seem to conflict with our intuitions? Consider for instance that a finite area can be bounded by an infinite perimeter.

Download connections template

SL 1.4

Content	Guidance, clarification and syllabus links
Financial applications of geometric sequences and series: • compound interest • annual depreciation.	Examination questions may require the use of technology, including built-in financial packages. The concept of simple interest may be used as an introduction to compound interest. Calculate the real value of an investment with an interest rate and an inflation rate. In examinations, questions that ask students to derive the formula will not be set. Compound interest can be calculated yearly, half-yearly, quarterly or monthly. **Link to**: exponential models/functions in topic 2.

Connections
Other contexts: Loans.

Links to other subjects: Loans and repayments (economics and business management).

Aim 8: Ethical perceptions of borrowing and lending money.

International-mindedness: Do all societies view investment and interest in the same way?

TOK: How have technological advances affected the nature and practice of mathematics? Consider the use of financial packages for instance.

Enrichment: The concept of e can be introduced through continuous compounding, $(1+\frac{1}{n})^n \rightarrow \mathrm{e}$, as $n \rightarrow \infty$, however this will not be examined.

Download connections template

SL 1.5

Content	Guidance, clarification and syllabus links
Laws of exponents with integer exponents.	**Examples:** $5^3 \times 5^{-6} = 5^{-3}, 6^4 \div 6^3 = 6, (2^3)^{-4} = 2^{-12},$ $(2x)^4 = 16x^4, 2x^{-3} = \frac{2}{x^3}.$
Introduction to logarithms with base 10 and e. Numerical evaluation of logarithms using technology.	Awareness that $a^x = b$ is equivalent to $\log_a b = x$, that $b > 0$, and $\log_e x = \ln x$.

Connections
Other contexts: Richter scale and decibel scale.

Links to other subjects: Calculation of pH and buffer solutions (chemistry)

TOK: Is mathematics invented or discovered? For instance, consider the number e or logarithms–did they already exist before man defined them? (This topic is an opportunity for teachers to generate reflection on “the nature of mathematics”).

Download connections template

SL 1.6

Content	Guidance, clarification and syllabus links
Simple deductive proof, numerical and algebraic; how to lay out a left-hand side to right-hand side (LHS to RHS) proof. The symbols and notation for equality and identity.	**Example:** Show that $\frac{1}{4}+\frac{1}{12}=\frac{1}{3}$. Show that the algebraic generalisation of this is $\frac{1}{m+1}+\frac{1}{m^2+m}\equiv\frac{1}{m}$ LHS to RHS proofs require students to begin with the left-hand side expression and transform this using known algebraic steps into the expression on the right-hand side (or vice versa). **Example:** Show that $(x-3)^2+5 \equiv x^2-6x+14$. Students will be expected to show how they can check a result including a check of their own results.

Connections
TOK: Is mathematical reasoning different from scientific reasoning, or reasoning in other Areas of Knowledge?

Download connections template

SL 1.7

Content	Guidance, clarification and syllabus links
Laws of exponents with rational exponents.	$a^{\frac{1}{m}} = \sqrt[m]{a}$, if m is even this refers to the positive root. For example: $16^{\frac{3}{4}} = 8$.
Laws of logarithms. $\log_a xy = \log_a x + \log_a y$ $\log_a \frac{x}{y} = \log_a x - \log_a y$	$y = a^x \Leftrightarrow x = \log_a y; \log_a a = 1, \log_a 1 = 0,$ $a, \ y \in \mathbb{N}, \ x \in \mathbb{Z}$ **Link to:** introduction to logarithms (SL1.5)

Content	Guidance, clarification and syllabus links
$\log_a x^m = m\log_a x$ for $a, \quad x, \quad y > 0$	**Examples:** $\frac{3}{4} = \log_{16}8, \log32 = 5\log2$ $\log24 = \log8 + \log3$ $\log_3\frac{10}{4} = \log_3 10 - \log_3 4$ $\log_4 3^5 = 5\log_4 3$ **Link to:** logarithmic and exponential graphs (SL2.9)
Change of base of a logarithm. $\log_a x = \frac{\log_b x}{\log_b a}$, for $a, \quad b, \quad x > 0$	**Examples:** $\log_4 7 = \frac{\ln7}{\ln4}$ $\log_{25}125 = \frac{\log_5 125}{\log_5 25} \quad (= \frac{3}{2})$
Solving exponential equations, including using logarithms.	**Examples:** $(\frac{1}{3})^x = 9^{x+1}, 2^{x-1} = 10.$ **Link to:** using logarithmic and exponential graphs (SL2.9).

Connections

Links to other subjects: pH, buffer calculations and finding activation energy from experimental data (chemistry).

TOK: How have seminal advances, such as the development of logarithms, changed the way in which mathematicians understand the world and the nature of mathematics?

Download connections template

SL 1.8

Content	Guidance, clarification and syllabus links
Sum of infinite convergent geometric sequences.	Use of $\lvert r \rvert < 1$ and modulus notation. **Link to**: geometric sequences and series (SL1.3).

Connections

TOK: Is it possible to know about things of which we can have no experience, such as infinity?

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SL 1.9

Content	Guidance, clarification and syllabus links
The binomial theorem: expansion of $(a+b)^n, n \in \mathbb{N}$.	Counting principles may be used in the development of the theorem.
Use of Pascal's triangle and ${}^n\mathrm{C}_r$.	${}^n\mathrm{C}_r$ should be found using **both** the formula and technology. **Example:** Find r when ${}^6\mathrm{C}_r = 20$, using a table of values generated with technology.

Connections

Aim 8: Ethics in mathematics–Pascal's triangle. Attributing the origin of a mathematical discovery to the wrong mathematician.

International-mindedness: The properties of "Pascal's triangle" have been known in a number of different cultures long before Pascal. (for example the Chinese mathematician Yang Hui).

TOK: How have notable individuals shaped the development of mathematics as an area of knowledge? Consider Pascal and "his" triangle.

Download connections template

AHL content

Recommended teaching hours: 20

The aim of the AHL content in the number and algebra topic is to extend and build upon the aims, concepts and skills from the SL content. It introduces students to some important techniques for expansion, simplification and solution of equations. Complex numbers are introduced and students will extend their knowledge of formal proof to proof by mathematical induction, proof by contradiction and proof by counterexample.

AHL 1.10

Content	Guidance, clarification and syllabus links
Counting principles, including permutations and combinations.	**Not required:** Permutations where some objects are identical. Circular arrangements.
Extension of the binomial theorem to fractional and negative indices, ie $(a+b)^n, \quad n \in \mathbb{Q}$.	$(a+b)^n = \quad (a(1+\frac{b}{a}))^n = \quad a^n(1+\frac{b}{a})^n, n \in \mathbb{Q}$ **Link to**: power series expansions (AHL5.19) **Not required:** Proof of binomial theorem.

Connections

Other contexts: Finding approximations to $\sqrt{2}$

Aim 8: How many different tickets are possible in a lottery? What does this tell us about the ethics of selling lottery tickets to those who do not understand the implications of these large numbers?

International-mindedness: The properties of "Pascal's triangle" have been known in a number of different cultures long before Pascal (for example the Chinese mathematician Yang Hui).

TOK: What counts as understanding in mathematics? Is it more than just getting the right answer?

Download connections template

AHL 1.11

Content	Guidance, clarification and syllabus links
Partial fractions.	Maximum of two distinct linear terms in the denominator, with degree of numerator less than the degree of the denominator. **Example:** $\frac{2x+1}{x^2+x-2} \equiv \frac{1}{(x-1)} + \frac{1}{(x+2)}$. **Link to:** use of partial fractions to rearrange the integrand (AHL5.15).

Connections

Download connections template

AHL 1.12

Content	Guidance, clarification and syllabus links
Complex numbers: the number i, where $\mathrm{i}^2 = -1$. Cartesian form $z = a + b\mathrm{i}$; the terms real part, imaginary part, conjugate, modulus and argument.	
The complex plane.	The complex plane is also known as the Argand diagram. **Link to:** vectors (AHL3.12).

Connections

Other contexts: Concepts in electrical engineering—impedance as a combination of resistance and reactance, also apparent power as a combination of real and reactive powers. These combinations take the form $a + b\mathrm{i}$.

TOK: How does language shape knowledge? For example, do the words "imaginary" and "complex" make the concepts more difficult than if they had different names?

Download connections template

AHL 1.13

Content	Guidance, clarification and syllabus links
Modulus–argument (polar) form: $z = r(\cos\theta + \mathrm{i}\sin\theta) = r\mathrm{cis}\theta$ Euler form: $z = re^{\mathrm{i}\theta}$ Sums, products and quotients in Cartesian, polar or Euler forms and their geometric interpretation.	The ability to convert between Cartesian, modulus-argument (polar) and Euler form is expected.

Connections

Other contexts: Concepts in electrical engineering–phase angle/shift, power factor and apparent power as a complex quantity in polar form.

TOK: Why might it be said that $\mathrm{e}^{\mathrm{i}\pi} + 1 = 0$ is beautiful? What is the place of beauty and elegance in mathematics? What about the place of creativity?

Download connections template

AHL 1.14

Content	Guidance, clarification and syllabus links
Complex conjugate roots of quadratic and polynomial equations with real coefficients.	Complex roots occur in conjugate pairs.
De Moivre's theorem and its extension to rational exponents.	Includes proof by induction for the case where $n \in \mathbb{Z}^+$; awareness that it is true for $n \in \mathbb{R}$.
Powers and roots of complex numbers.	**Link to:** sum and product of roots of polynomial equations (AHL 2.12), compound angle identities (AHL 3.10).

Connections

TOK: Could we ever reach a point where everything important in a mathematical sense is known? Reflect on the creation of complex numbers before their applications were known.

Enrichment: Can De Moivre's theorem be extended to all n?

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AHL 1.15

Content	Guidance, clarification and syllabus links
Proof by mathematical induction.	Proof should be incorporated throughout the course where appropriate. Mathematical induction links specifically to a wide variety of topics, for example complex numbers, differentiation, sums of sequences and divisibility.
Proof by contradiction.	**Examples:** Irrationality of $\sqrt{3}$; irrationality of the cube root of 5; Euclid's proof of an infinite number of prime numbers; if a is a rational number and b is an irrational number, then $a + b$ is an irrational number.
Use of a counterexample to show that a statement is not always true.	**Example:** Consider the set P of numbers of the form $n^2 + 41n + 41,\ \ n \in \mathbb{N}$, show that not all elements of P are prime. **Example:** Show that the following statement is not always true: there are no positive integer solutions to the equation $x^2 + y^2 = 10$. It is not sufficient to state the counterexample alone. Students must explain why their example is a counterexample.

Connections

Other contexts: The Four-colour theorem

International-mindedness: How did the Pythagoreans find out that $\sqrt{2}$ is irrational?

TOK: What is the role of the mathematical community in determining the validity of a mathematical proof? Do proofs provide us with completely certain knowledge? What is the difference between the inductive method in science and proof by induction in mathematics?

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AHL 1.16

Content	Guidance, clarification and syllabus links
Solutions of systems of linear equations (a maximum of three equations in three unknowns),	These systems should be solved using both algebraic and technological methods, for example row reduction or matrices.

Content	Guidance, clarification and syllabus links
including cases where there is a unique solution, an infinite number of solutions or no solution.	Systems which have no solution(s) are inconsistent. Finding a general solution for a system with an infinite number of solutions. **Link to:** intersection of lines and planes (AHL 3.18).

Connections
TOK: Mathematics, Sense, Perception and Reason: If we can find solutions in higher dimensions can we reason that these spaces exist beyond our sense perception?

Download connections template

Topic 2: Functions

Concepts

Essential understandings

Models are depictions of real-life events using expressions, equations or graphs while a function is defined as a relation or expression involving one or more variables. Creating different representations of functions to model the relationships between variables, visually and symbolically as graphs, equations and tables represents different ways to communicate mathematical ideas.

Suggested concepts embedded in this topic:

Representation, relationships, space, quantity, equivalence.

AHL: Systems, patterns.

Content-specific conceptual understandings:

- Different representations of functions, symbolically and visually as graphs, equations and tables provide different ways to communicate mathematical relationships.
- The parameters in a function or equation correspond to geometrical features of a graph and can represent physical quantities in spatial dimensions.
- Moving between different forms to represent functions allows for deeper understanding and provides different approaches to problem solving.
- Our spatial frame of reference affects the visible part of a function and by changing this “window” can show more or less of the function to best suit our needs.
- Equivalent representations of quadratic functions can reveal different characteristics of the same relationship.
- Functions represent mappings that assign to each value of the independent variable (input) one and only one dependent variable (output).

AHL

- Extending results from a specific case to a general form can allow us to apply them to a larger system.
- Patterns can be identified in behaviours which can give us insight into appropriate strategies to model or solve them.
- The intersection of a system of equations may be represented graphically and algebraically and represents the solution that satisfies the equations.

SL content

Recommended teaching hours: 21

The aim of the SL content in the functions topic is to introduce students to the important unifying theme of a function in mathematics and to apply functional methods to a variety of mathematical situations.

Throughout this topic students should be given the opportunity to use technology, such as graphing packages and graphing calculators to develop and apply their knowledge of functions, rather than using elaborate analytic techniques.

On examination papers:

- questions may be set requiring the graphing of functions that do not explicitly appear on the syllabus
- the domain will be the largest possible domain for which a function is defined unless otherwise stated; this will usually be the real numbers

Sections SL2.1 to SL2.4 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 2.1

Content	Guidance, clarification and syllabus links
Different forms of the equation of a straight line. Gradient; intercepts. Lines with gradients m_1 and m_2 Parallel lines $m_1 = m_2$. Perpendicular lines $m_1 \times m_2 = -1$.	$y = mx + c$ (gradient-intercept form). $ax + by + d = 0$ (general form). $y - y_1 = m(x - x_1)$ (point-gradient form). Calculate gradients of inclines such as mountain roads, bridges, etc.

Connections
Other contexts: Gradients of mountain roads, gradients of access ramps.

Links to other subjects: Exchange rates and price and income elasticity, demand and supply curves (economics); graphical analysis in experimental work (sciences group subjects).

TOK: Descartes showed that geometric problems could be solved algebraically and vice versa. What does this tell us about mathematical representation and mathematical knowledge?

Download connections template

SL 2.2

Content	Guidance, clarification and syllabus links
Concept of a function, domain, range and graph. Function notation, for example $f(x), \quad v(t), \quad C(n)$. The concept of a function as a mathematical model.	**Example:** $f(x) = \sqrt{2 - x}$, the domain is $x \leq 2$, range is $f(x) \geq 0$. A graph is helpful in visualizing the range.
Informal concept that an inverse function reverses or undoes the effect of a function. Inverse function as a reflection in the line $y = x$, and the notation $f^{-1}(x)$.	**Example:** Solving $f(x) = 10$ is equivalent to finding $f^{-1}(10)$. Students should be aware that inverse functions exist for one to one functions; the domain of $f^{-1}(x)$is equal to the range of $f(x)$.

Connections
Other contexts: Temperature and currency conversions**.**

Links to other subjects: Currency conversions and cost functions (economics and business management); projectile motion (physics).

Aim 8: What is the relationship between real-world problems and mathematical models?

International-mindedness: The development of functions by Rene Descartes (France), Gottfried Wilhelm Leibnitz (Germany) and Leonhard Euler (Switzerland); the notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries–how did the notation we use today become internationally accepted?

TOK: Do you think mathematics or logic should be classified as a language?

Download connections template

SL 2.3

Content	Guidance, clarification and syllabus links
The graph of a function; its equation $y = f(x)$.	Students should be aware of the difference between the command terms "draw" and "sketch".
Creating a sketch from information given or a context, including transferring a graph from screen to paper. Using technology to graph functions including their sums and differences.	All axes and key features should be labelled. This may include functions not specifically mentioned in topic 2.

Connections

Links to other subjects: Sketching and interpreting graphs (sciences group subjects, geography, economics).

TOK: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

Download connections template

SL 2.4

Content	Guidance, clarification and syllabus links
Determine key features of graphs.	Maximum and minimum values; intercepts; symmetry; vertex; zeros of functions or roots of equations; vertical and horizontal asymptotes using graphing technology.
Finding the point of intersection of two curves or lines using technology.	

Connections

Links to other subjects: Identification and interpretation of key features of graphs (sciences group subjects, geography, economics); production possibilities curve model, market equilibrium (economics).

International-mindedness: Bourbaki group analytical approach versus the Mandlebrot visual approach.

Use of technology: Graphing technology with sliders to determine the effects of altering parameters and variables.

Download connections template

SL 2.5

Content	Guidance, clarification and syllabus links
Composite functions.	$(f \circ g)(x) = f(g(x))$
Identity function. Finding the inverse function $f^{-1}(x)$.	$(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ The existence of an inverse for one-to-one functions. **Link to**: concept of inverse function as a reflection in the line $y = x$ (SL 2.2).

Connections

TOK: Do you think mathematics or logic should be classified as a language?

Download connections template

SL 2.6

Content	Guidance, clarification and syllabus links
The quadratic function $f(x) = ax^2 + bx + c$: its graph, y -intercept $(0, c)$. Axis of symmetry. The form $f(x) = a(x - p)(x - q)$, x-intercepts $(p, 0)$ and $(q, 0)$. The form $f(x) = a\,(x - h)^2 + k$, vertex (h, k).	A quadratic graph is also called a parabola. **Link to**: transformations (SL 2.11). Candidates are expected to be able to change from one form to another.

Connections

Links to other subjects: Kinematics, projectile motion and simple harmonic motion (physics).

TOK: Are there fundamental differences between mathematics and other areas of knowledge? If so, are these differences more than just methodological differences?

Download connections template

SL 2.7

Content	Guidance, clarification and syllabus links
Solution of quadratic equations and inequalities. The quadratic formula.	Using factorization, completing the square (vertex form), and the quadratic formula. Solutions may be referred to as roots or zeros.
The discriminant $\Delta = b^2 - 4ac$ and the nature of the roots, that is, two distinct real roots, two equal real roots, no real roots.	**Example**: For the equation $3kx^2 + 2x + k = 0$, find the possible values of k, which will give two distinct real roots, two equal real roots or no real roots.

Connections

Links to other subjects: Projectile motion and energy changes in simple harmonic motion (physics); equilibrium equations (chemistry).

International-mindedness: The Babylonian method of multiplication: $ab = \frac{(a+b)^2 - a^2 - b^2}{2}$. Sulba Sutras in ancient India and the Bakhshali Manuscript contained an algebraic formula for solving quadratic equations.

TOK: What are the key concepts that provide the building blocks for mathematical knowledge?

Use of technology: Dynamic graphing software with a slider.

Enrichment: Deriving the quadratic formula by completing the square.

Download connections template

SL 2.8

Content	Guidance, clarification and syllabus links
The reciprocal function $f(x) = \frac{1}{x},\ x \neq 0$: its graph and self-inverse nature.	
Rational functions of the form $f(x) = \frac{ax+b}{cx+d}$ and their graphs. Equations of vertical and horizontal asymptotes.	Sketches should include all horizontal and vertical asymptotes and any intercepts with the axes. **Link to:** transformations (SL2.11). Vertical asymptote: $x = -\frac{d}{c}$; Horizontal asymptote: $y = \frac{a}{c}$.

Connections

International-mindedness: The development of functions, Rene Descartes (France), Gottfried Wilhelm Leibniz (Germany) and Leonhard Euler (Switzerland).

TOK: What are the implications of accepting that mathematical knowledge changes over time?

Download connections template

SL 2.9

Content	Guidance, clarification and syllabus links
Exponential functions and their graphs: $f(x) = a^x, a > 0, f(x) = e^x$ Logarithmic functions and their graphs: $f(x) = \log_a x, x > 0, f(x) = \ln x, x > 0.$	**Link to**: financial applications of geometric sequences and series (SL 1.4). Relationships between these functions: $a^x = e^{x\ln a}; \log_a a^x = x, a, \quad x > 0, \quad a \neq 1$ Exponential and logarithmic functions as inverses of each other.

Connections

Links to other subjects: Radioactive decay, charging and discharging capacitors (physics); first order reactions and activation energy (chemistry); growth curves (biology).

Aim 8: The phrase "exponential growth" is used popularly to describe a number of phenomena. Is ths a misleading use of a mathematical term?

TOK: What role do "models" play in mathematics? Do they play a different role in mathematics compared to their role in other areas of knowledge?

Download connections template

SL 2.10

Content	Guidance, clarification and syllabus links
Solving equations, both graphically and analytically.	**Example:** $e^{2x} - 5e^x + 4 = 0$. **Link to:** function graphing skills (SL 2.3).
Use of technology to solve a variety of equations, including those where there is no appropriate analytic approach.	**Examples:** $e^x = \sin x$ $x^4 + 5x - 6 = 0$.
Applications of graphing skills and solving equations that relate to real-life situations.	**Link to:** exponential growth (SL 2.9)

Connections

Other contexts: Radioactive decay and population growth and decay, compound interest, projectile motion, braking distances.

Links to other subjects: Radioactive decay (physics); modelling (sciences group subjects); production possibilities curve model (economics).

TOK: What assumptions do mathematicians make when they apply mathematics to real-life situations?

Download connections template

SL 2.11

Content	Guidance, clarification and syllabus links
Transformations of graphs. Translations: $y = f(x) + b;\ \ y = f(x - a)$. Reflections (in both axes): $y = -f(x);\ \ y = f(-x)$. Vertical stretch with scale factor p: $y = pf(x)$. Horizontal stretch with scale factor $\frac{1}{q}$: $y = f(qx)$.	Students should be aware of the relevance of the order in which transformations are performed. Dynamic graphing packages could be used to investigate these transformations.
Composite transformations.	**Example:** Using $y = x^2$ to sketch $y = 3x^2 + 2$ **Link to:** composite functions (SL2.5). **Not required at SL**: transformations of the form $f(ax + b)$.

Connections

Links to other subjects: Shift in supply and demand curves (Economics); induced emf and simple harmonic motion (physics).

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AHL content

Recommended teaching hours: 11

The aim of the AHL functions topic is to extend and build upon the aims, concepts and skills from the SL content. It introduces students to useful techniques for finding and using roots of polynomials, graphing and interpreting rational functions, additional ways to classify functions, solving inequations and solving equations involving modulus notation.

HL students may be required to use technology to solve equations where there is no appropriate analytic approach.

AHL 2.12

Content	Guidance, clarification and syllabus links
Polynomial functions, their graphs and equations; zeros, roots and factors. The factor and remainder theorems.	
Sum and product of the roots of polynomial equations.	For the polynomial equation: $\sum_{r=0}^{n} a_r x^r = 0$, the sum is $\frac{-a_{n-1}}{a_n}$ the product is $\frac{(-1)^n a_0}{a_n}$ **Link to:** complex roots of quadratic and polynomial equations (AHL 1.14).

Connections

Links to other subjects: Modelling (sciences group subjects)

TOK: Is it an oversimplification to say that some areas of knowledge give us facts whereas other areas of knowledge give us interpretations?

Enrichment: Viete's theorem in full, "The equation that couldn't be solved" quadratic formula reducing a quadratic to a linear, Cardano and Bombelli.

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AHL 2.13

Content	Guidance, clarification and syllabus links
Rational functions of the form $f(x) = \frac{ax+b}{cx^2+dx+e}$, and $f(x) = \frac{ax^2+bx+c}{dx+e}$	The reciprocal function is a particular case. Graphs should include all asymptotes (horizontal, vertical and oblique) and any intercepts with axes. Dynamic graphing packages could be used to investigate these functions. **Link to:** rational functions (SL 2.8).

Connections

International mindedness: Bourbaki group analytical approach versus Mandlebrot visual approach.

TOK: Does studying the graph of a function contain the same level of mathematical rigour as studying the function algebraically? What are the advantages and disadvantages of having different forms and symbolic language in mathematics?

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AHL 2.14

Content	Guidance, clarification and syllabus links
Odd and even functions.	Even: $f(-x) = f(x)$ Odd: $f(-x) = -f(x)$ Includes periodic functions.
Finding the inverse function, $f^{-1}(x)$, including domain restriction.	
Self-inverse functions.	

Connections

International-mindedness: The notation for functions was developed by a number of different mathematicians in the 17th and 18th centuries. How did the notation we use today become internationally accepted?

TOK: If systems of notation and measurement are culturally and historically situated, does this mean mathematics cannot be seen as independent of culture?

Download connections template

AHL 2.15

Content	Guidance, clarification and syllabus links
Solutions of $g(x) \geq f(x)$, both graphically and analytically.	Graphical or algebraic methods for simple polynomials up to degree 3. Use of technology for these and other functions.

Connections

TOK: Are there differences in terms of value that different cultures ascribe to mathematics, or to the relative value that they ascribe to different areas of knowledge?

Download connections template

AHL 2.16

Content	Guidance, clarification and syllabus links
The graphs of the functions, $y = \lvert f(x) \rvert$ and $y = f(\lvert x \rvert), \quad y = \frac{1}{f(x)}, \quad y = f(ax+b), \quad y = [f(x)]^2$.	Dynamic graphing packages could be used to investigate these transformations.
Solution of modulus equations and inequalities.	**Example:** $\lvert 3x\arccos(x) \rvert > 1$

Connections

International-mindedness: The Bourbaki group analytic approach versus Mandlebrot visual approach.

Download connections template

Topic 3: Geometry and trigonometry

Concepts

Essential understandings:

Geometry and trigonometry allows us to quantify the physical world, enhancing our spatial awareness in two and three dimensions. This topic provides us with the tools for analysis, measurement and transformation of quantities, movements and relationships.

Suggested concepts embedded in this topic:

Generalization, space, relationships, equivalence, representation,

AHL: Quantity, Modelling.

Content-specific conceptual understandings:

- The properties of shapes depend on the dimension they occupy in space.
- Volume and surface area of shapes are determined by formulae, or general mathematical relationships or rules expressed using symbols or variables.
- The relationships between the length of the sides and the size of the angles in a triangle can be used to solve many problems involving position, distance, angles and area.
- Equivalent measurement systems, such as degrees and radians, can be used for angles to facilitate ease of calculation.
- Different representations of the values of trigonometric relationships, such as exact or approximate, may not be equivalent to one another.
- The trigonometric functions of angles may be defined on the unit circle, which can visually and algebraically represent the periodic or symmetric nature of their values.

AHL

- Position and movement can be modelled in three-dimensional space using vectors.
- The relationships between algebraic, geometric and vector methods can help us to solve problems and quantify those positions and movements.

SL content

Recommended teaching hours: 25

The aim of the SL content of the geometry and trigonometry topic is to introduce students to geometry in three dimensions and to non right-angled trigonometry. Students will explore the circular functions and use properties and identities to solve problems in abstract and real-life contexts.

Throughout this topic students should be given the opportunity to use technology such as graphing packages, graphing calculators and dynamic geometry software to develop and apply their knowledge of geometry and trigonometry.

On examination papers, radian measure should be assumed unless otherwise indicated.

Sections SL3.1 to SL3.3 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 3.1

Content	Guidance, clarification and syllabus links
The distance between two points in three-dimensional space, and their midpoint.	In SL examinations, only right-angled trigonometry questions will be set in reference to three-dimensional shapes.
Volume and surface area of three-dimensional solids including right-pyramid, right cone, sphere, hemisphere and combinations of these solids. The size of an angle between two intersecting lines or between a line and a plane.	In problems related to these topics, students should be able to identify relevant right-angled triangles in three-dimensional objects and use them to find unknown lengths and angles.

Connections

Other contexts: Architecture and design.

Links to other subjects: Design technology; volumes of stars and inverse square law (physics).

TOK: What is an axiomatic system? Are axioms self evident to everybody?

Download connections template

SL 3.2

Content	Guidance, clarification and syllabus links
Use of sine, cosine and tangent ratios to find the sides and angles of right-angled triangles.	In all areas of this topic, students should be encouraged to sketch well-labelled diagrams to support their solutions. **Link to:** inverse functions (SL2.2) when finding angles.
The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$; $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$. Area of a triangle as $\frac{1}{2}ab\sin C$.	This section does not include the ambiguous case of the sine rule.

Connections

Other contexts: Triangulation, map-making.

Links to other subjects: Vectors (physics).

International-mindedness: Diagrams of Pythagoras' theorem occur in early Chinese and Indian manuscripts. The earliest references to trigonometry are in Indian mathematics; the use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: Is it ethical that Pythagoras gave his name to a theorem that may not have been his own creation? What criteria might we use to make such a judgment?

Download connections template

SL 3.3

Content	Guidance, clarification and syllabus links
Applications of right and non-right angled trigonometry, including Pythagoras's theorem. Angles of elevation and depression.	Contexts may include use of bearings.

Content	Guidance, clarification and syllabus links
Construction of labelled diagrams from written statements.	

Connections

Other contexts: Triangulation, map-making, navigation and radio transmissions. Use of parallax for navigation.

Links to other subjects: Vectors, scalars, forces and dynamics (physics); field studies (sciences group subjects)

Aim 8: Who really invented Pythagoras's theorem?

Aim 9: In how many ways can you prove Pythagoras's theorem?

International-mindedness: The use of triangulation to find the curvature of the Earth in order to settle a dispute between England and France over Newton's gravity.

TOK: If the angles of a triangle can add up to less than 180°, 180° or more than 180°, what does this tell us about the nature of mathematical knowledge?

Download connections template

SL 3.4

Content	Guidance, clarification and syllabus links
The circle: radian measure of angles; length of an arc; area of a sector.	Radian measure may be expressed as exact multiples of π, or decimals.

Connections

Links to other subjects: Diffraction patterns and circular motion (physics).

International-mindedness: Seki Takakazu calculating π to ten decimal places; Hipparchus, Menelaus and Ptolemy; Why are there 360 degrees in a complete turn? Links to Babylonian mathematics.

TOK: Which is a better measure of angle: radian or degree? What criteria can/do/should mathematicians use to make such decisions?

Download connections template

SL 3.5

Content	Guidance, clarification and syllabus links
Definition of $\cos\theta, \sin\theta$ in terms of the unit circle.	Includes the relationship between angles in different quadrants. $\cos x = \cos(-x)$ **Examples:** $\tan(3\pi - x) = -\tan x$ $\sin(\pi + x) = -\sin x$
Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$.	The equation of a straight line through the origin is $y = x\tan\theta$, where θ is the angle formed between the line and positive x-axis.
Exact values of trigonometric ratios of $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$ and their multiples.	$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \cos\frac{3\pi}{4} = -\frac{1}{\sqrt{2}}, \tan\ \ 210° = \frac{\sqrt{3}}{3}$
Extension of the sine rule to the ambiguous case.	

Connections

International-mindedness: The first work to refer explicitly to the sine as a function of an angle is the Aryabhatiya of Aryabhata (ca 510).

TOK: Trigonometry was developed by successive civilizations and cultures. To what extent is mathematical knowledge embedded in particular traditions or bound to particular cultures? How have key events in the history of mathematics shaped its current form and methods?

Enrichment: The proof of Pythagoras' theorem in three dimensions.

Download connections template

SL 3.6

Content	Guidance, clarification and syllabus links
The Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$. Double angle identities for sine and cosine.	Simple geometrical diagrams and dynamic graphing packages may be used to illustrate the double angle identities (and other trigonometric identities).
The relationship between trigonometric ratios.	**Examples:** Given $\sin\theta$, find possible values of $\tan\theta$, (without finding θ). Given $\cos x = \frac{3}{4}$ and x is acute, find $\sin 2x$, (without finding x).

Connections

Download connections template

SL 3.7

Content	Guidance, clarification and syllabus links
The circular functions $\sin x$, $\cos x$, and $\tan x$; amplitude, their periodic nature, and their graphs Composite functions of the form $f(x) = a\sin(b(x + c)) + d$.	Trigonometric functions may have domains given in degrees or radians. **Examples:** $f(x) = \tan(x - \frac{\pi}{4})$, $f(x) = 2\cos(3(x - 4)) + 1$.
Transformations.	**Example:** $y = \sin x$ used to obtain $y = 3\sin 2x$ by a stretch of scale factor 3 in the y direction and a stretch of scale factor $\frac{1}{2}$ in the x direction. **Link to:** transformations of graphs (SL2.11).
Real-life contexts.	**Examples:** height of tide, motion of a Ferris wheel. Students should be aware that not all regression technology produces trigonometric functions in the form $f(x) = a\sin(b(x + c)) + d$.

Connections

Links to other subjects: Simple harmonic motion (physics).

TOK: Music can be expressed using mathematics. What does this tell us about the relationship between music and mathematics?

Download connections template

SL 3.8

Content	Guidance, clarification and syllabus links
Solving trigonometric equations in a finite interval, both graphically and analytically.	$2\sin x = 1, \quad 0 \le x \le 2\pi$ **Examples:** $2\sin 2x = 3\cos x, \quad 0° \le x \le 180°$ $2\tan(3(x-4)) = 1, \quad -\pi \le x \le 3\pi$
Equations leading to quadratic equations in $\sin x, \ \cos x$ or $\tan x$.	**Examples:** $2\sin^2 x + 5\cos x + 1 = 0$ for $0 \le x \le 4\pi$, $2\sin x = \cos 2x, \quad -\pi \le x \le \pi$ **Not required:** The general solution of trigonometric equations.

Connections

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AHL content

Recommended teaching hours: 26

The aim of the AHL content in the geometry and trigonometry topic is to extend and build upon the aims, concepts and skills from the SL content. It further explores the circular functions, introduces some important trigonometric identities, and introduces vectors in two and three dimensions. This will facilitate problem-solving involving points, lines and planes.

On examination papers radian measure should be assumed unless otherwise indicated.

AHL 3.9

Content	Guidance, clarification and syllabus links
Definition of the reciprocal trigonometric ratios $\sec\theta$, $\operatorname{cosec}\theta$ and $\cot\theta$. Pythagorean identities: $1 + \tan^2\theta = \sec^2\theta$ $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ The inverse functions $f(x) = \arcsin x$, $f(x) = \arccos x, \ f(x) = \arctan x$; their domains and ranges; their graphs.	

Connections

International-mindedness: The origin of degrees in the mathematics of Mesopotamia and why we use minutes and seconds for time; the origin of the word sine.

TOK: What is the relationship between concepts and facts? To what extent do the concepts that we use shape the conclusions that we reach?

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AHL 3.10

Content	Guidance, clarification and syllabus links
Compound angle identities. Double angle identity for tan.	Derivation of double angle identities from compound angle identities. **Link to:** De Moivre's theorem (AHL1.14).

Connections

Other contexts: Triangulation used by GPSs (global positioning systems); concepts in electrical engineering including generation of sinusoidal voltage.

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AHL 3.11

Content	Guidance, clarification and syllabus links
Relationships between trigonometric functions and the symmetry properties of their graphs.	$\sin(\pi - \theta) = \sin\theta$ $\cos(\pi - \theta) = -\cos\theta$ $\tan(\pi - \theta) = -\tan\theta$ **Link to**: the unit circle (SL3.5), odd and even functions (AHL2.14), compound angles (AHL3.10).

Connections

Links to other subjects: Simple harmonic motion graphs (physics)

TOK: Mathematics and knowledge claims: how can there be an infinite number of discrete solutions to an equation?

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AHL 3.12

Content	Guidance, clarification and syllabus links
Concept of a vector; position vectors; displacement vectors. Representation of vectors using directed line segments. Base vectors $i, \quad j, \quad k$. Components of a vector: $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$.	
Algebraic and geometric approaches to the following: • the sum and difference of two vectors • the zero vector $\boldsymbol{0}$, the vector $-\boldsymbol{v}$ • multiplication by a scalar, $k\boldsymbol{v}$, parallel vectors	Distance between points A and B is the magnitude of $\overrightarrow{\mathrm{AB}}$

Content	Guidance, clarification and syllabus links
• magnitude of a vector, $\|v\|$; unit vectors, $\frac{v}{\|v\|}$ • position vectors $\overrightarrow{OA} = a$, $\overrightarrow{OB} = b$ • displacement vector $\overrightarrow{AB} = b - a$ Proofs of geometrical properties using vectors.	

Connections
Links to other subjects: Vectors, scalars, forces and dynamics (physics).

Aim 8: Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb.

TOK: Vectors are used to solve many problems in position location. This can be used to save a lost sailor or destroy a building with a laser-guided bomb. To what extent does possession of knowledge carry with it an ethical obligation?

Download connections template

AHL 3.13

Content	Guidance, clarification and syllabus links
The definition of the scalar product of two vectors. The angle between two vectors. Perpendicular vectors; parallel vectors.	Applications of the properties of the scalar product $v \cdot w = w \cdot v$; $u \cdot (v + w) = u \cdot v + u \cdot w$; $(kv) \cdot w = k(v \cdot w)$; $v \cdot v = \|v\|^2$. $v \cdot w = \|v\|\|w\|\cos\theta$, where θ is the angle between v and w. For non-zero vectors $v \cdot w = 0$ is equivalent to the vectors being perpendicular; for parallel vectors $\|v \cdot w\| = \|v\|\|w\|$.

Connections
Links to other subjects: Forces and dynamics (physics).

TOK: The nature of mathematics: why this definition of scalar product?

Enrichment: Proof of the cosine rule using the dot product.

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AHL 3.14

Content	Guidance, clarification and syllabus links
Vector equation of a line in two and three dimensions: $r = a + \lambda b$.	Relevance of a (position) and b (direction). Knowledge of the following forms for equations of lines: Parametric form: $x = x_0 + \lambda l, y = y_0 + \lambda m, z = z_0 + \lambda n$. Cartesian form: $\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}$.
The angle between two lines.	Using the scalar product of the two direction vectors.
Simple applications to kinematics.	Interpretation of λ as time and b as velocity, with $\|b\|$ representing speed.

Connections
Other contexts: Modelling linear motion in three dimensions; navigational devices, for example GPS.

TOK: Why might it be argued that one form of representation is superior to another? What criteria might a mathematician use in making such an argument?

Download connections template

AHL 3.15

Content	Guidance, clarification and syllabus links
Coincident, parallel, intersecting and skew lines, distinguishing between these cases. Points of intersection.	Skew lines are non-parallel lines that do not intersect in three-dimensional space.

Connections
TOK: How can there be an infinite number of discrete solutions to an equation? What does this suggest about the nature of mathematical knowledge and how it compares to knowledge in other disciplines?

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AHL 3.16

Content	Guidance, clarification and syllabus links
The definition of the vector product of two vectors.	The vector product is also known as the "cross product". $v \times w = \|v\|\|w\|\sin\theta n$, where θ is the angle between v and w, and n is the unit normal vector whose direction is given by the right-hand screw rule.
Properties of the vector product.	$v \times w = -w \times v$; $u \times (v + w) = u \times v + u \times w$; $(kv) \times w = k(v \times w)$; $v \times v = 0$. For non-zero vectors $v \times w = 0$ is equivalent to the vectors being parallel.
Geometric interpretation of $\| v \times w \|$	Use of $\| v \times w \|$ to find the area of a parallelogram (and hence a triangle).

Connections
Links to other subjects: Magnetic forces and fields (physics).

TOK: To what extent is certainty attainable in mathematics? Is certainty attainable, or desirable, in other areas of knowledge?

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AHL 3.17

Content	Guidance, clarification and syllabus links
Vector equations of a plane: $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{b} + \mu\boldsymbol{c}$, where $\boldsymbol{b}$ and $\boldsymbol{c}$ are non-parallel vectors within the plane. $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is a normal to the plane and $\boldsymbol{a}$ is the position vector of a point on the plane. Cartesian equation of a plane $ax + by + cz = d$.	

Connections

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AHL 3.18

Content	Guidance, clarification and syllabus links
Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.	Finding intersections by solving equations; geometrical interpretation of solutions. **Link to**: solutions of systems of linear equations (AHL 1.16).

Connections
TOK: Mathematics and the knower: are symbolic representations of three-dimensional objects easier to deal with than visual representations? What does this tell us about our knowledge of mathematics in other dimensions?

Download connections template

Topic 4: Statistics and probability

Concepts

Essential understandings:
Statistics is concerned with the collection, analysis and interpretation of data and the theory of probability can be used to estimate parameters, discover empirical laws, test hypotheses and predict the occurrence of events. Statistical representations and measures allow us to represent data in many different forms to aid interpretation.

Probability enables us to quantify the likelihood of events occurring and so evaluate risk. Both statistics and probability provide important representations which enable us to make predictions, valid comparisons and informed decisions. These fields have power and limitations and should be applied with care and critically questioned to differentiate between the theoretical and the empirical/observed. Probability theory allows us to make informed choices, to evaluate risk, and to make predictions about seemingly random events.

Suggested concepts embedded in this topic:
Quantity, validity, approximation, generalization.

AHL: Change, systems.

Content-specific conceptual understandings:
- Organizing, representing, analysing and interpreting data and utilizing different statistical tools facilitates prediction and drawing of conclusions.
- Different statistical techniques require justification and the identification of their limitations and validity.
- Approximation in data can approach the truth but may not always achieve it.
- Some techniques of statistical analysis, such as regression, standardization or formulae, can be applied in a practical context to apply to general cases.
- Modelling through statistics can be reliable, but may have limitations.

AHL
- Properties of probability density functions can be used to identify measure of central tendency such as mean, mode and median.
- Probability methods such as Bayes theorem can be applied to real-world systems, such as medical studies or economics, to inform decisions and to better understand outcomes.

SL content
Recommended teaching hours: 27

The aim of the SL content in the statistics and probability topic is to introduce students to the important concepts, techniques and representations used in statistics and probability. Students should be given the opportunity to approach this topic in a practical way, to understand why certain techniques are used and to interpret the results. The use of technology such as simulations, spreadsheets, statistics software and statistics apps can greatly enhance this topic.

It is expected that most of the calculations required will be carried out using technology, but explanations of calculations by hand may enhance understanding. The emphasis is on understanding and interpreting the results obtained, in context.

In examinations students should be familiar with how to use the statistics functionality of allowed technology.

At SL the data set will be considered to be the population unless otherwise stated.

Sections SL4.1 to SL4.9 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 4.1

Content	Guidance, clarification and syllabus links
Concepts of population, sample, random sample, discrete and continuous data.	This is designed to cover the key questions that students should ask when they see a data set/analysis.
Reliability of data sources and bias in sampling.	Dealing with missing data, errors in the recording of data.
Interpretation of outliers.	Outlier is defined as a data item which is more than 1.5 × interquartile range (IQR) from the nearest quartile.

Content	Guidance, clarification and syllabus links
	Awareness that, in context, some outliers are a valid part of the sample but some outlying data items may be an error in the sample. **Link to**: box and whisker diagrams (SL4.2) and measures of dispersion (SL4.3).
Sampling techniques and their effectiveness.	Simple random, convenience, systematic, quota and stratified sampling methods.

Connections

Links to other subjects: Descriptive statistics and random samples (biology, psychology, sports exercise and health science, environmental systems and societies, geography, economics; business management); research methodologies (psychology).

Aim 8: Misleading statistics; examples of problems caused by absence of representative samples, for example Google flu predictor, US presidential elections in 1936, Literary Digest v George Gallup, Boston "pot-hole" app.

International-mindedness: The Kinsey report–famous sampling techniques.

TOK: Why have mathematics and statistics sometimes been treated as separate subjects? How easy is it to be misled by statistics? Is it ever justifiable to purposely use statistics to mislead others?

Download connections template

SL 4.2

Content	Guidance, clarification and syllabus links
Presentation of data (discrete and continuous): frequency distributions (tables).	Class intervals will be given as inequalities, without gaps.
Histograms. Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles, range and interquartile range (IQR).	Frequency histograms with equal class intervals. **Not required:** Frequency density histograms.
Production and understanding of box and whisker diagrams.	Use of box and whisker diagrams to compare two distributions, using symmetry, median, interquartile range or range. Outliers should be indicated with a cross. Determining whether the data may be normally distributed by consideration of the symmetry of the box and whiskers.

Connections

Links to other subjects: Presentation of data (sciences, individuals and societies).

International-mindedness: Discussion of the different formulae for the same statistical measure (for example, variance).

TOK: What is the difference between information and data? Does "data" mean the same thing in different areas of knowledge?

Download connections template

SL 4.3

Content	Guidance, clarification and syllabus links
Measures of central tendency (mean, median and mode). Estimation of mean from grouped data.	Calculation of mean using formula and technology. Students should use mid-interval values to estimate the mean of grouped data.
Modal class.	For equal class intervals only.
Measures of dispersion (interquartile range, standard deviation and variance).	Calculation of standard deviation and variance of the sample using only technology, however hand calculations may enhance understanding. Variance is the square of the standard deviation.
Effect of constant changes on the original data.	**Examples:** If three is subtracted from the data items, then the mean is decreased by three, but the standard deviation is unchanged. If all the data items are doubled, the mean is doubled and the standard deviation is also doubled.
Quartiles of discrete data.	Using technology. Awareness that different methods for finding quartiles exist and therefore the values obtained using technology and by hand may differ.

Connections

Other contexts: Comparing variation and spread in populations, human or natural, for example agricultural crop data, social indicators, reliability and maintenance.

Links to other subjects: Descriptive statistics (sciences and individuals and societies); consumer price index (economics).

International-mindedness: The benefits of sharing and analysing data from different countries; discussion of the different formulae for variance.

TOK: Could mathematics make alternative, equally true, formulae? What does this tell us about mathematical truths? Does the use of statistics lead to an over-emphasis on attributes that can be easily measured over those that cannot?

Download connections template

SL 4.4

Content	Guidance, clarification and syllabus links
Linear correlation of bivariate data. Pearson's product-moment correlation coefficient, r.	Technology should be used to calculate r. However, hand calculations of r may enhance understanding. Critical values of r will be given where appropriate. Students should be aware that Pearson's product moment correlation coefficient (r) is only meaningful for linear relationships.
Scatter diagrams; lines of best fit, by eye, passing through the mean point.	Positive, zero, negative; strong, weak, no correlation. Students should be able to make the distinction between correlation and causation and know that correlation does not imply causation.
Equation of the regression line of y on x.	Technology should be used to find the equation.

Content	Guidance, clarification and syllabus links
Use of the equation of the regression line for prediction purposes. Interpret the meaning of the parameters, a and b, in a linear regression $y = ax + b$.	Students should be aware: • of the dangers of extrapolation • that they cannot always reliably make a prediction of x from a value of y, when using a y on x line.

Connections

Other contexts: Linear regressions where correlation exists between two variables. Exploring cause and dependence for categorical variables, for example, on what factors might political persuasion depend?

Links to other subjects: Curves of best fit, correlation and causation (sciences group subjects); scatter graphs (geography).

Aim 8: The correlation between smoking and lung cancer was "discovered" using mathematics. Science had to justify the cause.

TOK: Correlation and causation–can we have knowledge of cause and effect relationships given that we can only observe correlation? What factors affect the reliability and validity of mathematical models in describing real-life phenomena?

Download connections template

SL 4.5

Content	Guidance, clarification and syllabus links
Concepts of trial, outcome, equally likely outcomes, relative frequency, sample space (U) and event. The probability of an event A is $\mathrm{P}(A) = \frac{n(A)}{n(U)}$. The complementary events A and A' (not A).	Sample spaces can be represented in many ways, for example as a table or a list. Experiments using coins, dice, cards and so on, can enhance understanding of the distinction between experimental (relative frequency) and theoretical probability. Simulations may be used to enhance this topic.
Expected number of occurrences.	**Example:** If there are 128 students in a class and the probability of being absent is 0.1, the expected number of absent students is 12.8.

Connections

Other contexts: Actuarial studies and the link between probability of life spans and insurance premiums, government planning based on likely projected figures, Monte Carlo methods.

Links to other subjects: Theoretical genetics and Punnett squares (biology); the position of a particle (physics).

Aim 8: The ethics of gambling.

International-mindedness: The St Petersburg paradox; Chebyshev and Pavlovsky (Russian).

TOK: To what extent are theoretical and experimental probabilities linked? What is the role of emotion in our perception of risk, for example in business, medicine and travel safety?

Use of technology: Computer simulations may be useful to enhance this topic.

Download connections template

SL 4.6

Content	Guidance, clarification and syllabus links
Use of Venn diagrams, tree diagrams, sample space diagrams and tables of outcomes to calculate probabilities.	
Combined events: $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$. Mutually exclusive events: $\mathrm{P}(A \cap B) = 0$.	The non-exclusivity of "or".
Conditional probability: $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$.	An alternate form of this is: $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A \mid B)$. Problems can be solved with the aid of a Venn diagram, tree diagram, sample space diagram or table of outcomes without explicit use of formulae. Probabilities with and without replacement.
Independent events: $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$.	

Connections

Aim 8: The gambling issue: use of probability in casinos. Could or should mathematics help increase incomes in gambling?

TOK: Can calculation of gambling probabilities be considered an ethical application of mathematics? Should mathematicians be held responsible for unethical applications of their work?

Download connections template

SL 4.7

Content	Guidance, clarification and syllabus links
Concept of discrete random variables and their probability distributions. Expected value (mean), for discrete data. Applications.	Probability distributions will be given in the following ways: X 1 2 3 4 5 $\mathrm{P}(X = x)$ 0.1 0.2 0.15 0.05 0.5 $\mathrm{P}(X = x) = \frac{1}{18}(4 + x)$ for $x \in \{1, 2, 3\}$ $\mathrm{E}(X) = 0$ indicates a fair game where X represents the gain of a player.

Connections

Other contexts: Games of chance.

Aim 8: Why has it been argued that theories based on the calculable probabilities found in casinos are pernicious when applied to everyday life (for example, economics)?

TOK: What do we mean by a "fair" game? Is it fair that casinos should make a profit?

Download connections template

SL 4.8

Content	Guidance, clarification and syllabus links
Binomial distribution. Mean and variance of the binomial distribution.	Situations where the binomial distribution is an appropriate model. In examinations, binomial probabilities should be found using available technology. **Not required:** Formal proof of mean and variance. **Link to:** expected number of occurrences (SL4.5).

Connections

Aim 8: Pascal's triangle, attributing the origin of a mathematical discovery to the wrong mathematician.

International-mindedness: The so-called "Pascal's triangle" was known to the Chinese mathematician Yang Hui much earlier than Pascal.

TOK: What criteria can we use to decide between different models?

Enrichment: Hypothesis testing using the binomial distribution.

Download connections template

SL 4.9

Content	Guidance, clarification and syllabus links
The normal distribution and curve. Properties of the normal distribution. Diagrammatic representation.	Awareness of the natural occurrence of the normal distribution. Students should be aware that approximately 68% of the data lies between $\mu \pm \sigma$, 95% lies between $\mu \pm 2\sigma$ and 99.7% of the data lies between $\mu \pm 3\sigma$.
Normal probability calculations.	Probabilities and values of the variable must be found using technology.
Inverse normal calculations	For inverse normal calculations mean and standard deviation will be given. This does not involve transformation to the standardized normal variable z.

Connections

Links to other subjects: Normally distributed real-life measurements and descriptive statistics (sciences group subjects, psychology, environmental systems and societies)

Aim 8: Why might the misuse of the normal distribution lead to dangerous inferences and conclusions?

International-mindedness: De Moivre's derivation of the normal distribution and Quetelet's use of it to describe *l'homme moyen*.

TOK: To what extent can we trust mathematical models such as the normal distribution? How can we know what to include, and what to exclude, in a model?

Download connections template

SL 4.10

Content	Guidance, clarification and syllabus links
Equation of the regression line of x on y.	
Use of the equation for prediction purposes.	Students should be aware that they cannot always reliably make a prediction of y from a value of x, when using an x on y line.

Connections

TOK: Is it possible to have knowledge of the future?

Download connections template

SL 4.11

Content	Guidance, clarification and syllabus links
Formal definition and use of the formulae: $\mathrm{P}(A\mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ for conditional probabilities, and $\mathrm{P}(A\mid B) = \mathrm{P}(A) = \mathrm{P}(A\mid B')$ for independent events.	An alternate form of this is: $\mathrm{P}(A \cap B) = \mathrm{P}(B)\mathrm{P}(A\mid B)$. Testing for independence.

Connections

Other contexts: Use of probability methods in medical studies to assess risk factors for certain diseases.

TOK: Given the interdisciplinary nature of many real-world applications of probability, is the division of knowledge into discrete disciplines or areas of knowledge artificial and/or useful?

Download connections template

SL 4.12

Content	Guidance, clarification and syllabus links
Standardization of normal variables (z- values).	Probabilities and values of the variable must be found using technology. The standardized value (z) gives the number of standard deviations from the mean.
Inverse normal calculations where mean and standard deviation are unknown.	Use of z-values to calculate unknown means and standard deviations.

Connections

Links to other subjects: The normal distribution (biology); descriptive statistics (psychology).

Download connections template

AHL content

Recommended teaching hours: 6

The aim of the AHL content in the statistics and probability topic is to extend and build upon the aims, concepts and skills from the SL content. Students are introduced to further conditional probability theory in the form of Bayes Theorem and properties of discrete and continuous random variables are further explored.

AHL 4.13

Content	Guidance, clarification and syllabus links
Use of Bayes' theorem for a maximum of three events.	**Link to:** independent events (SL4.6).

Connections

Other contexts: Use of probability methods in medical studies to assess risk factors for certain diseases.

TOK: Does the applicability of knowledge vary across the different areas of knowledge? What would the implications be if the value of all knowledge was measured solely in terms of its applicability?

Download connections template

AHL 4.14

Content	Guidance, clarification and syllabus links
Variance of a discrete random variable.	**Link to:** discrete random variables (SL 4.7)
Continuous random variables and their probability density functions.	$0 \leq f(x) \leq 1$, $\int_{-\infty}^{\infty} f(x)\mathrm{d}x = 1$ including piecewise functions.
Mode and median of continuous random variables.	For a continuous random variable, a value at which the probability density function has a maximum value is called a mode and for the median: $\int_{-\infty}^{m} f(x)\mathrm{d}x = \frac{1}{2}$.
Mean, variance and standard deviation of both discrete and continuous random variables.	Use of the notation $\mathrm{E}(X)$, $\mathrm{E}(X^2)$, $\mathrm{Var}(X)$, where $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ and related formulae. Use of $\mathrm{E}(X)$ for "fair" games.
The effect of linear transformations of X.	$\mathrm{E}(aX + b) = a\mathrm{E}(X) + b, \quad \mathrm{Var}(aX + b) = a^2\mathrm{Var}(X)$

Connections

Other contexts: Other discrete distributions, for example Poisson, may be appropriate for IA/toolkit and further investigation; expected value used in decision making in business, economics and life in general; expected gain to insurance companies.

TOK: Is mathematics more or less useful than other areas of knowledge for solving problems?

Enrichment: Is there a relationship between the interquartile range and the standard deviation for a normally distributed data set?

Download connections template

Topic 5: Calculus

Concepts

Essential understandings:

Calculus describes rates of change between two variables and the accumulation of limiting areas. Understanding these rates of change and accumulations allow us to model, interpret and analyze real-world problems and situations. Calculus helps us to understand the behaviour of functions and allows us to interpret the features of their graphs.

Suggested concepts embedded in this topic:

Change, patterns, relationships, approximation, generalization, space, modelling.

AHL: Systems, quantity.

Content-specific conceptual understandings:

- The derivative may be represented physically as a rate of change and geometrically as the gradient or slope function.
- Areas under curves can be can be approximated by the sum of the areas of rectangles which may be calculated even more accurately using integration.
- Examining rates of change close to turning points helps to identify intervals where the function increases/decreases, and identify the concavity of the function.
- Numerical integration can be used to approximate areas in the physical world.
- Mathematical modelling can provide effective solutions to real-life problems in optimization by maximizing or minimizing a quantity, such as cost or profit.
- Derivatives and integrals describe real-world kinematics problems in two and three-dimensional space by examining displacement, velocity and acceleration.

AHL

- Some functions may be continuous everywhere but not differentiable everywhere.
- A finite number of terms of an infinite series can be a general approximation of a function over a limited domain.
- Limits describe the output of a function as the input approaches a certain value and can represent convergence and divergence.
- Examining limits of functions at a point can help determine continuity and differentiability at a point.

SL content

Recommended teaching hours: 28

The aim of the SL content in the calculus topic is to introduce students to the concepts and techniques of differential and integral calculus and their applications.

Throughout this topic students should be given the opportunity to use technology such as graphing packages and graphing calculators to develop and apply their knowledge of calculus.

Sections SL5.1 to SL5.5 are content common to both Mathematics: analysis and approaches and Mathematics: applications and interpretation.

SL 5.1

Content	Guidance, clarification and syllabus links
Introduction to the concept of a limit.	Estimation of the value of a limit from a table or graph.

Content	Guidance, clarification and syllabus links
	Not required: Formal analytic methods of calculating limits.
Derivative interpreted as gradient function and as rate of change.	Forms of notation: $\frac{\mathrm{d}y}{\mathrm{d}x}$, $f'(x)$, $\frac{\mathrm{d}V}{\mathrm{d}r}$ or $\frac{\mathrm{d}s}{\mathrm{d}t}$ for the first derivative. Informal understanding of the gradient of a curve as a limit.

Connections

Links to other subjects: Marginal cost, marginal revenue, marginal profit, market structures (economics); kinematics, induced emf and simple harmonic motion (physics); interpreting the gradient of a curve (chemistry)

Aim 8: The debate over whether Newton or Leibnitz discovered certain calculus concepts; how the Greeks' distrust of zero meant that Archimedes' work did not lead to calculus.

International-mindedness: Attempts by Indian mathematicians (500-1000 CE) to explain division by zero.

TOK: What value does the knowledge of limits have? Is infinitesimal behaviour applicable to real life? Is intuition a valid way of knowing in mathematics?

Use of technology: Spreadsheets, dynamic graphing software and GDC should be used to explore ideas of limits, numerically and graphically. Hypotheses can be formed and then tested using technology.

Download connections template

SL 5.2

Content	Guidance, clarification and syllabus links
Increasing and decreasing functions. Graphical interpretation of $f'(x) > 0, f'(x) = 0, f'(x) < 0$.	Identifying intervals on which functions are increasing ($f'(x) > 0$) or decreasing ($f'(x) < 0$).

Connections

Download connections template

SL 5.3

Content	Guidance, clarification and syllabus links
Derivative of $f(x) = ax^n$ is $f'(x) = anx^{n-1}, n \in \mathbb{Z}$ The derivative of functions of the form $f(x) = ax^n + bx^{n-1} \ldots$ where all exponents are integers.	

Connections

TOK: The seemingly abstract concept of calculus allows us to create mathematical models that permit human feats such as getting a man on the Moon. What does this tell us about the links between mathematical models and reality?

Download connections template

SL 5.4

Content	Guidance, clarification and syllabus links
Tangents and normals at a given point, and their equations.	Use of both analytic approaches and technology.

Connections

Links to other subjects: Instantaneous velocity and optics, equipotential surfaces (physics); price elasticity (economics).

TOK: In what ways has technology impacted how knowledge is produced and shared in mathematics? Does technology simply allow us to arrange existing knowledge in new and different ways, or should this arrangement itself be considered knowledge?

Download connections template

SL 5.5

Content	Guidance, clarification and syllabus links
Introduction to integration as anti-differentiation of functions of the form $f(x) = ax^n + bx^{n-1} + \ldots$, where $n \in \mathbb{Z}, \quad n \neq -1$	Students should be aware of the link between anti-derivatives, definite integrals and area.
Anti-differentiation with a boundary condition to determine the constant term.	**Example:** If $\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + x$ and $y = 10$ when $x = 1$, then $y = x^3 + \frac{1}{2}x^2 + 8.5$.
Definite integrals using technology. Area of a region enclosed by a curve $y = f(x)$ and the x -axis, where $f(x) > 0$.	Students are expected to first write a correct expression before calculating the area, for example $\int_2^6 (3x^2 + 4)\mathrm{d}x$. The use of dynamic geometry or graphing software is encouraged in the development of this concept.

Connections

Other contexts: Velocity-time graphs

Links to other subjects: Velocity-time and acceleration-time graphs (physics and sports exercise and health science)

TOK: Is it possible for an area of knowledge to describe the world without transforming it?

Download connections template

SL 5.6

Content	Guidance, clarification and syllabus links
Derivative of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, e^x and $\ln x$. Differentiation of a sum and a multiple of these functions.	
The chain rule for composite functions.	**Example:** $f(x) = \mathrm{e}^{(x^2+2)}, \quad f(x) = \sin(3x - 1)$

Content	Guidance, clarification and syllabus links
The product and quotient rules.	**Link to:** composite functions (SL2.5).

Connections

Links to other subjects: Uniform circular motion and induced emf (physics).

TOK: What is the role of convention in mathematics? Is this similar or different to the role of convention in other areas of knowledge?

Download connections template

SL 5.7

Content	Guidance, clarification and syllabus links
The second derivative. Graphical behaviour of functions, including the relationship between the graphs of f, f' and f''.	Use of both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$. Technology can be used to explore graphs and calculate the derivatives of functions. **Link to:** function graphing skills (SL2.3).

Connections

Links to other subjects: Simple harmonic motion (physics).

Download connections template

SL 5.8

Content	Guidance, clarification and syllabus links
Local maximum and minimum points. Testing for maximum and minimum.	Using change of sign of the first derivative or using sign of the second derivative where $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum.
Optimization.	Examples of optimization may include profit, area and volume.
Points of inflexion with zero and non-zero gradients.	At a point of inflexion, $f''(x) = 0$ **and** changes sign (concavity change), for example $f''(x) = 0$ is not a sufficient condition for a point of inflexion for $y = x^4$ at $(0, 0)$. Use of the terms "concave-up" for $f''(x) > 0$, and "concave-down" for $f''(x) < 0$.

Connections

Other contexts: Profit, area, volume.

Links to other subjects: Velocity-time graphs, simple harmonic motion graphs and kinematics (physics); allocative efficiency (economics).

TOK: When mathematicians and historians say that they have explained something, are they using the word "explain" in the same way?

Download connections template

SL 5.9

Content	Guidance, clarification and syllabus links
Kinematic problems involving displacement s, velocity v, acceleration a and total distance travelled.	$v = \frac{ds}{dt}$; $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$ Displacement from t_1 to t_2 is given by $\int_{t_1}^{t_2} v(t)dt$. Distance between t_1 to t_2 is given by $\int_{t_1}^{t_2} \lvert v(t)\rvert dt$. Speed is the magnitude of velocity.

Connections

Links to other subjects: Kinematics (physics).

International-mindedness: Does the inclusion of kinematics as core mathematics reflect a particular cultural heritage? Who decides what is mathematics?

TOK: Is mathematics independent of culture? To what extent are we people aware of the impact of culture on what we they believe or know?

Download connections template

SL 5.10

Content	Guidance, clarification and syllabus links
Indefinite integral of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, $\frac{1}{x}$ and e^x.	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
The composites of any of these with the linear function $ax + b$.	**Example:** $f'(x) = \cos(2x + 3) \Rightarrow f(x) = \frac{1}{2}\sin(2x + 3) + C$
Integration by inspection (reverse chain rule) or by substitution for expressions of the form: $\int kg'(x)f(g(x))dx$.	**Examples:** $\int 2x(x^2 + 1)^4 dx$, $\int 4x\sin x^2 dx$, $\int \frac{\sin x}{\cos x}dx$.

Connections

Download connections template

SL 5.11

Content	Guidance, clarification and syllabus links
Definite integrals, including analytical approach.	$\int_a^b g'(x)dx = g(b) - g(a)$. The value of some definite integrals can only be found using technology. **Link to**: definite integrals using technology (SL5.5).

Content	Guidance, clarification and syllabus links
Areas of a region enclosed by a curve $y = f(x)$ and the x-axis, where $f(x)$ can be positive or negative, without the use of technology. Areas between curves.	Students are expected to first write a correct expression before calculating the area. Technology may be used to enhance understanding of the relationship between integrals and areas.

Connections

International-mindedness: Accurate calculation of the volume of a cylinder by Chinese mathematician Liu Hui; Ibn Al Haytham: first mathematician to calculate the integral of a function, in order to find the volume of a paraboloid.

TOK: Consider $f(x) = \frac{1}{x}$, $1 \le x \le \infty$. An infinite area sweeps out a finite volume. Can this be reconciled with our intuition? Do emotion and intuition have a role in mathematics?

Enrichment: Exploring numerical integration techniques such as Simpson's rule or the trapezoidal rule.

Download connections template

AHL content

Recommended teaching hours: 27

The aim of the AHL content in the calculus topic is to extend and build upon the aims, concepts and skills from the SL content. Further powerful techniques and useful applications of differential and integral calculus are introduced.

AHL 5.12

Content	Guidance, clarification and syllabus links
Informal understanding of continuity and differentiability of a function at a point.	In examinations, students will not be asked to test for continuity and differentiability.
Understanding of limits (convergence and divergence). Definition of derivative from first principles $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.	**Link to:** infinite geometric sequences (SL1.8). Use of this definition for polynomials only.
Higher derivatives.	Familiarity with the notations $\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$, $f^{(n)}(x)$. **Link to**: proof by mathematical induction (AHL 1.15).

Connections

Links to other subjects: Theory of the firm (economics).

International-mindedness: How the Greeks' distrust of zero meant that Archimedes' work did not lead to the Calculus; investigate attempts by Indian mathematicians (500-1000AD) to explain division by zero.

TOK: Does the fact that Leibniz and Newton came across the Calculus at similar times support the argument of Platonists over Constructivists?

Enrichment: Fundamental theorem of calculus.

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AHL 5.13

Content	Guidance, clarification and syllabus links
The evaluation of limits of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ and $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ using l'Hôpital's rule or the Maclaurin series.	The indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. For example: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$. **Link to:** horizontal asymptotes (SL2.8) .
Repeated use of l'Hôpital's rule.	

Connections

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AHL 5.14

Content	Guidance, clarification and syllabus links
Implicit differentiation. Related rates of change. Optimisation problems.	Appropriate use of the chain rule or implicit differentiation, including cases where the optimum solution is at the end point.

Connections

Other contexts: Links between mathematical and physical models.

TOK: Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However, some work was not possible until after Cauchy's work. What does this suggest about the nature of progress and development in mathematics? How might this be similar/different to the nature of progress and development in other areas of knowledge?

Download connections template

AHL 5.15

Content	Guidance, clarification and syllabus links
Derivatives of $\tan x$, $\sec x$, $\operatorname{cosec} x$, $\cot x$, a^x, $\log_a x$, $\arcsin x$, $\arccos x$, $\arctan x$.	
Indefinite integrals of the derivatives of any of the above functions. The composites of any of these with a linear function.	Indefinite integral interpreted as a family of curves. **Examples:** $\int \frac{1}{x^2 + 2x + 5} \mathrm{d}x = \frac{1}{2} \arctan \frac{(x+1)}{2} + C$ $\int \sec^2(2x+5) \mathrm{d}x = \frac{1}{2} \tan(2x+5) + C$
Use of partial fractions to rearrange the integrand.	$\int \frac{1}{x^2 + 3x + 2} \mathrm{d}x = \ln\left\|\frac{x+1}{x+2}\right\| + C$ **Link to:** partial fractions (AHL1.11)

Connections

TOK: Can a mathematical statement be true before it has been proven?

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AHL 5.16

Content	Guidance, clarification and syllabus links
Integration by substitution.	On examination papers, substitutions will be provided if the integral is not of the form $\int kg'(x)f(g(x))\mathrm{d}x$. **Link to:** integration by substitution (SL5.10).
Integration by parts.	**Examples:** $\int x\sin x\mathrm{d}x, \quad \int \ln x\mathrm{d}x, \quad \int \arcsin x\mathrm{d}x$
Repeated integration by parts.	**Examples:** $\int x^2\mathrm{e}^x\mathrm{d}x$ and $\int \mathrm{e}^x\sin x\mathrm{d}x$.

Connections

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AHL 5.17

Content	Guidance, clarification and syllabus links
Area of the region enclosed by a curve and the y-axis in a given interval. Volumes of revolution about the x-axis or y-axis.	

Connections
Other contexts: Industrial design.

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AHL 5.18

Content	Guidance, clarification and syllabus links
First order differential equations. Numerical solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$ using Euler's method.	$x_{n+1} = x_n + h$, where h is a constant.
Variables separable.	*Example*: the logistic equation $\frac{\mathrm{d}n}{\mathrm{d}t} = kn(a - n), \quad a, \quad k \in \mathbb{R}$ **Link to**: partial fractions (AHL1.11) and use of partial fractions to rearrange the integrand (AHL5.15).
Homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f(\frac{y}{x})$ using the substitution $y = vx$.	

Content	Guidance, clarification and syllabus links
Solution of $y' + P(x)y = Q(x)$, using the integrating factor.	

Connections
Other contexts: Newton's law of cooling, population growth, carbon dating.

Links to other subjects: Decay curves (physics); first order reactions (chemistry)

TOK: Does personal experience play a role in the formation of knowledge claims in mathematics? Does it play a different role in mathematics compared to other areas of knowledge?

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AHL 5.19

Content	Guidance, clarification and syllabus links
Maclaurin series to obtain expansions for $\mathrm{e}^x, \quad \sin x, \quad \cos x, \quad \ln(1+x), \quad (1+x)^p,$ $p \in \mathbb{Q}$.	
Use of simple substitution, products, integration and differentiation to obtain other series.	**Example**: for substitution: replace x with x^2 to define the Maclaurin series for e^{x^2}. **Example**: the expansion of $\mathrm{e}^x\sin x$.
Maclaurin series developed from differential equations.	

Connections
International-mindedness: Comparison of the Bourbaki to the Kerala School.

TOK: Is there always a trade-off between accuracy and simplicity?

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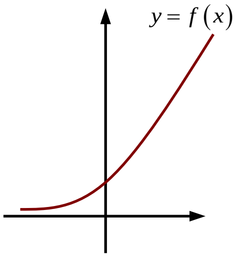
Topic 5 : Calculus

5.1 Differentiation

1. Limit of a Function

Before we study the concept of differentiation, let us examine what **the limit of a function** is.

Consider $f\left(x\right)=2^x$.



x	2^x
0	$2^0=1$
-1	$\frac{1}{2}$
-2	$\frac{1}{4}$
-3	$\frac{1}{8}$
-4	$\frac{1}{16}$

What do you observe about the values of $f\left(x\right)$ as x becomes more negative?

Mathematically, we express this behavior with the following notation : when $x\to-\infty$, $f\left(x\right)\to 0$.

More compactly, we write it as $\lim\limits_{x\to-\infty} 2^x=0$.

Note that 2^x is not equal to zero here. It **approaches** the value of 0 as x gets more and more negative.

Definition : If as x approaches a value a, $f(x)$ approaches a finite value l, then $\boldsymbol{l}$ is called **<u>the LIMIT of *f*(*x*) as *x* tends to *a*,</u>** and we write:

$$\text{As } x\longrightarrow a,\quad f(x)\longrightarrow l \qquad \text{OR} \qquad \lim_{x\to a} f\left(x\right)=l.$$

<u>Algebra of Limits</u> :

1. $\lim_{x\to a} c = c$, where c is a constant
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a} c\times f(x) = c\times\lim_{x\to a} f(x)$
4. $\lim_{x\to a}\left[f(x)\pm g(x)\right] = \lim_{x\to a} f(x)\pm\lim_{x\to a} g(x)$ provided the limits exist.
5. $\lim_{x\to a}\left[f(x)g(x)\right] = \lim_{x\to a} f(x)\lim_{x\to a} g(x)$ provided the limits exists.
6. $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ provided the limits exist.
7. $\lim_{x\to a}\left[f(x)\right]^n = \left[\lim_{x\to a} f(x)\right]^n$
8. $\lim_{x\to a} f(x) = f\left(\lim_{x\to a} x\right)$ provided $f(x)$ is continuous at $x=a$.

The proof the above results are beyond the scope of this syllabus. But you can apply them in the following questions.

Example 1: Find the following limits

(a) $\lim_{x\to 1}\left(3x^2+4x+5\right)$ **(b)** $\lim_{x\to 2}\frac{x^2-4}{x-2}$ **(c)** $\lim_{x\to 0}\frac{2x-3}{x+1}$

[(a) 12, (b) 4, (c) -3]

<u>Limits of a rational function as *x* approaches infinity</u>

In order to assess the behavior of a rational function as *x* approaches infinity, the approach involves dividing both the numerator and denominator by the highest power of *x* that appears in the denominator. This action helps determine which specific term within the entire expression dictates the function's behaviour as *x* becomes exceedingly large.

Example: $\lim_{x\to\infty}\frac{1}{x} = 0$, $\lim_{x\to\infty}\frac{2x}{4x^2} = \lim_{x\to\infty}\frac{\frac{2}{x}}{4} = \frac{0}{4} = 0$

Example 2: Find the following limits

(a) $\lim_{x\to\infty}\frac{2x^3+3x+4}{4x^3+5x^2+7}$ **(b)** $\lim_{x\to\infty}\frac{\sqrt{3x^4+7}}{x^2+3}$

(c) $\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}$ **(d)** $\lim_{x\to\infty}\left(\sqrt{16x^2+x}-8x\right)$

[(a) $\frac{1}{2}$, (b) $\sqrt{3}$, (c) $\frac{1}{4}$, (d) $-\infty$]

What do you observe when the power in the numerator is smaller than the power in the denominator? What happen when the powers are equal? What if the power in the denominator is smaller than the numerator such as $\lim\limits_{x\to\infty}\dfrac{x^3+2x^2+4}{3x^2-2}$?

When evaluating limits, you may check your answer by visiting the following link:
https://www.symbolab.com/solver/limit-calculator

Continuity

A function $f(x)$ is said to be ***continuous*** at a point $x=a$ if there is no 'break' or 'gap' in the graph of $y=f(x)$ at that point.

Definition : Let f be a real function and let c be in the domain of f.

Then f is continuous at c $\Leftrightarrow$ $\lim\limits_{x\to c} f(x)=f(c)$.

More precisely, if the left hand limit, right hand limit, and the value of the function at $x=c$ exist and are all equal to each other, then f is continuous at $x=c$.

i.e. $\lim\limits_{x\to c^-} f(x)=f(c)=\lim\limits_{x\to c^+} f(x)$

How is the more precise definition different from the first one given?

Continuity in an interval

1. f is continuous in an open interval $]a,b[$ $\Leftrightarrow$ f is continuous at every $c\in]a,b[$

2. f is continuous in a closed interval $[a,b]$ if and only if

  (a) f is continuous in $]a,b[$,

  (b) $\lim\limits_{x\to a^+} f(x)=f(a)$ and

  (c) $\lim\limits_{x\to b^-} f(x)=f(b)$

Graphically, a function f will be continuous at $x=c$ if there is no break in the graph of the function at the point $(c,f(c))$. In an interval, a function is continuous if there is no break in the graph of the function in the entire interval.

The function f will be discontinuous at $x=a$ in any of the following cases :

(a) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist but are not equal.

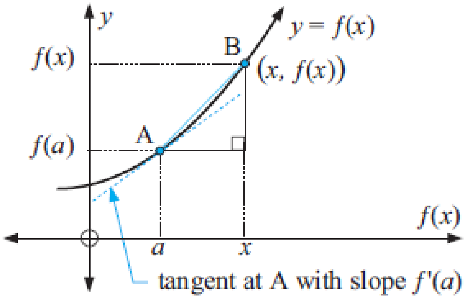
(b) $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and are equal but not equal to $f(a)$.

(c) $f(a)$ is not defined.

What is the difference between the terms undefined and indeterminate? Give your own examples to illustrate the differences if any.

2. Differentiation from First Principles

Consider a general function $y = f(x)$, a fixed point A(a, $f(a)$) and a variable point B(x, $f(x)$).



The slope of chord AB $= \dfrac{f(x)-f(a)}{x-a}$.

Now as B → A, $x \to a$

and the slope of chord AB → slope of tangent at A

So, $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$.

The slope of the tangent to $y = f(x)$ at $x = a$ is given by $f'(a) = \lim\limits_{x \to a} \dfrac{f(x)-f(a)}{x-a}$.

It is called the **gradient** of the function at $x = a$.

This is the slope of the curve at the point $x = a$, and is the instantaneous rate of change in y with respect to x at that point. Finding the slope using the limits of the function is referred to as the method '***first principles***'.

Example 3: Find the slope of the tangent to the curve $y = 2x^2 + 3$ at $x = 2$ from first principles.

Solution : [8]

Example 4: Find the derivative of $f(x) = \dfrac{8}{x}$ at $x = 2$ from first principles. [-2]

Solution :

Question 1: Find the slope of the tangent to the following curves from first principles.

(a) $f(x) = 1 - x^2$ at $x = 2$

(b) $f(x) = 2x^2 + 5x$ at $x = -1$

(c) $f(x) = \dfrac{2x-1}{x+3}$ at $x = -1$

(d) $f(x) = \dfrac{4}{x}$ at $x = 2$

(e) $f(x) = \dfrac{1}{x^2}$ at $x = 4$

Note : the derivative can be found from first principles using $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}$.

Example 5: Find the derivative of x^2 with respect to x using first principles. [2x]

Solution :

Question 2: Find the following derivatives with respect to x using first principles.

(a) $f\left(x\right)=\dfrac{1}{x}$ **(b)** $f\left(x\right)=\sqrt{x}$ **(c)** $f\left(x\right)=\dfrac{1}{x+2}$ **(d)** $f\left(x\right)=\sqrt{x+2}$

Solution :

Differentiability

Definition : A function f is **differentiable** at a point c if and only if $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ (or equivalently, $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$) exists.

f is **differentiable** on an open interval $]a,b[$ if and only if $\lim\limits_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$ exists $\forall c\in\,]a,b[$

Theorem : Let f be a function defined on the open interval $]a,b[$ and let $c\in\,]a,b[$.
Then, if f is differentiable at c, then f is continuous at c.

Proof : Suppose f is differentiable at $x=c$. Then the limit $\lim\limits_{x\to c}\dfrac{f(x)-f(c)}{x-c}$ exists.

$$\begin{aligned}\text{Then}\quad \lim_{x\to c}\left(f(x)-f(c)\right)&=\lim_{x\to c}\left((x-c)\frac{f(x)-f(c)}{x-c}\right)\\&=\lim_{x\to c}(x-c)\lim_{x\to c}\frac{f(x)-f(c)}{x-c}\\&=0\end{aligned}$$

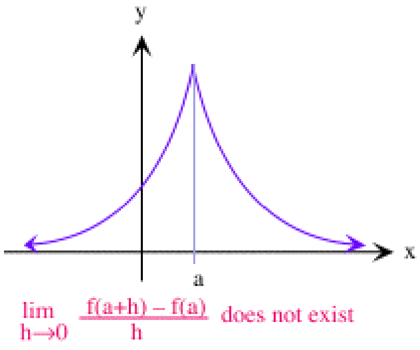
So $\lim\limits_{x\to c}f(x)=f(c)$ and thus f is continuous at $x=c$.

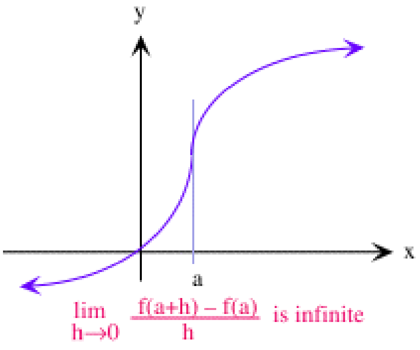
Note : A function can fail to be differentiable at $x=a$ if

(a) $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ does not exist or

(b) $\lim\limits_{h\to 0}\dfrac{f(a+h)-f(a)}{h}$ is infinite.

In the former case, there could be a cusp in the graph, and in the latter case, there could be a vertical tangent at $x=a$ [1]





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[1] https://www.zweigmedia.com/RealWorld/calctopic1/contanddiffb.html

3. l’Hopital’s Rule

l’Hopital’s rule is a tool that we can use to evaluate limits of **indeterminate form** e.g. $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$.

e.g. limits $\lim\limits_{x\to 0}\frac{\sin x}{x}$, $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$, $\lim\limits_{x\to\infty}\frac{2x+1}{x-3}$ are of an indeterminate form.

l’Hopital’s Rule

Let $f(x)$ and $g(x)$ be differentiable functions on an open interval $]a,b[$ (except possibly at $c\in\,]a,b[$). If

(a) $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = 0$ or $\pm\infty$,

(b) $g'(x)\neq 0$ for every $x\in\,]a,b[\,\backslash\{c\}$, and

(c) $\lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$ exists,

then $\lim\limits_{x\to c}\frac{f(x)}{g(x)} = \lim\limits_{x\to c}\frac{f'(x)}{g'(x)}$

Notes :

- be sure that the three conditions stated are met before using l’Hopital’s rule.
- on occasion, l’Hopital’s rule may not help and it is easier to attempt it using other means.

Example 6: Evaluate the following limits using l’Hopital’s rule

(a) $\lim\limits_{x\to 0}\frac{\sin x}{x}$ **(b)** $\lim\limits_{x\to 2}\frac{x^2-4}{x-2}$ **(c)** $\lim\limits_{x\to\infty}\frac{2x+1}{x-3}$

(d) $\lim\limits_{x\to 0}\frac{e^{2x}-1}{x}$ **(e)** $\lim\limits_{x\to 0}\frac{2\sin x-\sin 2x}{x-\sin x}$ **(f)** $\lim\limits_{x\to 0}\frac{1-\cos x}{x-x^3}$

(g) $\lim\limits_{x\to 9}\frac{x-9}{\sqrt{x}-3}$ **(h)** $\lim\limits_{x\to\infty}\frac{4+\ln x}{x^2-3}$ **(i)** $\lim\limits_{x\to\frac{\pi}{3}}\frac{3x-\pi}{1-2\cos x}$

[(a) 1, (b) 4, (c) 2, (d) 2 (e) 6 (f) 0 (g) 6 (h) 0 (i) $\sqrt{3}$]

Example 7: Find the derivative of a^x from first principles. $[a^x \ln a]$

Solution :

Example 8: Find the derivative of $\sin x$ from first principles. $[\cos x]$

[Hint : you need to use the factor formula $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$]

Solution :

Other Indeterminate Forms

l'Hopital's rule does not apply directly to other indeterminate forms. One needs to use clever algebraic manipulation to change the expression into one where l'Hopital's rule can be applied.

To use l'Hopital's rule for expressions of the form $0 \times \infty$, write the product $f \times g$ as $\frac{f}{1/g}$ or $\frac{g}{1/f}$.

Example 9: Evaluate **(a)** $\lim_{x \to 0^+} x \ln x$ **(b)** $\lim_{x \to -\infty} xe^x$ [(a) 0, (b) 0]

Solution :

4. Rules of Differentiation

Let u and v be two functions in x and k be a constant. Then

(1) $\frac{d}{dx}(k) = 0$

(2) $\frac{d}{dx}(ku) = k\frac{du}{dx}$

(3) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(4) Chain Rule : if $u = f(x)$ and $y = g(u)$ such that $y = g\left[f(x)\right]$ (composite function), then :

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

(5) Product Rule : If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

(6) Quotient Rule : If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

Investigate how you may use First Principles to prove the formulae listed above.

Example 11: Differentiate the following functions with respect to x.

(a) $y = \left(x^2 - 2x\right)^4$ **(b)** $y = \frac{4}{\sqrt{1-2x}}$

[(a) $8x^3(x-2)^3(x-1)$ (b) $\frac{4}{(1-2x)^{\frac{3}{2}}}$]

Solution :

Question 3: Differentiate the following functions with respect to x.

(a) $y = (5x+3)^6$ **(b)** $y = \frac{1}{(1-4x)^3}$ **(c)** $\frac{1}{\sqrt{2x^2+3}}$

Solution :

Question 4: Differentiate the following functions with respect to x.

(a) $y = \left(x^2+1\right)\left(x+3\right)^4$ **(b)** $y = x\sqrt{x+3}$ **(c)** $y = \frac{2x+1}{x^2+1}$ **(d)** $y = \frac{x}{\sqrt{4x+1}}$

Solution :

Differentiation of Standard Functions

Specific Cases	General Cases
$\frac{d}{dx}x^n = nx^{n-1}$	$\frac{d}{dx}\left[f(x)\right]^n = n\left[f(x)\right]^{n-1} \times f'(x)$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}\left[\ln f(x)\right] = \frac{1}{f(x)} \times f'(x)$
$\frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$	$\frac{d}{dx}\left[\log_a\left(f(x)\right)\right] = \frac{f'(x)}{f(x)\ln a}$
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}\left[e^{f(x)}\right] = e^{f(x)} \times f'(x)$
$\frac{d}{dx}a^x = a^x \ln a$	$\frac{d}{dx}\left[a^{f(x)}\right] = a^{f(x)} \ln a \times f'(x)$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}\left(\sin f(x)\right) = f'(x)\cos f(x)$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}\left(\cos f(x)\right) = -f'(x)\sin f(x)$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}\left(\tan f(x)\right) = f'(x)\sec^2 f(x)$
$\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	$\frac{d}{dx}\left(\operatorname{cosec} f(x)\right) = -f'(x)\operatorname{cosec} f(x)\cot f(x)$
$\frac{d}{dx}(\sec x) = \sec x \tan x$	$\frac{d}{dx}\left(\sec f(x)\right) = f'(x)\sec f(x)\tan f(x)$
$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\frac{d}{dx}\left(\cot f(x)\right) = -f'(x)\operatorname{cosec}^2 f(x)$

Example 12 : Differentiate the following functions with respect to x:

(a) $(2x+1)\ln x$ **(b)** 5^{1-2x}

Solution : [(a) $2\ln x + \frac{2x+1}{x}$ (b) $-10\ln 5\left(\frac{1}{25}\right)^x$]

Example 13 : Differentiate the following functions with respect to x :

(a) $\ln\left[(1-x)(x+2)\right]$ **(b)** $\ln(\ln x)$

Solution : [(a) $\frac{2x+1}{(x-1)(x+2)}$ (b) $\frac{1}{x\ln x}$]

Example 14 : Differentiate the following functions with respect to x :

(a) $(x+1)^2 e^{\frac{1}{x}}$ **(b)** $x\sqrt{4-x^2}$

Solution : [(a) $\frac{1}{x^2}e^{\frac{1}{x}}(x+1)(2x^2-x-1)$ (b) $\sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$]

Example 15 : Differentiate $\left(\log_8 x\right)^3$ with respect to *x*. $[\frac{3(\ln x)^2}{x\,(\ln 8)^3}]$

Solution :

Example 16 : Differentiate $\sqrt[3]{\frac{5-x}{x+3}}$ with respect to *x*. $[-\frac{8}{3}(5-x)^{-\frac{2}{3}}(x+3)^{-\frac{4}{3}}]$

Solution :

Example 17 : Differentiate the following functions with respect to *x*.

(a) $\cos^4(2x)$ **(b)** $\sec^3\left(\sqrt{1+2x}\right)$

[(a) $-8\cos^3(2x)\sin(2x)$ (b) $\frac{3\sec^3\sqrt{1+2x}\tan\sqrt{1+2x}}{\sqrt{1+2x}}$]

Solution :

Example 18 : Differentiate with respect to *x* :

(a) $e^{\sin\sqrt{x}}$ **(b)** $\ln\left(\operatorname{cosec} x+\cot x\right)$

[(a) $\frac{1}{2\sqrt{x}}e^{\sin\sqrt{x}}\cos\sqrt{x}$ (b) $-\operatorname{cosec} x$]

Solution :

Example 19 : Differentiate the following functions with respect to *x* :

(a) $\tan\left(3^x\right)$ **(b)** $\frac{\sin\left(\frac{1}{x}\right)}{\sqrt{1+x}}$

Solution : [(a) $3^x\cdot\ln 3\,(\sec^2(3^x))$ (b) $-\frac{(1+x)^{-\frac{3}{2}}}{2x^2}\left(2(1+x)\cos\frac{1}{x}+x^2\sin\frac{1}{x}\right)$]

Question 5: Differentiate the following exponential and logarithmic functions with respect to *x*.

(a) $y = 2^x$ **(b)** $y = 5^x$ **(c)** $y = x2^x$ **(d)** $y = x^3 6^{-x}$

(e) $y = \frac{2^x}{x}$ **(f)** $y = \frac{x}{3^x}$ **(g)** $y = \ln 7x$ **(h)** $y = \ln(2x+1)$

(i) $y = x^2 \ln x$ **(j)** $y = e^x \ln x$ **(k)** $(\ln x)^2$ **(l)** $y = \sqrt{x}\ln(2x)$

(m) $y = \ln\sqrt{1-2x}$ **(n)** $y = \ln\left(\frac{1}{2x+3}\right)$ **(o)** $y = \ln\left(x\sqrt{2-x}\right)$ **(p)** $y = \ln\left(\frac{x+3}{x-1}\right)$

Solution :

Question 6: Differentiate the following trigonometric functions with respect to *x*.

(a) $y = e^{2x}\tan x$ **(b)** $y = x\cos x$ **(c)** $y = \frac{\sin x}{x}$ **(d)** $y = x\tan x$

(e) $y = \cos^3 x$ **(f)** $y = \sqrt{\cos x}$ **(g)** $y = \cos x \sin 2x$ **(h)** $y = \cos^3(4x)$

Solution :

Question 7: Differentiate the following trigonometric functions with respect to *x*.

(a) $y = x\sec x$ **(b)** $y = e^x \cot x$ **(c)** $y = 4\sec 2x$ **(d)** $y = e^{-x}\cot\left(\frac{x}{2}\right)$

(e) $y = x^2 \operatorname{cosec} x$ **(f)** $y = x\sqrt{\operatorname{cosec} x}$ **(g)** $y = \ln(\sec x)$ **(h)** $y = x\operatorname{cosec}\left(x^2\right)$

(i) $y = \frac{\cot x}{\sqrt{x}}$

Solution :

5. Implicit Differentiation

For relations like $x^2 + xy - 5$, it is difficult or at times not possible to find $\frac{dy}{dx}$ by making *y* the subject of the formula. These types of relations are referred to as implicit relations.

To find $\frac{dy}{dx}$ in such cases,

1. Differentiate term by term using chain rule, product rule and quotient rule (eg $\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}$.)

2. Rearrange the terms in order to get an expression for $\frac{dy}{dx}$.

Example 20: Find $\frac{dy}{dx}$ if $3x^2 + y^2 = 2$. $[-\frac{3x}{y}]$

Solution :

Question 8: Find $\frac{dy}{dx}$ in each of the following:

(a) $x^3 + xy = 8$

(b) $4x^2 + y^2 = 5$

(c) $y^3 - \sqrt{x} = 3$

(d) $(x+3)(y+2) = \ln x$

(e) $xy = y^2 + 4$

(f) $e^y = (x-y)^2$

Solution :

Question 9: If $x^2 y + x\ln x - 6x = 0,$ show that $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = -\frac{1}{x}.$

Solution :

Question 10: If $e^x y = \cos x,$ show that $2y + 2\frac{dy}{dx} + \frac{d^2y}{dx^2} = 0.$

Solution :

Derivatives of inverse trigonometric functions

Specific Cases	General Cases
$\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\arcsin\left(f(x)\right)\right]=\frac{f'(x)}{\sqrt{1-\left(f(x)\right)^2}}$
$\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\left[\arccos\left(f(x)\right)\right]=-\frac{f'(x)}{\sqrt{1-\left(f(x)\right)^2}}$
$\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$	$\frac{d}{dx}\left[\arctan\left(f(x)\right)\right]=\frac{f'(x)}{1+\left(f(x)\right)^2}$

Question 11: Prove, using implicit differentiation, that

(a) $\frac{d}{dx}(\arcsin x)=\frac{1}{\sqrt{1-x^2}}$ **(b)** $\frac{d}{dx}(\arccos x)=-\frac{1}{\sqrt{1-x^2}}$ **(c)** $\frac{d}{dx}(\arctan x)=\frac{1}{1+x^2}$

[Hint for (a) : let $y=\arcsin x$]

Solution :

Question 12: Differentiate the following with respect to x

(a) $y=\arctan(2x)$ **(b)** $y=\arccos(3x)$ **(c)** $y=\arccos\left(\frac{x}{5}\right)$ **(d)** $y=\arctan x^2$

Solution :

Answers to Questions :

1. (a) -4 (b) 1 (c) $\frac{7}{4}$ (d) -1 (e) $-\frac{1}{32}$
2. (a) $-\frac{1}{x^2}$ (b) $\frac{1}{2\sqrt{x}}$ (c) $-\frac{1}{(x+2)^2}$ (d) $\frac{1}{2\sqrt{x+2}}$
3. (a) $30(5x+3)^5$ (b) $12(1-4x)^{-4}$ (c) $-2x\left(2x^2+3\right)^{-\frac{3}{2}}$
4. (a) $2(x+3)^3\left(3x^2+3x+2\right)$ (b) $\frac{3(x+2)}{2\sqrt{x+3}}$ (c) $\frac{2\left(1-x-x^2\right)}{\left(x^2+1\right)^2}$ (d) $\frac{2x+1}{(4x+1)\sqrt{4x+1}}$
5. (a) $2^x\ln 2$ (b) $5^x\ln 5$ (c) $2^x+x2^x\ln 2$ (d) $\frac{3x^2-x^3\ln 6}{6^x}$ (e) $\frac{x2^x\ln 2-2^x}{x^2}$ (f) $\frac{1-x\ln 3}{3^x}$ (g) $\frac{1}{x}$ (h) $\frac{2}{2x+1}$ (i) $2x\ln x+x$ (j) $e^x\ln x+\frac{e^x}{x}$ (k) $\frac{2\ln x}{x}$ (l) $\frac{\ln(2x)+2}{2\sqrt{x}}$ (m) $\frac{-1}{1-2x}$ (n) $\frac{-2}{2x+3}$ (o) $\frac{1}{x}-\frac{1}{2(2-x)}$ (p) $\frac{1}{x+3}-\frac{1}{x-1}$
6. (a) $2e^{2x}\tan x+\frac{e^{2x}}{\cos^2 x}$ (b) $\cos x-x\sin x$ (c) $\frac{x\cos x-\sin x}{x^2}$ (d) $\tan x+\frac{x}{\cos^2 x}$ (e) $-3\sin x\cos^2 x$ (f) $-\frac{\sin x}{2\sqrt{\cos x}}$ (g) $-\sin x\sin 2x+2\cos x\cos 2x$ (h) $-12\sin 4x\cos^2 4x$
7. (a) $\sec x+x\sec x\tan x$ (b) $e^x\left(\cot x-\operatorname{cosec}^2 x\right)$ (c) $8\sec 2x\tan 2x$ (d) $-e^{-x}\left(\cot\left(\frac{x}{2}\right)+\frac{1}{2}\operatorname{cosec}^2\left(\frac{x}{2}\right)\right)$ (e) $x\operatorname{cosec}x(2-x\cot x)$ (f) $\sqrt{\operatorname{cosec}x}\left[1-\frac{x}{2}\cot x\right]$ (g) $\tan x$ (h) $\operatorname{cosec}x^2\left[1-2x^2\cot\left(x^2\right)\right]$ (i) $-\frac{\cos x\sin x+2x}{2x\sqrt{x}\sin^2 x}$
8. (a) $\frac{-3x^2-y}{x}$ (b) $-\frac{4x}{y}$ (c) $\frac{1}{6\sqrt{x}y^2}$ (d) $\frac{1-xy-2x}{x(x+3)}$ (e) $\frac{y}{2y-x}$ (f) $\frac{2(x-y)}{e^y+2x-2y}$ 12. (a) $\frac{2}{1+4x^2}$ (b) $-\frac{3}{\sqrt{1-9x^2}}$ (c) $-\frac{1}{\sqrt{25-x^2}}$ (d) $\frac{2x}{1+x^4}$
TOK (Mathematics and Knowledge Claims): Euler was able to make important advances in mathematical analysis before calculus had been put on a solid theoretical foundation by Cauchy and others. However some work was not possible until after Cauchy's work. What does this tell us about the importance of proof and the nature of mathematics?
TOK (Mathematics and the real world): The seemingly abstract concept of the calculus allows us to create mathematical models that permit human feats such as getting a man on the moon. What does this tell us about the links between mathematical models and physical reality?

Reflect on what you have learn about the meaning of limits, continuity and differentiability.

How is differentiation valid if the first principle considers the limit as h tends to zero. What happens at the discontinuity 0? Does it matter to the correctness of the solution?

When the limit of a function is indeterminate, what does it mean?

Topic 5: Calculus

WS 5.1 – Limits & Differentiation

1. Evaluate the following limits:

(a) $\lim\limits_{u\to 1}\dfrac{u^2-1}{u-1}$ **(b)** $\lim\limits_{x\to 4}\dfrac{2-\sqrt{x}}{4-x}$ **(c)** $\lim\limits_{t\to\infty}\left(\sqrt{t}-\sqrt{t+2}\right)$

(d) $\lim\limits_{x\to\infty}\dfrac{x^3+2x-10}{2x^3}$ **(e)** $\lim\limits_{x\to\infty}\left(x-\sqrt{x^2-x}\right)$ **(f)** $\lim\limits_{t\to 0}\left(\dfrac{1}{t\sqrt{1+t}}-\dfrac{1}{t}\right)$

2. Use l'Hopital's rule to find the following limits:

(a) $\lim\limits_{x\to 5}\dfrac{x^2-25}{x-5}$ **(b)** $\lim\limits_{x\to\infty}\dfrac{x^3-4x^2+5x}{2x^3-3x+1}$ **(c)** $\lim\limits_{x\to 0}\dfrac{\tan x^2}{x}$

(d) $\lim\limits_{x\to 0}\dfrac{\sin 6x}{3x}$ (e) $\lim\limits_{x\to 0}\dfrac{\sin^2 x}{x^2}$ **(f)** $\lim\limits_{x\to\infty}x^3e^{-x^2}$

(g)* $\lim\limits_{x\to 0}\dfrac{x2^x}{2^x-1}$ **(h)*** $\lim\limits_{x\to 1}(x-1)\tan\dfrac{\pi x}{2}$ **(i)*** $\lim\limits_{x\to -\infty}x^2e^{x-1}$

3. Differentiate the following functions **using first principles**

(a) $10x^2$ **(b)** $\dfrac{1}{1-x}$

4. Differentiate the following with respect to x

(a) $\ln\left(\dfrac{x-1}{3-x}\right)$ **(b)** $\arcsin\left(\dfrac{1}{x}\right)$ **(c)** $e^{-2x}\left(2\cos 3x-3\sin 2x\right)$

5. Find $\dfrac{dy}{dx}$ using implicit differentiation

(a) $(\sin x)(\sin y)=1$ **(b)** $e^{3x}=\cos(x+2y)$

6*. If $y=\sin(m\arcsin x)$, show that $\left(1-x^2\right)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0.$

7. Differentiate the following with respect to x

(a) $\arcsin\sqrt{1-x^2}$ **(b)** $\dfrac{\cos x}{1-\sin x}$ **(c)** $\arctan\left(\dfrac{1+x}{1-x}\right)$

Answers:

1. (a) 2, (b) 0.25, (c) 0, (d) 0.5 (e) 0.5 (f) -0.5

2. (a) 10 (b) $\frac{1}{2}$ (c) 0 (d) 2 (e) 1

   (f) 0 (g) $\frac{1}{\ln 2}$ (h) $-\frac{2}{\pi}$ (i) 0

3. (a) $20x$ (b) $\frac{1}{(1-x)^2}$

4. (a) $\frac{2}{(x-1)(3-x)}$, (b) $-\frac{1}{|x|\sqrt{x^2-1}}$ (c) $e^{-2x}\left(6\sin 2x-6\cos 2x-6\sin 3x-4\cos 3x\right)$

5. (a) $-\cot x \tan y$ (b) $-\frac{1}{2}\left(\frac{3e^{3x}}{\sin(x+2y)}+1\right)=-\frac{1}{2}\left(\frac{3e^{3x}+\sin(x+2y)}{\sin(x+2y)}\right)$

7. (a) $\frac{-x}{|x|\sqrt{1-x^2}}$ (b) $\frac{1}{1-\sin x}$ (c) $\frac{1}{1+x^2}$

Topic 5 : Calculus

5.2 Applications of Differentiation

1. Tangents and Normals

Let $y = f(x)$ be a differentiable function and let $P(a,b)$ be a point on the graph of $y = f(x)$. Then

- The tangent to the graph of $y = f(x)$ at P is given by $y - b = f'(a)(x - a)$
- The normal to the graph of $y = f(x)$ at P is given by $y - b = -\dfrac{1}{f'(a)}(x - a)$

NB : the normal at P is perpendicular to the tangent at P.

Example 1 : Find the equation of the tangent to the curve $x^3y^2 = \cos(\pi y)$ at the point $(-1,1)$.

[$2y = 3x + 5$]

2. Curve Sketching

Definitions : Let f be a differentiable function with domain D_f. Then

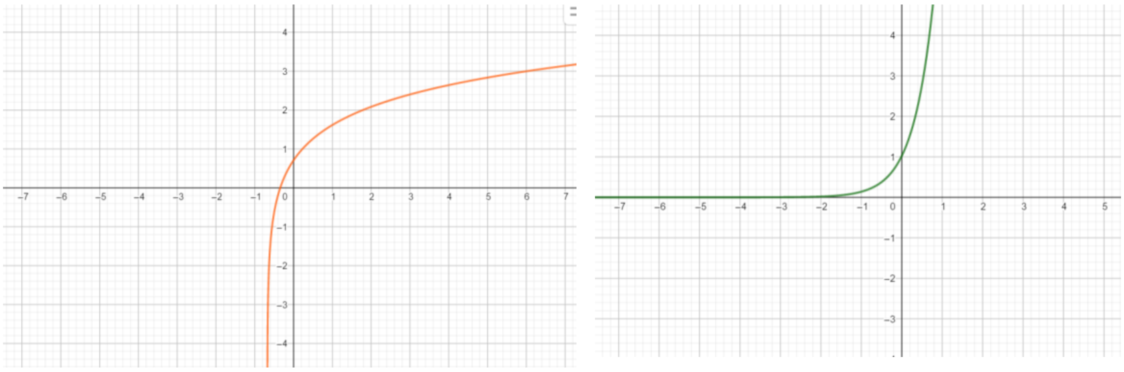
(a) f is increasing if and only if $f'(x) > 0$ for every $x \in D_f$.

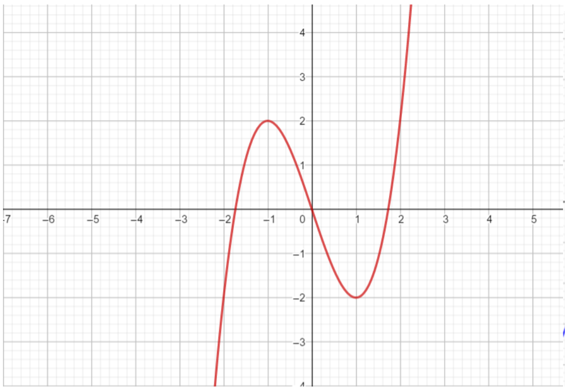
(b) f is decreasing if and only if $f'(x) < 0$ for every $x \in D_f$.

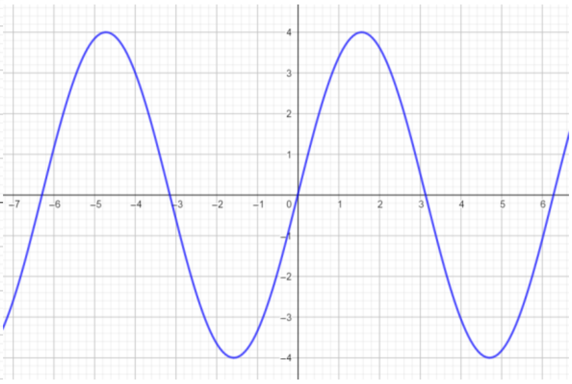
(c) f is concave up if and only if $f''(x) > 0$ for every $x \in D_f$.

(d) f is concave down if and only if $f''(x) < 0$ for every $x \in D_f$.

Identify from the following curves the properties that are listed in the above definitions. Which region of the curves have an increasing gradient change? What curvature does that correspond to?







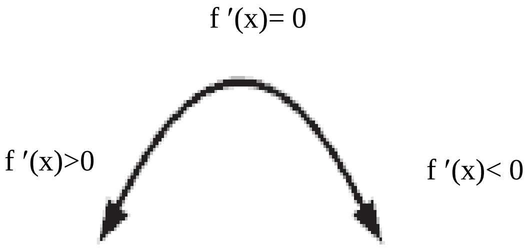
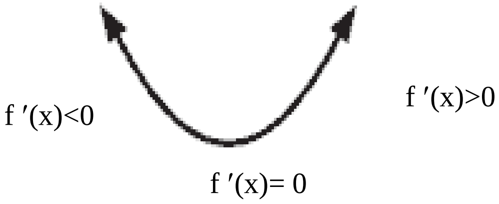
Definition : Let f be a differentiable function with domain D_f. Then the point $\left(a, f(a)\right)$ is called

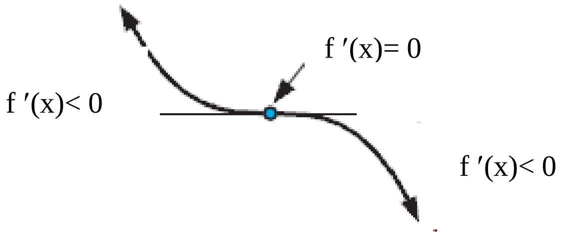
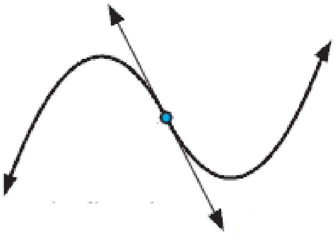
(a) a **(local) maximum point** of f if f is increasing for $x < a$ and f is decreasing for $x > a$.

(b) a **(local) minimum point** of f if f is decreasing for $x < a$ and f is increasing for $x > a$.

Identify from the above descriptions the minimum and maximum points on the curves illustrated above.

Shape of a Curve –

Concave-down	 At the stationary point, $f'(x)=0$. Change is observed in the sign of $f'(x)$. It is a local maximum as $f''(x)<0$.
Concave-up	 At the stationary point, $f'(x)=0$. Change is observed in the sign of $f'(x)$. It is a local minimum as $f''(x)>0$.
A <ins>**Point of Inflexion**</ins> is a point on the curve at which there is a change in concavity. We have a point of inflexion at $x=a$ if $f''(a)=0$ **and** $f''(x)$ changes sign on either side of $x=a$. [**NB :** $f''(a)=0$ is not a sufficient condition (Example: y = x^4.)]	

Stationary Inflexion	 No change in sign of $f'(x)$ is observed. $f'(x)=0$ and $f''(x)=0$.
Non-stationary Inflexion	The gradient of the tangent is not zero at this point i.e $f'(x)\neq 0$ but f ′′ (x) = 0 

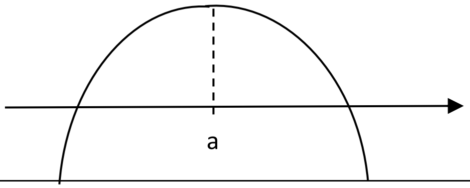
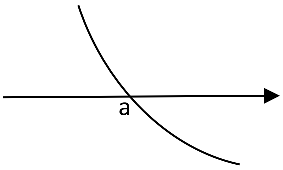
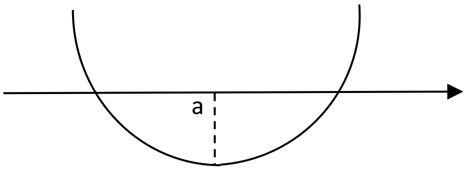
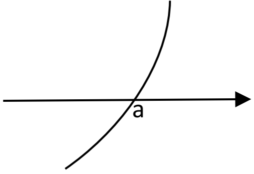
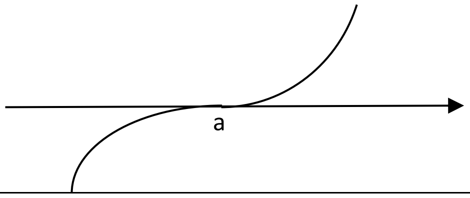
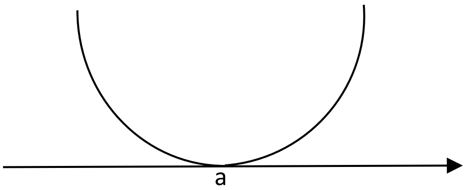
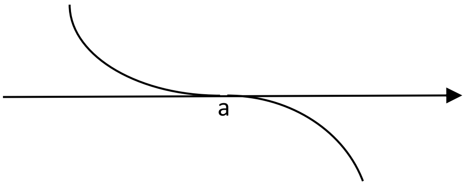
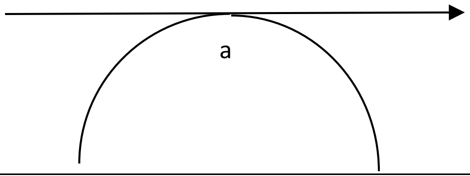
TOK : When mathematicians and historians say that they have explained something, are they using the word “explain” in the same way?

Example 2 (non-GDC): For (a) $f(x)=x^3-3x+2$ (b) $f(x)=3x^4-16x^3+24x^2-9$,

(i) find and classify all points all stationary points;
(ii) find and classify all points of inflexion;
(iii) find the intervals where the function is increasing/decreasing;
(iv) find the intervals where the function is concave up/down.
(v) Hence, sketch the graph showing **all** important features.

Sketching the graph of $y = f'(x)$

Often, you may be required to sketch the graph of $f'(x)$ **from the graph of $f(x)$ alone.** The important thing to note is: the **stationary points** on the graph of $f(x)$ correspond to the ***x*-intercepts** on the graph of $f'(x)$.

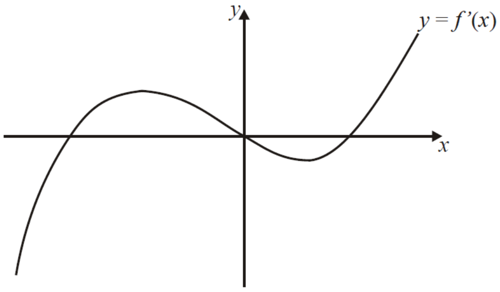
	Graph of $f(x)$	**Graph of $f'(x)$**
	(x, y)	$\left(x, \frac{dy}{dx}\right)$
Stationary points $f'(x) = 0$	**Maximum** point at *x* = a 	*x*-intercept $(\mathrm{a}, 0)$ from **positive** to **negative** y-coordinates 
	Minimum point at *x* = a 	*x*-intercept $(\mathrm{a}, 0)$ from **negative** to **positive** y-coordinates 
	Rising stationary point of inflexion 	*x*-intercept $(\mathrm{a}, 0)$, **Minimum** point 
	Falling stationary point of inflexion 	*x*-intercept $(\mathrm{a}, 0)$, **Maximum** point 
Asymptotes	Vertical Asymptote	Vertical Asymptote, unchanged
	Horizontal Asymptote (*y* = a)	Horizontal Asymptote (*y* = 0)
Points of Inflexion	Non-stationary points of inflexion	Inconclusive, can be maximum, minimum or vertical asymptote.
	Stationary points of inflexion	The maximum/minimum point lies on the *x*-axis.

Example 3 : Sketch the graph of $f\ '\left(x\right)$ for the following functions:

(a) $f(x)=-\left(x+3\right)\left(x-3\right)$ **(b)** $f(x)=\left(x-1\right)^{2}\left(x+1\right)$ **(c)** $f(x)=\left(x-2\right)^{3}\left(x+1\right)^{2}$

(d) $f(x)=\cos x$ **(e)** $f(x)=\tan x$ **(f)** $f(x)=2-\dfrac{1}{x-1}$

Example 4: The diagram shows the graph of $y=f\,'\left(x\right)$.



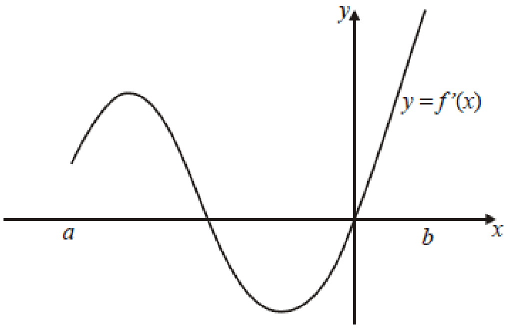
Indicate, and label clearly, **on the graph**

(a) the points where $y=f\left(x\right)$ has minimum points;

(b) the points where $y=f\left(x\right)$ has maximum points;

(c) the points where $y=f\left(x\right)$ has points of inflexion;

Example 5: The diagram shows a sketch of the graph of $y=f\,'\left(x\right)$ for $a\le x\le b$.



On the grid below, which has the same scale on the *x*-axis, draw a sketch of the graph of $y=f\left(x\right)$ for $a\le x\le b$, given that $f\left(0\right)=0$ and $f\left(x\right)\ge 0$ for all *x*. Indicate clearly on your graph the minimum points, maximum points and points of inflexion (if any).

Anglo-Chinese School (Independent)

Year 6 IBDP Mathematics: Analysis and Approaches HL

Topic 5: Calculus

WS 5.2: Applications of Differentiation

Tangents and Normals

1. Prove that the gradient of the curve $2xy = 3x^2 - 2$ is always greater than $\frac{3}{2}$. Find the equation of the normal to the curve at the point where $x = \frac{1}{2}$.

2. Find the equation of the normal to the curve $x^2 - 2xy + 2y^2 = 25$ at the point $(7,3)$. Find two points on the curve such that the normal to the curve at these two points are parallel to the y-axis.

3*. Given that $y^2 - 5xy + 8x^2 = 2$, prove that $\frac{dy}{dx} = \frac{5y-16x}{2y-5x}$. The distinct points P and Q on the curve each have x-coordinate 1. Find the point where the normal to the curve at P and Q meet.

4. Find the equations of the tangents to the curve $y = x^2 + 2x + 4$ which pass through the origin.

5*. The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point $(1,7)$. Find the values of a and b.

6. The normal to the curve $y = \frac{k}{x} + \ln x^2$ for $x \neq 0, k \in \mathbb{R}$, at the point where $x = 2$, has equation $3x + 2y = b$, where $b \in \mathbb{R}$. Find the exact value of k.

Maxima and minima

7. The sum of twice one number and five times a second number is 70. What number should be selected so that the product of the numbers is as large as possible?

8. A piece of wire 20 cm long is to be cut into two pieces, one forming a circle and the other a square. For the sum of the two areas enclosed by the wires to be minimal, prove that the radius of the circle is $\frac{10}{4+\pi}$.

9. Consider an isosceles triangle with sides 5, 5 and 6. Find the dimensions of the rectangle of the largest area that can be inscribed in the triangle such that one side is along the base of length 6.

10. Without using a GDC, find the *x*-coordinates of the points at which the graph of $y = \dfrac{x^2}{2x-1}$ has either a local maximum or a local minimum and distinguish between them.

11. Water exits an inverted conical tank (radius 3m and height 8m) at a constant rate of $0.2\,\text{m}^3\,/\,\text{minute}$. If the surface of the water has radius r,

(a) Find $V(r)$, the volume of the water remaining

(b) Find the exact rate at which the surface radius is changing at the instant when the height of water is 5 m.

12. Consider the curve with equation $f(x) = e^{-2x^2}$ for $x < 0$. Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

13. If $V(x) = \pi x^2\sqrt{100-4x^2}$, where $x \geq 0$, find the exact value of x to give the maximum value of V and state this maximum value of V.

14*. The function f is defined by $f(x) = \dfrac{x^2}{2^x}$, for $x > 0$.

(a) **(i)** Show that $f'(x) = \dfrac{2x - x^2\ln 2}{2^x}$.

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(b) **(i)** Find the exact value of x satisfying the equation $f'(x) = 0$

(ii) Show that this value gives a maximum value for $f(x)$.

(c) Find the *x*-coordinates of the two points of inflexion on the graph of f.

Answers :

1. $44y + 8x + 51 = 0$

2. $4y + x = 19;\ (5,5), (-5,-5)$

3. $\left(\dfrac{1}{7}, \dfrac{15}{7}\right)$

4. $y = 6x, y = -2x$

5. $a = -4$ and $b = 18$

6. $\dfrac{4}{3}$

7. 17.5, 7

9. 3, 2

10. x=0(max), x=1(min)

11(a) $V(r) = \dfrac{8}{9}\pi r^3 m^3$ **(b)** $\dfrac{dr}{dt} = -\dfrac{8}{375\pi}$ *m*/minute.

12. $\left(-0.5, \dfrac{1}{\sqrt{e}}\right)$

13. $x = \sqrt{\dfrac{50}{3}} = 5\sqrt{\dfrac{2}{3}},\quad v_{\max} = \dfrac{500\pi}{3\sqrt{3}}$

14. (a)(ii) $\dfrac{x^2(\ln 2)^2 - 4x\ln 2 + 2}{2^x}$ **(b) (i)** $x = \dfrac{2}{\ln 2}$, (c) 0.845, 4.93

Topic 5 : Calculus

5.3 Integration

1. Indefinite Integrals

From differentiation, we know that $\frac{d}{dx}x^3 = 3x^2, \frac{d}{dx}\left(x^3+7\right)=3x^2$.
The reverse process of differentiation is called **integration** or **anti-differentiation**.
Note from above, the integral of $3x^2$ can be either $\left(x^3+7\right)$ or x^3, or more.

If the functions $F(x)$ and $f(x)$ are related by $\frac{d}{dx}F(x)=f(x)$, we say that $f(x)$ is the ***derivative*** of $F(x)$.

Notation : The indefinite integral of $f(x)$ with respect to *x*, $\int f(x)\ dx = F(x)+C$, where $f(x)$ is the integrand and C is the constant of integration.

Properties of indefinite integrals

(A) $\int kf(x)\ dx = k\int f(x)\ dx$ where *k* is a constant.

(B) If *f* and *g* are defined on the same interval, then
$\int\left[af(x)\pm bg(x)\right]\ dx = a\int f(x)\ dx \pm b\int g(x)\ dx,$ where *a* and *b* are constants.

2. Standard Integrals

Algebraic Functions

Differentiation	**Integration**
$\frac{d}{dx}\left(x^{n+1}\right)=(n+1)x^n$	$\int x^n\ dx = \frac{x^{n+1}}{n+1}+C,$ where $n\neq -1.$
$\frac{d}{dx}(Ax+B)^{n+1}=A(n+1)(Ax+B)^n$	$\int (Ax+B)^n\ dx = \frac{(Ax+B)^{n+1}}{A(n+1)}+C,$ where $n\neq -1.$
$\frac{d}{dx}\left[f(x)\right]^{n+1}=(n+1)\left[f(x)\right]^n\left[f'(x)\right]$	$\int\left[f(x)\right]^n\left[f'(x)\right]\ dx = \frac{\left[f(x)\right]^{n+1}}{n+1}+C$, where $n\neq -1.$

Example 1: Find **(a)** $\int 4+\sqrt{1-2x}-x^5-\frac{1}{2x^2}\ dx$ **(b)** $\int x(x-3)^3\ dx$

[(a) $4x-\frac{(1-2x)^{\frac{3}{2}}}{3}-\frac{x^6}{6}+\frac{1}{2x}+C$ (b) $\frac{x^5}{5}-\frac{9x^4}{4}+9x^3-\frac{27x^2}{2}+C$]

Trigonometric Functions

Differentiation	Integration
$\frac{d}{dx}\sin(Ax+B)=A\cos(Ax+B)$	$\int\cos(Ax+B)\ dx=\frac{1}{A}\sin(Ax+B)+C$
$\frac{d}{dx}\cos(Ax+B)=-A\sin(Ax+B)$	$\int\sin(Ax+B)\ dx=-\frac{1}{A}\cos(Ax+B)+C$
$\frac{d}{dx}\tan(Ax+B)=A\sec^2(Ax+B)$	$\int\sec^2(Ax+B)\ dx=\frac{1}{A}\tan(Ax+B)+C$
$\frac{d}{dx}\sec(Ax+B)=A\sec(Ax+B)\tan(Ax+B)$	$\int\sec(Ax+B)\tan(Ax+B)\ dx=\frac{1}{A}\sec(Ax+B)+C$
$\frac{d}{dx}\text{cosec}(Ax+B)$ $=-A\text{cosec}(Ax+B)\cot(Ax+B)$	$\int\text{cosec}(Ax+B)\cot(Ax+B)\ dx$ $=-\frac{1}{A}\text{cosec}(Ax+B)+C$
$\frac{d}{dx}\cot(Ax+B)=-A\text{cosec}^2(Ax+B)$	$\int\text{cosec}^2(Ax+B)\ dx=-\frac{1}{A}\cot(Ax+B)+C$

Example 2: Integrate the following with respect to *x*:

(a) $\sin\left(2x+\frac{\pi}{2}\right)$ **(b)** $6\sec^2 3x$ **(c)** $\text{cosec}\left(\frac{\pi}{4}-3x\right)\cot\left(\frac{\pi}{4}-3x\right)$

[(a) $-\frac{1}{2}\cos\left(2x+\frac{\pi}{2}\right)+C$ (b) $2\tan 3x+C$ (c) $\frac{1}{3}\text{cosec}\left(\frac{\pi}{4}-3x\right)+C$]

Solution:

Example 3: Find **(a)** $\int\cos^2 3x\ dx$ **(b)** $\int\tan^2\frac{x}{2}\ dx$

[(a) $\frac{1}{12}\sin 6x+\frac{1}{2}x+C$ (b) $2\tan\frac{x}{2}-x+C$]

Solution:

Special Forms For Integration

Differentiation	Integration
$\frac{d}{dx}\left(\frac{\left[f(x)\right]^{n+1}}{n+1}\right)=\left[f(x)\right]^{n}f'(x)$	$\int\left[f(x)\right]^{n}f'(x)\ dx=\frac{\left[f(x)\right]^{n+1}}{n+1}+C \qquad (n\neq -1)$
$\frac{d}{dx}\left(\ln f(x)\right)=\frac{f'(x)}{f(x)}$	$\int\frac{f'(x)}{f(x)}\ dx=\ln\left\|f(x)\right\|+C$
$\frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}f'(x)$	$\int e^{f(x)}f'(x)\ dx=e^{f(x)}+C$

Example 4: Find **(a)** $\int xe^{x^2}\ dx$ **(b)** $\int \sin x e^{\cos x}\ dx$ **(c)** $\int\frac{3x}{x^2+1}\ dx$ **(d)** $\int\tan x\ dx$

[(a) $\frac{1}{2}e^{x^2}+C$ (b) $-e^{\cos x}+C$ (c) $\frac{3}{2}\ln\left|x^2+1\right|+C$ (d) $-\ln\left|\cos x\right|+C$]

Solution:

Integration Using Partial Fractions

Note that we can use partial fractions to decompose complicated rational functions into simpler expressions.

Example 5: Find **(a)** $\int\frac{3}{(x+1)(x-1)}\ dx$ **(b)** $\int\frac{x+3}{x(x+1)}\ dx$

[(a) $\frac{3}{2}\ln\left|\frac{x-1}{x+1}\right|+c$ (b) $\ln\left|\frac{x^3}{(x+1)^2}\right|+c$]

Solution:

Inverse Trigonometric Functions

Differentiation	Integration
$\frac{d}{dx}\arcsin\frac{Ax}{a}=\frac{1}{A}\frac{1}{\sqrt{a^2-(Ax)^2}}$	$\int\frac{1}{\sqrt{a^2-x^2}}\ dx=\arcsin\frac{x}{a}+C$
$\frac{d}{dx}\arctan\frac{x}{a}=\frac{a}{a^2+x^2}$	$\int\frac{1}{a^2+x^2}\ dx=\frac{1}{a}\arctan\frac{x}{a}+C$

Note that the derivative of $\arccos x$ is $-\frac{1}{\sqrt{1-x^2}}$ which is very similar to the derivative of $\arcsin x$.

Example 6: Find **(a)** $\int\frac{1}{4+9x^2}\ dx$ **(b)** $\int\frac{7}{\sqrt{1-4x^2}}\ dx$ **(c)** $\int\frac{2}{x^2+2x+5}\ dx$

[(a) $\frac{1}{6}\arctan\frac{3x}{2}+C$ (b) $\frac{7}{2}\arcsin 2x+C$ (c) $\arctan\frac{x+1}{2}+C$]

Solution:

3. Integration by Substitution

This method is used in cases of *non-standard integrals*. It introduces a suitable substitution to simplify a 'difficult term' in the integrand. Here are some suggestions of possible substitutions to integrate the non-standard integrals :

When a function contains	**Try substituting**
$\sqrt{f(x)}$	$u = f(x)$
$\ln x$	$u = \ln x$
e^x	$u = e^x$
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $x = a\cos\theta$
$a^2 + x^2$ or $\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Example 7: By using a suitable substitution, integrate the following functions with respect to *x*:

(a) $x\sqrt{x+2}$ **(b)** $\dfrac{\sqrt{x^2-9}}{x}$ **(c)** $\dfrac{1}{\sqrt{9-4x^2}}$ **(d)** $\dfrac{(\ln x)^3+1}{x}$ **(e)** $\sqrt{4-x^2}$

[(a) $\dfrac{2(x+2)^{\frac{5}{2}}}{5} - \dfrac{4(x+2)^{\frac{3}{2}}}{3} + C$ (b) $\sqrt{x^2-9} - 3\text{arccos}\dfrac{3}{x} + C$

(c) $\dfrac{1}{2}\text{arcsin}\left(\dfrac{2x}{3}\right) + C$ (d) $\dfrac{(\ln x)^4}{4} + \ln x + C$ (e) $\dfrac{x\sqrt{4-x^2}}{2} + 2\text{arcsin}\dfrac{x}{2} + C$]

Solution:

Note : don't forget to do re-substitution!

4. Integration by Parts

Let u and v be two functions of x.

Then the derivative of uv with respect to x gives $\frac{d}{dx}uv = u\frac{dv}{dx} + v\frac{du}{dx}$.

Now integrating both sides with respect to x gives $uv = \int u\frac{dv}{dx}\ dx + \int v\frac{du}{dx}\ dx$

Rearranging the terms, we have $\boxed{\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx}$

Integrating using this formula is called **integration by parts**.

A way to choose "u" is by

Decreasing priority to be chosen as u →

L	**I**	**A**	**T**	**E**
Logarithmic e.g. ln, lg	Inverse trig e.g. $\arcsin x, \arctan x$	Algebraic e.g. $x^2, \sqrt{x}$	Trigonometric e.g. $\sin x, \tan x$	Exponential e.g. e^x

Notes

1. **Integrating a Single Function**: If a single function $f(x)$ cannot be integrated directly, take $u = f(x)$ and $\frac{dv}{dx} = 1$.
2. LIATE rule provides one of the ways to solve the integration, but it may not be the only way
3. LIATE rule works for most of the integrations in the IB exam, but there are some counterexamples. For example, $\int x^3 \sin x^2\ dx \quad (u = x^2)$ and $\int \sin^n x\ dx \quad (u = \sin^{n-1} x)$.[1]

Example 8 (Integrating a single function and reduction of powers): Find

(a) $\int \ln x\ dx$ **(b)** $\int \arcsin x\ dx$ **(c)** $\int x^2 \sin x\ dx$

[(a) $x\ln x - x + C$ (b) $x\arcsin x + \sqrt{1-x^2} + C$ (c) $2x\sin x + 2\cos x - x^2\cos x + C$]

Solution:

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[1] Taken from Oxford textbook for IB Maths HL (2012)

Example 9 (Integrating Cyclical Functions): Find (a) $\int e^x \cos x \ dx$ (b) $\int e^x \sin 2x \ dx$

[(a) $\frac{1}{2}e^x\left(\sin x+\cos x\right)+C$ (b) $\frac{e^x}{5}\left(\sin 2x-2\cos 2x\right)+C$]

Solution:

5. Definite Integrals

Definition : The definite integral from *a* to *b* of $f\left(x\right)$ is defined as :

$$\boxed{\int_a^b f\left(x\right)\,dx=\left[F\left(x\right)\right]_a^b=F\left(b\right)-F\left(a\right)}$$

where $F\left(x\right)=\int f\left(x\right)\ dx$ and *a* and *b* are the limits of the integration.

Notes:

- In **definite** integration, the constant of integration is ignored.
- The definite integral can be found this way only if the function $f\left(x\right)$ to be integrated is **defined for every value of *x* from *a* to *b*.** i.e. the function $f\left(x\right)$ must be continuous over the interval from $x=a$ to $x=b$.
  e.g. $\int_{-1}^{1}\frac{1}{x}\,dx$ does not make sense.
- The GDC can be very helpful to find definite integrals, especially if it is not easy to find the anti-derivative by hand. Ensure that your GDC is in radian if the function involved trigonometric functions.

Example 10: Find **(a)** $\int_1^4 \frac{1}{\left(x+3\right)^3}\,dx$ **(b)** $\int_0^{\frac{\pi}{4}}\cos^2 3x\,dx$ [(a) 0.0210, (b) 0.309]

Solution:

5.1 Definite integrals with Substitution

We need to change the limits when we do a substitution.

Example 11: Using the substitution $x = \frac{1}{u}$, find $\int_{\frac{1}{2}}^{1} \frac{dx}{x\sqrt{4x^2-1}}$. Leave your answer in exact form. [$\frac{\pi}{3}$]

Solution:

Example 12: Find $\int_{0}^{1} \frac{dx}{\left(1+x^2\right)^2}$. Leave your answer in exact form. [$\frac{1}{4}+\frac{\pi}{8}$]

Solution:

5.2 Definite integrals with Integration by Parts

Apply the limits throughout the integral: $\int_a^b u\frac{dv}{dx}\,dx = \left[uv\right]_a^b - \int_a^b v\frac{du}{dx}\,dx$

Example 13: Find **(a)** $\int_0^1 xe^x\,dx$ **(b)** $\int_0^{\frac{\pi}{2}} x\cos x\,dx$ [(a) 1 (b) $\frac{\pi}{2}-1$]

Solution:

5.3 Simple Applications of Definite Integration

Example 14: Find the curve which passes through the point $(0,-1)$ and whose gradient at (x,y) is $\cos^2 x$. [$y=\frac{1}{2}(x+\sin x\cos x)-1$]

Solution:

Example 15 The gradient of a curve is $\tan 2x$ and the curve passes through the point $\left(\frac{\pi}{6},0\right)$. Find the equation of the curve. [$y=-\frac{1}{2}\left(\ln\left|\cos 2x\right|+\ln 2\right)$]

Solution:

Example 16 : The rate of change of the area, A cm^2, of a circle is $2e^{-t}\cos t$. Find A in terms of t if the area of the circle is 11 cm^2 when $t=0$. [$A=e^{-t}(\sin t-\cos t)+12$]

Solution:

Topic 5: Calculus

WS 5.3 (a): Basic Integration Techniques 1

Find :

1. $\int 6\sqrt{x}\ dx$

2. $\int \frac{5}{\sqrt{x}}\ dx$

3. $\int \sqrt{x}(1-x)\ dx$

***4.** $\int 2x(x^2+8)^6\ dx$

***5.** $\int \frac{x}{\sqrt{x^2+1}}\ dx$

6. $\int \sqrt{e^x}\ dx$

7. $\int 4e^{5-3x}\ dx$

8. $\int e^{-x}(3-4e^{2x})^2\ dx$

9. $\int 2x^{\frac{7}{5}} - 3x^{-\frac{11}{9}} + 16x^{15}\ dx$

10. $\int \frac{(\sqrt{x}+1)^3}{\sqrt{x}}\ dx$

11. $\int \frac{x^3+5x^2-2}{x^2}\ dx$

12. $\int (2x-1)^{10}\ dx$

13. $\int \frac{4}{1+3x}\ dx$

14. $\int -\frac{1}{1-2x}\ dx$

15. $\int \frac{5}{6-7x}\ dx$

16. $\int \frac{1}{(11-3x)^3}\ dx$

17. $\int \frac{1}{x} + \frac{4}{1-x}\ dx$

18. $\int \sqrt{x+1}\ dx$

***19.** $\int \frac{x^3}{x-1}\ dx$

***20.** $\int \frac{x^2-2x-1}{x+2}\ dx$

Answers:

1. $4\sqrt{x^3}+c$

2. $10\sqrt{x}+c$

3. $\frac{2}{3}\sqrt{x^3} - \frac{2}{5}\sqrt{x^5} + c$

4. $\frac{(x^2+8)^7}{7}+c$

5. $\sqrt{x^2+1}+c$

6. $2\sqrt{e^x}+c$

7. $-\frac{4}{3}e^{5-3x}+c$

8. $-9e^{-x} - 24e^x + \frac{16e^{3x}}{3} + c$

9. $\frac{5}{6}x^{\frac{12}{5}} + \frac{27}{2x^{\frac{2}{9}}} + x^{16} + c$

10. $\frac{x^2}{2} + 2\sqrt{x^3} + 3x + 2\sqrt{x} + c$
$\left(\text{or } \frac{1}{2}(\sqrt{x}+1)^4 + c\right)$

11. $\frac{1}{2}x^2 + 5x + \frac{2}{x} + c$

12. $\frac{1}{22}(2x-1)^{11}+c$

13. $\frac{4}{3}\ln|1+3x|+c$

14. $\frac{1}{2}\ln|1-2x|+c$

15. $-\frac{5}{7}\ln|6-7x|+c$

16. $\frac{1}{6(11-3x)^2}+c$

17. $\ln|x| - 4\ln|1-x| + c$

18. $\frac{2}{3}\sqrt{(x+1)^3}+c$

19. $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x-1| + c$

20. $\frac{x^2}{2} - 4x + 7\ln|x+2| + c$

Topic 5: Calculus

WS 5.3 (b): Basic Integration Techniques 2

Find:

1. $\int \cos x\sqrt{1-2\sin x}\ dx$

2. $\int \sec^3 x\tan x\ dx$

3. $\int \tan^2 x\ dx$

4. $\int \cos^2 2x\ dx$

5. $\int \sin\frac{x}{2}\cos\frac{x}{2}\ dx$

6. $\int \frac{3}{1-\sin^2 x}\ dx$

7. $\int \sec x\left(\sec x-\tan x\right)\ dx$

8. $\int \tan^2\left(2x-3\right)\ dx$

9. $\int \frac{1}{\cos^2 3x}\ dx$

10. $\int 5\sec\left(\frac{\pi}{4}-x\right)\tan\left(\frac{\pi}{4}-x\right)\ dx$

11. $\int 2\operatorname{cosec}3x\cot 3x\ dx$

12. $\int 5\cos\left(5-\frac{x}{2}\right)\ dx$

13. $\int \cos 3x+3\sin 4x\ dx$

14. $\int 2\sin^2\frac{\pi}{2}x\ dx$

15. $\int \frac{5\sin^3 x}{\left(1+\cos x\right)\left(1-\cos x\right)}\ dx$

16. $\int \frac{2-7x\cos x}{x}\ dx$

17. $\int \frac{1}{1+9x^2}\ dx$

18. $\int \frac{3}{\sqrt{5-4x^2}}\ dx$

19. $\int \frac{3}{\sqrt{16-\left(3t-1\right)^2}}\ dt$

20.* $\int \frac{1}{25x^2-10x+10}\ dx$

21.* $\int \frac{x^3}{\left(x+1\right)\left(x-3\right)}\ dx$

Answers:

1. $-\frac{1}{3}\left(1-2\sin x\right)^{\frac{3}{2}}+c$

2. $\frac{\sec^3 x}{3}+c$

3. $\tan x-x+c$

4. $\frac{1}{8}\sin 4x+\frac{x}{2}+c$

5. $-\frac{1}{2}\cos x+c$

6. $3\tan x+c$

7. $\tan x-\sec x+c$

8. $\frac{1}{2}\tan\left(2x-3\right)-x+c$

9. $\frac{1}{3}\tan 3x+c$

10. $-5\sec\left(\frac{\pi}{4}-x\right)+c$

11. $-\frac{2}{3}\operatorname{cosec}3x+c$

12. $-10\sin\left(5-\frac{x}{2}\right)+c$

13. $\frac{1}{3}\sin 3x-\frac{3}{4}\cos 4x+c$

14. $x-\frac{1}{\pi}\sin\pi x+c$

15. $-5\cos x+c$

16. $2\ln\left|x\right|-7\sin x+c$

17. $\frac{1}{3}\arctan 3x+c$

18. $\frac{3}{2}\arcsin\left(\frac{2x}{\sqrt{5}}\right)+c$

19. $\arcsin\left(\frac{3t-1}{4}\right)+c$

20. $\frac{1}{15}\arctan\left(\frac{5x-1}{3}\right)+c$

21. $\frac{x^2}{2}+2x+\frac{1}{4}\ln\left|x+1\right|+\frac{27}{4}\ln\left|x-3\right|+c$

Topic 5: Calculus

WS 5.3 (c): Further Integration Techniques (by substitution, by parts) & Definite Integration

1. Find the following integrals by making the substitution suggested :

(a) $\int x\left(x^2-3\right)^4\ dx,\qquad u=x^2-3$ **(b)** $\int \frac{1}{x^2\sqrt{25-x^2}}\ dx,\qquad x=5\sin\theta$

(c) $\int \frac{1}{\sin 2\theta}\ d\theta,\qquad t=\tan\theta$ **(d)** $\int \frac{1}{x\sqrt{x^2-4}}\ dx,\qquad x=\frac{1}{t}$

2. By using a suitable substitution, show that the exact value of $\int_0^{\frac{\sqrt{3}}{2}} \sqrt{4-4y^2}\ dy$ is $\frac{\sqrt{3}}{4}+\frac{\pi}{3}$.

3. By using the substitution $x=\sec^2\theta,$ or otherwise, show that $\int_1^2 \frac{1}{x^2\sqrt{x-1}}\ dx=\frac{1}{2}+\frac{\pi}{4}$.

4. **(a)** Find the values of A and B such that $\frac{2}{u^2-1}=\frac{A}{u-1}+\frac{B}{u+1}$.

(b)* Hence, by using a suitable substitution, show that $\int_2^7 \frac{1}{(x+1)\sqrt{x+2}}\ dx=\ln\frac{3}{2}$.

5. Integrate the following functions with respect to *x*.

(a) x^2e^x **(b)** $x^2\cos x$ **(c)** $x^n\ln x$ **(d)** $(\ln 2x)^2$

(e) $\arccos x$ **(f)** $(\arcsin x)^2$ **(g)** $e^{3x}\cos 4x$ **(h)** $2e^x\sin x\cos x$

6. Find the curve which passes through the point $(2,5)$ and whose gradient at (x,y) is x^3.

7. Find the curve which passes through the point $(0,1)$ and whose gradient at (x,y) is $\sin x+\cos x$.

8.* If $\int_a^b x^3\ dx=0$ and $\int_a^b x^2\ dx=\frac{2}{3}$, find the values of *a* and of *b*.

9. Without using a calculator, simplify $\int_0^2 \frac{1}{4x^2-1}\ dx$

10. Given that $\frac{x^2-1}{x^2+1}\equiv A+\frac{B}{x^2+1}$, find *A* and *B*. Hence, or otherwise, solve $\int_0^1 \frac{x^2-1}{x^2+1}\ dx$.

11.* Given that $\frac{1}{x^2-a^2}\equiv\frac{A}{x-a}+\frac{B}{x+a}, a\neq 0,$ find *A* and *B* in terms of *a*. Hence, prove that

$$\int \frac{1}{x^2-a^2}\ dx=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+c\,.$$

Answers:

<table>
<tr><td colspan="4">1. (a) $\frac{1}{10}\left(x^2-3\right)^5+C$ (b) $\frac{-\sqrt{25-x^2}}{25x}+C$ (c) $\frac{1}{2}\ln\left|\tan\theta\right|+C$ (d) $-\frac{1}{2}\arcsin\left(\frac{2}{x}\right)+C$</td></tr>
<tr><td>4. (a) $A=1, B=-1$</td><td colspan="3">5. (a) $e^x\left(x^2-2x+2\right)+C$ (b) $\left(x^2-2\right)\sin x+2x\cos x+C$</td></tr>
<tr><td colspan="4">5. (c) $\frac{x^{n+1}}{\left(n+1\right)^2}\left[\left(n+1\right)\ln x-1\right]+C$ (d) $x\left(\ln 2x\right)^2-2x\ln 2x+2x+C$ (e) $x\arccos x-\sqrt{1-x^2}+C$
(f) $x\left(\arcsin x\right)^2+2\arcsin x\sqrt{1-x^2}-2x+C$ (g) $\frac{e^{3x}}{25}\left(4\sin 4x+3\cos 4x\right)+C$</td></tr>
<tr><td>(h) $\frac{1}{5}e^x\left(\sin 2x-2\cos 2x\right)+C$</td><td>6. $y=\frac{x^4}{4}+1$</td><td>7. $y=-\cos x+\sin x+2$</td><td>8. $a=-1, b=1$</td></tr>
<tr><td>9. $\frac{1}{4}\ln\frac{3}{5}$</td><td>10. $A=1, B=-2; 1-\frac{\pi}{2}$</td><td>11. $A=\frac{1}{2a}, B=-\frac{1}{2a}$</td><td></td></tr>
</table>

Topic 5 : Calculus

5.4 Area and Volume of Revolution

1. Areas

1.1 Areas about the *x*-axis

Fig 1

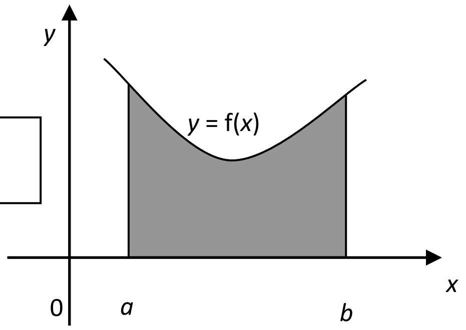
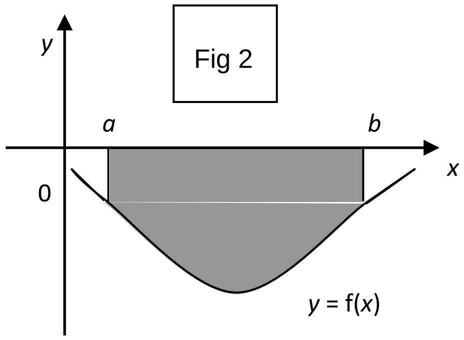


Fig 2



Referring to **Fig 1**:

The area bounded by the curve $y = f(x)$, the *x*-axis and the lines $x = a$ and $x = b$ is given by

$$\int_a^b f(x)\,dx \quad \text{or} \quad \int_a^b y\,dx \qquad \text{where } y = f(x).$$

If the curve is below the *x*-axis as in **Fig 2**, then $\int_a^b y\,dx$ is **<u>negative</u>**.

If the curve $y = f(x)$ cuts the *x* axis at $x = b$ as shown in **Fig 3**,

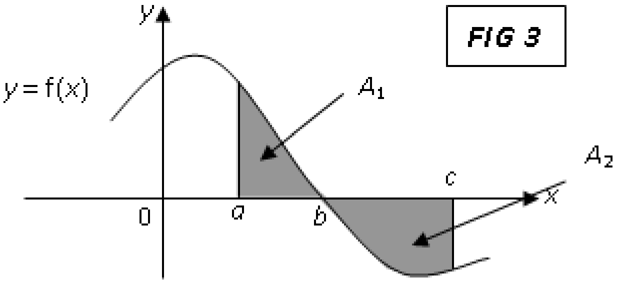


FIG 3

The area under the curve between $x = a$ and $x = c$ is

$$\int_a^b y\,dx + \left|\int_b^c y\,dx\right| \qquad \left[\ \int_b^c y\,dx \text{ is negative}\right]$$

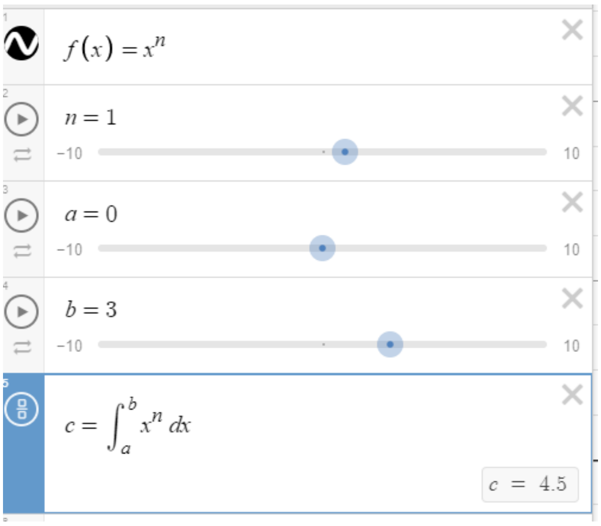
Alternatively, the total area between $x = a$ and $x = c$ is $\int_a^c |f(x)|\,dx$.

This is especially useful if you are working out the area using a Graphic Display Calculator where working out the above integral is particularly easy.

Toolkit : Areas under Graph

1. Open the area under graph applet at https://www.desmos.com/calculator/x4f54hxohc

2. On the sliders in the left hand pane, set $n = 1, a = 0, b = 3$.



$f(x) = x^n$

$n = 1$

$a = 0$

$b = 3$

$c = \int_a^b x^n \, dx$

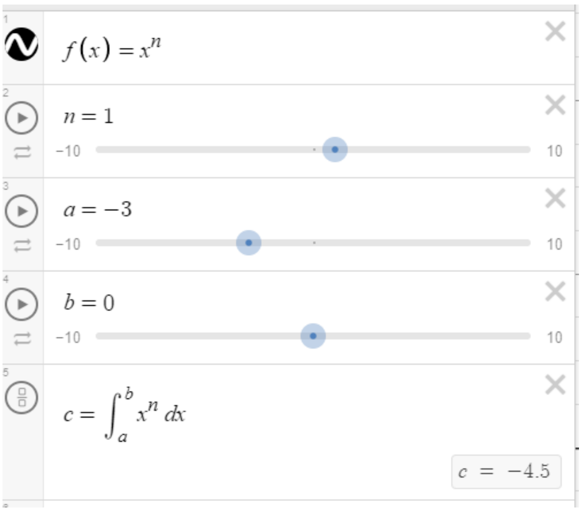
$c = 4.5$

Sketch the graph obtained and calculate the area of the shaded region by geometry.

Write down the value of c which is defined as $\int_a^b x^n \, dx$ What can you say about the quantities?

Test your conjecture by adjusting the slider to vary the values of a and b. For this step, keep your values of a and b bigger than zero.

3. On the sliders, set $n = 1$, $a = -3, b = 0$.



$f(x) = x^n$

$n = 1$

$a = -3$

$b = 0$

$c = \int_a^b x^n \, dx$

$c = -4.5$

Sketch the graph obtained in the space below and calculate the area of the shaded region by geometry.

Write down the value of c which is defined as $\int_a^b x^n \, dx$ What can you say about the quantities?

Test your conjecture by adjusting the slider to vary the values of a and b. For this step, keep your values of a and b smaller than zero.

4. Keep the value of $n = 1$, but adjust the values of n to be any value.
What do you observe about the value of c vs the area measured?

5. Repeat steps 1 to 4 but with three other different values of n.
Write down your observations and learning points and share it with a classmate.

6. **Reflection & Debrief**
What is the difference between the area under graph and $\int_a^b f(x) \, dx$?
Why is it important to sketch the graph of $y = f(x)$ before finding the area under a graph?

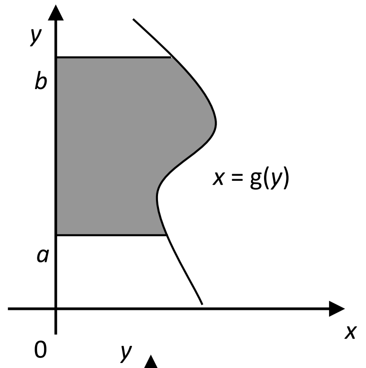
Example 1: Find the area enclosed by the curve $y = x(x-1)(x-2)$ and the x-axis. [0.5]

Solution:

Example 2: Find the area bounded by the *x*-axis, the *y*-axis, the curve $y = e^{2x}$ and the line $x = 2$.

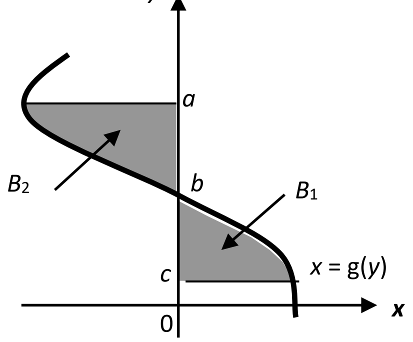
Solution: $[\frac{e^4 - 1}{2}]$

1.2 Areas about the *y*-axis



The area bounded by the curve *x* = g(*y*), the *y*-axis and the lines *y* = *a* and *y* = *b* is given by:

$$\int_a^b g(y)\,\mathrm{d}y \quad \text{or} \quad \int_a^b x\,\mathrm{d}y$$



Similarly, area B$_1$ has a positive value while area

B$_2$ has a negative value.

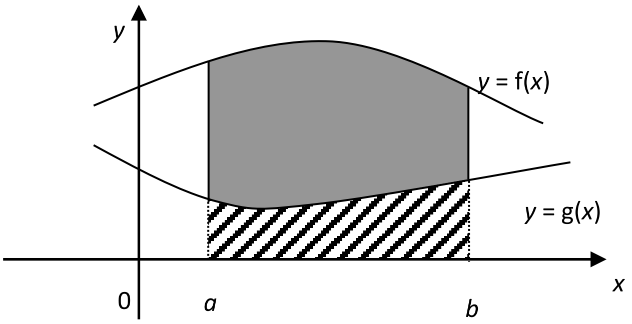
Hence, area = $\displaystyle\int_c^b x\,\mathrm{d}y + \left|\int_b^a x\,\mathrm{d}y\right|$.

Similarly, if you are using a GDC, the area above is simply $\int_c^a |x|\,dy$

Example 3: Sketch the graph of the curve $y^2 = 4(x-2)$. Find the area enclosed by the curve $y^2 = 4(x-2)$, the *y*-axis and the lines $y = \pm 1$. $[\frac{25}{6}]$

Solution:

1.3 Area between Two Curves



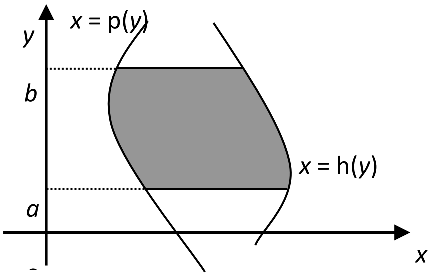
Area bounded between 2 curves y = f(x) and y = g(x) from $x = a$ to $x = b$ is given by,

$$A=\int_a^b\left[f(x)-g(x)\right]dx$$

Note that we always subtract the equation of the bottom curve from the equation of the top curve.

Similarly, the area bounded between 2 curves x = p(y) and x = h(y) is given by:

$$\int_a^b [h(y)-p(y)]\,\mathrm{d}y$$



Example 4: Find the area bounded by the curve $y=2x-x^2$ and the straight line $y=-x$. [4.5]

Solution:

<u>TOK</u>: Consider $f(x)=\frac{1}{x}, 1\leq x<\infty$. An infinite area sweeps out a finite volume. Can this be reconciled with our intuition? Do emotion and intuition have a role in mathematics?

2. Volumes

2.1 Rotation about *x*-Axis

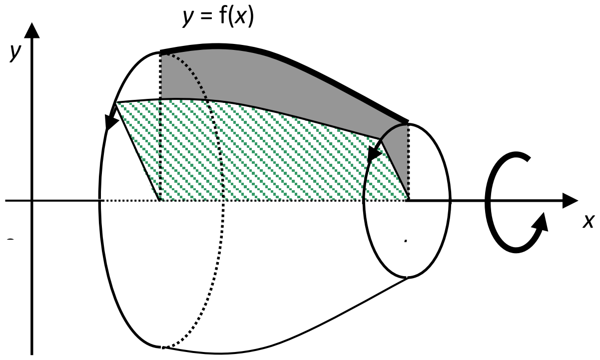


FIG 4

If the shaded area under the curve $y = \mathrm{f}(x)$ between $x = a$ and $x = b$ is rotated through 360° or 2π about the **x-axis** , a solid is generated as shown in **Fig 4**.

Volume generated about *x*-axis:

$$V = \int_a^b \pi\, y^2 \,\mathrm{d}x \quad \text{or} \quad \int_a^b \pi[\mathrm{f}\,(x)]^2 \,\mathrm{d}x$$

Example 5: The region *R* is bounded by the part of the curve $y = (x-2)^{\frac{3}{2}}$ for which $2 \le x \le 4$, the *x*-axis, and the lines $x = 2$ and $x = 4$. Find, in terms of π, the volume of the solid obtained when *R* is rotated through four right angles about the *x*-axis. [4π]

Solution:

2.2 Rotation about the *y*-Axis

Similarly, if the shaded area under the curve $x = f(y)$ between $y=a$ and $y=b$ is rotated through 360^o or 2π about the ***y*-axis**,

Volume of solid generated about *y*-axis :

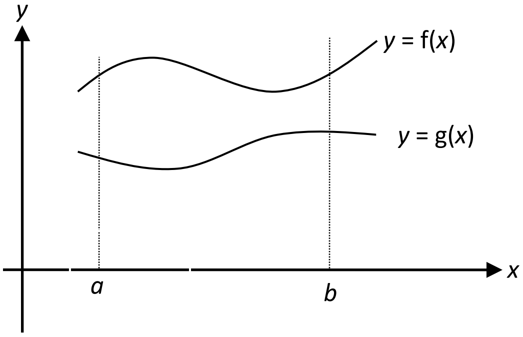
$$V = \int_a^b \pi \, x^2 \mathrm{d}y \quad \text{or} \quad \int_a^b \pi \, [f(y)]^2 \mathrm{d}y$$

Example 6: Calculate the volume of the solid generated when the region bounded by $y = x^2$ and $y = 2 - x^2$ is rotated through π radians about the *y*-axis. [π]

Solution:

2.3 Volume between Two Curves

Let f and g be two functions defined on [*a*, *b*] with $f(x) \ge g(x) \ge 0$ for each *x* in [*a*, *b*] .
Let *R* be the region bounded by $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$.



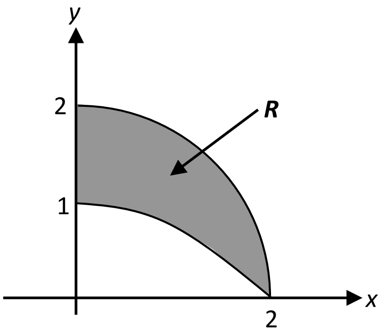
Volume of solid obtained by rotating the region *R* about the *x* -axis through 360°

$$= \int_a^b \pi[f(x)]^2 \mathrm{d}x - \int_a^b \pi[g(x)]^2 \mathrm{d}x = \pi \int_a^b \left([f(x)]^2 - [g(x)]^2 \right) \mathrm{d}x$$

NOTE:

1. Volume obtained is **NOT** $\pi \int_a^b \left(f(x) - g(x) \right)^2 \, dx$
2. Similarly, volume of solid obtained by rotating the region bounded by $x = f(y)$, $x = g(y)$ and the lines $y = c$ and $y = d$ about the $y-$ axis through 360° is $\pi \int_c^d \left([f(y)]^2 - [g(y)]^2 \right) \mathrm{d}y$.

Example 7: The diagram shows the region *R* in the first quadrant enclosed by the curves $y = \frac{1}{4}\left(4 - x^2\right), y = \frac{1}{2}\left(4 - x^2\right)$ and the y-axis. Calculate the volume of revolution formed when *R* is rotated through $360°$ about the *y*-axis. [2π]



Solution:

Anglo-Chinese School (Independent)

Year 6 IBDP Mathematics: Analysis and Approaches HL

Topic 5: Calculus

WS 5.4: Area and Volume of Revolution

1. Find the area bounded by $y = x^3 - 4x$, the *x*-axis, $x = 1$ and $x = 2$.

2. Find the area of the region bounded by $y = \dfrac{3}{x}, y = \sqrt{x},$ the *y*-axis and $y = 3$.

3. Find the exact area of the region enclosed by the curves $y = 4 - x$ and $y = \dfrac{2}{x-1}$.

4. Find the area of the region bounded by

(i) $f(x) = e^x - e^{-x}$ the *x*-axis, the line $x = -1$ and the line $x = 1$.

(ii) $f(x) = \dfrac{1}{x-1} + 1,$ the *x*-axis, the line $x = -1$ and the line $x = 0.5$.

(iii) $y = \cos 2x + 1$, the *x*-axis, the line $x = 0$ and the line $x = \dfrac{\pi}{2}$.

(iv) $y = x(x+1)(x-2)$ and the *x*-axis.

(v) $y = |2x - 1|$, the *x*-axis, the line $x = -1$ and the line $x = 2$.

(vi) $y = |2x| - 1$, the *x*-axis, the line $x = -1$ and the line $x = 2$.

5. The area of the region enclosed by the curve $y^2 = 4ax$ and the line $x = a$ is ka^2 square units, where $a > 0$. Find the value of k.

6. Find the volume of the solid formed by revolving the following curves about the *x*-axis by 2π.

(i) $y = x^{\frac{3}{2}}, \ 1 \le x \le 4$

(ii) $y = \sqrt{25 - x^2}, \ 0 \le x \le 5$

(iii) $y = \cos 2x, \ 0 \le x \le \dfrac{\pi}{4}$

(iv) $y = \sec x, \ 0 \le x \le \dfrac{\pi}{3}$

7.* A hemispherical bowl of radius 8 cm contains water to a depth of 3 cm. Given that the equation of a circle at centre (a, b) and radius *r* is $(x-a)^2 + (y-b)^2 = r^2,$ what is the volume of water in the bowl?

8. Find the volume of the solid formed when

(i) $y = \sqrt{x}$ between $y = 1$ and $y = 4$ is revolved about the *y*-axis by 2π;

(ii) $y = \ln x$ between $y = 0$ and $y = 2$ is revolved about the *y*-axis by 2π.

9. Find the volume of the solid formed by revolving the region enclosed by the curve with equation $f\left(x\right)=\sqrt{25-x^2}$ and the line $g\left(x\right)=3$ about the *x*-axis.

10.* The volume of the solid formed when the region bounded by the curve $y=e^x-k$, the *x*-axis and the line $x=\ln 3$ is rotated about the *x*-axis is $\pi\ln 3$ cubic units. Given that $0<k<3,$ find the value of k.

Answers:

1. 2.25 square units

2. 3.20 square units

3. $\frac{3}{2}-2\ln 2$ square units

4. **(i)** $2\left(e+e^{-1}-2\right)$ square units **(ii)** 0.5 square units

(iii) $\frac{\pi}{2}$ square units **(iv)** $\frac{37}{12}$ square units

(v) 4.5 square units **(vi)** 3 square units

5. $k=\frac{8}{3}$

6. **(i)** $\frac{255\pi}{4}$ **(ii)** $\frac{250\pi}{3}$ **(iii)** $\frac{\pi^2}{8}$ **(iv)** $\sqrt{3}\pi$

7. 63π

8. **(i)** $\frac{1023\pi}{5}$ **(ii)** $\frac{\pi}{2}\left(e^4-1\right)$

9. $\frac{256}{3}\pi$

10. $k=1$

Topic 5 : Calculus

5.5 Kinematics

1. Relationship between Acceleration, Velocity and Displacement

Let a, v, s and t denote acceleration, velocity, displacement and time respectively. Then

(a) Velocity is the rate of change of displacement, $v = \frac{ds}{dt}$

(b) Velocity-time graph

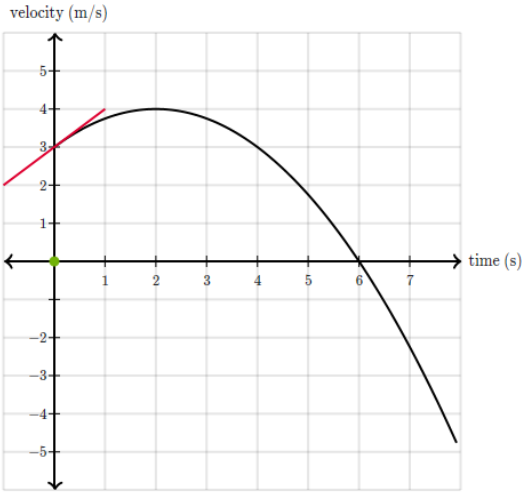


Figure 1 Velocity-Time Graph[1]

(i) The gradient of the velocity-time graph gives the acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$

(ii) The area under the velocity-time graph gives the displacement $s(t) = \int v(t)\ dt$

(iii) For the example above, the total displacement from t = 0 to t = 7 is given by $\int_0^7 v(t)\ dt$

(iv) However, the total distance from t = 0 to t = 7 is given by $\int_0^7 |v(t)|\ dt$

---

[1] Velocity-Time Graph taken from https://www.khanacademy.org/science/physics/one-dimensional-motion/acceleration-tutorial/a/what-are-velocity-vs-time-graphs

(v) At $v(t)=0$, the particle undergoes a change in direction when the graph crosses the time axis, in the example above, the particle changes direction at t = 6s

(vi) At the maximum velocity (stationary point) of the velocity-time graph, the acceleration is zero since $a(t)=\frac{dv}{dt}=0$

In addition, using chain rule of differentiation, we have: $a=v\frac{dv}{ds}$

Proof: $a=\frac{dv}{dt}=\frac{ds}{dt}\frac{dv}{ds}=v\frac{dv}{ds}$

2. Useful Relationship Between Velocity, Speed, and Distance

(a) Speed = $|v|$

(b) Distance = $\int \text{Speed } dt=\int |v|\, dt$

(c) Average speed from T_1 to $T_2=\frac{\text{Total Distance}}{\text{Total Time}}=\frac{\int_{T_1}^{T_2}|v(t)|\, dt}{T_2-T_1}$

Example 1: The velocity of a particle at *t* seconds is given by $v=(2t-1)^3$. Given that the initial displacement is 2 cm, find

(a) its displacement at *t* seconds [$x=\frac{(2t-1)^4}{8}+\frac{15}{8}$]

(b) the total distance travelled in the first 3 seconds. [78.25]

Solution:

Example 2: A particle is initially at the origin and moving to the right at 4cm s^{-1}. It accelerates with time according to the equation $a(t)=3-6t$ cm s^{-2}.

(a) Find the velocity function of the particle and sketch its graph for $0 \le t \le 4$.

(b) Find the total displacement and the total distance travelled in the first 4 seconds of motion.

[(a) $v(t)=3t-3t^2+4$ (b) –24 cm, 36.4 cm]

Solution:

Example 3: A particle moves such its velocity v ms^{-1} is related to its displacement *s* m by the equation $v=\arcsin\left(\sqrt{s}\right)$. Find the acceleration of the particle when $s=0.1\text{m}$. [0.536 ms^{-2}]

Solution:

Example 4 (try this after learning variables separable differential equations):

A particle moves in a line with velocity v ms^{-1}. Its acceleration is given by $a=-\left(v^2+4\right)\text{ms}^{-2}$, for $\left(0 \le t \le 1\right)$. At time $t=0$, the particle has velocity 2ms^{-1}. Show that the velocity *v* at time *t* is given by $v=2\tan\left(\dfrac{\pi-8t}{4}\right)$.

Solution:

Topic 5: Calculus

WS 5.5: Kinematics

1. A particle moves along the *x*-axis so that at time *t* seconds its position is given by $x = t^3 - 6t^2 - 36t$ metres.

(a) Find the maximum speed and maximum velocity in the first seven seconds of motion

(b) Find the average speed in the first seven seconds of motion.

2. A particle *P* has velocity $v = t^2 - t - 2 \text{ cm s}^{-1}$. Find

(a) the time when *P* is at rest,

(b) the distance traveled by the particle in the first 3 seconds of the motion,

(c) the displacement of the particle at the end of 3 seconds.

3. The velocity of a particle traveling in a straight line is given by $v = 50 - 10e^{-0.5t} \text{ ms}^{-1}$, where *t* is the time in seconds and $t \geq 0$.

(a) State the initial velocity of the particle.

(b) Find the velocity of the particle after 3 seconds.

(c) How long would it take for the particle's velocity to increase to 45 ms^{-1}?

(d) What happens to *v* as $t \to \infty$?

(e) Show that the particle's acceleration is always positive.

(f) Sketch the graph of v against t.

(g) Find the total distance traveled by the particle in the first 3 seconds of motion.

4. A body has an initial velocity of 20 m/s. It moves in a straight line with acceleration function $a(t) = 4e^{-\frac{t}{20}}$ ms^{-2}, at time *t* seconds.

(a) Show that, as *t* increases, the body approaches a limiting velocity.

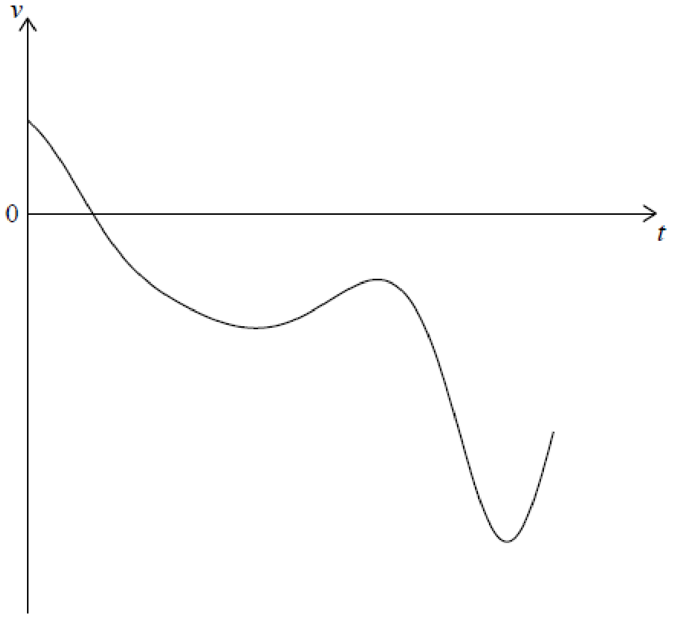
(b) Find the total distance travelled in the first 10 seconds of motion.

5.* Two particles *P* and *Q* move in the positive direction on the *x*-axis, *P* with constant acceleration 2ms^{-2} and *Q* with constant acceleration 1 ms^{-2}. At time $t = 0$, P is projected from the origin *O* with speed 1 m/s and at time $t = 4$, Q is projected from *O* with speed 16 m/s.

(a) Find the times between which *Q* is ahead of *P*.

(b) Find the distance from *O* when *Q* overtakes *P*, and the distance from *O* when *P* overtakes *Q*.

6. A particle P moves along a straight line. The velocity v ms^{-1} of P after t seconds is given by $v(t) = 7\cos(t) - 5t^{\cos(t)}$, for $0 \le t \le 7$. The following diagram shows the graph of v.



(a) Find the initial velocity of P.
(b) Find the maximum speed of P.
(c) Write down the number of times that the acceleration of P is 0 ms^{-2}.
(d) Find the acceleration of P when it changes direction.
(e) Find the total distance travelled by P. [2018MayTZ2 SLP2]

7. Xavier, the parachutist, jumps out of a plane at a height of h metres above the ground. After free falling for 10 seconds his parachute opens. His velocity, v ms^{-1}, t seconds after jumping from the plane, can be modelled by the function

$$v(t) = \begin{cases} 9.8t, & 0 \le t \le 10 \\ \dfrac{98}{\sqrt{1+(t-10)^2}}, & t > 10 \end{cases}$$

His velocity when he reaches the ground is 2.8 ms^{-1}.

(a) Find his velocity when $t = 15$.
(b) Calculate the vertical distance Xavier travelled in the first 10 seconds.
(c) Determine the value of h. [2017MayTZ1 HLP2]

Answers:

1(a) 48 m/s, 27 m/s **(b)** $\dfrac{229}{7}$ m/s **2(a)** 2 **(b)** $\dfrac{31}{6}$ cm **(c)** -1.5 cm
3(a) 40 m/s **(b)** 47.8 m/s **(c)** 1.39 seconds **(d)** $v \to 50$ **(g)** 134 m
4. 370 m **5(a)** $8 < t < 14$ **(b)** 72 m, 210 m
6(a) 7 **(b)** 24.7 **(c)** 3 **(d)** –9.25 **(e)** 63.9
7(a) 19.2 m/s **(b)** 490 m **(c)** 906 m

Topic 5: Calculus

5.6 Differential Equations

1. Introduction to Differential Equations

A **differential equation** is one that relates the derivative of an unknown function to an independent variable *x* and the dependent variable *y*.

Examples of differential equations are:

(i) $\frac{dy}{dx} = \cos x$ (ii) $x^2 \frac{dy}{dx} = y^2 + 2$ (iii) $\frac{d^2 y}{dx^2} = 6x - 2$

A **first order differential equation** has only one derivative and it is a first derivative.
Examples (i) and (ii) above are examples of first order differential equation.

A **solution of a differential equation** is a function $y(x)$ which satisfies the differential equation for all values of *x* in the domain.

In this topic, we will be exploring different methods of solving first order differential equations.

2. Direct Integration

Differential equations of the form $\frac{dy}{dx} = f(x)$ can be solved by direct integration.
Here, $y = \int f(x)\, dx + c$ is the **general solution** for y.
A **particular solution** can be found by substituting a given value of *x* and *y* in order to find the unknown *c*.

Example 1: Find the general solution of the following differential equations.

(a) $\frac{dy}{dx} = 3x^2 - 4$ **(b)** $\frac{1}{x}\frac{dy}{dx} = \frac{1}{x^2+1}$.

[(a) $y = x^3 - 4x + c$ (b) $y = \frac{1}{2}\ln\left(x^2+1\right) + c$]

Solution:

3. Variable Separable Differential Equations

Variable separable differential equations are of the form $\frac{dy}{dx} = f(x)g(y)$.

If the differential equation is of this form, then we can rearrange:

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

$$\Rightarrow \quad \int \frac{1}{g(y)}\frac{dy}{dx}\,dx = \int f(x)\,dx$$

$$\Rightarrow \quad \int \frac{1}{g(y)}\,dy = \int f(x)\,dx \qquad (\text{by chain rule})$$

Example 2: Find the general solution of the following differential equations.

(a) $3\frac{dy}{dx} = 5y^2$ **(b)** $\left(1+x^2\right)\frac{dy}{dx} = 1+y^2$ **(c)** $\frac{dy}{dx} - y\left(1+y^2\right) = 0$

[(a) $\frac{1}{y} = -\frac{5}{3}(x+c)$ (b) $y = \tan\left(\arctan x + c\right)$ (c) $\frac{y^2}{1+y^2} = Ae^{2x}$]

Solution:

Example 3: Solve the differential equation $\frac{dy}{dx} = y$ for y which satisfies the condition $y(1) = -2e$.

Solution: $\left[y = -2e^x\right]$

4. Homogenous Differential Equations

A homogeneous differential equation is of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

The total power of x and y are equal for every term in the equation.

These are examples of homogeneous differential equations:

(i) $\frac{dy}{dx} = \frac{3x+y}{x-y}$ (ii) $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ (iii) $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2+yx^2}$

We can convert homogeneous differential equations into variable separable differential equations, $\frac{dy}{dx} = f\left(\frac{y}{x}\right) = f(v)$, by using the substitution $y = vx$, where v is a function of x.

Differentiating $y = vx$ with respect to x, we get $\frac{dy}{dx} = v + x\frac{dv}{dx}$. Combining both equations, we get $v + x\frac{dv}{dx} = g(v)$ which is a variable separable differential equation.

Example 4:

(a) Using the substitution $y = vx$, where v is a function of x, solve $\frac{dy}{dx} = \frac{x+2y}{x}$.

(b) Find the particular solution if $y = \frac{2}{3}$ when $x = 3$.

[(a) $y = Ax^2 - x$ (b) $y = \frac{11}{27}x^2 - x$]

Solution:

Example 5:

Using the substitution $y = vx$, where v is a function of x, solve $x\frac{dy}{dx} - y = \sqrt{x^2 - y^2}$.

$[y = x\sin(\ln|x| + c)]$

5. Linear First Order Differential Equations

The standard form of a linear first order differential equation is $\frac{dy}{dx}+P(x)y=Q(x)$. The equation contains the variable *y* and its first derivative. *P*(*x*) and *Q*(*x*) are either numeric constants or function of *x*. It is generally not variable separable.

To solve for *y*, we first let $I(x)$ be the **integrating factor** that will help us solve the equation.

Multiply throughout by $I(x)$ and we get

$$I(x)\frac{dy}{dx}+I(x)P(x)y=I(x)Q(x)$$

Now, if we choose $I(x)$ such that $I'(x)=I(x)P(x)$, note that the equation can be re-written as

$$I(x)\frac{dy}{dx}+I'(x)y=I(x)Q(x)$$

$$\Rightarrow \quad \frac{d}{dx}\left(I(x)\cdot y\right)=I(x)Q(x) \qquad \text{by using Product Rule of Differentiation in reverse order}$$

$$\Rightarrow \quad I(x)y=\int I(x)Q(x)\,dx$$

And hence $y=\frac{1}{I(x)}\int I(x)Q(x)\,dx$.

This is of course provided we know what the integrating factor $I(x)$ is. To find $I(x)$, consider

$$I'(x)=I(x)P(x) \qquad (\text{by construction})$$

$$\Rightarrow \quad \frac{I'(x)}{I(x)}=P(x)$$

$$\Rightarrow \quad \int\frac{I'(x)}{I(x)}\,dx=\int P(x)\,dx$$

$$\Rightarrow \quad \ln\left|I(x)\right|=\int P(x)\,dx$$

$$\Rightarrow \quad I(x)=e^{\int P(x)\,dx} \qquad \text{or} \qquad -e^{\int P(x)\,dx}$$

However we can just take $\boxed{I(x)=e^{\int P(x)\,dx}}$ without loss of generality (because of symmetry).

Summary

To solve a first order linear differential equation of the form $\frac{dy}{dx}+P(x)y=Q(x)$,

1. Find the integrating factor $I(x)$ using the formula $I(x)=e^{\int P(x)\,dx}$
2. Multiply throughout by the integrating factor and observe that the LHS can be re-written as the derivative of the product $I(x)y$
3. Integrate both sides directly.

Example 6: Solve the differential equation $\frac{dy}{dx}+3x^2y=6x^2$. [$y=2+ce^{-x^3}$]

Solution:

Example 7: Solve the differential equation $\cos x\frac{dy}{dx}-2y\sin x=3$ for $x\in\left]-\frac{\pi}{2},\frac{\pi}{2}\right[$ where $y=1$ when $x=0$. [$y=\sec^2 x\left(3\sin x+1\right)$]

Solution:

6. Word Problems Involving Differential Equations

Example 8: The gradient of a curve at the point (x, y) is proportional to y^2. The curve passes through the points $(0, -2)$ and $\left(1, \frac{2}{3}\right)$. Find the equation of the curve. [$y = \frac{2}{4x-1}$]

Solution:

Example 9: Newton's law of cooling states that the rate of loss of temperature of a cooling body is proportional to the difference in temperature between the body $(\theta°)$ and its surroundings $(\theta_s°)$.

(a) Express this law in the form of a differential equation, using the symbols *t* for time and *θ* for the temperature of the body.

(b) Solve this equation for *θ* in terms of *t*, given that the room temperature is 20°C and that it takes a particular body 12 minutes to cool from 100°C to 50°C.

(c) Hence find the time taken by the body to cool from 50°C to 25°C.

[(a) $\frac{d\theta}{dt} = -k(\theta - \theta_s)$ (b) $\theta = 20 + 80e^{-\frac{1}{12}\ln\frac{8}{3}t}$

(c) 21.9 mins]

Solution:

TOK : Does personal experience play a role in the formation of knowledge claims in mathematics? Does it play a different role in mathematics compared to other areas of knowledge?

7. Approximating Solutions of Differential Equations

To find approximations to the solutions of differential equations, especially when the equation cannot be solved explicitly, we can make use of Euler's Method, or Maclaurin's series.

7.1 Euler's Method of Approximation

This method is an iterative method that is used to approximate the coordinates of another point on the curve. It assumes that the curve is approximately linear for **small increments of *x***. We will begin at an initial value $\left(x_0, y_0\right)$ and progressively "move closer" to our intended value $\left(x_n, y_n\right)$, by adding increments of *h* to x_0 and evaluating what the corresponding values of *y* are. This is a numerical approximation to the solution for *y*.

Euler's Method of Approximation:

Given a differential equation $\frac{dy}{dx} = f\left(x, y\right)$ with initial condition $\left(x_0, y_0\right)$, and a constant **step length** *h*:

1. Find the gradient $m_0 = f\left(x_0, y_0\right)$.
2. Calculate $x_1 = x_0 + h$. To calculate y_1, we find the corresponding increment of *y*, k_0 using the formula $k_0 = m_0 h$.
3. Repeat Step 1 to find $m_1 = f\left(x_1, y_1\right)$ and Step 2 to find $\left(x_2, y_2\right) = \left(x_1 + h, y_1 + k_1\right)$, where $k_1 = m_1 h$.
4. Repeat Step 3 as many times as necessary to reach the target point $\left(x_n, y_n\right)$.

For clarity, we normally present the above in the form of a table consisting the columns for x_n, y_n and m_n.

Example 10: Use Euler's Method with a step length of 0.25 to estimate the value of $y\left(1\right)$ for the differential equation $\frac{dy}{dx} = x - y^2 + 2, y\left(0\right) = 1$, to 3 significant figures. [1.63]

Solution:

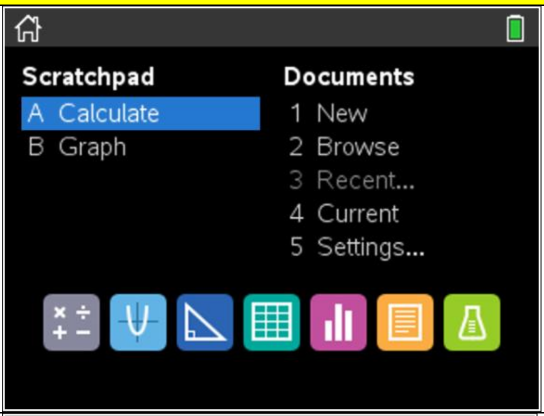
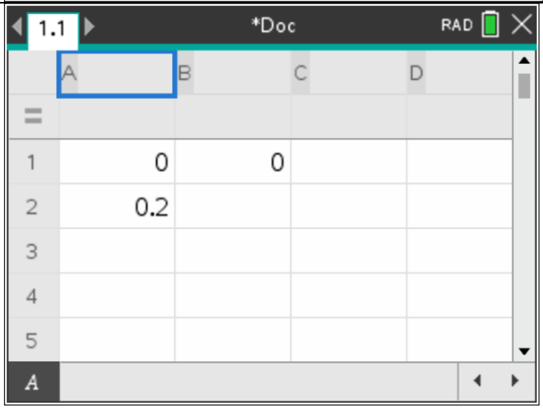
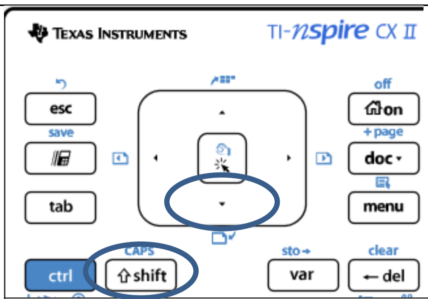
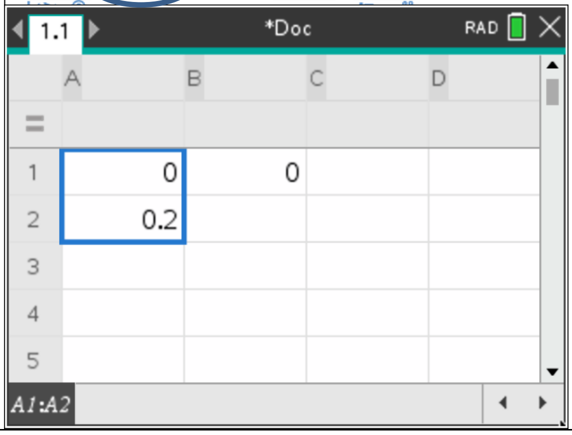
Activity

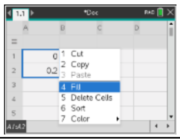
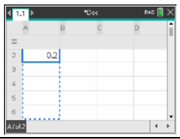
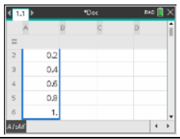
Explore the simple Programming Features in TiNSpire to code the Euler's Method of Approximation for Example 9 initially with a step length of 0.25. Explore the result when you vary the step length from 0.05 to 0.25. What can you conclude?

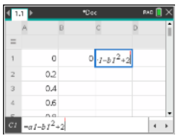
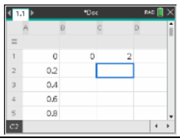
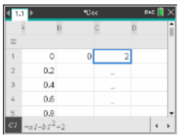
Apply the programming ideas with TiNspire on some of your assignment questions.

Euler's Method on GDC (Sample Solution)

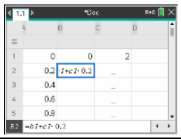
e.g. It is given that $\frac{dy}{dx} = x - y^2 + 2$, where $y(0) = 0$. Use Euler's Method with step length $h = 0.2$ to find $y(1)$.

Step	Screenshot
Open the "Lists and Spreadsheets" option ▦	
In column A, key in the initial value of x and the next value after 1 step In column B, key in the initial value of y. Here in this example, • $x_0 = 0$ • $y_0 = 0$ • $h = 0.2$ so the next value of x is $x_1 = 0 + 0.2 = 0.2$.	
At cell A1, hold down the shift ⇧shift key and select cell A2 by pressing the down key.	 

Step	Screenshot
Move your cursor to the bottom right hand corner of the blue box generated (selection of both cells) Press ctrl + menu and select "Fill"	
Drag the cursor downward until the appropriate number of cells are filled.	
Press enter	

Step	**Screenshot**
In column C, compute the initial value of $\frac{dy}{dx}$ $\left(m_0\right)$ by referencing the formula for $\frac{dy}{dx}$ and making use of cells A1 and B1 to compute it with the appropriate formula. (here the formula for $\frac{dy}{dx}=x-y^2+2$ as given) $\therefore m_0 = x_0 + y_0^2 + 2$	 1 | 0 | 0 | $\cdot 1-b1^{2}+2$ 2 | 0.2 3 | 0.4 4 | 0.6 C1 $=a1-b1^{2}+2$
Press enter	 1 | 0 | 0 | 2 2 | 0.2 3 | 0.4 4 | 0.6 5 | 0.8 C2
By using a similar method to generate the sequence of x values, generate the sequence of $\frac{dy}{dx}$ values	 1 | 0 | 0 | 2 2 | 0.2 | | _ 3 | 0.4 | | _ 4 | 0.6 | | _ 5 | 0.8 C1 $=a1-b1^{2}+2$

<table>
<tr><th>Step</th><th>Screenshot</th></tr>
<tr><td>In cell B2, compute the value of y_1 by making use of cell B1 and C1 and using the formula $y_1 = y_0 + m_0 h$ (note: must use the same sequence if not you will encounter an error).

($h = 0.2$ here)</td><td>

<table>
<tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr>
<tr><td>=</td><td></td><td></td><td></td><td></td></tr>
<tr><td>1</td><td>0</td><td>0</td><td>2</td><td></td></tr>
<tr><td>2</td><td>0.2</td><td>·1+c1· 0.2</td><td>_</td><td></td></tr>
<tr><td>3</td><td>0.4</td><td></td><td>_</td><td></td></tr>
<tr><td>4</td><td>0.6</td><td></td><td>_</td><td></td></tr>
<tr><td>5</td><td>0.8</td><td></td><td>_</td><td></td></tr>
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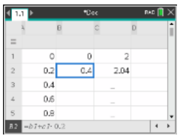
B2 =b1+c1· 0.2</td></tr>
<tr><td>Press enter</td><td>

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<tr><td>2</td><td>0.2</td><td>0.4</td><td>2.04</td><td></td></tr>
<tr><td>3</td><td>0.4</td><td></td><td>_</td><td></td></tr>
<tr><td>4</td><td>0.6</td><td></td><td>_</td><td></td></tr>
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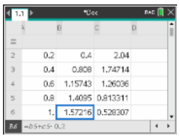
<tr><td>Similarly, generate the sequence of y values</td><td>

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<tr><td>4</td><td>0.6</td><td>1.15743</td><td>1.26036</td><td></td></tr>
<tr><td>5</td><td>0.8</td><td>1.4095</td><td>0.813311</td><td></td></tr>
<tr><td>6</td><td>1.</td><td>1.57216</td><td>0.528307</td><td></td></tr>
</table>
B6 =b5+c5· 0.2</td></tr>
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The answer for $y(1) = 1.57$

For a video tutorial, please scan this QR code:

Topic 5: Calculus

WS 5.6: First Order Differential Equations

Variable-Separable Differential Questions

1. Solve the following differential equations

(a) $y\dfrac{dy}{dx}=\sin x$ **(b)** $\dfrac{1}{x}\dfrac{dy}{dx}=\dfrac{1}{x^2+1}$ **(c)** $\dfrac{y^2}{x^3}\dfrac{dy}{dx}=\ln x$

(d) $e^x\dfrac{dy}{dx}=\dfrac{x}{y}$ **(e)** $v^2\dfrac{dv}{dt}=\left(2+t\right)^3$ **(f)** $\tan x\dfrac{dy}{dx}=2y^2\sec^2 x$

2. Find the particular solution of the following differential equations:

(a) $\dfrac{y}{x}\dfrac{dy}{dx}=\dfrac{y^2+1}{x^2+1};\ y=0$ when $x=1$

(b) $e^t\dfrac{ds}{dt}=\sqrt{s};\ s=4$ when $t=0$

(c) $x^2\dfrac{dy}{dx}+1+y^2=0$; $y=1$ when $x=\dfrac{4}{\pi}$

(d) $\left(1+\cos 2x\right)\dfrac{dy}{dx}=y\sin 2x$; $y=2$ when $x=\dfrac{\pi}{4}.$

3. A curve passing through the points $\left(1,2\right)$ and $\left(\dfrac{1}{4},-10\right)$ has a gradient which is inversely proportional to x^2. Find the equation of the curve.

4.* A tank contains 500 litres of brine in which 75 kg of salt is dissolved. Brine containing 3 kg of salt per 10 litres of water flows in at a rate of 20 litres per minute, and the mixture flows out at the same rate.

(i) Show that, for $t\ge 0,\dfrac{dx}{dt}=\dfrac{150-x}{25},$ where *x* is the amount of dissolved salt in the tank at time $t\ge 0.$

(ii) When will there be 125 kg of dissolved salt in the tank?

(iii) How much dissolved salt is in the tank after a long time?

(iv) State an assumption used in the model.

5. The current *I* in an electric circuit at time *t* satisfies the differential equation $4\dfrac{dI}{dt}=2-3I$.

(a) Find *I* in terms of *t*, given that $I=2$ when $t=0.$

(b) State what happens to the current in this circuit for large values of *t*

6.* At time *t*, the radius of a spherical balloon is *r* cm. The balloon is being inflated so that the rate of increase of its radius is inversely proportional to the square root of its radius. When $t=1, r=4$, and at this instant, the radius is increasing at 1.5cm s^{-1}.

(a) Form a differential equation and solve for *r* in terms of *t*.

(b) Determine how much air was in the balloon initially?

Homogenous Equations

7. **(a)** Given that $\frac{dy}{dx}=\frac{-2xy}{x^2+y^2}$, use the substitution $y=vx$, where v is a function of x to show that $x\frac{dv}{dx}=-\frac{3v+v^3}{1+v^2}$.

(b) Hence, show that the solution for the given differential equation is $3x^2y+y^3=c$ where *c* is an arbitrary constant.

8. Find the solution of the differential equation

(a) $\frac{dy}{dx}=\frac{x^2+y^2}{xy}$

(b) $(x+y)\frac{dy}{dx}=x-y$

(c) $x(x+y)\frac{dy}{dx}+y(3x+y)=0$

Integrating Factor Method

9. Solve the following differential equations using the Product Rule of Differentiation

$$\frac{d}{dx}(f(y)g(x))=g(x)f'(y)\frac{dy}{dx}+f(y)g'(x)$$

(a) $x\frac{dy}{dx}+y=e^x$

(b) $\cos x\frac{dy}{dx}-y\sin x=x^2$

(c) $\frac{1}{x}\frac{dy}{dx}-\frac{y}{x^2}=\sin x$

10. Solve the following differential equations by using an Integrating Factor

(a) $x\frac{dy}{dx}+y=x^2+1$

(b) $\frac{dy}{dx}-\frac{y}{x}=-xe^{-x}$

(c) $x^2\frac{dy}{dx}+2xy=\cos x$

(d) $x\frac{dy}{dx}+2y=\frac{\sin x}{x}$

(e) $x\frac{dy}{dx}=y-x^2e^{-x}$

11.* The velocity of a car at time *t* (in seconds) and at displacement *x* (in m) is given by $v=x+4t$, for $0\le t\le 3$. Given that $x=0$ when $t=0$, find:

(a) The displacement *x* in terms of *t* only

(b) The acceleration of the car at time *t* = 2 s

12.* **(a)** Differentiate the expression $x^2\tan y$ with respect to x, where y is a function of x.

(b) Hence solve the differential equation $x^2\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$ given that $y(1)=0$. Give your answer in the form of $y=f(x)$.

Numerical Solution Using Euler Method

13. For the following questions, use Euler's method to find the approximate value of y, using the given step size and initial conditions.

(a)	$\frac{dy}{dx}=x^2+y^2$	at $x=0.4$	with step size 0.1, given $y(0)=1.$
(b)	$\frac{dy}{dx}=2xy$	at $x=3,$	with step size 0.4, given $y(1)=2.$
(c)	$\frac{dy}{dx}=e^x+2y^2$	at $x=0.3,$	with step size 0.1, given $y(0)=1.$
(d)	$\frac{dy}{dx}=\sin(x+y)-e^x$	at $x=0.5,$	with step size 0.1, given $y(0)=4.$

14. Consider the differential equation $\frac{dy}{dx}=2x+y-1$ with $y(0)=1$.

(a) Using Euler's Method with increments of 0.2, find an approximate value for y when $x=1$.

(b) Explain how Euler's method could be improved to provide better approximation.

(c) Solve the differential equation to find an exact value for y when $x=1$.

Answers:

1. (a) $\frac{1}{2}y^2=-\cos x+c$ (b) $y=\frac{1}{2}\ln\left(x^2+1\right)+c$ (c) $\frac{1}{3}y^3=\frac{1}{4}x^4\ln x-\frac{1}{16}x^4+c$

(d) $\frac{1}{2}y^2=-xe^{-x}-e^{-x}+c$ (e) $\frac{1}{3}v^3=\frac{1}{4}(2+t)^4+c$ (f) $-\frac{1}{2y}=\ln\left|\tan x\right|+c$

2. (a) $2y^2=x^2-1$ (b) $2\sqrt{s}=-\frac{1}{e^t}+5$ (c) $y=\tan\frac{1}{x}$ (d) $y=\frac{2}{\sqrt{1+\cos 2x}}$

3. $y=-\frac{4}{x}+6$

4. (ii) 27.5 minutes (iii) 150 kg (iv) The salt entering and flowing out of the tank are uniformly dissolved.

5. (a) $I=\frac{2}{3}\left(1+2e^{-\frac{3t}{4}}\right)$ (b) $I\to\frac{2}{3}$ as $t\to\infty$

6. (a) $r=(4.5t+3.5)^{\frac{2}{3}}$ (b) 51.3

8. (a) $y^2=x^2\left(\ln x^2+A\right)$ (b) $x^2-2xy-y^2=A$ (c) $2x^2y^2+4x^3y=A$

9. (a) $y=\frac{1}{x}\left(e^x+c\right)$ (b) $y=\sec x\left(\frac{1}{3}x^3+c\right)$ (c) $y=cx-x\cos x$

10. (a) $y=\frac{1}{3}x^2+1+\frac{c}{x}$ (b) $y=x\left(e^{-x}+c\right)$ (c) $y=\frac{\sin x}{x^2}+\frac{c}{x^2}$

(d) $x^2y=c-\cos x$ (e) $y=x\left(e^{-x}+c\right)$

11. (a) $x=4(e^t-t-1)$ (b) $4e^2$

12. (a) $2x\tan y+x^2\sec^2 y\frac{dy}{dx}$ (b) $y=\arctan\left(\frac{x^2}{4}-\frac{1}{4x^2}\right)$

13. (a) 1.57 (b) 158 (c) 2.48 (d) 3.06

14. (a) 1.98 (b) By reducing the step length (c) $y=2\left(e^x-x\right)-1$, 2.44

Topic 5: Calculus

5.7 Maclaurin Series

Consider the convergent geometric series $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ where $|r| < 1$.

If we let a = 1 and r = x, then we have $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ where $|x| < 1$.

We can represent the function $f(x) = \frac{1}{1-x}$ where $|x| < 1$ with the power series $\sum_{n=0}^{\infty} x^n$ where $|x| < 1$

Definition: The ***Maclaurin series*** expansion of $f(x)$ is a series of ascending powers of x of the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + ...+, \qquad |x| < R,$$

where R is the radius of convergence and it is half the interval of x.

A power series can be added, differentiated and integrated just like any polynomial.

To determine the coefficients c_n, follow the steps below:

i) First evaluate the series at $x = 0$. We have $f(0) = c_0$

ii) Then differentiate $f(x)$ with respect to x:

$$f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

Therefore, when $x = 0,\ c_1 = f'(0)$

iii) Differentiating again, $f''(x) = 2c_2 + 3(2)c_3 x + \dots$

$$\therefore f''(0) = 2c_2 \qquad \Rightarrow \qquad c_2 = \frac{f''(0)}{2}$$

iv) Continuing in this manner, we find that $f^{(n)}(0) = n! c_n$

$$\therefore c_n = \frac{f^{(n)}(0)}{n!} \quad \text{where } 0!=1 \text{ and } f^{(0)}(x) = f(x)$$

v) So if $f(x) = \sum_{n=0}^{\infty} c_n x, \quad |x| < r$

$$\text{Then } f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

An example of a Maclaurin series is the Binomial theorem for negative / fractional indices:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots \text{ for } |x| < 1$$

Example 1: Find the first four terms of the Maclaurin series for $f(x) = e^x$.

Solution:

Example 2: Find the first three terms of the Maclaurin series for $f(x) = \cos x$.

Solution:

Example 3: Find the first three terms of the Maclaurin series $f(x) = \ln(1 + x)$.

Solution:

The formula booklet gives the Maclaurin series of these special functions:	
- $e^x = 1 + x + \frac{x^2}{2!} + \dots$ - $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$	- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ - $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ - $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} + \dots$

Example 4: Find the first four terms of the Maclaurin series for $f(x) = x^4 e^{-3x^2}$.

Solution: $\left[x^4 - 3x^6 + \frac{9}{2}x^8 - \frac{9}{2}x^{10} + ...\right]$

Example 5:

(a) Find the first three terms of the Maclaurin series for $\ln\left(1+e^x\right)$.

(b) Hence, determine the value of $\lim_{x\to 0}\frac{2\ln\left(1+e^x\right) - x - \ln 4}{x^2}$.

Solution: $\left[\text{(a) } \ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + ... \text{ (b) } \frac{1}{4}\right]$

Binomial Theorem for Negative and Fractional Indices (An Extension of Binomial Theorem)

Suppose $n \in \mathbb{Q}$. The Binomial Theorem can be generalized to the following:

$$(a+b)^n = a^n\left(1+n\left(\frac{b}{a}\right)+\frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2+...\right)$$

Notes:

(a) This is an infinite series.

(b) This is only valid for $\left|\frac{b}{a}\right|<1$.

(c) The derivation makes use of the Maclaurin series expansion

Example 6: Write down and simplify the first three terms, in ascending power of x, in the Extended Binomial expansion of $\sqrt{1+x}$. State the values of x for which the expansion is valid.

Solution: $[1+\frac{1}{2}x-\frac{1}{8}x^2+\ldots,\ |x|<1]$

Example 7: Write down the first three terms of the binomial expansion of $\frac{1}{2+x}$, in ascending powers of x. State the values of x for which the expansion is valid. $[\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}+\ldots,|x|<2\ ]$

Solution:

Example 8: Write down the first three terms of the binomial expansion of $\frac{1}{(1-x)^2}$ in descending powers of x. State the values of x for which the expansion is valid. $[\frac{1}{x^2}+\frac{2}{x^3}+\frac{3}{x^4}+\cdots,|x|>1]$

Solution:

Example 9: Write down and simplify the first four terms, in ascending power of *x*, in the expansion of $\left(8-9x\right)^{\frac{1}{3}}$. Hence, by finding a suitable value of *x*, find a rational approximation to $\sqrt[3]{-10}$. Why is this not a good approximation?

Solution: $\left[-\frac{65}{32}\right]$

Some Useful Results

- $\left(1-x\right)^{-1}=1+x+x^2+\ldots+x^r+\ldots$ for $\left|x\right|<1$.
- $\left(1+x\right)^{-1}=1-x+x^2-\ldots+\left(-1\right)^r x^r+\ldots$ for $\left|x\right|<1$.

These are derived from the sum to infinity of a GP.

Evaluation of Limits using Maclaurin Series

Maclaurin series may be used as another method to evaluate limits where $x\to 0$.

Example 10: Use Maclaurin series to evaluate $\lim\limits_{x\to 0}\frac{\arctan x}{x}$. [1]

Solution:

Solving Differential Equations using Maclaurin Series

Another method to approximate the solution to differential equations is to assume that there is a solution in the form of a Maclaurin series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots$ and differentiate repeatedly to find each term.

Example 11: Use Maclaurin series to solve the differential equation $\frac{dy}{dx} = x - y^2 + 2$, where $y(0) = 1$, giving your answer up to and including the term in x^3. [$y = 1 + x - \frac{1}{2}x^2 + 0x^3 + \dots$]

Solution:

Topic 5: Calculus

WS 5.7: Maclaurin Series

1. Find the Maclaurin series for $\tan x$, up to and including the term in x^3. Hence, find the approximate value of $\tan 0.2$.

2. Prove that $\sin\left(\frac{\pi}{6}+x\right) \approx \frac{1}{2}+\frac{1}{2}\sqrt{3}x-\frac{1}{4}x^2+\ldots$

3. Find the Maclaurin series of $\frac{x-7}{x^2-x-2}$, up to and including the term in x^2.

4. Given that $f(x)=\arctan x$, find

(i) $f'(x)$;

(ii) the first three terms, in ascending power of *x*, of the Maclaurin series for $f'(x)$.

(iii) Hence, find the Maclaurin series of $f(x)$.

5. Use the fact that $\frac{d}{dx}(\arctan 3x)=\frac{3}{1+9x^2}$ to find the coefficient of the third non-zero term of the Maclaurin series representation of $\arctan 3x$.

6.* **(i)** By successively differentiating $(1+x)^n$, find the Maclaurin series for $(1+x)^n$ up to and including the term in x^3.

(ii) Obtain the expansion of $(4-x)^{\frac{3}{2}}(1+2x^2)^{\frac{3}{2}}$, up to and including the term in x^3.

(iii) Find the set of values of x for which the expansion in **(ii)** is valid.

7. Use the Maclaurin series of $\ln(1+x)$ to show that $\ln\left(\frac{1+x}{1-x}\right) \approx 2x+\frac{2}{3}x^3+\frac{2}{5}x^5+\ldots$ and state the range of x for which the expansion is valid.

8. Evaluate the following limits using Maclaurin series.

(a) $\lim_{x\to 0}\frac{\cos x-1}{x^2}$ **(b)** $\lim_{x\to 0}\frac{xe^x}{\ln(1+x)}$ **(c)** $\lim_{x\to 0}(e^x-1)\cot x$

9. For the following questions, use a Maclaurin series (to the term in x^3) to approximate the solution of the differential equations, using the initial conditions

(a) $\frac{dy}{dx} = y^2 - x$ given $y(0) = 1$

(b) $\frac{dy}{dx} = y + xy^2$ given $y(0) = 2$

(c) $\frac{d^2y}{dx^2} = xy$ given $y(0) = y'(0) = 1$

10.* Consider the differential equation $\frac{dy}{dx} + y\tan x = 2\cos^4 x$ given that $y = 1$ when $x = 0$.

(a) Solve the differential equation, giving your answer in the form $y = f(x)$.

(b) Using differentiation, show that $\frac{d^2y}{dx^2} + y = -10\sin x\cos^3 x$.

(c) Hence find the first four terms of the Maclaurin series for y.

11. Obtain the first three non-zero terms of the expansion of

(a) $(1+2x)^{-1}$ **(b)** $\frac{x}{\sqrt{1+2x}}$

in **(i)** ascending powers of x and **(ii)** descending powers of x.

In each case, find the range of x for which the expansion is valid.

12. Use the binomial theorem to evaluate $\sqrt{25.1}$ to five decimal places.

13. Expand $\sqrt{\frac{2+x}{1-4x}}$ as a series in ascending powers of x up to and including the term in x^2.
State the range of x for which the expansion is valid.

14.* (a) Given that $y = \frac{1}{\sqrt{1+2x} + \sqrt{1+x}}$ where $|x| < \frac{1}{2}$, show that, provided $x \neq 0$,

$$y = \frac{1}{x}\left[\sqrt{1+2x} - \sqrt{1+x}\right].$$

(b) Using this second form of y, express y as a series of ascending powers of x up to and including the term in x^2.

(c) Hence, show that by putting $x = \frac{1}{100}$, $\frac{10}{\sqrt{102} + \sqrt{101}} = \frac{79407}{160000}$.

15.* Expand $(8+4x)^{\frac{1}{3}}(1-x)^{\frac{1}{4}}$ in ascending powers of x up to and including the term in x^2 inclusive.
Give the range of values of x for which the expansion is valid.

Answers:

1. $\tan x \approx x + \frac{1}{3}x^3 + \ldots$, 0.203

3. $\frac{7}{2} - \frac{9}{4}x + \frac{23}{8}x^2 + \ldots$

4. (i) $\frac{1}{1+x^2}$ (ii) $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, $\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{2k+1} + c$

5. $\frac{243}{5}$

6. (i) $1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \ldots$
   (ii) $8 - 3x + \frac{387}{16}x^2 - \frac{1151}{128}x^3 + \ldots$
   (iii) $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$

7. $|x| < 1$

8. (a) $-\frac{1}{2}$ (b) 1 (c) 1

9. (a) $y = 1 + x + \frac{x^2}{2} + \frac{2x^3}{3}$ (b) $y = 2 + 2x + 3x^2 + \frac{11}{3}x^3$ (c) $1 + x + \frac{1}{6}x^3$

10. (a) $y = \sin 2x - \frac{2}{3}\cos x \sin^3 x + \cos x$ (c) $y = 1 + 2x - \frac{1}{2}x^2 - 2x^3$

11a. (i) $1 - 2x + 4x^2, |x| < \frac{1}{2}$ (ii) $\frac{1}{2}x^{-1} - \frac{1}{4}x^{-2} + \frac{1}{8}x^{-3} + \ldots, |x| > \frac{1}{2}.$

11b. (i) $x - x^2 + \frac{3}{2}x^3, |x| < \frac{1}{2}$ (ii) $\frac{1}{\sqrt{2}}x^{\frac{1}{2}}\left(1 - \frac{1}{4}x^{-1} + \frac{3}{32}x^{-2}\right), |x| > \frac{1}{2}.$

12. 5.00999

13. $\sqrt{2} + \frac{9}{2\sqrt{2}}x + \frac{207}{16\sqrt{2}}x^2, |x| < \frac{1}{4}$

14. (b) $\frac{1}{2} - \frac{3}{8}x + \frac{7}{16}x^2$

15. $2 - \frac{1}{6}x - \frac{47}{144}x^2$; $|x| < 1$

Topic 1: Number and Algebra

1.3 Permutations and combinations

1. Introduction

In our daily lives, we may encounter problems of the following types:
(a) How many ways are there to arrange 3 different books A, B and C on a shelf?
(b) How many ways are there to seat 7 people in a row?

For problem (a), we may list out all the possible cases :

ABC , ACB , BAC , BCA , CAB , CBA

Therefore, there are 6 different ways to arrange 3 different books on a shelf.

Note that this method is neither clever nor efficient because if there are more books, say, 10, it will be very tedious to list all the possible cases.

Hence we shall now formulate a general method to tackle problems of this nature.

2. Basic Counting Principles

Multiplication Principle
Suppose a process consists of m steps, such that the first step can be performed in n_1 ways, the second step can be performed in n_2 ways, … , the m^{th} step can be performed in n_m ways. Then the number of ways of performing the process is $n_1 \times n_2 \times n_3 \times \cdots \times n_m$.

Example 1: There are four roads from town A to town B, three roads from town B to town C. How many ways are there if one is to travel from A to C through B? [12]

Addition Principle
Suppose a process consists of m different mutually exclusive cases, such that the first case can be performed in n_1 ways, the second case can be performed in n_2 ways, … , the m^{th} case can be performed in n_m ways. Then the number of ways of performing the process is $n_1 + n_2 + n_3 + \cdots + n_m$.

Example 2: How many whole numbers greater than 6000 can be formed with the digits 3, 4, 6, 8 and 9 if a digit cannot occur more than once in a number? [192]

3. Permutations (Arrangements)

A permutation is an ordered arrangement of a number of objects. It is a selection and arrangement of objects with order taken into account.

The number of ways of arranging n different objects in a straight line is $n!$

Example 3: How many ways are there to arrange 4 letters A, B, C and D in a straight line? [24]

4. Permutation of *r* objects from *n* unlike objects

The number of permutations of r objects from n unlike objects is given by

$$^{n}P_{r} = n(n-1)(n-2)\ldots(n-r+1) = \frac{n!}{(n-r)!}$$

Example 4: How many 3-letter code words can be formed from the letters of the word SCOTLAND? How many of these 3-letter code words do not contain any vowel at all? [336; 120]

Example 5: 12 horses run in a race. The published results list the horses finishing first, second and third. Assuming that there are no dead heats, find the number of different published results. [1320]

5. Permutations with restrictions

Example 6: How many 3-digit numbers (first digit not 0) can be formed from the digits 0 to 9,

(i) if repetition is allowed,

(ii) if repetition is not allowed? [900; 648]

Example 7: How many even numbers greater than 30000 can be formed with digits 2, 3, 4, 8, 9 if no figure is repeated?

Solution:

Case 1: Last digit is '2'

4	**3**	**2**	**1**	**1**

Number of arrangements = 24

Case 2: Last digit is '4' or '8'

3	**3**	**2**	**1**	**2**

Number of arrangements = 36

Therefore, total number of even numbers greater than 30000 = 24 + 36 = 60

Example 8: How many ways can 6 men and 2 boys be arranged in a queue if

(i) there are no restrictions,
(ii) the boys are standing next to each other,
(iii) the boys are to be separated? [40320; 10080; 30240]

Example 9: 4 different consonants and 3 different vowels are to be arranged in a row. Find the number of arrangements if

(i) the 3 vowels are to be placed together,
(ii) the first and last places are to be consonants. [720; 1440]

6. Combinations (Selections)

A combination is an unordered selection of a number of objects from a given set.
The order of selection is not important and we are concerned only with the number of objects in each group.

The number of combinations of r objects from n unlike objects is

$$ {}^{n}C_{r} = \frac{n!}{r!\left(n-r\right)!} $$

Note that ${}^{n}P_{r} = {}^{n}C_{r} \times r!$

Example 10: In how many ways can a set of 3 boys be selected from 9,

(i) if there are no restrictions,
(ii) if the eldest is included,
(iii) if the eldest is excluded? [84; 28; 56]

Example 11: Out of a group of 5 boys and 4 girls, in how many ways can a party of 4 be selected to include at least 2 boys? [105]

Example 12: A box contains 7 balls: 3 red, 2 black, 1 white and 1 green. In how many ways can 3 balls be chosen? [11]

Example 13: A boy drew 2 parallel lines on a piece of paper. He then marked 5 points on the first line and 7 points on the other. How many different triangles can be formed by joining these points as their vertices? [175]

Example 14: How many selections can be made from 4 different books when any number of books (except 0) can be taken at a time? [15]

Solution

Method 1

Since all the 4 books are different, each book can either be selected or not selected. Therefore, number of ways for each book = 2. Number of ways for 4 different books = $2^4 = 16$.But the above includes the case where none of the books are selected. Hence total number of ways = $16 - 1 = 15$

Method 2

Either 1, 2, 3 or 4 books may be selected.

Therefore, total number of ways = ${}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4 = 15$

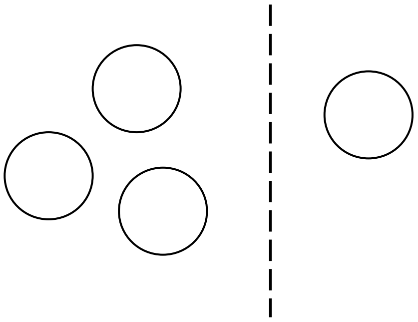
7. Complementary Counting

Complementary counting is counting the complement of the set we want to count, and subtracting that from the total number of possibilities, or the universal set for that particular problem. A useful hint that complementary counting may lead to a quick solution is the phrase "*at least*".

Example 15: In how many ways can six different coins be divided between two students such that each student receives at least one coin?

Solution

We need to divide the coins between the two students, so first we draw a line to separate the coins.



Each coin has two possibilities and belongs to one of the students, so six coins have in total $2^6 = 64$ possibilities, including two cases where all six coins belong to one student only.

Therefore, the required answer is given by 64 – 2 = 62.

Topic 1: Number and Algebra

WS 1.3 – Permutations and combinations

1. Calculate the number of ways each of the following choices can be made.

- **(a)** 4 books are to be chosen from a list of 9 titles.
- **(b)** 15 people have sent in winning entries for a magazine competition, and 3 are to be chosen and placed in order of merit so as to receive the 1st, 2nd and 3rd prizes.
- **(c)** A committee of 3 people comprising the president, secretary and treasurer are to be chosen from 8 possible candidates.

2. How many different numbers of 4 digits can be formed from the digits 0, 1, 2, …, 9 if repetition is **(a)** allowed, **(b)** not allowed?

3. A team of 6 people is to be chosen from a list of 10 candidates. Find in how many ways this can be done,

- **(a)** if the order of the people in the team does not matter,
- **(b)** if the team consists of 6 people in a definite order.

4. Find the number of ways in which 6 boys and 6 girls can be seated in a row such that

- **(a)** no two girls may sit together,
- **(b)** boys and girls sit alternately,
- **(c)** all the girls sit together and all the boys sit together,
- **(d)** all the girls are never together.

5. Three girls and three boys enter a railway compartment in which there are six seats altogether, three on each side.

- **(a)** In how many different ways can the seats be occupied?
- **(b)** If the girls all sit on one side and the boys sit opposite, in how many different ways can the seats be occupied?
- **(c)** If the girls enter the compartment first and occupy three of the four corner seats and the boys then follow them and occupy the remaining seats, in how many different ways can the seats be occupied?

6. A car can hold 3 people in the front seat and 4 in the back seat. In how many ways can 7 people be seated in the car if John and Samantha must sit in the back seat and there is only one driver?

7. A cultural event is to showcase 2 song items, 2 plays and 4 dance items.

- **(a)** Find the number of possible arrangements.
- **(b)** Find the number of arrangements
  - **(i)** which begin with a song item
  - **(ii)** in which the 2 plays are not consecutive,
  - **(iii)** in which the 4 dance items are separated.

8. **(a)** Calculate the number of ways of selecting 2 points from 6 distinct points.
(b) Six distinct points are marked on each of two parallel lines. Calculate the number of
  (i) distinct quadrilaterals which may be formed using 4 of the 12 points as vertices,
  (ii) distinct triangles which may be formed using 3 of the 12 points as vertices.

9. n_1, n_2 and n_3 points are given on the sides [*BC*], [*CA*] and [*AB*] respectively of the triangle ΔABC . Find the number of triangles formed by taking these given points as vertices of a triangle.

10. A squad consists of 8 batsmen, 6 bowlers and 2 wicket keepers.
(a) Find the number of ways in which a team of 6 batsmen, 4 bowlers and a wicket keeper may be selected.
(b) Find the number of ways in which
  (i) the team may be selected if it is to include 4 specified batsmen and 2 specified bowlers.
  (ii) the 6 batsmen may be selected from the 8 available, given that 2 particular batsmen may not be selected together.

11. A committee of 5 members is to be formed from 5 couples. Find the number of committees if
(a) the selection is random (i.e. there are no restrictions)
(b) a particular couple is in the committee,
(c) there are more men than women.

12. A committee of four children is chosen from eight children. The two oldest children cannot be both chosen. Find the number of ways the committee may be chosen.

13. In how many ways can a jury of 12 be selected from 9 men and 6 women such that there are at least 6 men and no more than 4 women on the jury.

14. John and Mary are members of a group of eight boys and two girls.
(a) In how many ways can they all be seated in a row if John and Mary always sit together? (Leave your answer in factorial form)
(b) In how many ways can the EXCO committee of five be chosen from the same group if:
  (i) both John and Mary are on it,
  (ii) either John or Mary is on it but not both,
  (iii) at least one girl is on it?

15. A cricket training squad consists of 4 bowlers, 8 batsmen, 2 wicket keepers and 4 fielders. From this squad a team of 11 players is to be selected. In how many ways can this be done if the team must consist of 3 bowlers, 5 batsmen, 1 wicket keeper and 2 fielders?

16. A girl wishes to phone a friend but cannot remember the exact number. She knows it is a five-digit number, which is even, and that it consists of the digits 2, 3, 4, 5 and 6 in some order. Using this information, find the largest number of different wrong telephone numbers she could try.

17. The number 105840 can be expressed in prime factors $2^4\times3^3\times5^1\times7^2$. Excluding 1 and 105840, how many positive integers are factors of 105840?
[Hint: one factor of 105840 is 90, which can be expressed as $2^1\times3^2\times5^1\times7^0$.]

18. Show that ${}^{n+1}C_3-{}^{n-1}C_3=\left(n-1\right)^2$. Hence find n if ${}^{n+1}C_3-{}^{n-1}C_3=16$.

19. Find n and r if ${}^{n}C_r:{}^{n}C_{r+1}:{}^{n}C_{r+2}=1:2:3$.

20. If ${}^{n}C_1,{}^{n}C_2$ and ${}^{n}C_3$ are in AP, find the value of n.

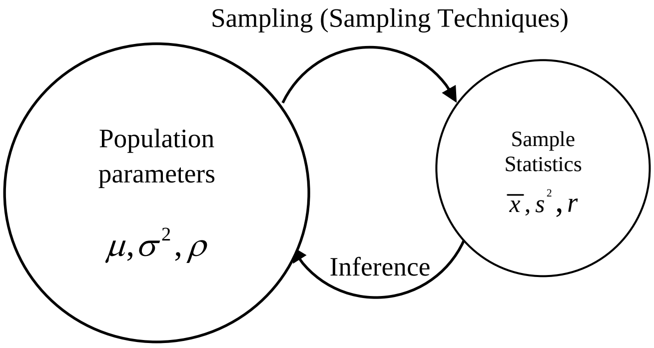
Answers:

1. (a) 126 (b) 2730 (c) 336
2. (a) 9000 (b) 4536
3. (a) 210 (b)151200.
4. (a) 3628800 (b) 1036800
   (c) 1036800 (d) 475372800
5. (a) 720 (b) 72 (c) 144
6. 288
7. 40320 (a) 10080
   (b) 30240 (c) 2880
8. 15 (a) 225 (b) 180
9. ${}^{n_1+n_2+n_3}C_3-\left({}^{n_1}C_3+{}^{n_2}C_3+{}^{n_3}C_3\right)$
10. 840 (a) 72 (b) 13
11. (a) 252 (b) 56 (c) 126
12. 55
13. 155
14. (a) 2x9!
    (b) (i) 56 (ii) 140 (iii) 196
15. 2688
16. 71
17. 118
    (Hint: $\left(4+1\right)\times\left(3+1\right)\times\left(1+1\right)\times\left(2+1\right)-2$)
18. 5
19. n=14, r=4.
20. 7

Topic 4: Statistics and probability

4.1 Descriptive Statistics

Concepts of Population and Sample



A **population** consists of the totality of the observations with which we are concerned. The **size** of the population is the number of observations in the population.

As it is often impossible or impractical to observe the entire set observations that make up the population, for example, due to high costs or time constraint, we must depend on a subset of observations from the population to help us make inferences concerning that population. This subset of observations from the population is called a **sample**.

The **sampling frame** is a list of the items or people (within the target population) where you take your sample. A **sampling unit** is a single item from the sampling frame that is chosen to be sampled. A **sampling variable** is the variable that is being investigated, which you measure from each sampling unit, it can be height, weight, age or whether the person smokes. Each sampling variable can take a range of possible **sampling values**.

A **biased** sample is any sampling procedure that produces inferences that consistently overestimate or underestimate some characteristic of the population. To eliminate possibility of bias in the sampling procedure, we choose a **random sample** such that the observations are made independently and at random.

Sampling Techniques

Sampling Techniques	Probability Sampling	Simple Random Sampling
		Stratified Random Sampling
		Cluster Sampling*
		Systematic Sampling
		Multi-stage Sampling*
	Non-Probability Sampling	Quota Sampling
		Snowball Sampling*
		Judgement Sampling *
		Convenience Sampling

Methods marked by * are not required in the syllabus.

Probability sampling means that every item/member in the population has an equal chance of being included in the sample. Probability sampling has the greatest freedom from bias but may be the costliest and/or time consuming.

Non-Probability Sampling is normally used in case study research design and qualitative research. These tend to focus on small samples and are intended to examine a real-life phenomenon. It is not used to make statistical inference to the wider population. Normally a clear rationale is needed for inclusion of some individuals rather than others.

A Biased or Fair Sample

In many scenarios, a population will encompass subgroups, each requiring proportional representation within the sample. For instance, consider a survey conducted in a school where half the population comprises females, and gender significantly influences the survey findings. If the sample includes a higher proportion of males compared to females, then the sample does not accurately represent the population. In this case, the sample is considered biased rather than fair.

Bias in a sample signifies that certain characteristics of the population are either excessively emphasized or underrepresented. Employing some form of randomization in the sampling method is one approach to mitigate bias and ensure a fair representation.

Types of Sampling

We will be exploring 5 types of sampling for the syllabus. Each method has it owns advantages and disadvantages. The decision on which sampling to be used should depend on the situations.

1. **Simple Random sampling**: each member of the population has an equal chance of being chosen as a sampling unit. For example, to find the average height of *n* Y5 students in ACS, we assign serial numbers (1 to *n*) to each student to make a sampling frame and use computer to randomly draw 100 numbers (100 students in the sample). The disadvantage is often that the list of the entire population (also called the central list) is not available or subject to constant change.

Example: Joshua wants to create a sample by randomly selecting six students from his class of 27. To do this, he assigns each student in his class a number from 1 to 27. Afterward, he utilizes the random number generator on his calculator, which produces six random numbers ranging from 0 to 1:

$$0.551, 0.713, 0.243, 0.299, 0.841, 0.458$$

Can you assist Joshua in converting these values into numbers between 1 and 27, allowing him to effectively pick individuals for his sample?

Solution:

2. **Systematic sampling**: It is the sampling method where every n^{th} case after a random start point is selected. This is normally used as an alternative to simple random sampling for its simplicity in application.
   For example, to find the average height of a Y5 student in ACS, the students are arranged by date of birth and every 10^{th} student is chosen after selecting a random start point. If the sample frame has been ordered according to some relevant characteristics such as age, years of services, this procedure produces a spread of values for the characteristics- essentially a kind of informal stratification.

Example: There are 500 Y5 students in ACS. The Year Director wants to obtain a systematic sample of 50 students to do a survey on the effectiveness of PC lesson? Describe an effective way how this can be achieved.

Solution:

3. **Convenience sampling** is selecting participants because they are often readily and easily available to the interviewer (thus the name “convenience”).
   For example, you interview students from your own class since it is easier to meet up and they are more likely to help you. Another example is to interview the parents from your close friends. This is normally viewed as a bias and it is not a true representation of the wider population.

4. **Stratified sampling:** In this approach, a sample is created by choosing equivalent percentages from every subgroup (or stratum) present in a population. This method is especially fitting when distinct subgroups are well-defined, each demanding sufficient inclusion.
   For example, if the sampling frame consists of 3 strata, children, youth and adults, and 15% are youth. Then the sample will be designed such that 15% of the sample are youth. So, if 100 samples are drawn, 15 should be drawn from the youth.
   Stratified sampling is often used when there is a lot of variation within a population, and one needs to ensure that each stratum is adequately represented. Although stratified sampling might entail higher costs and a lengthier process compared to alternative methods, it can yield the most accurate depiction of the broader population.

Example: ACS wants to create a stratified sample of 100 students to do a survey regarding the food variety in the SAC. The targeted students for this survey are the Y1 to Y4 students. There are 560 Y1 students, 525 Y2 students, 630 Y3 students and 620 Y4 students. How many students from each level should be selected to form the sample?

Solution:

5. **Quota sampling**: Quota sampling is a non-random sampling technique akin to stratified sampling. In this approach, researchers establish the specific number of participants needed from each subgroup, or stratum, to make the sample. This predetermined "quota" is often fulfilled through convenience sampling. Once the required number of respondents from a particular stratum is reached, any additional participants from that same stratum are excluded from consideration. For example, in market research at a shopping centre, the interviewers may have a target to interview 3 groups, children, youth and adult. If 15% are youth, similar to stratified sampling, 15% of the total sample should be drawn from the youth group. The main difference then, is that quota sampling uses non-probability sampling within each group to choose the samples (usually **convenience sampling**) and stratified sampling uses probability method within each stratum (can be **simple random sampling** or **systematics sampling**). This sampling method is used frequently in street interviews, where the interviewer has a counter to track the number of samples in each group but interview the people, he/she encounters.

Class activity: discuss the relative advantages and disadvantages of the different sampling techniques.

We normally try to minimise sources of bias in designing a survey. **Sources of bias** in sampling include:

- some members of the population may be excluded from the sampling frame.
  e.g. a survey will be conducted by visiting the homes of the members of the population as registered on their NRIC. However, this will exclude people who are homeless.
- Non-response
  e.g. a survey form may be sent out on Google Forms, and left to the students to do. However, not everyone will respond and those who did not reply were not represented in the results.
- Bad design
  Questions may be unclear and may lead respondents to give wrong information.
- Bias by the respondent
  This is the hardest bias to eliminate. Respondents may not want to tell the truth in response to your questions.
- Unrepresentative data
  If the sampling method is chosen wrongly, the results will be skewed. For example, if one is to study the average height of students in ACS, but the sample consists of students in the basketball team (sample by convenience or vicinity), it is likely that the average height found will be higher than the actual value.

Continuous and Discrete Random Variables

Quantitative variables
(a) are those that can be given a numerical basis (as opposed to qualitative variables such as taste, smoking or non-smoking)
(b) can be measured or counted
(c) can be sub-divided into **discrete** variables and **continuous** variables

Discrete Variables

(a) can only take on **fixed** values (there are "gaps" between values)

e.g. number of apples in a box (0, 1, 2, 3, …), daily attendance at a cinema, number of eggs laid per month by 1000 hens, shoe sizes (6, $6\frac{1}{2}$, 7, …)

(b) are restricted to the fixed values, e.g. cannot have 123.4 people attending a cinema show, or have 43.19 apples in a box, shoe size $6\frac{1}{4}$ etc.

Continuous Variables

(a) can take any value (no "gap", but maybe within a certain range)

(b) not restricted (to whole numbers or integer values) e.g. heights, weights, temperatures

Frequency Distributions

We often display data in terms of frequency distributions. Some examples are as follows:
(a) Frequency distribution for **discrete** data:

Mark	0	1	2	3	4	5	6	7	8	9	10	**Total**
Frequency	2	3	1	5	0	4	3	6	1	2	3	30

Note: Discrete data can be grouped into 'classes', but then **the original information is lost**, as seen below:

Mark	0 – 1	2 – 3	4 – 5	6 – 7	8 or more	**Total**
Frequency	5	6	4	9	6	30

(b) Frequency distribution for **continuous** data

Note: Continuous data cannot assume exact values, but can be within a certain range or measured to a certain degree of accuracy, such as time taken by each student in a class of 30 to complete a task.

Time Taken (min)	1-10	11-20	21-30	31-40	41-50	51-60	**Total**
Frequency	3	2	4	10	6	5	30

Grouped Frequency Distribution

Raw data gathered can be confusing without a clear pattern, so we arrange the data in an orderly way. One way is to group the data into a number of **class intervals** of equal size if possible. Steps for grouping a set of data into class intervals are:

(1) Identify largest & smallest observations.

(2) Find difference between largest and smallest observations.

(3) Decide on no. of class intervals (dep. on amt. of data). Choose 5 to 20 class intervals.

(4) Choose 1st class interval to include smallest observation, the last include largest.

Interval Limits and Boundaries

Age (Years)	10-14	15-19	20-24	25-29	30-34	35-39	**Total**
Frequency	2	5	7	11	1	4	30

For a class interval 25 – 29, the **lower interval limit** is 25, **upper interval limit** is 29.

For measurements recorded to the nearest whole number, the interval 25 – 29 will have original measurement x where $24.5 \le x < 29.5$, and the interval 30 – 34 will have $29.5 \le x < 34.5$.

$24.5 \le x < 29.5$ and $29.5 \le x < 34.5$ are called the **interval boundaries**.

Interval Width Boundaries

The **interval length** or **interval width** of a class interval is the difference between its upper and lower class boundaries. For example, the width of the class 25 – 29 is 29.5 – 24.5 = 5 units.

Mid-interval Values

For a class interval 25 – 29, the **mid-interval value** is the average of the upper and lower limit of the interval which is $\frac{25+29}{2} = 27$

Measures of Central Tendency

Central tendency is a descriptive measure of a set of data that shows where the central point in the distribution is located. The central point is called an **average** which is a single number that represents the entire set of data.

(a) **Mode** = a score or value that occurs most frequently in a set of data

e.g. For the set of numbers 11, 8, 6, 5, 5, 3, 3, 3, 2 and 1, the mode is 3

Modal Class

From the grouped frequency distribution, we can identify the class with the highest frequency, called the **modal class**.

Example: The annual salaries of the employees of a factory are summarized in the table given below.

Salary (in thousand dollars)	Frequency
$10 \le x < 20$	35
$20 \le x < 30$	42
$30 \le x < 40$	58
$40 \le x < 50$	14
$50 \le x < 60$	3

State the modal class and its interval width.

***Solution*:**

(b) **Median** = a score or value that occupies the middle position in a distribution of scores
(i) the middle observation if the no. of observations is odd
(ii) the mean of the 2 middle observations if the no. of observations is even

Note: A set of ungrouped data must first be ranked in order (e.g. from the largest to the smallest)

e.g. For the set of numbers 3, 9, 15, 19, 21, 24 and 27, the median is 19

e.g. For the set of numbers 8, 6, 5, 4, 3 and 3, the median is (5+4)/2 = 4.5

(c) **Mean**

In a population, the **mean** μ is the one single parameter which describes the measure of central location of the data. The formula is given by

$$\mu = \frac{\sum_{i=1}^{k} f_i x_i}{n} \quad \text{or} \quad \bar{x} = \frac{\sum fx}{\sum f}$$

where f_i describes the frequency of x_i, n is the total size of the population $n = \sum_{i=1}^{k} f_i$, and k is the total number of different types of observations.

Note: Mean
(i) takes every score in a distribution into account
(ii) is the most stable measure of central tendency from sample to sample.
(iii) provides basis for many statistical comparisons (unlike the mode and median which, once computed, very little can be done)

e.g. For the set of numbers 9, 7, 5, 4, 3 and 2, the mean is (9+7+5+4+3+2)/6 = 5

Example: Discrete Data in a frequency table

An editor reads through a 250-page manuscript. The number of typographical errors found on each page are summarized in the table below.

No. of errors	0	1	2	3	4
No. of pages	61	109	53	23	4

Calculate the mean number of errors found per page. [1.2]

$$\overline{x} = \frac{\sum fx}{\sum f} =$$

Calculating Mean of Grouped Data

To **estimate** the **mean** of grouped data (since we do not know the exact value of data), we represent all values in a class interval by the **mid-value** of the interval. Generally, the mean of a set of n numbers grouped into class intervals is $\bar{x} = \frac{\sum fx}{\sum f}$, where x represents the mid-value of the class interval, f is the frequency of the interval.

Example: Grouped Data (introduce use GDC)

The table shows results of a survey on the prices of an article sold in different shops.

Price (cents)	90-94	95-99	100-104	105-109	110-114	115-119	120-124
Frequency	4	11	15	24	18	9	3

Estimate the mean of the distribution.

***Solution*:**

Class interval	**Mid-value (x)**	**Frequency (f)**	fx
90-94			
95-99			
100-104			
105-109			
110-114			
115-119			
120-124			
		$\sum f =$	$\sum fx =$

The mean cost is $\bar{x} = \frac{\sum fx}{\sum f} =$

Measuring Degree of Dispersion

However, knowledge of the central tendency alone does not by itself give an adequate description of the data. We need to know *how the observation spread out from the average*. It is possible to have 2 sets of observations with the same central tendency that differ considerably in the variability of their measurements about the average.

Range

The range of a set of data is the difference between the maximum and minimum value.

Interquartile Range

The median divides the ordered data into two halves. It is possible to divide the data into quarters (quantiles), tenths (deciles) and hundredth (percentiles).

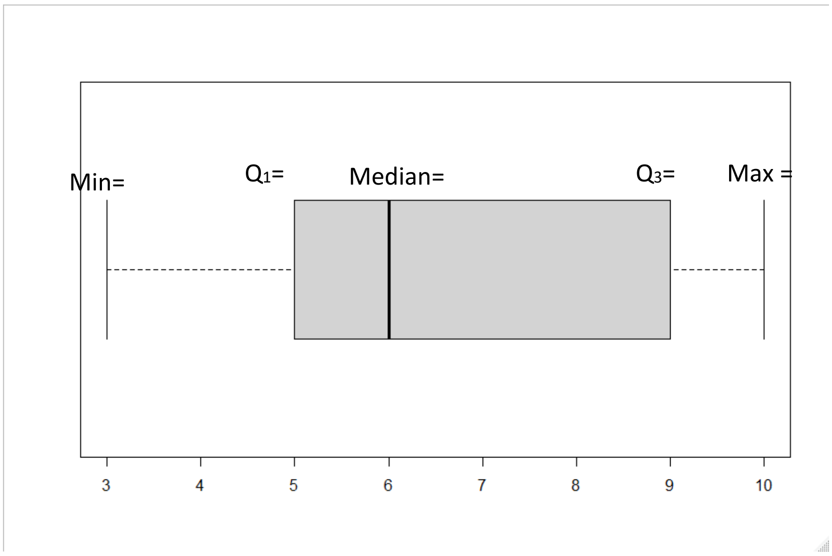
There are three quartiles: the lower quartile $\left(Q_1\right)$, median $\left(Q_2\right)$ and the upper quartile $\left(Q_3\right)$.

Interquartile range (IQR) = $Q_3 - Q_1$

Example: The following data are the numbers of students who go on to pursue degree programs related to Mathematics from School A from 2008 to 2018.

3, 4, 5, 5, 6, 6, 7, 8, 9, 9, 10

Find the range and interquartile range. Illustrate the data using a box-and-whisker plot.



Outliers

In your data, you may have outliers, which are extreme data values that skewed the data. To decide whether a given data is an outlier, we use 1.5 times of IQR as a gauge.

A data is an outlier if it is more than $Q_3 + 1.5 \times IQR$ or less than $Q_1 - 1.5 \times IQR$.

The outliers are often **indicated individually using crosses** and the GDC will automatically recognize any outliers in the box-and-whiskers plot.

Example:

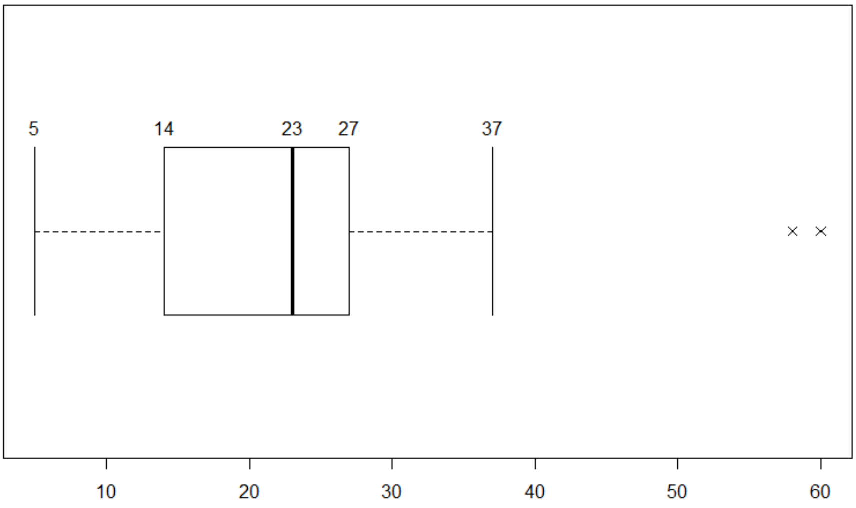
(a) The numbers of words in the first 18 sentences of Chapter 1 of *A Tale of Two Cities* by Charles Dickens are as follows:

118, 39, 27, 13, 49, 35, 51, 29, 68, 54, 58, 42, 16, 221, 80, 25, 41, 33

Illustrate the data using a box-and-whiskers plot and calculate the range and interquartile range of the data. [Ans: IQR=29; range = 67]

(b) The numbers of words in the first 18 sentences of Chapter 1 of *War and Peace* by Leo Tolstoy are counted and presented as a box-and-whiskers plot below.

14, 58, 5, 14, 24, 24, 9, 23, 60, 16, 27, 35, 24, 37, 23, 8, 20, 10



(c) Comment on the differences between the two books for the number of words in the first 18 sentences.

Variance

This is where the concept of the **population variance** comes in. The variance of a population is a measure of the spread of the data, thus reflecting the variability of the data. The variance gives an idea of how much variability from the mean there is in the data by measuring the average of the squares of the deviations of each data point from the population mean.

The **larger** the population variance, the more spread out the data is. The **smaller** the population variance, the more tightly clustered the data points are about the mean.

The formula for population variance (denoted by notation σ^2 this is sigma-squared) is

$$\sigma^2 = \frac{\sum_{i=1}^{k} f_i \left(x_i - \mu\right)^2}{n} = \frac{\sum_{i=1}^{k} f_i x_i^2}{n} - \mu^2 \text{ or } \sigma^2 = \frac{\sum x_i^2}{n} - \mu^2$$

where f_i describes the frequency of x_i, $n = \sum_{i=1}^{k} f_i$ is the total size of the population, and k is the total number of different types of observations.

Standard Deviation

The **standard deviation**, denoted by σ is the positive square root of the variance.

[i.e. standard deviation = $\sqrt{\text{variance}}$]

Standard deviation can be calculated using the Graphic calculator.

Note: When calculating mean/variance, if class intervals are given in the question, the mid-point of an interval is taken to represent the interval.

Example 1: Consider the data set $\{k-4, k-1, k, k+3, k+5\}$ where $k \in$ □ .

(a) Find the mean of this data set.

(b) If an error was spotted and each value in the data above increases by 2, find the mean of the new data set in terms of *k*. [Ans: $k + \frac{3}{5}$; $k + \frac{13}{5}$]

Example 2: Each day, the number of diners, x, in a restaurant was recorded and the following grouped frequency distribution table was obtained for a period of 300 days in the year.

No. of diners in restaurant, x_i	16 – 20	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45	46 – 50
No. of days, f_i	67	74	38	39	42	22	18

Calculate

(i) the mean number of diners per day

(ii) the variance of the population and the standard deviation of the data set.

[Ans: 28.9; 85.8; 9.24]

Topic 4: Statistics and probability

WS 4.1 – Descriptive Statistics

1. The length of life (to nearest hour) of each of 50 electric bulbs is recorded in the table below. Calculate the median length of life.

Length of life (h)	660-670	670-680	680-690	690-700	700-710
Frequency	3	7	20	17	3

Calculate

(a) Mean

(b) Variance

(c) Standard Deviation using the data above.

2. **(a)** For a set of 10 numbers, $\sum x$ = 290 and $\sum x^2$ = 8469. Find the mean and the variance.

(b) The numbers *a*, *b*, 8, 5, 7 have a mean of 6 and a variance of 2. Find the values of *a* and *b*, if *a* > *b*.

(c) Find the mean and standard deviation of the set of integers 1, 2, 3, …, 20.

3. For a set of 20 numbers $\sum x = 300$ and $\sum x^2 = 5500$. For a second set of 30 numbers, $\sum x = 480$ and $\sum x^2 = 9600$. Find the mean and the standard deviation of the combined set of 50 numbers.

4. The score for a round of golf for each of 50 club members was noted. Find the mean score for a round and the standard deviation.

Score, x	66	67	68	69	70	71	72	73
Frequency, f	2	5	10	12	9	6	4	2

5. For a set of observations, $\sum f = 20$, $\sum fx^2 = 16143$, $\sum fx = 563$. Find the values of the mean and the standard deviation.

6. The table shows the times taken on 30 consecutive days for a coach to complete one journey on a particular route. Tines have been given to the nearest minute. Find the mean time for the journey and the standard deviation.

Time (min)	60-63	64-67	68-71	72-75	76-79
Frequency	1	3	12	10	4

7. **(a)** Find the median, mean and standard deviation of the set of numbers 3, 5, 12, 1, 6, 3 and 12.

(b) A set of digits consists of *m* zeros and *n* ones.

(i) Find *M*, the mean of this set, and show that the standard deviation, *S*, is given by $\dfrac{\sqrt{(mn)}}{(m+n)}$.

(ii) If $M = S$, find the value of the median.

8. Clare wants to open an online shop selling accessories. She decides to conduct an online survey, and gets five of her friends to help her. Each of them is asked to question 20 people between the ages of 20 to 25 and 20 people between the ages of 26 to 40.

(a) What kind of sampling is Clare using?

(b) Is the sample obtained through this method a random sample? Why?

(c) Comment on the appropriateness of the sample obtained.

9. A school is required to obtain the views of students about a school policy. It is decided to do this by means of selecting students to join a focus group discussion. Describe briefly how you would select the students for this using

(a) simple random sampling and (b) stratified sampling.

State, with a reason, which of these two sampling methods you consider to be more appropriate for this situation.

Answers

1. (a) 687 (b) 92 (c) 9.59

2. (a) 29, 5.9 (b) 6, 4 (c) 10.5, 5.77

3. 15.6, 7.66

4. 69.3, 1.7

5. 28.15, 3.84

6. 71.2, 3.82

7. (a) 5, 6, 4.07 (b) (i) $\dfrac{n}{m+n}$ (b) (ii) $\dfrac{1}{2}$

8. (a) Quota sampling

(b) Not exactly, as even though the survey is online, it is also restricted to those within the circle of friends of the five interviewers.

(c) Not appropriate. The sample obtained may not be representative of preferences of people from all types of age groups. It also does not take gender into account.

9. (a) In simple random sampling, we will select the sample at random.

(b) In stratified sampling, the students in the school will be categorized according to different groups, say level and stream of study (Y1 Exp, Y1 IP etc), to form strata. The required sample size for each stratum will then be calculated. Finally, the pupils in each stratum are selected using simple random sampling.

Stratified sampling is more appropriate as it will provide a good mix of the students across the whole school.

Topic 4: Statistics and probability

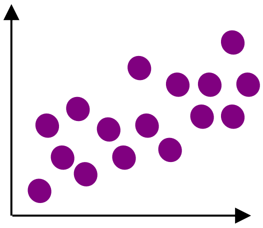
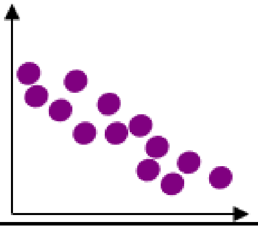
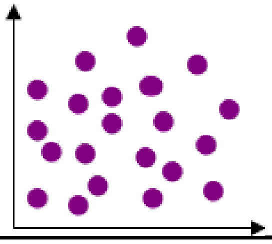
4.2 Correlation and regression

1. Correlation

We now introduce a measure of association called the **Pearson's product-moment correlation coefficient** r. It will help us to measure the **correlation** between two variables from a sample. The population correlation coefficient is called rho and it represented by the Greek letter ρ.

The **correlation** between two random variables X and Y is the measure of the *degree of linear association* between the two variables.

The following section demonstrates scatter diagrams of examples of different types of data:

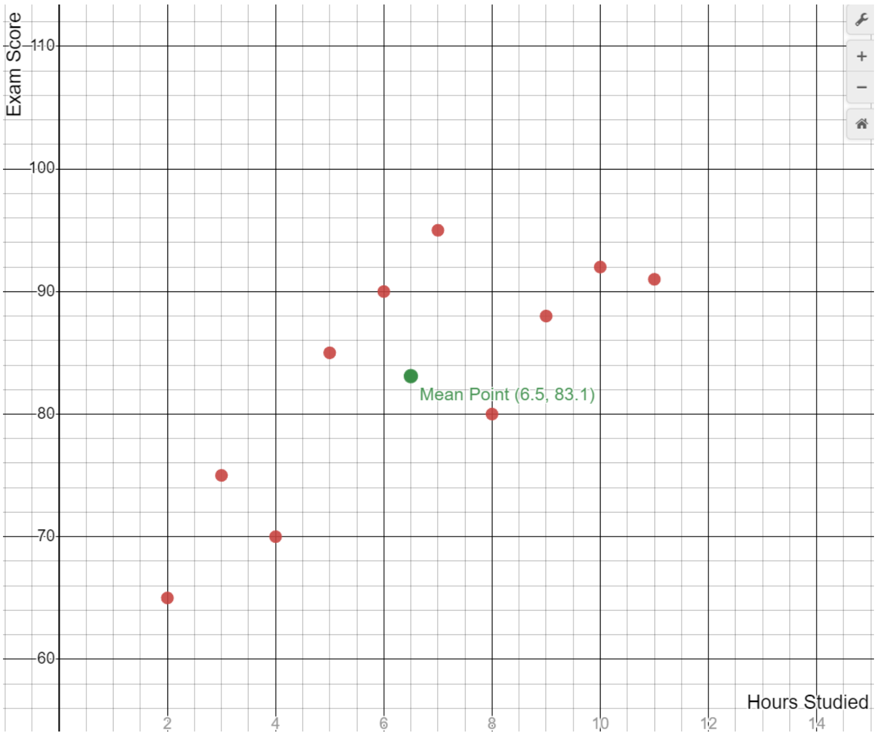
Type	Example	Comments
Positive Correlation $r > 0$		In a scatter plot, when an increase in one variable is generally associated with an increase on the average in the second variable, we say that the two variables are 'positively related' or 'positively correlated'.
Negative Correlation $r < 0$		If an increase in one variable causes a decrease on the average in the second variable, we then say that the two variables are 'negatively correlated'.
Zero Correlation $r \approx 0$		Note that if $\rho = 0$, it does not necessarily imply that there is no relation between the variables. All we can say is that there is no linear correlation between the two variables.

Estimating the line of Best Fit by eye

Consider the following data from a sample of 10 students regarding the number of hours studied and their examination score.

Hours Studied (x)	Exam Score (y)
2	65
3	75
4	70
5	85
6	90
7	95
8	80
9	88
10	92
11	91

The scatter plot is shown below:



To draw the line of best fit, we try to draw the line through as many data points as possible but in reality, we may not even get the line to pass through two of these points. However, there is a mean point, $\left(\bar{x},\bar{y}\right)$, which all line of best fit **MUST** pass through. Try and draw the line of best fit in the diagram above bearing in mind we should try to minimise the square of the vertical distance between the line and the data points.

The regression line for the above is $y = 2.62x + 66.0$. Now, you can try and draw the regression line using the given equation. To draw a straight line, we only require two points. Since we already have the mean point, we can use the vertical-intercept (y-intercept) as the second point. Did the line you draw earlier matches the second line? It is not easy to get the line of best fit without the equation.

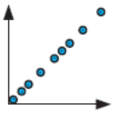
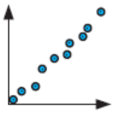
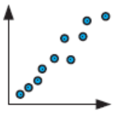
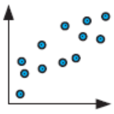
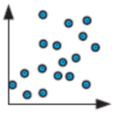
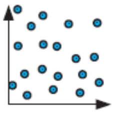
1.2 Pearson's Product-Moment Correlation Coefficient, *r*

(Also known as Sample Product-Moment Correlation Coefficient)

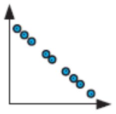
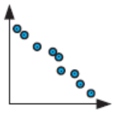
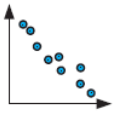
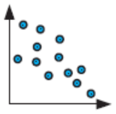
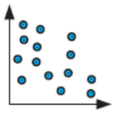
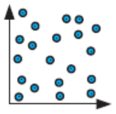
To calculate the Pearson's Product-Moment Correlation Coefficient¸ *r*, we can use the GDC (refer to page 124)

Once you have the value of *r*, you can describe the correlation between the two variables. Refer to the table below. Do note that this only shows the correlation between the two variables and correlation does not necessarily mean causation. For example, you may observe that there is a correlation between drinking coffee and high blood pressure. However, does it mean that drinking coffee is the cause of blood pressure in the respondents?

Positive correlation

$r = 1$	perfect positive correlation	
$0.95 \leqslant r < 1$	very strong positive correlation	
$0.87 \leqslant r < 0.95$	strong positive correlation	
$0.5 \leqslant r < 0.87$	moderate positive correlation	
$0.1 \leqslant r < 0.5$	weak positive correlation	
$0 \leqslant r < 0.1$	no correlation	

Negative correlation

$r = -1$	perfect negative correlation	
$-1 < r \leqslant -0.95$	very strong negative correlation	
$-0.95 < r \leqslant -0.87$	strong negative correlation	
$-0.87 < r \leqslant -0.5$	moderate negative correlation	
$-0.5 < r \leqslant -0.1$	weak negative correlation	
$-0.1 < r \leqslant 0$	no correlation	

Example 1: Find the correlation coefficient for the pairs of data given below and comment on the relationship between the two variables.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

[0.977]

Example 2: The rate of growth y of a particular organism is thought to depend in some way on the temperature t. The table below shows the results of 10 experiments.

Temp t	6	9	11	14	17	22	26	29	32	34
Rate y	5	13	15	21	20	24	19	16	10	7

Calculate the sample product moment correlation coefficient and comment on the two variables.

[–0.0117]

Example 3: A group of 12 children participated in a psychological study designed to access the relationship, if any, between the age x years, and the average total sleep time (ATST), y minutes. To obtain a measure for ATST, recordings were taken for each child on five consecutive nights and then averaged. The results obtained are shown below.

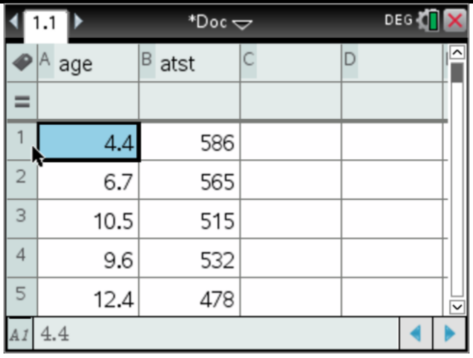
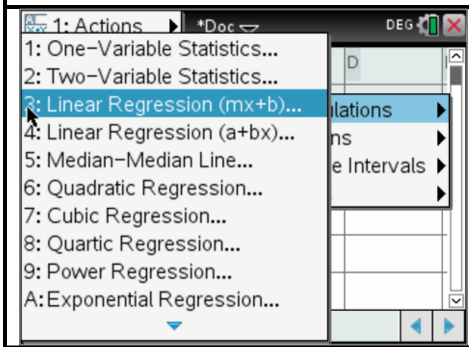
Child	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**
Age (x)	4.4	6.7	10.5	9.6	12.4	5.5	11.1	8.6	14.0	10.1	7.2	7.9
ATST (y)	586	565	515	532	478	560	493	533	575	490	530	515

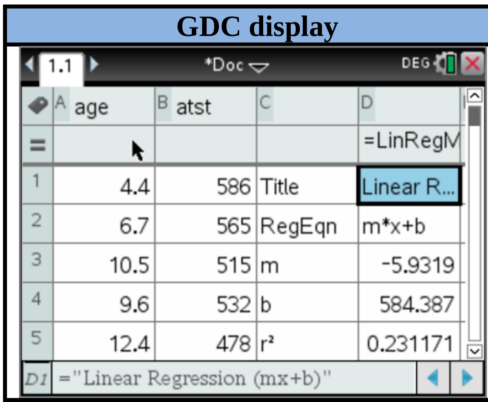
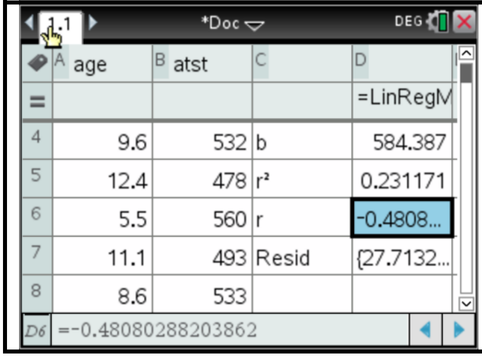
(i) Find the sample product-moment correlation coefficient between x and y.

(ii) Plot the data on a scatter diagram. Discuss, briefly, whether or not your conclusion in (i) should be amended. [–0.481, weak negative correlation]

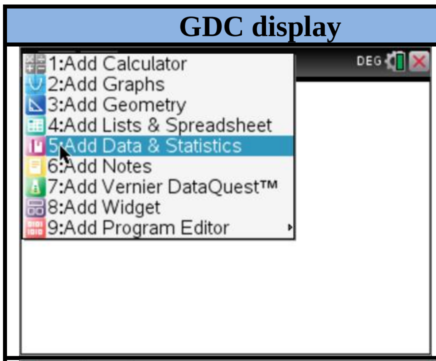
1.3 Using GDC to obtain the value of *r*

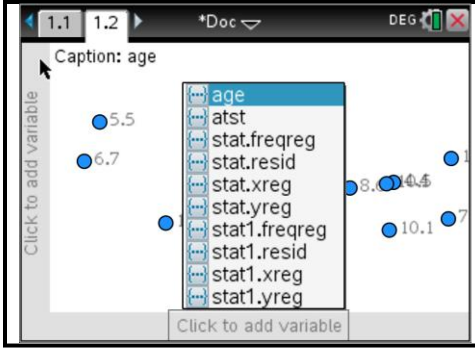
You may run **Linear Regression** from the spreadsheet page instead to obtain the *r* value, along with the coefficients of the best-fit line.

GDC display	Main GDC keys	Remarks
	on	Use the spreadsheet, 4th icon at the bottom of the screen
 A age, B atst 1: 4.4, 586 2: 6.7, 565 3: 10.5, 515 4: 9.6, 532 5: 12.4, 478 A1 4.4	menu 4 1 3	Enter values of *x* into Column A and values of *y* into Column B. Name the columns “age” and “atst” (in the top row) *you can also use “x” and “y” but it sometimes can be confusing if further operations such as regression is done.
	menu 4: Statistics 1: Stat Calculations 3: Linear Regression	You can choose either (mx+b) or (a+bx), just be careful which value is the gradient and which is the y-intercept for your answer.
 Linear Regression (mx+b) X List: 'age Y List: 'atst Save RegEqn to: f1 Frequency List: 1 Category List: Include Categories: 		Fill up the dialogue window by selecting appropriate x and y variable. Make sure that the answer is shown on column c[] onwards. Take note of the function name “f1”, where the regression equation is stored. You can call up the function to do substitution, when you are doing prediction.

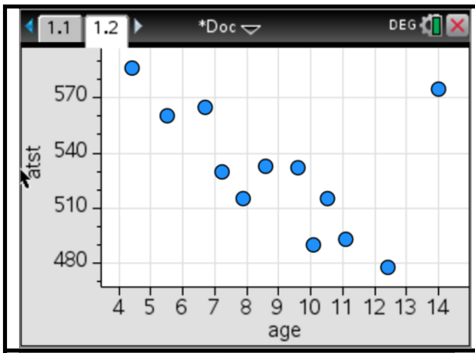
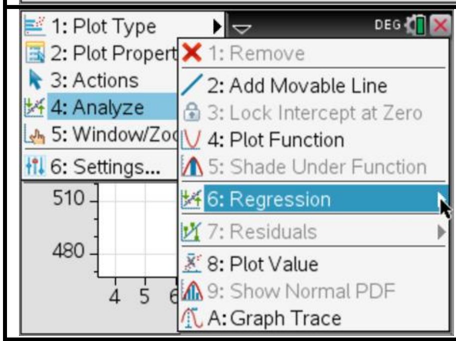
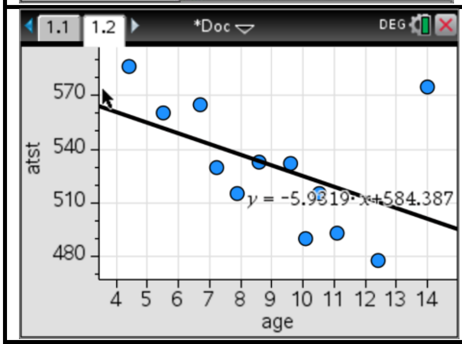
<table>
<tr><th>GDC display</th><th>Main GDC keys</th><th>Remarks</th></tr>
<tr><td>
<table>
<tr><th></th><th>A age</th><th>B atst</th><th>C</th><th>D</th></tr>
<tr><td>=</td><td></td><td></td><td></td><td>=LinRegM</td></tr>
<tr><td>1</td><td>4.4</td><td>586</td><td>Title</td><td>Linear R...</td></tr>
<tr><td>2</td><td>6.7</td><td>565</td><td>RegEqn</td><td>m*x+b</td></tr>
<tr><td>3</td><td>10.5</td><td>515</td><td>m</td><td>-5.9319</td></tr>
<tr><td>4</td><td>9.6</td><td>532</td><td>b</td><td>584.387</td></tr>
<tr><td>5</td><td>12.4</td><td>478</td><td>r²</td><td>0.231171</td></tr>
</table>
D1 ="Linear Regression (mx+b)"</td><td></td><td>Read off answers by scrolling down to the r row.</td></tr>
<tr><td>
<table>
<tr><th></th><th>A age</th><th>B atst</th><th>C</th><th>D</th></tr>
<tr><td>=</td><td></td><td></td><td></td><td>=LinRegM</td></tr>
<tr><td>4</td><td>9.6</td><td>532</td><td>b</td><td>584.387</td></tr>
<tr><td>5</td><td>12.4</td><td>478</td><td>r²</td><td>0.231171</td></tr>
<tr><td>6</td><td>5.5</td><td>560</td><td>r</td><td>-0.4808...</td></tr>
<tr><td>7</td><td>11.1</td><td>493</td><td>Resid</td><td>{27.7132...</td></tr>
<tr><td>8</td><td>8.6</td><td>533</td><td></td><td></td></tr>
</table>
D6 =-0.48080288203862</td><td></td><td>A more accurate value of r can be read off from the row at the bottom of the window.</td></tr>
</table>

1.4 Using GDC to obtain a scatter diagram

<table>
<tr><th>GDC display</th><th>Main GDC keys</th><th>Remarks</th></tr>
<tr><td>
</td><td>ctrl + doc▾
(to add a new page)

Select “Add Data & Statistics”</td><td></td></tr>
<tr><td>
</td><td></td><td>If you named your columns, the data should be shown as random scattered points already.

Go the axes and click to add the appropriate variables for the *x*- and *y*- axis.</td></tr>
</table>

		You can now sketch the scatter plot based on the GDC graph.
	menu 4 6 1	You can also display the linear regression line on the scatter plot by selecting linear regression mx+b or a+bx (discussed in the next part)
		

2. Regression

When we first meet Statistics, we encountered random quantities (random variables) one at a time. Soon however we need to handle more than one random quantity at a time. Already we have to think about how they are related to each other.

There are linear and non-linear regressions, but for the IB syllabus, we will only focus on linear regression, where we model the relationship between two variables in a straight line.

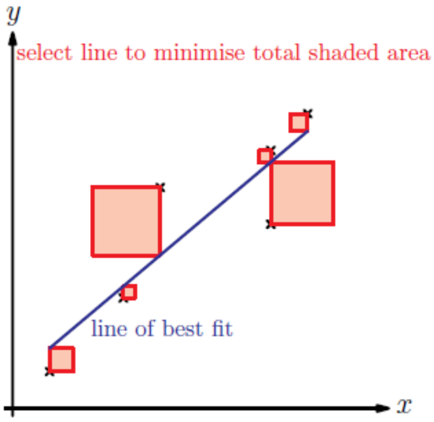
Some Definitions

- y is called the **response variable** and x the **explanatory variable.**

  We know more about y knowing x than not knowing x. Thus, knowledge of x explains, or accounts for, part but not all of the variability we see in y.
- Another name for x is the predictor variable. We may wish to use x to predict y (the prediction will be an uncertain one, to be sure, but better than nothing. There is information content in x about y, and we want to use this information).
- A third name for x is **the regressor, or regressor variable**. We will turn to the reason for this name in a later part of the notes.

We have studied the correlation coefficient, r, a numerical measure of the **degree** of **linear relationship** between two variables. Now, we will study a different problem where it is **known** that the **relationship** between two variables is linear, and we wish to determine that relationship by calculation. There are several ways of fitting the straight line. We have already discussed this at the beginning of this topic.

2.1 Regression line of y on x



When x is the controlled (independent) variable and y is the dependent variable, the line of closest fit is called the ***regression line*** of $\boldsymbol{y}$ on $\boldsymbol{x}$.

We make use of the ***least squares method*** (proof not required) to find the regression line of y on x, given by $y = a + bx$.

2.2 Regression line of x on y

When y is the controlled (independent) variable and x is the dependent variable, the line of closest fit is called the regression line of $\boldsymbol{x}$ on y.

Similarly, we use the least squares method to find the regression line of x on y, given by $x = c + dy$.

Notes

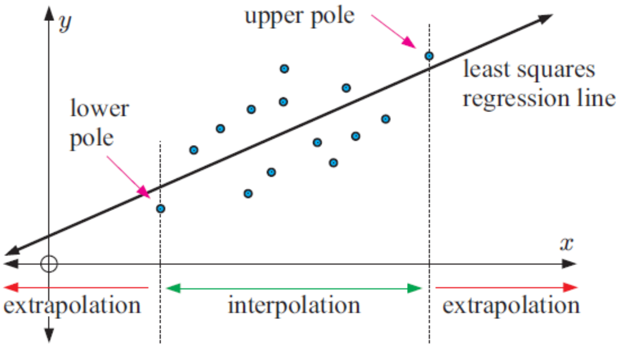
[1] In general, the two regression lines will be different. You can use the GDC to find the regression lines of *y*-on-*x* and *x*-on-*y*. Refer to page 125-126.

[2] Both regression lines pass through the point $(\bar{x},\bar{y})$. In another word, their intersection point is $(\bar{x},\bar{y})$.

[3] Using the regression line of *y*-on-*x*, we can predict *y* values given the *x* values. Similarly, using the regression line of *x*-on-*y*, we can predict *x* values given the *y* values.

2.3 Reliability of Regression Equation for Prediction

- Strength of correlation
- Interpolation vs extrapolation



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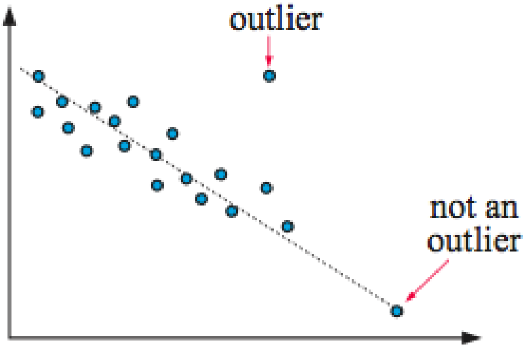
- Outliers



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Example 4: The score for Paper I and Paper II of the Mathematics HL Preliminary examination of 13 students are given in the following table:

Student	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**	**K**	**L**	**M**
Paper I (x)	76	86	46	104	76	49	62	82	54	86	70	70	92
Paper II (y)	70	71	34	102	80	58	54	77	64	94	61	78	73

(i) Plot the scatter diagram for the data.

(ii) Find the equation of the regression line of y on x and the equation of the regression line of x on y.

(iii) Student N did Paper I and scored 62 but was absent for Paper II. Estimate the score that student N would have obtained if he sat for Paper II.

(iv) Unlike student N, student O was absent for Paper I instead of Paper II in which he scored 89. Estimate the score that student O would have obtained if he sat for Paper I.

(v) Student P did Paper I and scored 105 but was absent for Paper II. Estimate the score that student N would have obtained if he sat for Paper II. Is this estimate accurate?

[(ii) $y = 8.36 + 0.847x, x = 14.8 + 0.831y$

(iii) 60.9 (iv) 88.7; (v) 97.2. No, we have extrapolated from the range of the given data]

Topic 4: Statistics and probability

WS 4.2: Correlation and regression

1. An anemometer is used to estimate wind speeds by observing the rotational speed of its vanes. This speed is converted to wind speed by means of an equation obtained from calibrating the instrument in a wind tunnel. In this calibration process, the wind speed is fixed precisely and the resulting anemometer speed is noted. For a particular anemometer, this process produced the following set of data.

Actual wind speed (m/s), *s*	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
Anemometer (revs/min), *r*	30	38	48	58	68	80	92	106	120	134

(i) Obtain the equation of the estimated least squares regression line of r on s.

(ii) If the actual wind speed is 1.65 m/s, use the equation of the regression line found in (i) to estimate the rotational speed of the anemometer.

(iii) Demonstrate, using the above regression line as an example, that it is unwise to extrapolate beyond the range of data.

2. The age of many types of trees can be determined by counting the growth rings in a cross-section of the trunk once it has been felled. A random sample of ten oak trees in a timber yard was taken and the age of each tree determined. The average girth of each trunk was also measured and the data concerning ages and girth is shown below.

Age (*x* years)	20	23	30	38	39	45	45	48	55	71
Girth (*y* units)	26	30	45	48	46	60	64	68	70	92

(i) Obtain the equation of the estimated regression line of girth on age.

(ii) Estimate the average girth of the trunk from a 35-year-old oak tree.

3. The table below shows the engine capacity and price of new cars in a certain country in January 2014.

Car model	**A**	**B**	**C**	**D**	**E**	**F**	**G**	**H**	**I**	**J**
Engine capacity, *x* (thousand cc)	1.0	1.3	1.8	2.2	2.0	0.6	0.7	1.5	1.5	1.7
Price, *y* (hundred dollars)	4.0	4.2	5.2	7.0	7.0	2.2	2.2	4.2	3.0	6.2

(i) Calculate the linear product-moment correlation coefficient, r.

(ii) Find the least squares regression line of price on engine capacity.

4. The yield (per square metre) of a crop, *c*, is believed to depend on the May rainfall, *m*. For 9 regions, records are kept of the average values of *c* and *m*, and these are recorded below:

c	8.3	10.1	15.2	6.4	11.8	12.2	13.4	11.9	9.9
m	14.7	10.4	18.8	13.1	14.9	13.8	16.8	11.8	12.2

(i) Find the equation of the appropriate regression line.
(ii) Find *r*, the linear product-moment correlation coefficient between *c* and *m*.
(iii) In a tenth region, the average May rainfall was 14.6. Estimate the average yield of the crop for that region, giving your answer correct to one decimal place.

5. The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of *x* on *y*. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of *a* and the value of *b*.
(b) Let L_2 be the regression line of *y* on *x*. The line L_1 and L_2 pass through the same point with coordinates (p, q). Find the value of *p* and the value of *q*.
(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's marks on the first test. Comment on the reliability of this estimation.

Answers:

1. (i) $r = 116\,s - 90.8$; (ii) 101 rev / min
2. $y = 1.29x + 1.51$, $y = 46.7$
3. (i) $r = 0.901$,
   (ii) $y = 3.04x + 0.174$
4. (i)$c = 2.48 + 0.607m$ (ii) 0.593 (iii) 11.4
5. (a) $a = 1.29, b = -10.4$ (b) $p = 28.7, q = 30.3$ (c) 27.1

Topic 4: Statistics and probability

4.3 Probability

1. Probability

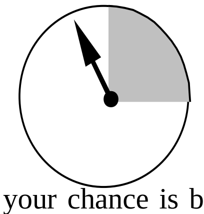
Probability is a measure of the likelihood of an event occurring.

An event that is more likely to occur has a higher probability than one that is less likely to occur.

Consider the following situations. Try to guess which situation will give you the best chance:

(a) Getting a number "6" when you roll a fair standard six-sided die;

(b) The pointer in the diagram being in the shaded region when the pointer pivoted at the centre is able to rotate freely on a disc as shown.



From the above situations, the most likely notion that you have used to evaluate your chance is by looking at all the possible situations and weighing it against your case. This will form the basis of an understanding of probability.

Related Concepts and Definitions

1. **Experiment or trial**: Used to describe any process that generates raw data. Some common examples include throwing a die, tossing a coin etc.
2. **Outcome**: Possible results in an experiment. Example: {1} is one outcome of tossing a die and {Heads} is one outcome of tossing a coin.
3. **Sample Space** (ε, pronounced as "epsilon"): The set of all possible outcomes of an experiment. Example: ε = {1, 2, 3, 4, 5, 6} is the sample space when you toss a fair 6-sided die.
4. Fair / unbiased / random: Each outcome is **equally likely** to occur. When you toss a fair 6-sided die, the chance of obtaining any of the numbers {1, 2, 3, 4, 5, 6} is the same, i.e. the outcomes are equally likely.
5. $n(\varepsilon)$: Denotes the number of elements in set ε if ε is finite.

Examples:

Throwing a die, ε = {1, 2, 3, 4, 5, 6};

Tossing 2 coins, ε = {(H, H), (H, T), (T, H), (T, T)}

Each outcome is called a **sample point**.

An **event** A is a subset of the sample space ε such that

$$\therefore \underbrace{\boldsymbol{n}(A)}_{\text{No. of sample points in } A} \le \underbrace{\boldsymbol{n}(\varepsilon)}_{\text{No. of sample points in } \varepsilon}$$

When the experiment is **fair**,

$$\boxed{\text{Probability of the event } A,\ P(A) = \frac{n(A)}{n(\varepsilon)}}$$

Notes:

1. The calculation of $n(A)$ and $n(\varepsilon)$ can be done by using Permutation and Combination (P&C).

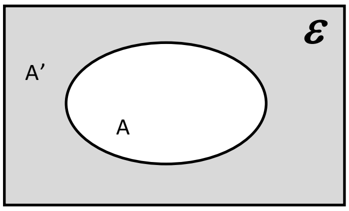
2. Listing / counting of **ALL** possible cases is important as $n(A)$ is divided by $n(\varepsilon)$.
   For questions involving permutations/combinations, $n(\varepsilon)$ = Number of permutations/combinations without restrictions.

3. $\mathbf{0 \le P(A) \le 1}$ for any event **A**.

4. When ***P*(*A*) = 0**, ***A*** is an **impossible event**. In particular ***A*** = ∅
   When ***P*(*A*) = 1**, which is a **sure (absolutely certain) event**. In particular ***A*** = $\boldsymbol{\varepsilon}$, which implies that total probability is 1.

5. If ***A'*** denotes the ***Complement*** of event ***A***, then ***P*(*A'*) = 1 – *P*(*A*)**

6. The Venn Diagram is useful visually in illustrating the relation between events.



Example 1: Two dice are thrown. What is the probability that the sum of the numbers on the 2 dice is 10? [$\frac{1}{12}$]

Solution:

A table of outcomes is usually useful in solving such problems.

		Die 1					
	Sum (+)	**1**	**2**	**3**	**4**	**5**	**6**
Die 2	**1**	2	3	4	5	6	7
	2		4	5	6	7	8
	3			6	7	8	9
	4				8	9	10
	5					10	11
	6						12

Example 2: An experiment involving tossing a coin and rolling a fair die was conducted. Find the probabilities of: **(a)** tossing a head, **(b)** getting a tail and a 5, **(c)** getting a tail or a 5.

[$\frac{1}{2}$; $\frac{1}{12}$; $\frac{7}{12}$]

Example 3: What is the probability to draw three white marbles from a bag of 11 marbles with only 5 white marbles without replacement?

[$\frac{2}{33}$]

Solution

Total number of ways to draw three marbles =

Total number of cases with 3 white marbles =

$\therefore$ Required probability =

Example 4: A committee of 5 is to be formed from a group of 9 people consisting of 3 boys, 4 girls and a brother-sister pair. What is the probability that

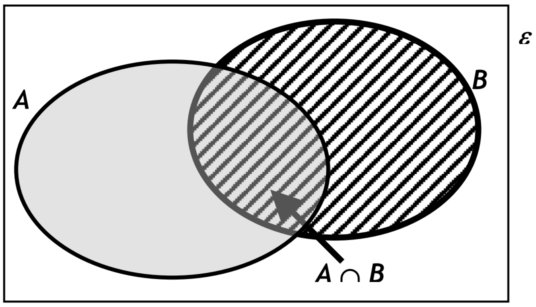
(i) the committee will include the brother-sister pair;

(ii) a particular girl is not in the committee together with a particular boy? $[\frac{5}{18} ; \frac{13}{18}]$

2. Laws of Probability

2.1 Addition Law

In a Venn Diagram, we can see that in general,



$A \cap B \neq \varnothing$

If ***A*** and ***B*** are 2 events of the same experiment, then $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

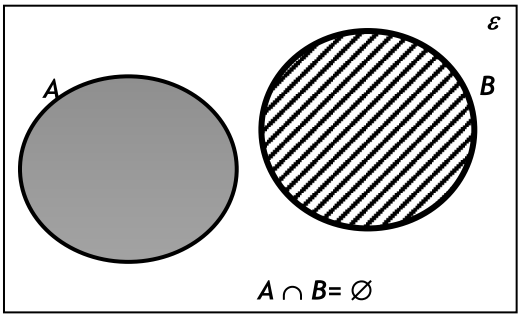
In set notation: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Intuitively, we may think of $A \cup B$ as the total area ● of the shaded region. $A \cap B$ is counted twice when you add up the areas ***A*** and ***B*** individually. That is why you need to subtract the area $A \cap B$ **.**

2.2 Mutually Exclusive Events

Events which cannot occur simultaneously are said to be ***mutually exclusive.*** (i.e. events with no points in common). In set notation, $P(A \cap B) = 0$.

In a Venn Diagram:



Examples

(a) A coin is tossed. The event, tossing a head or a tail, are mutually exclusive (both events cannot occur in one throw)

(b) A card is drawn from a pack. The events, drawing a heart and drawing an ace, are ***not*** mutually exclusive. (ie. You can draw the ace of heart.)

Hence, if two events are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$

An extension: If events $A_1, A_2, ...A_n$ are **mutually exclusive** events, we will have

$$P(A_1 \cup A_2 \cup ... \cup A_n) = P(A_1) + P(A_2) + ... + P(A_n)$$

Note: We use this concept of mutually exclusive events when we consider cases and add up the total probability based on the probabilities of the individual cases.

Example 5: A fair die is thrown. Find the probability of the following events:

(i) 1 or 2 **(ii)** an even number **(iii)** an even number or a number less than 4.

[$\frac{1}{3}$; $\frac{1}{2}$; $\frac{5}{6}$]

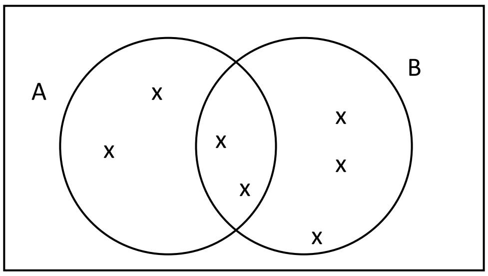
Solution: **(i)** P(1 or 2) =

(ii) P(an even number) =

(iii) P("an even number" or "number less than 4")

2.3 Exhaustive Events

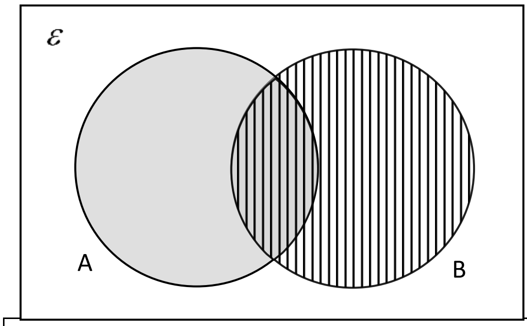
Two events, A and B, are said to be exhaustive if together they include all possible outcomes in the sample space, i.e. $n((A \cup B)') = 0$



When A and B are exhaustive, $P(A \cup B) = 1$

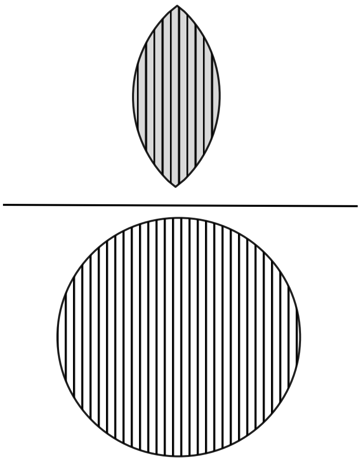
Example 6: Given $P(X) = \frac{4}{5}$, $P(Y) = \frac{1}{2}$ and $P(X \cap Y) = \frac{3}{10}$, show that the events X and Y are exhaustive.

3. Conditional Probability



Graphically,

$P(A \mid B) =$



If A and B are two events and $P(A) \neq 0$ and $P(B) \neq 0$, then the conditional probability of A, given that B has already occurred is written as $P(A \mid B)$.

By definition, $$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

Note: It is useful to understand how conditional probabilities can be represented on probability tree diagrams. In fact, probability tree diagrams are often very helpful for solving of conditional probability problems.

Example 7: A dice is thrown once. If A is the event 'the number is less than 4' and B is the event 'the number is odd'. Find $P(A|B)$. [$\frac{2}{3}$]

Example 8: A and B are events, and A' denotes the complementary event to A. The following probabilities are given: $P(A) = 0.4$, $P(B|A) = 0.7$, $P(A' \cap B) = 0.3$.

Find the probabilities:

(i) $P(A \cap B)$ **(ii)** $P(B)$ **(iii)** $P(A \cup B)$ **(iv)** $P(A|B)$

[0.28; 0.58; 0.7; $\frac{14}{29}$]

3.1 Independent Events

Let A and B be 2 events. If the probability of occurrence of one of them is not influenced by the occurrence of the other, then events A and B are **independent**; and

$$\boxed{P(A|B) = P(A) = P(A|B') \text{ and } P(B|A) = P(B).}$$

Recall $P(A \cap B) = P(A)P(B|A) = P(B)P(A|B)$

Therefore $\boxed{P(A \cap B) = P(A) \times P(B)}$ ***Multiplication Law***

This is also the rule to check whether 2 events are independent.

NOTE:

1. If events A and B are independent, then

$$\left.\begin{array}{l} A' \text{ and } B \\ A \text{ and } B' \\ A' \text{ and } B' \end{array}\right\} \text{are also independent.}$$

Independence and mutual exclusivity are two *separate* concepts – do not confuse them!

Example 9: If two fair dice are thrown, find the probability of

- **(i)** an even number with the first dice and a 3 or a 5 with the second dice;
- **(ii)** at least one even number appearing on the two dice.
- **(iii)** the difference between the two numbers is 2.

[$\frac{1}{6}$; $\frac{3}{4}$; $\frac{2}{9}$]

4. Bayes' Theorem

4.1 Introduction

In conditional probability we tried to solve the following question:

Given that event A has happened, what is the probability that event B will occur?

$$P(B \mid A)$$

For Bayes' Theorem, we are effectively asking the "reverse" question:

Given that event B is the outcome, what is the probability that event A happened before event B?

$$P(A \mid B)$$

The classic application of Bayes' Theorem is in calculating the probabilities of "False Positive" and "False Negative" in the field of medical testing.

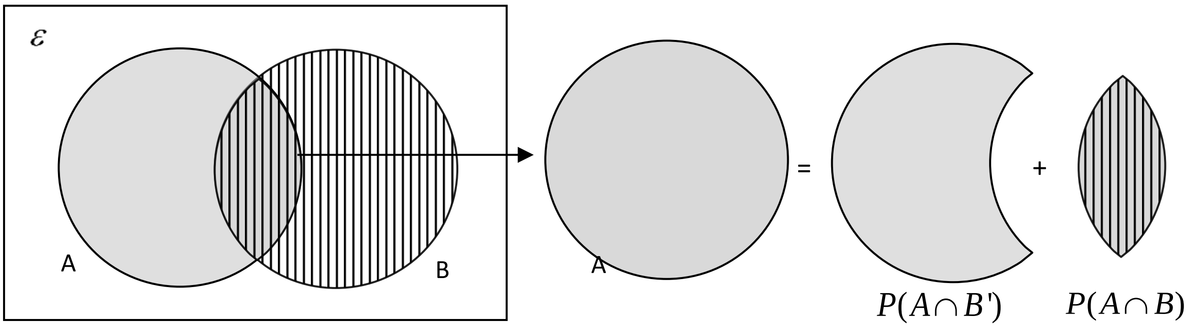
4.2 Total Probability Theorem

Theorem:

Let *A* and *B* be two events. Then $P(A) = P(B)P(A \mid B) + P(B')P(A \mid B')$.

The total probability theorem states that the whole is the sum of its parts.

A simple illustration of the general idea is shown below.



Translating the diagram into probability statements that uses the fact $A \cap B$ and $A \cap B'$ are mutually exclusive:

$$P(A) = P(A \cap B) + P(A \cap B')$$

$$= P(B) \times P(A \mid B) + P(B') \times P(A \mid B')$$

The result can be generalized easily to m 'slices' as follows. Suppose that $B_1,\ B_2, B_3, \ldots, B_m$ are m mutually exclusive and exhaustive events in the sample space ε. Let A be some other event. A formal statement of the total probability theorem is that for these events:

$$P(A) = \sum_{i=1}^{m} P(A \cap B_i) = \sum_{i=1}^{m} P(B_i) \times P(A \mid B_i)$$

Example 10: *[Graham Upton et. al.]* Of those students who do well in Physics, 80% also do well in Mathematics. Of those who do not do well in Physics, only 30% do well in Mathematics. If 40% do well in Physics, what proportion do well in Mathematics? [0.5]

Solution:

Define the events A, B_1 and B_2 as follows:

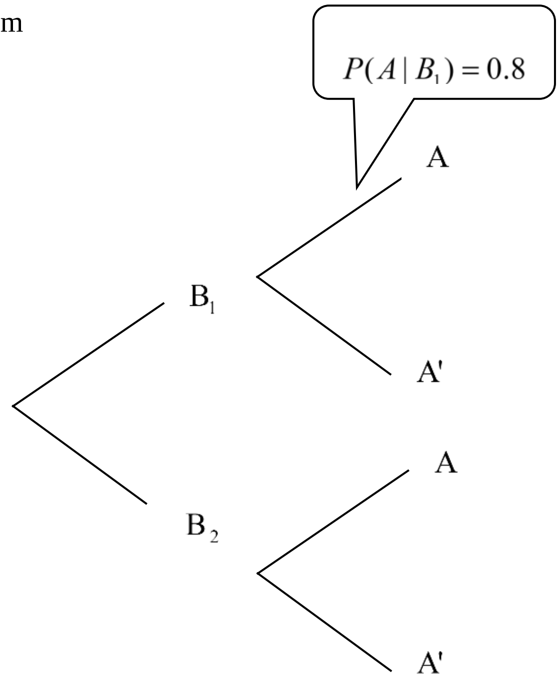
A: does well in Mathematics

B_1: does well in Physics

B_2 : does not do well in Physics

(a) Using Total Probability Theorem

(b) Using tree diagram

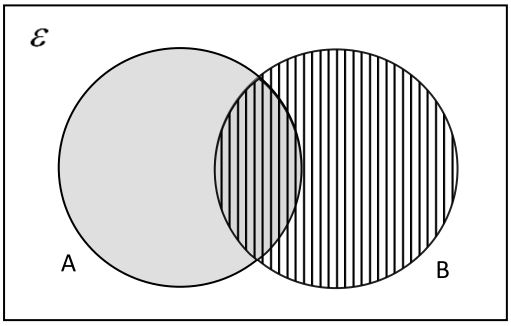


Tree diagram is a very useful tool. Do take note for second layer onwards, the probabilities are conditional.

4.3 Bayes' Theorem

Bayes' Theorem: Let *A* and *B* be events. Then $P(B\,|\,A)=\dfrac{P(B)\times P(A\,|\,B)}{P(B)\times P(A\,|\,B)+P(B')\times P(A\,|\,B')}$.

Proof:



Since

$$P(A\cap B)=P(A)\times P(B\,|\,A)=P(B)\times P(A\,|\,B)$$

and

$$P(B\,|\,A)=\frac{P(B\cap A)}{P(A)}=\frac{P(A\cap B)}{P(A)}$$

Substitute $P(A)$ using total probability theorem,

$$P(B\,|\,A)=\frac{P(B)\times P(A\,|\,B)}{P(B)\times P(A\,|\,B)+P(B')\times P(A\,|\,B')}$$

And to state formally, if there were m alternative previous events that could have happened, namely $\mathrm{B}_1,\ \mathrm{B}_2,\mathrm{B}_3,...,\mathrm{B}_m$ and they are **independent** and **exhaustive**, then

$$P(B_i|\mathrm{A})=\frac{P(B_i)\times P(A\,|\,B_i)}{P(A)}=\frac{P(B_i)\times P(A\,|\,B_i)}{\sum_{i=1}^{m}P(\mathrm{B}_i)\times P(\mathrm{A}|\mathrm{B}_i)}$$

Example 11:[1] It is given that B_1 and B_2 are mutually exclusive and **exhaustive**, and $P(A\,|\,B_1)=0.3,\ \ P(A\,|\,B_2)=0.4, P(B_1)=0.4.$ Find

(a) $P(B_1\,|\,A)$

(b) $P(B_2\,|\,A)$

(c) $P(B_1\,|\,A')$

(d) Can you write down the value of $P(B_2\,|\,A')$ given what you have observed in the relationship between part (a) and (b). [Ans: (a)1/3(b)2/3(c)0.4375 (d)0.5625]

False Positive and False Negative:

		Test Outcome	
		Positive	**Negative**
Reality	**Positive**	True Positive	***False Negative***
	Negative	***False Positive***	True Negative

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[1] Graham et. al. 1996, *Understanding Statistics*, Oxford University Press.

Example 12:[2]

A new blood test has been designed to detect a form of cancer. The probability that the test correctly identifies someone with the cancer is 0.97 and the probability that the test correctly identifies someone without the cancer is 0.93. Approximately 0.1% of the general population are known to contract this cancer.

When a patient has a blood test, the test results are positive for the cancer. Find the probability that the patient has the cancer. [Ans: 97/7090]

(a) Using Bayes' Theorem, (b) Using tree diagram,

Example 13: *[Haese & Harris]*

12% of the over-60 population of a country have lung cancer. Of those with lung cancer, 50% were heavy smokers, 40% were moderate smokers and 10% were non-smokers. Of those without lung cancer, 5% were heavy smokers, 15% were moderate smokers and 80% were non-smokers. A member of the over-60 population of Agento is chosen at random. Find the probability that the person

(a) was a heavy smoker;
(b) has lung cancer given the person was a moderate smoker;
(c) has lung cancer given the person was a non-smoker.

[Ans: (a) 0.104 (b) 4/15 (c) 3/179]

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[2] Haese & Harris, *Higher Level Mathematics for International Baccalaureate*, 3rd Edition.

Topic 4: Statistics and probability

WS 4.3(a): Probability

1. Two balls are taken at random from a box containing three black, three red and three yellow balls. Find the probability that

(a) both are red,

(b) neither of the balls removed are red,

(c) at least one is red.

2*. The probability of an archer hitting the bull's eye with any one shot is $\frac{1}{5}$. Find the probability that

(a) he hits the bull's eye with his second shot,

(b) he hit the bull's eye exactly once in three shots,

(c) he hit the bull's eye at least once in four shots.

3. Two coins are tossed. One coin is fair and the other is biased so that throwing a head is three times as likely as throwing a tail. Find the probability that

(a) on one toss of both coins, they both land head up,

(b) on two tosses of both coins, two tails are thrown both times.

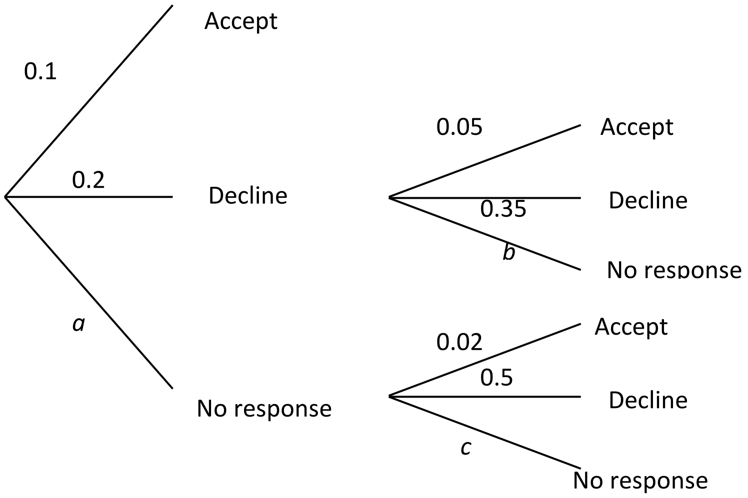
4*. A shelf has fifteen paperback and twelve hardback novels on it. Three novels are selected at random, one by one, without replacement. Find the probability that

(a) all the novels selected are paperbacks,

(b) the second hardback novel is removed on the third selection.

5. A sales company telephones a large number N of potential customers with a special offer. The proportion of these customers who accept the offer is 10% and the proportion who decline the offer is 20%. The rest make no response. Those who decline the offer are telephoned again, and, of these, 5% accept the offer, 35% decline the offer and the rest make no response to this second call. Those who make no response to the first call are telephoned again and, of these, 2% accept the offer, 50% decline the offer, and the rest again make no response to this second call.

The information is shown on the tree diagram below.



(a) Find the values of a, b and c.

(b) Find the probability that a customer, randomly chosen from the N potential customers,

- **(i)** declines the offer in both calls,
- **(ii)** accepts the offer,
- **(iii)** declines the offer in the second call, given that the offer was not accepted in the first call.

(c) Two customers are chosen at random from the N potential customers. Find the probability that both of them are telephoned twice and both make no response to the second call.

6. Let E and F be events with $P(E)=\frac{3}{5}, P(F)=\frac{3}{10}$ and $P(E \cap F)=\frac{1}{5}$. Are the events E and F independent events?

7. For events A and B, it is known that $P(A)=P(B)$ and $P(A \cap B)=0.1$ and $P(A \cup B)=0.7$. Find $P(A)$.

8. Given that $P(A')=\frac{2}{3}, P(B)=\frac{1}{2}$ and $P(A \cap B)=\frac{1}{12}$, find $P(A \cup B)$.

9. If A and B are independent events with $P(A)=0.3$ and $P(B)=0.4$.Find

(a) $P(A\cap B)$

(b) $P(A\cup B)$

(c) $P(A/B)$

(d) $P(B/A)$

10. X and Y are events such that $P(X)=\frac{2}{5}, P(X\,|\,Y)=\frac{1}{2}$ and $P(Y\,|\,X)=\frac{2}{3}$. Find

(a) $P(X\cap Y)$

(b) $P(Y)$

(c) $P(X\cup Y)$

11. $P(A)=\frac{1}{3}, P(B\,|\,A)=\frac{1}{4}$ and $P(B'\,|\,A')=\frac{4}{5}$. By drawing a tree diagram, find

(a) $P(B'\,|\,A)$

(b) $P(A\cap B)$

(c) $P(B)$

(d) $P(A\cup B)$

12. If $P(A)=\frac{7}{13}, P(B)=\frac{9}{13}$ and $P(A\cap B)=\frac{4}{13}$, find $P(A\,|\,B)$.

13. A couple has two children. What is the probability that both the children are boys given that at least one of them is a boy?

14. Ten cards numbered 1 to 10 are placed in a box, mixed up thoroughly and then one card is drawn randomly. If it is known that the number on the card is more than 3 , what is the probability that it is an even number?

15. In a school, there are 1000 students out of which 430 are girls. It is known that out of the 430 girls, 10% study in year 6. What is the probability that a student selected randomly studies in year 6 given that the chosen student is a girl?

16. A die is thrown three times. Events A and B are defined as follows:

A: 4 turns up in the third throw

B: 6 turns up in the first throw and 5 in the second throw.

Find the probability of A given that B has already occurred.

17. A die is thrown twice. What is the probability that the number 4 has appeared at least once given that the sum of the numbers appearing is a 6?

18. Consider the following experiment. A coin is tossed once. If the coin shows a head, it is tossed again and the experiment stops. If it shows tail, then a die is thrown once and the experiment stops. Find the conditional probability of the event that "the die shows a number greater than 4" given that "there is at least one tail".

19. Bag A contains 3 red and 4 black balls while another bag B contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and it is found to be red. Find the probability that it was drawn from bag B.

20. In a factory which manufactures bolts, machines A, B and C manufacture 25%, 35% and 40% of the bolts respectively of the bolts respectively. Of their outputs 5, 4 and 2 percent are respectively defective bolts. A bolt is drawn at random from the product and is found to be defective. What is the probability that it is manufactured by the machine B.

21. Suppose that the reliability of a HIV test is specified as follows:

> Of the people having HIV, in 90% cases the test detects the disease but in 10% cases the disease goes undetected. Of the people free of HIV, 99% of the cases are judged HIV negative but 1% are diagnosed as showing HIV positive. From a large population of which only 0.1% have HIV, one person is selected at random, given the HIV test, and the pathologist reports him/her as HIV positive.

What is the probability that the person actually has HIV?

Answers:

1. (a) $\frac{1}{12}$ (b) $\frac{5}{12}$ (c) $\frac{7}{12}$ 2. (a) $\frac{1}{5}$ (b) $\frac{48}{125}$ (c) $\frac{369}{625}$ 3. (a) $\frac{3}{8}$ (b) $\frac{1}{64}$

4. (a) $\frac{7}{45}$ (b) $\frac{44}{195}$ 5. (a) $a = 0.7, b = 0.6, c = 0.48$ (b) (i) 0.07 (ii) 0.124 (iii) $\frac{7}{15}$ (c) 0.208

6. Not independent events 7. 0.4 8. 0.75 9. (a) 0.12 (b) 0.58 (c) 0.3 (d) 0.4

10. (a) $\frac{4}{15}$ (b) $\frac{8}{15}$ (c) $\frac{2}{3}$ 11. (a) $\frac{3}{4}$ (b) $\frac{1}{12}$ (c) $\frac{13}{60}$ (d) $\frac{7}{15}$

12. $\frac{4}{9}$ 13. $\frac{1}{3}$ 14. $\frac{4}{7}$ 15. 0.1

16. $\frac{1}{6}$ 17. $\frac{2}{5}$ 18. $\frac{2}{9}$ 19. $\frac{35}{68}$

20. $\frac{28}{69}$ 21. $\frac{10}{121}$

Topic 4: Statistics and probability

WS 4.3(b) : Bayes' Theorem

1. Events B_1, B_2, B_3 are exhaustive ($\sum_{i=1}^{3} P(B_i) = 1$).

Given that $P(B_1) = 0.35$, $P(B_2) = 0.4$, $P(A | B_1) = 0.7$, $P(A | B_2) = 0.85$ and $P(A | B_3) = 0.78$,

(i) Draw a tree diagram to help you find the probability $P(A)$

(ii) Find the probability $P(B_2 | A)$ using Bayes' theorem.

2. A pair of fair dice is thrown. Given that at least one of them shows a 2, what is the probability of the total being 7?

3. Mr. Tan's gardener is not dependable; the probability that he will forget to water the rosebush during Mr Tan's absence is $\frac{2}{3}$. The rosebush is in questionable condition anyhow; if watered, the probability for its withering is $\frac{1}{2}$; if it is not watered, the probability for its withering is $\frac{3}{4}$. Upon returning Mr. Tan finds that the rosebush has withered. What is the probability that the gardener did not water the rosebush?

4. A department store employs four gift wrappers at Christmas time. Betty, who works long hours, wraps 36% of all packages and fails to remove the price tag 2% of the time and Susan who also works long hours, wraps 40% of all packages and fails to remove the price tag 3% of the time. The other two, Carol and Janet, work relatively shorter hours; Carol wraps 10% of all packages and fails to remove the price tag 6% of the time, and Janet wraps 14% of all packages and fails to remove the price tag 8% of the time.

What is the probability that a gift bought at this store which did not have its price tag removed was wrapped by Betty?

5. Randy has enrolled as a freshman at a local Management University and that the probability that he will get a scholarship is 0.35. If he gets a scholarship the probability that he will graduate is 0.82 and if he does not get a scholarship the probability that he will graduate is only 0.44.

i) What is the probability that he will graduate?

ii) Suppose that years later we hear that Randy graduated from the given university. What is the probability that he did get the scholarship?

6. In a T-maze, a rat is given food if it turns left and an electric shock if it turns right. On the first trial there is a fifty-fifty chance that a rat will turn either way; then, if it receives food on a given trial the probability that it will turn left on the next trial is 0.72 and if it receives electric shock on a given trial the probability that it will turn left on the next trial is 0.84.

- **i)** What is the probability that a rat will turn left on the second trial?
- **ii)** Assuming that the probability of a rat turning left on any given trial depends only on what it did in the preceding trial, find the probability that a rat will turn left on the third trial.
- **iii)** With reference to part (i), suppose that a rat turns left on the second trial. What is the probability that it had turned left on the first trial?

7. The odds are 3 to 2 that a famous Kenyan distance runner will enter the Army Marathon. If he does not enter, the probability that the European champion will win is 0.66, but if he enters, the probability that the European champion will win is only 0.18.

- **i)** What is the probability that the European champion will win the race?
- **ii)** If the European champion won the race, what odds should we give that the famous Kenyan runner did not enter?

8. The probability that a bus from Geylang to Toa Payoh will leave on time is 0.80, and the probability that it will leave on time and also arrive on time is 0.72.

- **i)** What is the conditional probability that a bus which leaves on time will also arrive on time?
- **ii)** If the probability that such a bus will arrive on time is 0.75, what is the conditional probability that a bus which arrives on time also left on time?
- **iii)** Suppose the probability that a bus which leaves Geylang late will arrive on time in Toa Payoh is 0.78. What is the probability that any one of these buses from Geylang to Toa Payoh will arrive on time?

9. A new blood test has been shown to be effective in the early detection of a disease. The probability that the blood test correctly identifies someone with this disease is 0.99, and the probability that the blood test correctly identifies someone without that disease is 0.95. The incidence of this disease in the general population is 0.0001.

A doctor administered the blood test to a patient and the test result indicated that this patient had the disease. What is the probability that the patient has the disease?

10. A car is made in three versions: 2-door, 4-door and hatchback. The proportions of the three types made are 25%, 40% and 35% respectively. Each version of the car has either a 1400cc engine or a 1600cc engine. Of the 2-door version, 70% have 1400cc engines. The proportions of 1400cc engines for the 4-door and hatchback versions are 40% and 35% respectively.

In a publicity stunt, the car makers choose an owner at random to receive a prize of free car servicing. Given that the owner's car has a 1600cc engine, find the probability that it is a 2-door car.

Answers:

1.(i) 0.78   (ii) 0.436

2. $\frac{2}{11}$

3.0.75

4. 0.198

5.(i) 0.573   (ii) 0.501

6.(i) 0.78   (ii) 0.746   (iii) 0.462

7.(i) 0.372 (ii) 0.710 or odds of roughly 7 to 3

8.(i) 0.90 (ii) 0.96 (iii) 0.876

9.0.00198

10.0.138

Topic 4: Statistics and probability

4.4 Discrete Random Variables

1. Discrete Random Variables

When a random experiment is carried out, we are often interested in the numbers that are associated to the outcomes/events involving an element of chance or interested in the values of a random variable.

By assigning a real number x_r to **each** event E_r in the sample space S, we then have a function X defined at all points of the sample space. (i.e. X takes the value x_r when the event E_r occurs.)

This function X is called a **random variable.**

Generally, statisticians use a capital letter to represent a random variable and a lower-case letter, to represent one of its values.

- ***X*** represents the random variable *X* and ***P(X)*** represents the probability distribution of ***X.***
- ***P(X = x)*** refers to the probability when the random variable ***X*** is equal to a particular value, denoted by ***x***. For example, *P(X = 1)* refers to the probability when the random variable *X* is equal to *1*. Generally, ***P(X = x)*** is a function which generates the probability at any valid value of ***X.***
- The sum of probabilities $\sum_{\text{all } r} P\left(X = x_r\right) = 1$ for all values of *x* in the event space.

For example, when an unbiased coin is tossed twice in succession and we are interested in the number of heads obtained, then:

Outcome	**Probability**	**Number of heads**
TT	$\frac{1}{4}$	
TH	$\frac{1}{4}$	
HT	$\frac{1}{4}$	
HH	$\frac{1}{4}$	

Define X to be "the number of heads obtained", then we note that X takes values $\{0,1,2\}$ and $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$ and $P(X = 2) = \frac{1}{4}$.

We notice that

(i) X takes exact values : 0 , 1 , 2. (ii) Each value of X has some probability between 0 & 1.

(iii) $P\left(X = 0\right) + P\left(X = 1\right) + P\left(X = 2\right) = 1$

In this case, we called X a **discrete random variable.**

Discrete Random Variable

A discrete random variable *X* has the following properties:

(i) it can only assume values $x_1, x_2, x_3, \ldots, x_n, \ldots$

(ii) the probabilities associated are such that

$0 \le P\left(X = x_i\right) \le 1$ for all $i = 1, 2, \ldots, n, \ldots$

(iii) $\sum_{\text{all } x} P(X = x) = 1$

Example 1 : Let *X* be the number of 4's obtained when 2 fair dice are thrown. Then

Outcome	**Probability**	**Number of 4's**
$(4', 4')$	$\frac{25}{36}$	0
$(4', 4)$	$\frac{5}{36}$	1
$(4, 4')$	$\frac{5}{36}$	1
$(4, 4)$	$\frac{1}{36}$	2

So, *X* takes values 0, 1 and 2 and the **probability distribution** can be expressed in what is called a probability distribution table:

x	0	1	2
$P\left(X = x\right)$	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

Check that: $\sum_{\text{all } x} P(X = x) = 1$

$\therefore$ *X* is a discrete random variable.

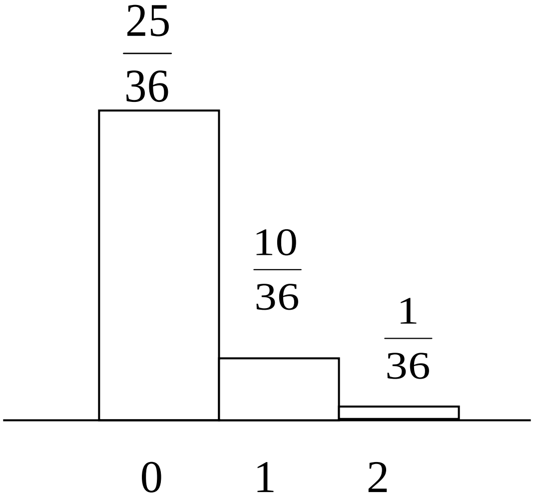
2. Probability Density Function

The ***probability distribution*** of X is a table listing all possible values X can take on, together with the associated probabilities as in **Example 1**.

The function which is responsible for allocating probabilities is known as the ***probability density function (p.d.f.)*** of X. Sometimes, the pdf can be expressed as a formula.

The graph of the probability distribution of a discrete random variable is in the form of a **histogram**:

From **Example 1:**



Example 2: The p.d.f. of a discrete random variable *Y* is given by $P(Y=y)=cy^2$, for *y* = 1, 2, 3, 4 and 5 and *c* is a constant.

(i) Tabulate the probability distribution of *Y*.

(ii) Find the value of *c*. [Ans: $\frac{1}{55}$]

(iii) Find the values of (a) P($Y < 2$) , (b) P($Y \leq 2$) , (c) P($1 < Y \leq 3$) , (d) P($Y \geq 3$) .

[Ans: (a) $\frac{1}{55}$ (b) $\frac{1}{11}$ (c) $\frac{13}{55}$ (d) $\frac{10}{11}$]

Solution:

(i) The probability distribution of *Y* is:

y	1	2	3	4	5
$P(Y=y)$	c	$4c$	$9c$	$16c$	$25c$

3. Cumulative Distribution Function

Given a frequency distribution, the corresponding cumulative frequencies can be obtained by summing up all the frequencies up to a certain value. Similarly, if X is a discrete random variable, then the corresponding cumulative probabilities are obtained by summing up all the probabilities up to a certain value.

If X is a discrete random variable with pdf $P(X=x)$ for $x = x_1, x_2, x_3, ..., x_n$, then the **cumulative distribution function (c.d.f)** is given by $F(x) = P(X \le x) = \sum_{x=x_1}^{x_n} P(X=x)$.

4. Expectation or E(X)

Experimental Approach

Suppose we throw an unbiased dice 120 times and record the results:

Score, x	1	2	3	4	5	6	
Frequency, f	15	22	23	19	23	18	Total: 120

Then the mean score obtained is

$$\bar{x} = \frac{\sum fx}{\sum f} =$$

In the **long run**, what would we expect the mean score to be?

Theoretical Approach

For the discrete random variable X where X is "the number on the die", the probability distribution is

Score, x	1	2	3	4	5	6
$P(X=x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

Then the expected mean, $E(X)$, is obtained by $\sum_{i=1}^{6} x_i P(X = x_i)$

$=$

$=$

Important Note: If we have a statistical experiment, *an experimental approach* will result in *a frequency distribution* and *a mean value, a theoretical approach* results in *a probability distribution* and *an expected value.*

Thus:

Let *X* be a discrete random variable. The **expectation** (or **expected value**) of *X*, written as μ or $E\left(X\right)$, is defined by

$$\boxed{\mu = E(X) = \sum_{\text{all } x} xP(X = x)}$$

In general,

Let *X* be a discrete random variable. If $g\left(X\right)$ is any function of X , then in general

$$\boxed{E\left[g\left(X\right)\right] = \sum_{\text{all } x} g\left(x\right)P\left(X = x\right)}$$

Examples:

$$E(X^2) = \sum_{\text{all } x} x^2 P(X = x)$$

$$E(\sqrt{X}) = \sum_{\text{all } x} \sqrt{x}P(X = x)$$

$$E\left(\left|X\right|\right) = \sum_{\text{all } x} \left|x\right| P(X = x)$$

Example 3

Find $E\left(X\right)$ for the following probability distributions of *X*.

x	-2	-1	0	1	2
$P\left(X = x\right)$	0.3	0.1	0.2	0.1	0.3

Solution:

$$E(X) = \sum_{\text{all } x} xP(X = x) = 0$$

Note: In this case, we observe symmetry in the distribution, then $E\left(X\right)$ can be obtained by symmetry

i.e. $E\left(X\right) = 0$ (by symmetry).

5. Variance of *X*

Consider the discrete random variable *X* and let $E(X)=\mu$. The variance of *X*, written as σ^2 or $Var(X)$, is given by:

$$\sigma^2 = E(X-\mu)^2 = E(X^2)-\mu^2 = E(X^2)-(E(X))^2$$

The standard deviation of *X* is $\sigma = \sqrt{Var(X)}$. Do note that $\sigma \geq 0$.

Example 4

Find $Var(X)$ for the following probability distribution:

x	–3	–2	0	2	3
$P(X=x)$	0.3	0.3	0.2	0.1	0.1

[Ans: 4.2]

Example 5

The discrete random variable *X* has probability density function $P(X=x)=k|x|$, where $x=-3,-2,-1,0,1,2$ and 3. Find

(i) the value of the constant *k*;

(ii) $E(X)$;

(iii) $E(X^2)$;

(iv) the standard deviation of *X*;

(v) P(*X* is within 1 standard deviation of the mean)

[Ans: (i) $\frac{1}{12}$ (ii) 0 (iii) 6 (iv) $\sqrt{6}$ (v) $\frac{1}{2}$]

Solution:

$P(X=x)=k|x|$, $\quad x=-3,-2,-1,0,1,2$ and 3

i.e.

x	–3	–2	- 1	0	1	2	3
$P(X=x)$	$3k$	$2k$	k	0	k	$2k$	$3k$

6. Linear Transformations on *X*

Consider the random variable *X* and let *a* and *b* be non-zero constants.

Then $\boxed{\mathrm{E}(aX+b)=a\,\mathrm{E}(X)+b \text{ and } \mathrm{Var}(aX+b)=a^2\,\mathrm{Var}(X).}$

Proof :

$$\begin{aligned}
\mathrm{E}(aX+b) &= \sum_{\text{all } x}(ax+b)P(aX+b=ax+b) \\
&= \sum_{\text{all } x}(ax+b)P(X=x) \\
&= a\sum_{\text{all } x}xP(X=x)+b\sum_{\text{all } x}P(X=x) \\
&= a\,\mathrm{E}(X)+b(1)=a\,\mathrm{E}(X)+b
\end{aligned}$$

$$\begin{aligned}
\mathrm{Var}(aX+b) &= \mathrm{E}\left((aX+b)^2\right)-\left[\mathrm{E}(aX+b)\right]^2 \\
&= \mathrm{E}\left(a^2X^2+2abX+b^2\right)-\left(a\,\mathrm{E}(X)+b\right)^2 \\
&= a^2\,\mathrm{E}\left(X^2\right)+2ab\,\mathrm{E}(X)+b^2-\left(a^2\left[\mathrm{E}(X)\right]^2+2ab\,\mathrm{E}(X)+b^2\right) \\
&= a^2\,\mathrm{E}\left(X^2\right)-a^2\left[\mathrm{E}(X)\right]^2 \\
&= a^2\,\mathrm{Var}(X)
\end{aligned}$$

This result applies to both discrete and continuous random variables.

Topic 4: Statistics and probability

WS 4.4: Discrete Random Variables

1. The probability density function of the discrete random variable X is given by

$P(X=x)=a\left(\frac{3}{4}\right)^x$ for $x=0,1,2,\ldots$

(i) Find the value of the constant *a*.

(ii) Find the values of $P(X<3)$ and $P(X>2)$.

2. The probability distribution of a random variable X is as shown in the table:

x	1	2	3	4	5
$P(X=x)$	$\frac{1}{10}$	$\frac{3}{10}$	y	$\frac{2}{10}$	$\frac{1}{10}$

(i) Find the value of *y*.

(ii) Find the value of $E(X)$.

3. An unbiased die is in the form of a regular tetrahedron and has its faces numbered 1, 2, 3, 4 and when the die is thrown onto a horizontal table, the number of the face in contact with the table is noted. Two such dice are thrown and the score X is found by multiplying these numbers together.

(i) Obtain the probability distribution of *X*

(ii) Derive the values of **(a)** $P(X>8)$; **(b)** $E(X)$; **(c)** $Var(X)$.

4. A tetrahedral die with its faces numbered 1, 2, 3 and 4 is thrown and the number of the face on which it lands is noted.

(a) The 'score' is double the number on which it lands. Find the expectation and variance of the 'score'.

(b) A new experiment is set up where the 'score' is the sum of the numbers obtained when the die is thrown twice. Find the expectation and the variance of this new 'score'.

5. *X* and *Y* are independent random variables with probability density functions as shown:

x	0	1	2
$P(X=x)$	0.2	0.6	0.2

y	1	2	3
$P(Y=y)$	0.3	0.4	0.3

Construct the probability distribution of $X-Y$ and find

(a) $E(X-Y)$ **(b)** $Var(X-Y)$

6. A fair coin and a fair six-faced die are thrown simultaneously. The random variable *X* is defined as follows:

If the coin shows a head, then *x* is the score on the die.
If the coin shows a tail, then *x* is twice the score on the dice.

(i) Find μ, the expected value of *X* and show that $P(X < \mu) = \frac{7}{12}$.

(ii) Show that $Var(X) = \frac{497}{48}$.

7*. The probability distribution of a discrete random variable *X* is given by $P(X = x) = kx$ where $x = 1, 2, 3, \ldots .n.$

(a) Find *k* in terms of *n*.

(b) Find, in terms of *x* and *n*, an expression for $P(X \le x)$, the cumulative distribution function of *X*.

8*. The probability distribution of a discrete random variable *X* is given by $P(X = x) = k\left(\frac{2}{3}\right)^x$ where $x = 0, 1, 2, 3, \ldots .$

(a) Show that $P(X \le x) = 3k\left[1 - \left(\frac{2}{3}\right)^{x+1}\right]$, where $x = 0, 1, 2, 3, \ldots$

(b) Hence or otherwise, find the value of *k*.

9*. The discrete random variable *X* has cumulative distribution function given by $P(X \le x) = kx$ where $x = 1, 2, 3, \ldots, n.$

(a) Find the value of *k* in terms of *n*.

(b) Find the probability distribution of *X*, $P(X = x)$, in terms of *n*.

Answers:

1. (i) $\frac{1}{4}$ (ii) $\frac{37}{64}$, $\frac{27}{64}$

2. (i) $\frac{3}{10}$ (ii) $2\frac{9}{10}$

3. (i)

x	1	2	3	4	6	8	9	12	16
$P(X=x)$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$

(ii) (a) $\frac{1}{4}$ (b) $\frac{25}{4}$ (c) $\frac{275}{16}$

4. (a) 5, 5 (b) 5, 2.5

5.

r	-3	-2	-1	0	1
$P(X-Y=r)$	0.06	0.26	0.36	0.26	0.06

5. (a) -1 (b) 1

6. (i) $\mu = E(X) = 5\frac{1}{4}$

7. (a) $k = \frac{2}{n(n+1)}$ (b) $P(X \le x) = \frac{x(x+1)}{n(n+1)}, \quad x = 1, 2, 3, \ldots, n$

8. (b) $\frac{1}{3}$

9. (a) $k = \frac{1}{n}$ (b) $P(X = x) = \frac{1}{n}, \quad x = 1, 2, 3, \ldots, n$

Topic 4: Statistics and probability

4.5 Binomial Distribution

1. Probability Distribution of a Binomial Random Variable

(a) The experiment consists of ***n* repeated trials**.

(b) The repeated trials are **independent**.

(c) Each trial has two possible outcomes: either a '**success**' or a '**failure**'.

(d) The probability of success is the **same** in each trial.

We define a binomial experiment as which possesses the above **four** properties.

Let X be the **number of successes** in n trials of a binomial experiment and let the probability of success be p.

X is then called a **binomial random variable**.

Notation: $X \sim B(n, p)$ where n and p are called the parameters of the distribution.

The **probability distribution function** (p.d.f.) of the binomial random variable X (which is the **number of successes** in n **independent** trials) is given by:

$$P(X = k) = {}^{n}C_{k}\, p^{k} q^{n-k} = {}^{n}C_{k}\, p^{k} \left(1 - p\right)^{n-k}, \qquad k = 0, 1, 2, \ldots n$$

The use of this formula is not required for exams

As an illustration, let's say we throw a die, and we are interested to find the *probability of obtaining a six* when the die is thrown 2 times.

Let X be the number of sixes obtained when a die is thrown 2 times.

In this experiment:

(a) The experiment consists of **2 repeated trials**.

(b) The repeated trials are **independent**.

(c) Each trial has two possible outcomes: either a '**success**' (a six) or a '**failure**' (non-six).

(d) The probability of success is the **same** in each trial (probability of getting a six is always $\frac{1}{6}$).

Hence, $X \sim B\left(2, \frac{1}{6}\right)$.

Not required to know.

GDC syntax. No need to show in your working.

$$P\left(\text{getting one six}\right) = P(X = 1) = \boxed{{}^{2}C_{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{1}} = \boxed{\text{binompdf}\left(2, \frac{1}{6}, 1\right)} \approx 0.278$$

2. Mean of a Binomial Random Variable

If the random variable X is such that $X \sim B(n, p)$ then

$\boxed{E(X) = np \quad \text{and} \quad Var(X) = npq = np(1-p)}$ where $q = 1 - p$

From the probability distribution table of the previous illustration, we note that

$P(X = k) = {}^nC_k p^k q^{n-k} = {}^2C_k \left(\frac{1}{6}\right)^k \left(1 - \frac{1}{6}\right)^{2-k}$ which is the p.d.f. of a binomial variable.

X	**0**	**1**	**2**
$f(x) = P(X = x)$	$\left(\frac{5}{6}\right)^2$	$\left(\frac{5}{6}\right)\left(\frac{1}{6}\right) + \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)$	$\left(\frac{1}{6}\right)^2$

We can see that $X \sim B\left(2, \frac{1}{6}\right)$.

$$E(X) = \sum_{\text{all } x} xP(X = x)$$

$$= 0\left(\frac{5}{6}\right)^2 + 1\left[2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\right] + 2\left(\frac{1}{6}\right)^2$$

$$= \frac{1}{3}$$

$$= (2)\left(\frac{1}{6}\right) = np$$

$Var(X) = E\left(X^2\right) - [E(X)]^2$ ---------------proof of the formula for $Var(X)$ is shown in 4.6

$$= \left\{0 + 1\left[2\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\right] + 4\left(\frac{1}{6}\right)^2\right\} - \left(\frac{1}{3}\right)^2$$

$$= \frac{5}{18}$$

$$= (2)\left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = npq$$

In general, the calculation is done using Binomial Pdf and Binomial Cdf function in the GDC.

Mode of Binomial Distribution

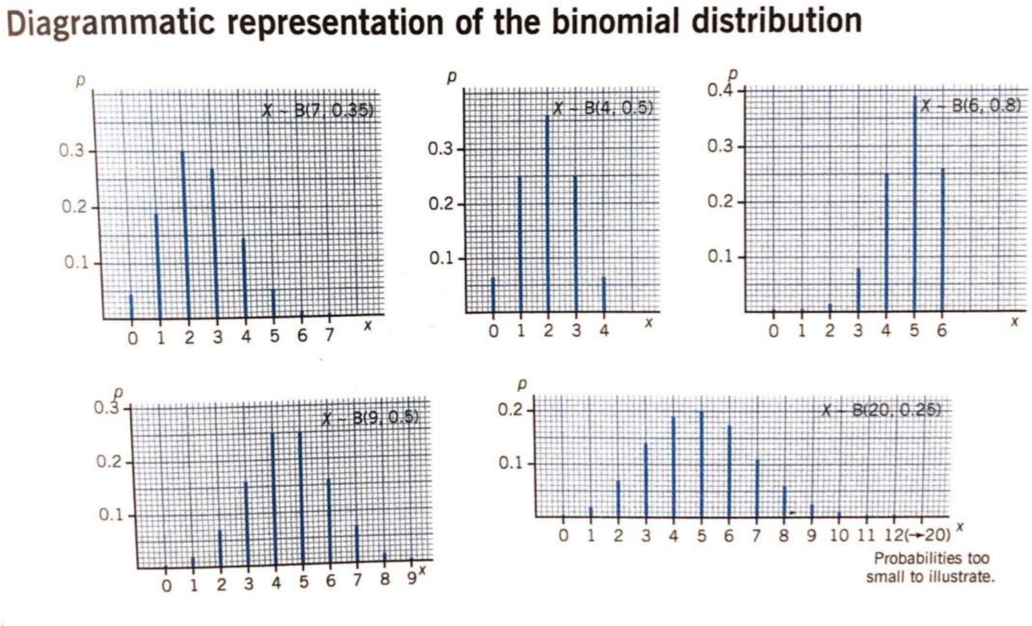


Figure from “ A Concise Course in Advance Level Statistics” By J. Crawshaw and J. Chambers

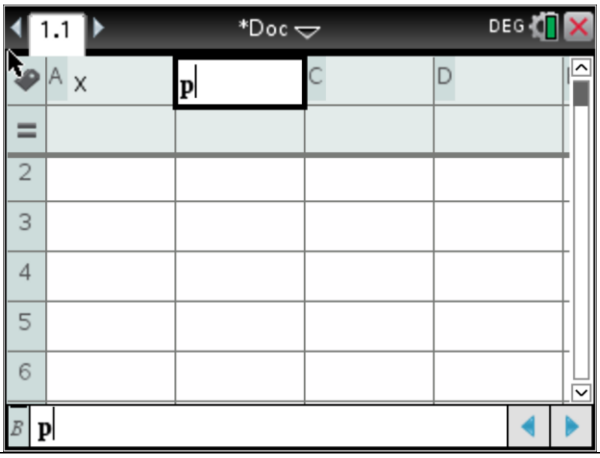
The **mode** is the value of *X* that is **most likely to occur** (with highest probability).

If $p = 0.5$ and *n* is even, there is only one mode at the median position $\frac{n}{2}$. The exception is when $p = 0.5$ and *n* is odd, then there are two modes which are at $\frac{n-1}{2}$ and $\frac{n+1}{2}$.

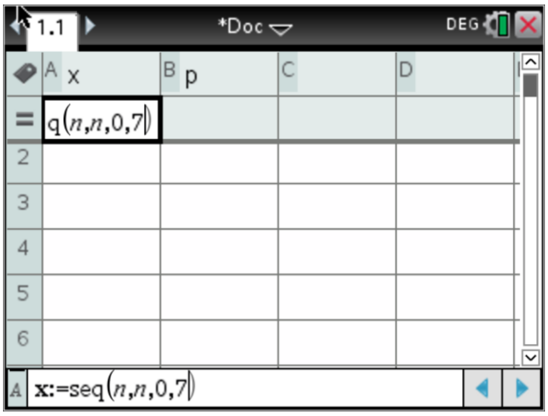
If $p \neq 0.5$, one needs to consider the probabilities of values of *X* close to E(*X*) and use the GDC table function to check the value of *X* that has the highest probability.

Usually, the distribution has only one mode.

GDC steps to list each x with its probability, for example, $X \sim B(7, 0.35)$.

1. Open List & Spreadsheet on a new page. Name first column “x” for the number of successes and second column “p” for probability associated with each x.	

2. Populate the column of x with numbers running from 0 to 7 by keying in formula seq(n,n,0,7) into the second row of first column.



x:=seq(n,n,0,7)

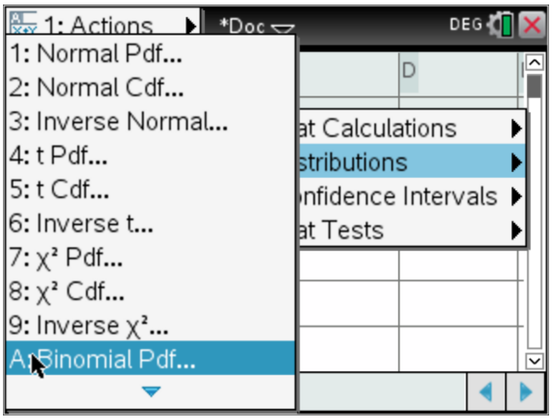
3. Hit “enter” and the first column will filled up with all the X values for the binomial distribution.

=seq(n,n,0

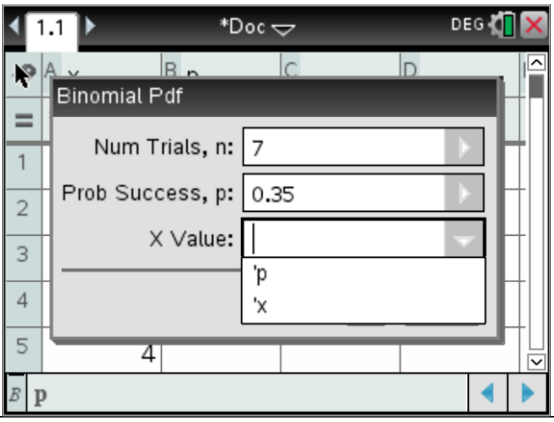
	A x
1	0
2	1
3	2
4	3
5	4

A1 =0

4. Key in the formula to calculate probability of each x value in the second row of second column by calling up Binompdf function under “menu” / “4: Statistics”/”2:Distributions”/”A: Binomial Pdf…”



5. Fill in the value of n and p in the dialogue box, this p refers to the probability of success of the binomial distribution. For the X value, you want to use the x from the first column so click the triangle and select ‘x which represents the “x” column.



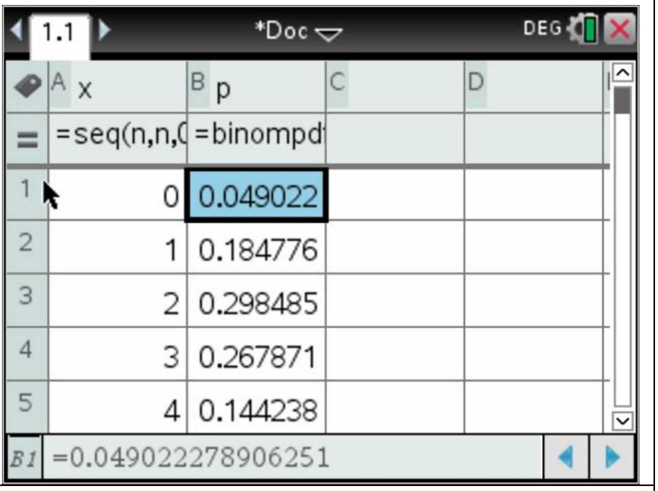
Num Trials, n: 7

Prob Success, p: 0.35

X Value:

B p

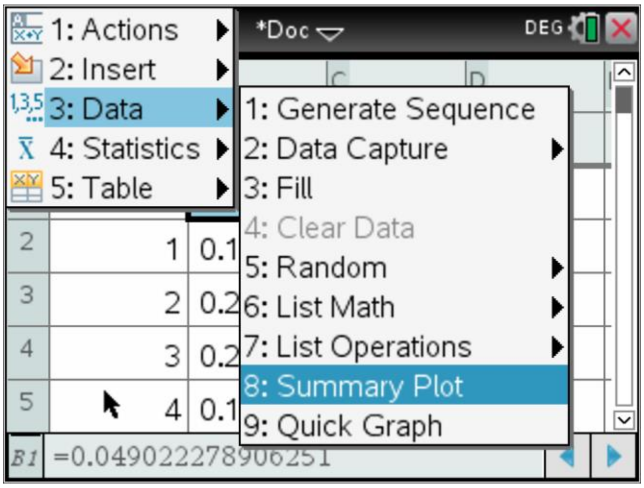
6. You can scroll down to see all the probabilities associated with each number of success in the binomial distribution $X \sim B(7, 0.35)$.



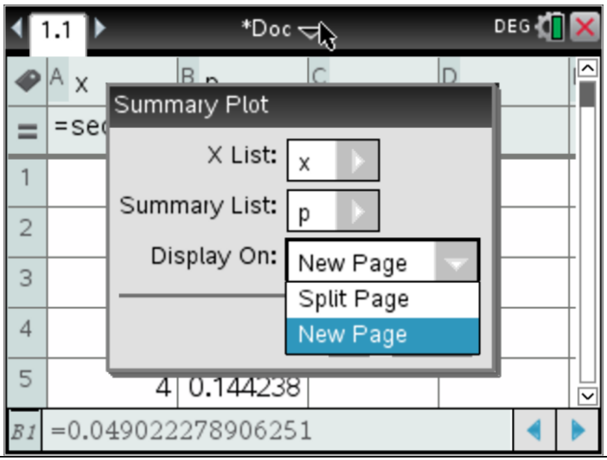
	A x	B p	C	D
=	=seq(n,n,(	=binompd		
1	0	0.049022		
2	1	0.184776		
3	2	0.298485		
4	3	0.267871		
5	4	0.144238		

B1 =0.049022278906251

7. If your n is large and you would like to find the mode quickly, you can convert the table into a probability histogram by “menu”/3:Data/8: Summary Plot.



8. For the dialogue box, choose “x” column as X List and Summary List as the “p” column. You may choose to display both table and graph together by using “Split Page”. I generally prefer separate page so I will be using “New Page”.



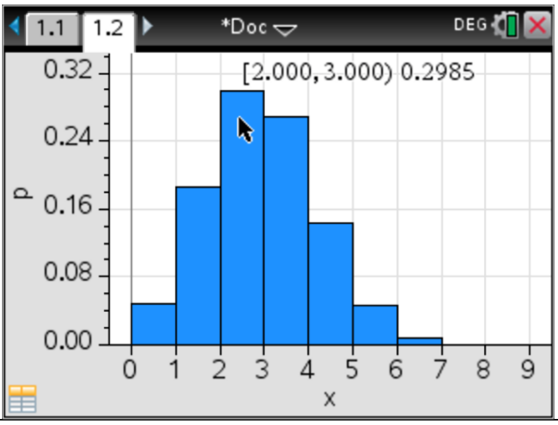
Summary Plot

X List: x

Summary List: p

Display On: New Page

9. You can confirm that $x = 2$ has the highest probability and therefore the mode is $x = 2$. Take note of the notation used by the GDC, which is $[2.000, 3.000)$, that means the horizontal line is equal to 2 but not 3.



Example 1: The probability that a pen drawn at random from a box of pens is defective is 0.5. If a sample of 7 pens is taken, find the probability that it will contain

(a) no defective pens;
(b) 5 or 6 defective pens;
(c) less than 3 defective pens;
(d) the most likely number of defective pens in the sample of 7 pens;
(e) the expected number of defective pens in the sample and their standard deviation.

[Ans: (a)(i) 0.00781 (ii) 0.219 (iii) 0.227 (b) 3 and 4 (c) 3.5, 1.32]

Example 2: The probability that a target is hit is 0.3. All shots are independent of each other. Find the least number of shots which should be fired if the probability that the target is hit at least once is greater than 0.95. [Ans: 9]

GDC Method:

1. The GDC has a function called “Inverse Binomial N” where it returns the answer for total number of trials n, when the following are given:
    a. Probability of success: p
    b. Cumulative probability $P(X \le x)$

In order to rewrite into the format that GDC can solve, we start out with

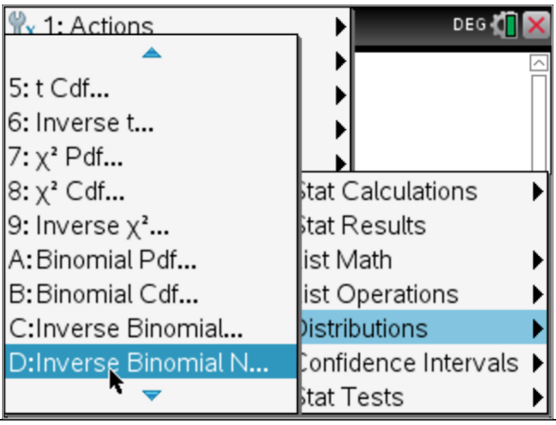
$P(X \ge 1) > 0.95$

And rewrite into

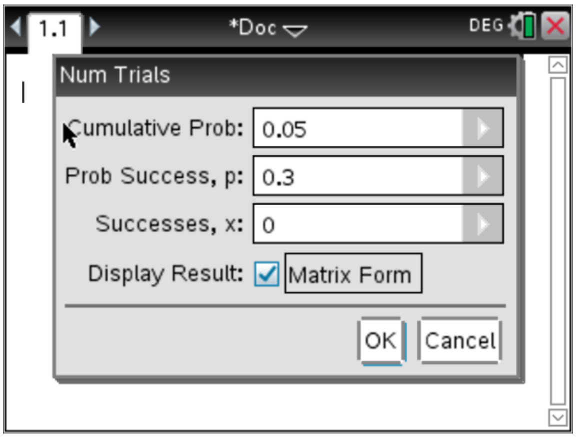
$P(X \le 0) \le 0.05$

*Note that the upper limit must be inclusive in the cumulative probability.

2. Open a new “Calculator” page and go to “menu”/ 6: Statistics/5: Distribution/D: Inverse Binomial N…



3. Fill in the cumulative probability which is the maximum value of probability allowed, and probability of success p. The successes x is the upper limit of x for the cumulative probability. You may want to tick Matrix form, where they show you the two values of n before and after surpassing 0.05 the required value.


Num Trials

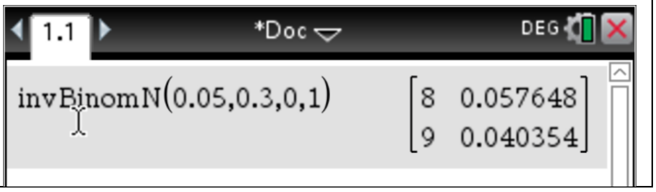
Cumulative Prob: 0.05

Prob Success, p: 0.3

Successes, x: 0

Display Result: ☑ Matrix Form

4. As shown in the result, when $n=8, P(X \le 0)=0.0577$ and when $n=9, P(X \le 0)=0.0404$ so n = 9 is the solution.


invBinomN(0.05,0.3,0,1) $\begin{bmatrix} 8 & 0.057648 \\ 9 & 0.040354 \end{bmatrix}$

Example 3: A crossword puzzle is published in ‘The Times’ each day of the week, except Sunday. A man can complete, on average, 8 out of 10 of the crossword puzzles.

(a) Find the expected value and the standard deviation of the number of completed crosswords in a given week.

(b) Find the probability that he will complete at least 5 in a given week.

(c) Given that he completes the puzzle on Monday, find, the probability that he will complete at least 4 in the rest of the week.

(d) Find the probability that, in a period of four weeks, he completes 4 or less in only one of the four weeks.

[Ans: (a) 4.8, 0.980 (b) 0.655 (c) 0.737 (d) 0.388]

Topic 4: Statistics and probability

WS 4.5: Binomial Distribution

1. For each of the experiments described below, *X* denotes the random variable 'the number of black marbles that are being drawn out of 10 trials'. State, giving a reason, whether *X* follows a binomial distribution.

Experiment 1:	A bag contains 10 black and 20 white marbles which are selected at random, one at a time with replacement.
Experiment 2:	This experiment is a repeat of Experiment 1 except that the bag contains 10 black, 20 white and 30 red marbles.
Experiment 3:	This experiment is a repeat of Experiment 1 except that marbles are not replaced after selection.

2. Assuming that a couple is equally likely to produce a girl or a boy, find the probability that in a family of 5 children there will be more boys than girls.

3. A coin is biased so that it is twice as likely to show heads as tails. Find the probability that in 5 tosses of the coin,

(a) exactly 3 heads are obtained,

(b) more than 3 heads are obtained.

4. Given that $X \sim B(n, 0.6)$ and $P(X < 1) = 0.0256$, find the value of *n*.

5. The probability that a bullet will hit the target is 0.85. All shots are assumed to be independent of each other. Find the minimum number of bullets that would need to be shot to ensure that the probability of getting more than 8 bullets hitting the target is greater than 90%.

6. Of the articles from a certain production line, 10% are defective. If a sample of 25 articles is taken, find the expected number of defective articles and the standard deviation.

7. The probability that an apple, picked at random from a sack, is bad is 0.05. Find the standard deviation of the number of bad apples in a sample of 15 apples.

8. *X* is a random variable such that $X \sim B(n, p)$. Given that $E(X) = 2.4$ and $p = 0.3$, find n and the standard deviation of X.

9. In a group of people, the expected number who wears glasses is 2 and the variance is 1.6. Find the probability that

(a) a person chosen at random from the group wears glasses;

(b) 6 people in the group wear glasses.

10. If the random variable *X* is such that $X \sim B(10, p)$ where $p < 0.5$ and $Var(X) = \frac{15}{8}$, find

(a) *p*;

(b) $E(X)$;

(c) $P(X = 2)$.

11. The random variable *X* is a Binomial random variable with mean 2 and variance 1.6. Find

(a) the most likely value of *X*,

(b) $P(X < 6)$.

12. If $X \sim B(n, 0.3)$ and $P(X \geq 1) > 0.8$, find the least possible value of n.

Answers:

2\. 0.5

3\. (a) 0.329 (b) 0.461

4\. 4

5\. 12

6\. 2.5, 1.5

7\. 0.844

8\. 1.30

9\. (a) 0.2 (b) 0.00551

10\. (a) 0.25 (b) 2.5 (c) 0.282

11\. (a) 2 (b) 0.994

12\. 5

Topic 4: Statistics and probability

4.6 Continuous Random Variables (HL only)

1. Probability Density Function

A continuous random variable is a theoretical representation of a continuous variable such as height, mass or time.

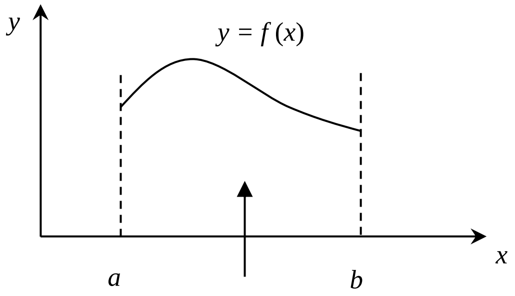
A **continuous random variable** is specified by its probability density function (**p.d.f.**) written as $f(x)$.

If X is a continuous random variable with probability density function $f(x)$ valid over the range $a \le x \le b$, then

(i) $f(x) \ge 0$

(ii) $\int_{\text{all } x} f(x) \; dx = 1$

i.e. $\int_a^b f(x)dx = 1$

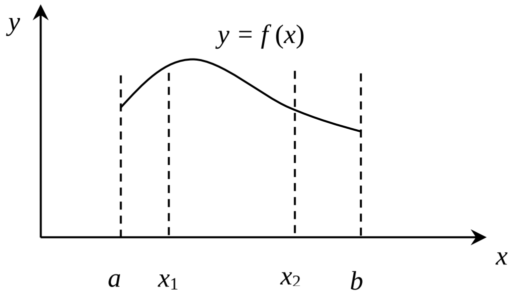


Area under the curve $y = f(x)$ represents the probability.

(iii) $P(X \le x_1) = \int_a^{x_1} f(x)dx$, where $a \le x_1 \le x_2 \le b$.

(iv) $P(X \ge x_2) = \int_{x_2}^{b} f(x)dx$, where $a \le x_1 \le x_2 \le b$.

(v) $P(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)dx$, where $a \le x_1 \le x_2 \le b$.



Note:

$$P(x_1 \le X \le x_2) = P(X = x_1) + P(x_1 < X < x_2) + P(X = x_2) = 0 + P(x_1 < X < x_2) + 0$$

$$= P(x_1 < X < x_2)$$

$$\therefore P(x_1 \le X \le x_2) = P(x_1 < X < x_2) = P(x_1 \le X < x_2) = P(x_1 < X \le x_2)$$

Example 1: The continuous random variable *X* has probability density function $f(x) = kx^2$, where $0 \le x \le 2$.

(i) Find the value of the constant k.

(ii) Sketch the graph of $y = f(x)$.

(iii) Find $P(X \ge 1)$.

(iv) Find $P(0.5 \le X < 1.5)$.

[Ans: (i) $\frac{3}{8}$ (iii) 0.875 (iv) 0.406]

2. Expectation (Mean) of a Continuous Random Variable

If *X* is a continuous random variable with probability density function $f(x)$, then the expectation (mean) of *X* is μ or $E(X)$ given by

$$\mu = E(X) = \int_{\text{all } x} xf(x)\,dx.$$

If $g(x)$ is any function of *X*, then we have

$$E\left[g(x)\right] = \int_{\text{all } x} g(x)f(x)\,dx.$$

In particular, we have $E\left(X^2\right) = \int_{\text{all } x} x^2 f(x)\,dx$.

Note: Recall that $Var(X) = E\left(X^2\right) - \left[E(X)\right]^2$.

3. Variance of Continuous Random Variables

Its variance, σ^2 or $Var(X)$ is given by

$$\begin{aligned}\sigma^2 &= Var(X)\\ &= E(X-\mu)^2\\ &= E\left(X^2\right) - \left(E(X)\right)^2\\ &= \int_{\text{all } x} x^2 f(x)\,dx - \left[\int_{\text{all } x} xf(x)\,dx\right]^2\end{aligned}$$

Note: The standard deviation of *X* is $\sigma = \sqrt{Var(X)}$.

Example 2: The continuous random variable *X* has probability density function

$$f(x) = \begin{cases} 4k & , \quad \frac{2}{3} \le x < 2 \\ kx(4-x) & , \quad 2 \le x \le 4 \\ 0 & , \quad \text{otherwise} \end{cases} \quad \text{where } k \text{ is a constant.}$$

(a) Find the value of the constant k.

(b) Find $E(X)$.

(c) Find $E(X^2)$.

(d) Find $Var(X)$ and the standard deviation of *X*.

[Ans: (a) $\frac{3}{32}$ (b) 2.04 (c) 4.86 (d) 0.695, 0.833]

4. Mode of Continuous Random Variable

The **mode** is the value of *X* for which $f(x)$ is greatest, in the given range of *X*. We find it by either

(a) sketching the graph of $y = f(x)$

or

(b) finding the *x*-coordinate of the stationary point by putting $f'(x) = 0$ and check that it gives maximum value of $f(x)$.

Example 3: The continuous random variable *X* has probability density function given by $f(x)$ where

$$f(x) = \begin{cases} \frac{3}{4}(1-x^2), & -1 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}.$$

(a) Sketch $y = f(x)$ and hence find the mode of *X*.

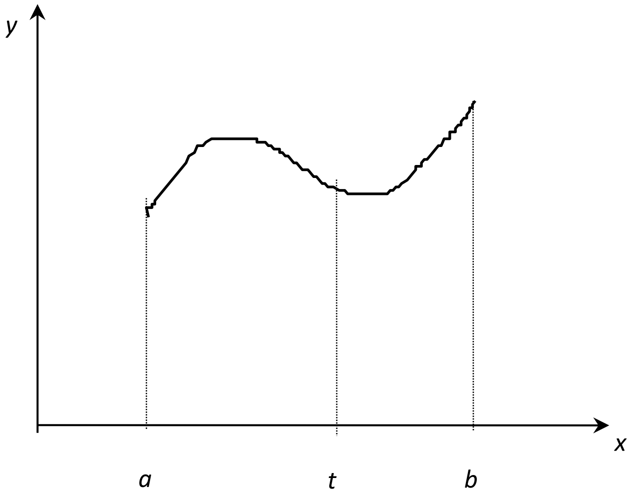
(b) Find $P(|X| < 0.5)$.

[Ans: (a) 0 (b) 0.688]

5. Cumulative Distribution Function

If *X* is a continuous random variable with probability density function $f(x)$ defined for $a \le x \le b$, then the cumulative distribution function (C.D.F) is given by $F(t)$ where the interval is taken over all values of $x \le t$.

$$F(t) = P(X \le t) = \int_a^t f(x)\,dx \ .$$



Note:

(i) $F(b) = \int_a^b f(x)dx = 1;$

$F(a) = \int_a^a f(x)dx = 0.$

(ii) $P(X \le x_1) = F(x_1)$, $P(X \le x_2) = F(x_2)$ and

$P(x_1 \le X \le x_2) = F(x_2) - F(x_1)$

6. The Median and Interquartile Range

The median (or 50th percentile), usually denoted by *m*, is the value of *X* such that the line $x = m$ splits the area under the curve $y = f(x)$ into two halves.

i.e. $\int_a^m f(x)dx = 0.5$ or $F(m) = 0.5$

The lower quartile (or 25th percentile), *r*, is the value of *X* such that $\int_a^r f(x)dx = 0.25$ or $F(r) = 0.25$.

The upper quartile (or 75th percentile), *s*, is the value of *X* such that $\int_a^s f(x)dx = 0.75$ or $F(s) = 0.75$.

In general, the k^{th} **percentile**, *t*, is the value of *X* such that $\int_a^t f(x)dx = \frac{k}{100}$ or $F(t) = \frac{k}{100}$.

The **interquartile range = upper quartile – lower quartile**.

Example 4: If X is a continuous random variable with probability density function $f(x)=\frac{1}{8}x$, where $0 \le x \le 4$.

(a) Find the cumulative distribution function $F(x)$ and sketch $y=F(x)$.

(b) Find the median *m*, the upper quartile *s* and the lower quartile *r*.

(c) Find $P(0.3 < X \le 1.8)$ [Ans: (a) $F(x)=\frac{x^2}{16}, 0 \le x \le 4$ (b) $\sqrt{8}, \sqrt{12}, 2$ (c) 0.197]

Obtaining p.d.f from C.D.F

$$\text{c.d.f. } F(t) = \int_a^t f(x)dx, \quad a \le t \le b$$

$$\therefore \quad \text{p.d.f. } f(x) = \frac{d}{dx}F(x) = F'(x)$$

Example 5: The continuous random variable *X* has cumulative distribution function $F(x)$ where

$$F(x) = \begin{cases} 0 & , \quad x < 0 \\ k\,x^3 & , \quad 0 \le x \le 3 \\ 1 & , \quad x \ge 3 \end{cases}$$

(a) Find the value of the constant *k*.

(b) Find the probability density function $f(x)$ and sketch the graph of $y=f(x)$.

[Ans: (a) $\frac{1}{27}$ (b) $f(x)=\frac{x^2}{9}, 0 \le x \le 3$]

Linear Transformations on *X*

Consider the continuous random variable X and let a, b be non-zero constants.

Then $\boxed{\mathrm{E}(aX+b)=a\,\mathrm{E}(X)+b \text{ and } \mathrm{Var}(aX+b)=a^2\,\mathrm{Var}(X).}$

Proof:

$$
\begin{aligned}
\mathrm{E}(aX+b) &= \int_{\text{all } x} (ax+b)f(x)\ dx \\
&= \int_{\text{all } x} axf(x)+bf(x)\,dx \\
&= a\,\mathrm{E}(X)+b
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Var}(aX+b) &= \mathrm{E}\left((aX+b)^2\right)-\left[\mathrm{E}(aX+b)\right]^2 \\
&= \mathrm{E}\left(a^2X^2+2abX+b^2\right)-\left(a\,\mathrm{E}(X)+b\right)^2 \\
&= a^2\,\mathrm{E}\left(X^2\right)+2ab\,\mathrm{E}(X)+b^2-\left(a^2\left[\mathrm{E}(X)\right]^2+2ab\,\mathrm{E}(X)+b^2\right) \\
&= a^2\,\mathrm{E}\left(X^2\right)-a^2\left[\mathrm{E}(X)\right]^2 \\
&= a^2\,\mathrm{Var}(X)
\end{aligned}
$$

This result applies to both discrete and continuous random variables.

Topic 4: Statistics and probability

WS 4.6: Continuous Random Variables

1. The continuous random variable *X* has probability density function given by

$$f(x)=\begin{cases} k, & 0\le x<2 \\ k(2x-3), & 2\le x\le 3 \\ 0, & \text{otherwise}\end{cases}$$

a. Find the value of the constant *k*.

b. Sketch the graph of $y=f(x)$.

c. Find $P(X\le 1)$.

d. Find $P(X>2.5)$.

e. Find $P(1\le X\le 2.3)$.

2. The continuous random variable *X* has probability density function given by

$$f(x)=\begin{cases} k(2-x), & \text{for } 0\le x\le 2 \\ 0, & \text{otherwise}\end{cases}$$

, where k is a constant.

a. Find the value of *k*.

b. Find the cumulative distribution function of *X*.

c. The continuous random variable *Y* is given by $Y=1-\frac{1}{2}X$.

i. Show that $P(Y<y)=y^2$, where $0\le y\le 1$.

ii. Deduce the probability density function of *Y* and hence, or otherwise, show that $E(Y)=\frac{2}{3}$.

3. The continuous random variable X has probability density function given by

$$f(x)=\begin{cases} \frac{\pi}{20}\cos\left(\frac{\pi x}{20}\right), & \text{for } 0\le x\le 10 \\ 0, & \text{otherwise}\end{cases}$$

a. Find the cumulative distribution $F(x)$ and sketch $y=F(x)$.

b. Find the median *m*.

4. The continuous random variable X has a probability density function given by

$$f(x)=\begin{cases} 4k & , \quad 0\le x\le 1 \\ k(x-3)^2 & , \quad 1\le x\le 3 \\ 0 & , \quad \text{otherwise} \end{cases}$$

a. Find the value of *k* and hence show that the median of *X* is $\frac{5}{6}$.

b. Show that $E(X)=0.9$ and deduce the value of *c*, given that $E(X+c)=3E(X-c)$.

c. Given that x_1 and x_2 are two independent observations of *X*, find the probability that

i. both are greater than 1,
ii. one is less than 0.5 and the other is greater than 2.

d. Find the cumulative distribution function of *X*.

5. The continuous random variable *X* has cumulative distribution function given by

$$F(x)=\begin{cases} 0 & , \quad x\le 1 \\ k(x-1)^3 & , \quad 1<x<3 \\ 1 & , \quad x\ge 3 \end{cases}$$

a. Find the value of *k*.

b. Find in a form not involving *k*, the lower and upper quartile of *X*, giving your answers correct to 3 significant figures.

c. Find the probability density function of *X*.

d. Find $E(X-1)$, and hence write down the value of $E(X)$.

6. The continuous random variable *X* is such that

$$P(X>x)=\begin{cases} 1 & , \quad x\le 0 \\ k(3-x)^3 & , \quad 0<x<3 \\ 0 & , \quad x\ge 3 \end{cases}$$

Show that $P(X>1)=\frac{8}{27}$.

Find the probability density function of *X*, and hence find $E(X)$.

7. The continuous random variable *X* has cumulative distribution function given by

$$F(x) = \begin{cases} 0 & , \quad x \le 0 \\ \sqrt{x} & , \quad 0 < x < 1 \\ 1 & , \quad x \ge 1 \end{cases}$$

a. Find the median of *X*.

b. Find $f(x)$, the probability density function of *X*. Hence:

i. Show that $E(X) = \frac{1}{3}$.

ii. Find $Var(X)$.

c. Show that the median of $\sqrt{X}$ and the mean of $\sqrt{X}$ are equal.

8. The continuous random variable *T* has probability density function given by

$$f(t) = \begin{cases} k(t-2), & \text{for } 2 \le t \le 4 \\ 0, & \text{otherwise} \end{cases},$$

Where *k* is a constant.

a. Find, in terms of *k*, the cumulative distribution function of *T*.

b. Hence or otherwise, find the value of *k*.

c. Show that $P(T > 3) = 0.75$.

d. The event $T > 3$ is denoted by *A*, and the event $2T > 3$ is denoted by *B*. Find $P(A \cup B)$ and $P(A \cap B)$.

9. The continuous random variable *X* has probability density function given by

$$f(x) = \begin{cases} k(2-x), & \text{for } 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$, where *k* is a constant.

a. Find, in terms of *k*, the cumulative distribution function.

b. Hence or otherwise, find the value of *k*.

Answers

1. (a) $k = \frac{1}{4}$ (c) $\frac{1}{4}$ (d) $0.3125 = \frac{5}{16}$ (e) $0.3475 = \frac{139}{400}$

2. (a) $k = \frac{1}{2}$ (b) $F(x) = \begin{cases} 0 & , \quad x < 0 \\ x - \frac{x^2}{4} & , \quad 0 \le x \le 2 \\ 1 & , \quad x > 2 \end{cases}$; (c) (ii) $g(y) = \begin{cases} 2y & , \quad 0 \le y \le 1 \\ 0 & , \quad \text{otherwise} \end{cases}$

3. (a) $F(x) = \sin\left(\frac{\pi x}{20}\right), 0 \le x \le 10$ (b) $\frac{10}{3}$

4. (a) $k = \frac{3}{20}$ (b) $c = \frac{9}{20}, \frac{4}{25}, \frac{3}{100}$ (c) $F(x) = \begin{cases} 0.6x & , \quad 0 \le x \le 1 \\ 1 + 0.05(x-3)^3 & , \quad 1 \le x \le 3 \end{cases}$

5. (a) 0.125 (b) 2.26, 2.82 (d) 1.5, 2.5 6. $\frac{3}{4}$ 7. $\frac{1}{4}, \frac{4}{45}$

8. (a) $F(t) = \frac{k}{2}(t-2)^2, 2 \le t \le 4$ (b) $k = 0.5$ (d) 1, 0.75 9. (a) $k\left(2x - \frac{x^2}{2}\right)$ (b) $k = \frac{1}{2}$

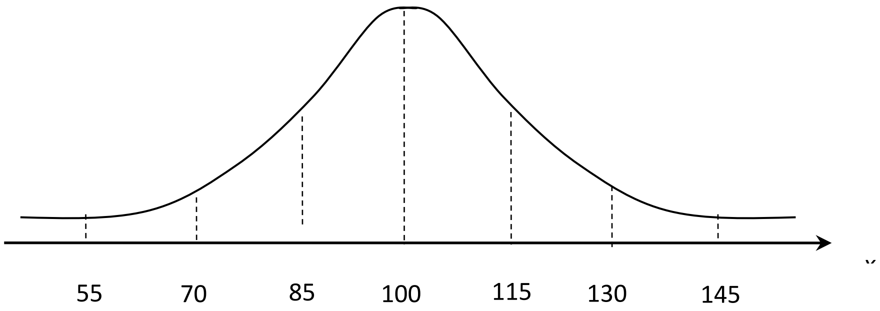
Topic 4: Statistics and probability

4.7 Normal Distribution

1. The Normal Distribution

Many measured quantities in the natural sciences follow a normal distribution, for example heights, masses, ages, random errors, I. Q. scores, examination results and the normal distribution is one of the most important continuous distributions.

The curve that represents the normal distribution is the famous bell-shaped curve that is extensively used like in the representation of I.Q. scores. The average I.Q. score is 100.



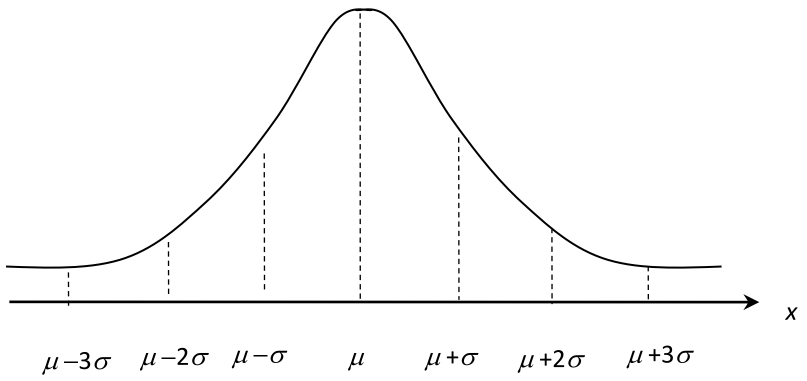
In general, the normal curve of a normal random variable X is characterised by the two parameters of X, namely the **mean** μ and **variance** σ^2**.** In symbol : $X \sim N\left(\mu, \sigma^2\right)$. Note that X is a continuous random variable.

The probability density function (pdf) for the normal distribution is given by

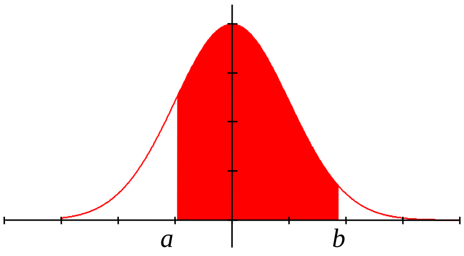
$$f\left(x\right)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{\left(x-\mu\right)^2}{2\sigma^2}}$$

The function cannot be integrated manually using other well-known functions, therefore, in most cases, the use of the GDC is required in order to calculate the required probabilities.

Properties of the Normal Curve



- Bell-shaped and symmetrical about $x=\mu$.
- Maximum value of the curve occurs when $x=\mu$, therefore ***mode* = *median* = *mean* = μ.**
- Points of inflexion (non-stationary) are at $x=\mu-\sigma$ and at $x=\mu+\sigma$.
- Probability that the normal random variable X is within 1 standard deviation of the mean, i.e. $P(|X-\mu|<\sigma)=0.683$ or $P(\mu-\sigma<X<\mu+\sigma)=0.683$.
- Also, we have $P(|X-\mu|<2\sigma)=0.954$ and $P(|X-\mu|<3\sigma)=0.998$.
- The range of the distribution is therefore approximately 6 standard deviations.



- The area under the curve from *a* to *b* represents $P(a\le X\le b)$.

2. Standard Normal Distribution

If the normal distribution has $\mu=0$ and $\mu^2=1$, it is called a **standard** normal distribution and is denoted as $Z\sim N(0,1)$. The probability density function (pdf) for the standard normal distribution is given by $f(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

In general, any normal distribution $X\sim N(\mu,\sigma^2)$ can be transformed into the standard normal $Z\sim N(0,1)$ by using the transformation $Z=\dfrac{X-\mu}{\sigma}$.

For example, if $X \sim N\left(\mu, \sigma^2\right)$ and we want to find $P\left(x_1 \geq X \leq x_2\right)$, then X must first be transformed to $Z \sim N(0,1)$. Then $P\left(x_1 \geq X \leq x_2\right) = P\left(z_1 \geq Z \leq z_2\right)$ where $z_1 = \dfrac{x_1 - \mu}{\sigma}$ and $z_2 = \dfrac{x_2 - \mu}{\sigma}$.

Why do we need to standardize?

1. Standardizing allows us to have a more meaningful understanding of the data.
2. When the mean, μ, or/and variance, σ^2 is unknown, standardizing allows us to use the inverse standard normal function in GDC to calculate the unknown normal distribution parameters.

Note: Some advantages of using $Z \sim N(0,1)$:

- i. Symmetry about 0.
- ii. We can express Z in terms of μ and σ
- iii. If $P\left(Z > z_1\right) = P\left(Z < z_2\right)$, we know that $z_1 = -z_2$. (try sketching the curve to see!)
- iv. If $P\left(Z > z_1\right) > 0.5$ we can conclude that $z_1 < 0$.

Example 1 (Calculating Probabilities using Ti Nspire NormalCdf function)

If $X \sim N(50,100)$ use your GDC to find

(a) $P\left(X < 62\right)$

(b) $P\left(X > -45\right)$

(c) $P\left(-45 < X < 62\right)$

(d) $P\left(|X| > 45\right)$

(e) $P\left(|X - 30| \geq 5\right)$

[Ans: (a) 0.885 (b) 1.00 (c) 0.885 (d) 0.691 (e) 0.939]

Example 2 (Calculating unknown value of random variable given probability using Ti Nspire invNorm function)
Given $X \sim N(50,100)$, find

(i) a if $P\left(X < a\right) = 0.925$

(ii) b if $P\left(X > b\right) = 0.113$

(iii) the range of values of c if $P\left(X > c\right) < 0.123$

(iv) d if $P\left(\left|X - 50\right| \geq d\right) = 0.125$

[Ans: (i) 64.4 (ii) 62.1 (iii) *c* > 61.6 (iv) 15.3]

Example 3 (Calculating unknown mean and variance by using Standard Normal Z~N(0,1) or Ti Nspire method)

Given $X \sim N\left(\mu, \sigma^2\right)$, find μ and σ^2 if $P\left(X < 38.14\right) = 0.38$ and $P\left(X > 49.87\right) = 0.05$.

[Ans: $\mu = 40.0, \sigma^2 = 36.1$]

Example 4 (Calculating unknown mean and variance by using Standard Normal Z~N(0,1) or Ti Nspire method)

A certain product is put into cans by a canning machine. The mass of product in a can is normally distributed with mean *m* and standard deviation 0.5 g. The value *m* can be adjusted by the operation of the machine. The cans are labelled as containing at least 300g of product, and the operator wishes to set the mean so that not more than 2 can in 1000 on average will contain less than 300 g of product. Find the least value of *m* which will enable this to be achieved. [Ans: 301g]

Topic 4: Statistics and probability

WS 4.7: Normal Distribution

1. X is a normal variable with mean μ and standard deviation σ. It is given that $P(X > 128) = 0.15$ and that $P(X > 97) = 0.875$. Calculate μ and σ.

2. Circular metal tokens are used to operate a washing machine in a Laundromat. The diameters of the tokens are known to be normally distributed. Only tokens with diameters between 1.94 and 2.06 cm and operate the machine.

(a) Find the mean and the standard deviation of the distribution given that 2% of the tokens are too small and 3% are too large.

(b) Find the probability that less than two tokens out of 20 will not operate the machine.

3. If $X \sim N(84,12)$, find h if

(a) $P(X > h) = 0.9255$

(b) $P(|X - 84| \le h) = 0.4028$

4. If $X \sim N(k, k^2)$, find

(a) **(i)** $P(X > 0)$ **(ii)** $P\left(X < \frac{k}{2}\right)$

(b) The constant a such that $P(|X - k| < ak) = 0.9$.

5. Let X denote a random variable giving a value with distribution $X \sim N(-5,9)$. Find the probability that

(a) an item chosen at random will have a positive value,

(b) out of 10 items chosen at random, just 4 will have a positive value.

6. A random variable Y has a normal distribution.

(a) Calculate the mean and the standard deviation, given that $P(Y > 5.82) = 0.04$ and $P(Y < -3.74) = 0.18$.

(b) The random variable R, also normally distributed, has the same standard deviation as Y, but its mean is unknown. Find the greatest possible value of $P(-3.74 < R < 5.82)$.

7. The quality of articles manufactured in a certain process is classified as ‘poor’, ‘fair’ or ‘good’ on the basis of a measured quantity Y which may be assumed to have a normal distribution with mean 50 units and standard deviation 5 units. Articles are ‘poor’ if $Y < 44$ and the proportion of ‘fair’ and ‘good’ are equal. Calculate the boundary value of Y separating ‘fair’ from ‘good’.

8. Before joining the Mensa Society, every candidate is given an intelligence test which, applied to the general public, would give a normal distribution of IQs with mean 100 and standard deviation 20. The candidate is not admitted unless his IQ as given by the test is at least 130.

(a) Estimate the median IQ of the members of the Mensa Society, as that their IQ distribution is representative of that of the part of the population having IQs greater than or equal to 130.

(b) What IQ would be expected to be exceeded by one member in ten of the society?

9. If X follows a normal distribution such that $P\left(X > 10\right) = 0.1$ and $P\left(9 < X < 10\right) = 0.2$, calculate its mean and standard deviation.

Answers:

1. 113.2, 14.2
2. (a) 2.00, 0.0305 (b) 0.736
3. (a) 79.0 (b) 1.83
4. (a) (i) 0.841 (ii) 0.309 (b) 1.64
5. (a)0.0478 (b) 0.000817
6. (a) -0.458, 3.59 (b) 0.818
7. 50.7
8. (a) 137 (b) 149.5
9. 8.31, 1.32

Preparing for HL Paper 3

This paper is for 1 hour 15 mins, worth 55 marks, representing 20% of the final mark. There will be two compulsory extended-response problem-solving questions, and the marks for each question will normally lie in the range 23 to 32.

Questions require extended response involving sustained reasoning. Individual questions will develop from a single theme where the emphasis is on problem solving, leading to a generalization or the interpretation of a context. There is also an emphasis on the use of technology, and thus a GDC is required for this paper, but not every question part will necessarily require its use.

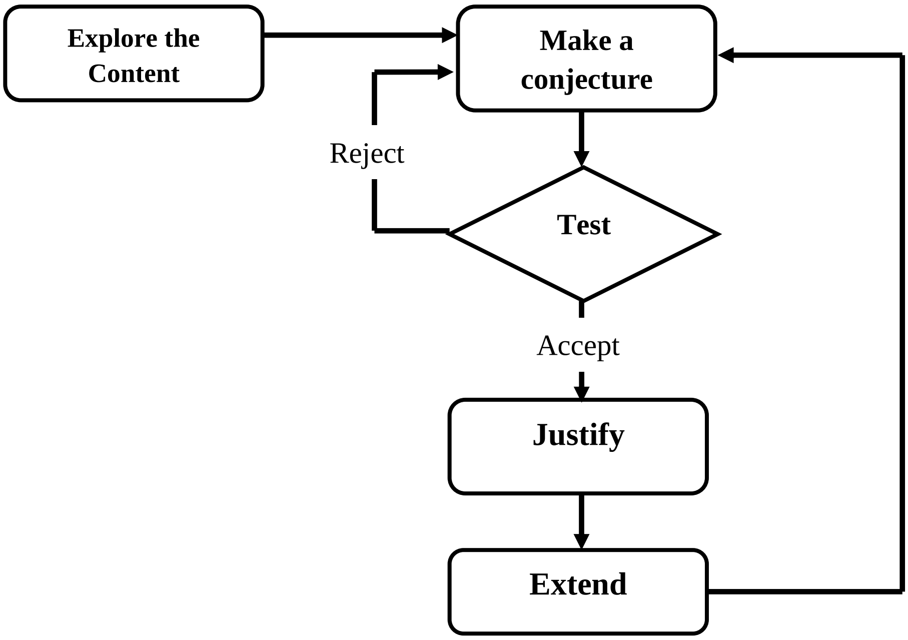
The first part of each question will be on syllabus content leading to the problem-solving context. Therefore, knowledge of all syllabus topics is required for this paper. The questions will usually include either unfamiliar mathematics, or familiar mathematics used in unfamiliar settings. Hence, it is possible that the question may include topics outside of the syllabus. In these cases, sufficient support will be given in the question to ensure all parts are accessible.

The emphasis of the paper is to discover general patterns, to verify them, and to informally justify or prove the results.

Students will be required to

- Recall, select and use their knowledge of mathematical skills, results and models in both abstract and real-world contexts to solve problems
- Investigate unfamiliar situations
- Organize and analyze information
- Make conjectures
- Draw conclusions and test their validity

Inquiry approach to Problem Solving



Advice for Students

1. The IB guide says that "the questions will be structured so that failure to do an early part of the question will not normally prevent a student from accessing later parts. Often, the early parts will be "show that", which enable students to carry the result of one part into subsequent parts." If you encounter a difficulty while doing the question, you should look for points where you can "get back in".
2. Use the five minute reading time to carefully read through the whole question, as trickier parts may come earlier and later parts may be a simple application of the result derived.
3. Keep an open-mind for the paper and read the question carefully. If something doesn't work, try it again. Keep the problem-solving cycle in mind.
4. When encountering unfamiliar definitions, substitute values into the definition or formulas given to get a better understanding of the concepts (if not already required by parts of the question).
5. Remember that knowledge can come from all parts of the syllabus and thus students may need to make connections between different parts of the syllabus.
6. Remember that the method to disprove a statement is to provide a counterexample.
7. Review the main methods of proof before the examinations:
   a. Direct algebraic proof
   b. Proof by mathematical induction
   c. Proof by contradiction
8. Always look out for patterns so you can make conjectures.
9. In class, don't be afraid to take risks and try different possibilities. Do more practice papers and persevere through the problems.

Here are some shorter problems for class discussion. There are longer questions in the worksheets and in the practice papers.

Example 1: IB Nov 2009 Maths HL P1 Qn 12

A tangent to the graph of $y = \ln x$ passes through the origin.

(a) Sketch the graphs of $y = \ln x$ and the tangent on the same set of axes, and hence find the equation of the tangent.

(b) Use your sketch to explain why $\ln x \le \frac{x}{e}$ for $x > 0.$

(c) Show that $x^e \le e^x$ for $x > 0.$

(d) Determine which is larger, π^e or e^π .

Example 2: IB May 2009 Maths HL P1 TZ1 Qn 13 Part A

If z is a non-zero complex number, we define $L(z)$ by the equation

$$L(z) = \ln|z| + i\arg(z), 0 \le \arg(z) < 2\pi.$$

(a) Show that when z is a positive real number, $L(z) = \ln z.$

(b) Use the equation to calculate

(i) $L(-1)$ (ii) $L(1-i)$ (iii) $L(-1+i)$

(c) Hence show that the property $L(z_1z_2) = L(z_1) + L(z_2)$ does not hold for all values of z_1 and z_2.

Example 3: Consider the function $f(n) = 2^{3n+1} + 5$ where $n \in \Box^{+}$

(a) Write down the values of $f(1), f(2), f(3), f(15)$ and $f(20)$

(b) By noting the common factor of all the numbers found in **(a)**, make a conjecture for the divisibility of $f(n)$ for every $n \in \mathbb{Z}^{+}$.

(c) Verify the truth of your conjecture by considering $f(2022)$.

(d) Prove your conjecture.

Example 4:

(a) Without using a calculator, determine which of the following numbers is bigger

(i) 2^3 or 3!

(ii) 5^9 or 9! (Hint: consider whether $\frac{5^9}{9!}$ is greater than or less than 1)

(iii) 50^{99} or 99!

(b) Hence, make a conjecture for an inequality that describes the situation above.

(c) Construct an example to verify that your conjecture is true.

(d) Prove your conjecture.

Example 5: Let $S_n = \sum_{r=1}^{n} \frac{1}{r}$ where $n \in \mathbb{Z}^{+}$.

(a) Write down the value of $S_1, S_4,$ and S_8

(b) Verify that $S_4 - S_2 > \frac{1}{2}$ and $S_8 - S_4 > \frac{1}{2}$

(c) Hence prove that $S_8 > \frac{3}{2}$.

(d) Prove that $S_{2^k} - S_{2^{k-1}} > \frac{1}{2}$ for any $k \in \mathbb{Z}^{+}$.

(e) Hence verify that $S_{2^k} > 1 + \frac{k}{2}$ for any $k \in \mathbb{Z}^{+}$.

(f) Determine whether the harmonic series, $S_\infty = \sum_{r=1}^{\infty} \frac{1}{r}$, is convergent or divergent.

Paper 3 Practice Questions

Question	Topics and concepts
1	Polynomials and binomial theorem
2	Telescoping Series and Method of difference
3	Rational functions and partial fractions
4	Integration and Reduction formula
5	Riemann Sums
6	Riemann Sums and Maclaurin Series
7	Taylor Series
8	Complex numbers and roots of unity
9	Hyperbolic functions
10	Monte Carlo method for estimating π
11	Probabilities and Fibonacci Sequence
12	Rotations and parametric equations

1. This question asks you to investigate the factors of a polynomial.

(a) Consider the expansion of $E = \dfrac{(1+x)^n}{1-x}$ where $n \in \mathbb{Z}^{+}$.

Using the binomial theorem for ascending powers, show that the first four terms of E is $1+\left(1+\binom{n}{1}\right)x+\left(1+\binom{n}{1}+\binom{n}{2}\right)x^2+\left(1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}\right)x^3.$

(b) Let the coefficient of x^r in the expansion of E be denoted by $P_r(n)$.

Show that $P_3(n)=1+n+\dfrac{n(n-1)}{2!}+\dfrac{n(n-1)(n-2)}{3!}$

(c) Now generalise and consider the expansion of E when $n=-1$. By using the Factor Theorem, or otherwise, deduce that when r is odd, $n+1$ is a factor of the polynomial $P_r(n)$.

(d) When r is even, show that $n+1$ is a factor of the polynomial $P_r(n)-1$.

(e) Use the results from part (c) and (d) to determine the first 6 terms of $\dfrac{(1+x)^{-1}}{1-x}$.

2. This question asks you to investigate telescoping series and method of difference.

(a) Give that $u_n = e^{nx} - e^{(n+1)x}$,

(i) Find $u_1 + u_2 + u_3$.

(ii) Find $\sum_{r=1}^{N} u_r$ in terms of N and x.

(iii) Determine the set of values of x for which sum to infinity ($\sum_{r=1}^{\infty} u_r$) exists

(iv) Hence, find the value of the sum to infinity for cases where it exists.

(b) Show that $\frac{2^{n+1}}{n+1} - \frac{2^n}{n} = \frac{2^n(n-1)}{n(n+1)}$, where n is a positive integer. Hence, or otherwise, show that $\sum_{n=1}^{N} \frac{2^n}{n+1} - \sum_{n=1}^{N} \frac{2^n}{n(n+1)} = \frac{2^{N+1}}{N+1} - 2$.

3. This question asks you to investigate rational functions and partial fractions.

The indefinite integral $I = \int \frac{P(x)}{x^3+1} dx$, where $P(x)$ is a polynomial in x.

(a) Find I when $P(x) = x^2$.

(b) By writing $x^3 + 1 = (x+1)(x^2 + Ax + B)$, where A and B are constants, find I when

(i) $P(x) = x^2 - x + 1$

(ii) $P(x) = x + 1$

(c) Using the results of part (i) and (ii), or otherwise, find I when $P(x) = 1$.

4. This question asks you to investigate reduction formula and definite integrals.

(a) On the same set of axes, sketch and label $y = (1+x^2)^{-1}$, $y = (1+x^2)^{-2}$ and $y = (1+x^2)^{-3}$ for $0 < x < 1$.

(b) Find the exact values of $\int_0^1 (1+x^2)^{-1}\ dx$

(c) By substituting $x = \tan\theta$, or otherwise, find the exact value of $\int_0^1 (1+x^2)^{-2}\ dx$

Let $I_n = \int_0^1 (1+x^2)^{-n}\ dx$.

(d) (i) By expressing $(1+x^2)^{-n}$ as $(1+x^2)^{-n}(1)$, or otherwise, show that

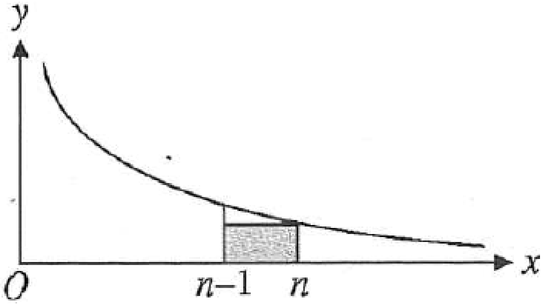
$$I_{n+1} = \left(1 - \frac{1}{2n}\right) I_n + \frac{2^{-n-1}}{n} \text{ for } n \geq 1.$$

(ii) Find the exact value of $\int_0^1 (1+x^2)^{-3}\ dx$

(e) Find the exact value of $\int_0^1 (x^2 - 2x + 2)^{-3}\, dx$.

[adapted from IBO Training Resources by Daniel Hwang]

5. This question asks you to investigate areas using Riemann sums.



The diagram shows a sketch of the graph of $y = \frac{1}{\sqrt{x}}$. By considering the shaded rectangle and the area of the region between the graph and the x-axis for $n-1 \le x \le n$, where $n \ge 1$, show that

$$\frac{1}{\sqrt{n}} < 2\left(\sqrt{n} - \sqrt{n-1}\right).$$

(a) Deduce that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}$.

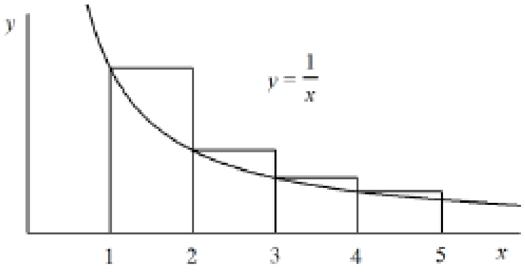
(b) Show also that $\frac{1}{\sqrt{n}} > 2\left(\sqrt{n+1} - \sqrt{n}\right)$.

(c) Deduce that $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 2\sqrt{n+1} - 2$.

(d) Hence find a value of N for which $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + ... + \frac{1}{\sqrt{n}} > 1000$.

6. This question asks you to investigate the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$.

(a) By considering the graph below, or otherwise, show that $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.



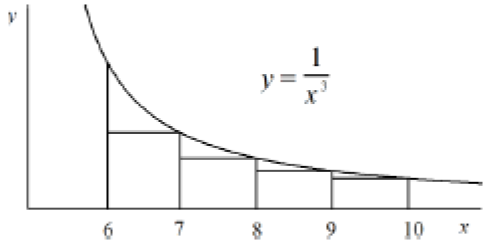
(b) (i) By considering the Maclaurin's series of $\frac{\sin x}{x}$, or otherwise, find the values of x that satisfy the equation $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots=0$

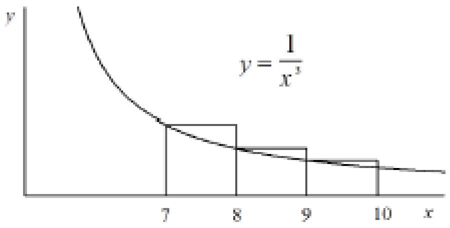
Given that $1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+\cdots=\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{(2\pi)^2}\right)\left(1-\frac{x^3}{(3\pi)^2}\right)\ldots$

(ii) By equating the coefficients of x^2 or otherwise, find the exact value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(iii) Find the exact value of $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$.

(c) (i) By considering the graphs below, or otherwise, show that $\frac{1}{98}<\sum_{n=7}^{\infty}\frac{1}{n^3}<\frac{1}{72}$.





(ii) Hence, approximate $\sum_{n=1}^{\infty}\frac{1}{n^3}$ to three significant figures.

[adapted from IBO Training Resources by Daniel Hwang]

7. This question asks you to investigate solving differential equations and Taylor series

The Taylor Series of *f*(*x*) that is infinitely differentiable about *x* = *a*, is of the form:

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)(x-a)^2}{2!}+\frac{f'''(a)(x-a)^3}{3!}+\ldots$$

Note: when *a* = 0, we have the Maclaurin Series.

(a) Determine the first three derivatives of the function $f(x)=x(\ln x-1)$.

(b) Find the first three non-zero terms of the Taylor series for *f*(*x*) about $x=1$.

8. This question asks you to investigate complex numbers and roots of unity.
Answers obtained only by technology are not allowed.

(a) Let $z_1 = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

(i) Show that z_1 is a root of $z^7 - 1 = 0$.

(ii) Show that z_1 is a root of $z^6 + z^5 + z^4 + z^3 + z^2 + z + 1 = 0$.

(iii) Simplify $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$.

(iv) Simplify $\cos\frac{\pi}{7} + \cos\frac{3\pi}{7} + \cos\frac{5\pi}{7} + \cos\frac{7\pi}{7}$

(b) (i) Solve the equation $\sin 7\theta = 0$.

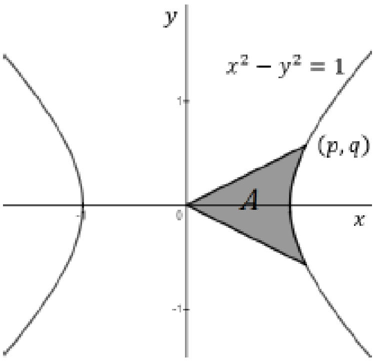
(ii) Show that $\frac{\sin 7\theta}{\sin\theta} = 64\cos^6\theta - 80\cos^4\theta + 24\cos^2\theta - 1$ for $\sin\theta \neq 0$.

(iii) Solve $64x^6 - 80x^4 + 24x^2 - 1 = 0$.

(iv) Deduce that $\left(\cos\frac{\pi}{7}\right)\left(\cos\frac{2\pi}{7}\right)\left(\cos\frac{3\pi}{7}\right)\ldots\left(\cos\frac{7\pi}{7}\right)$.

[adapted from IBO Training Resources by Daniel Hwang]

9. This question asks you to investigate the hyperbolic functions cosh and sinh.



Draw line segments from $(0,0)$ to (p, q) and $(p, -q)$ on the right side of the hyperbola $x^2 - y^2 = 1$. Call the area of the enclosed region A if $q \geq 0$ or $-A$ if $q < 0$.

Given that $p = \cosh A$ and $q = \sinh A$.

(a) Show that $A = 2\int_0^q \left(\sqrt{y^2+1} - \frac{\sqrt{q^2+1}}{q} y \right) dy$.

(b) (i) Differentiate $f(y) = y\sqrt{y^2+1} + \ln(y + \sqrt{y^2+1})$ with respect to y and simplify.

(ii) Hence, or otherwise, show that $A = \ln\left(q + \sqrt{q^2+1}\right)$.

(c) Show that $\sinh A = \frac{e^A - e^{-A}}{2}$ and $\cosh A = \frac{e^A + e^{-A}}{2}$.

(d) (i) Show that $\cosh^2 x - \sinh^2 x = 1$.

(ii) Show that $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$.

(iii) Given that $\tanh x = \frac{\sinh x}{\cosh x}$, find $\frac{d}{dx}\tanh x$ and simplify your answer.

The inverse of sinh, denoted by $\sinh^{-1}\square$, is defined by $\sinh^{-1}(\sinh x) = x$.

(e) Show that $\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c$.

[adapted from IBO Training Resources by Daniel Hwang]

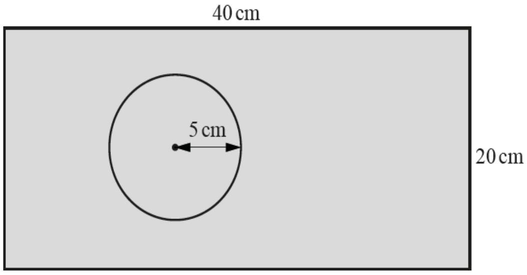
10. This question asks you to investigate Monte Carlo method to estimate the value of π .

Lee draws circles on pieces of paper. He drops grains of rice at random onto the pieces of paper. He counts the number of grains of rice inside each circle.

(a) Lee draws a circle of radius 5 cm on a rectangular piece of paper measuring 40 cm by 20 cm.

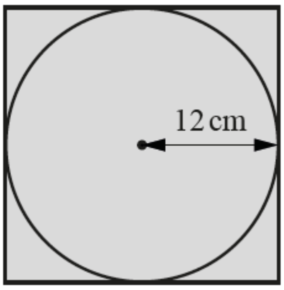
The probability p that a grain of rice lands inside the circle is given by $p = \frac{\text{area of circle}}{\text{area of paper}}$.

Show that $p \approx 0.098$ for this piece of paper.

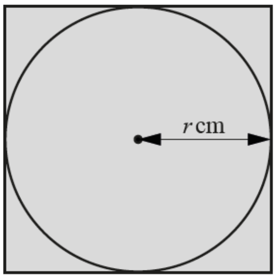


(b) Lee draws a circle of radius 12 cm on a different piece of paper. The circle touches all four edges of the paper.

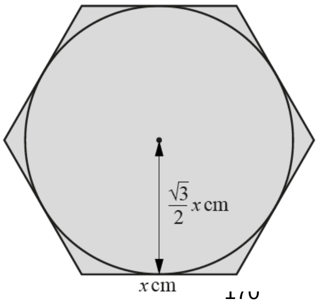
Lee drops 50 grains of rice at random onto the piece of paper. He removes the 50 grains of rice and drops another 50 grains of rice at random onto the piece of paper. The combined number of grains of rice inside the circle is 78. Use the formula $p = \frac{\text{area of circle}}{\text{area of paper}}$ to estimate the value of π .



(c) Lee draws a circle of radius r cm on a different piece of paper. The circle touches all four edges of the paper. Suggest, for any value of r, an equation to estimate the value of π in terms of p.



(d) Lee draws a circle on a piece of paper in the shape of a regular hexagon of side length x cm. The circle touches all six edges of the paper. Find an estimate for the value of π when $x = 30$ and $p = 0.905$.



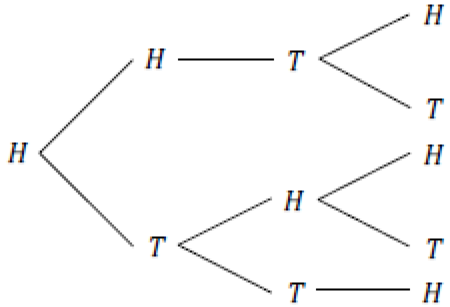
(e) Show that, for any value of r, an estimate for $\pi = kp$ where $k = a\sqrt{b}$, a and b are integers.
[Adapted from IGCSE 0607_s18_qp_63]

11. This question asks you to investigate the probabilities and Fibonacci sequences.

Let D_n be the number of sequences of n tosses starting with a head **without** three consecutive heads or tails.

(a) Find D_1 and D_2

The three diagram shows sequences of four tosses starting with a head **without** three consecutive heads or tails.



(b) Explain why $D_n = D_{n-1} + D_{n-2}$ for $n \geq 3$. (Hint: look into Fibonacci Sequence)

(c) Find the probability of getting at least three consecutive heads or tails in 10 tosses.
[adapted from IBO Training Resources by Daniel Hwang]

12. This question asks you to investigate the rotations and parametric equations.

(a) The line $y = 0$ is rotated θ radians anti-clockwise about the origin. Find the equation of the new line in terms of y, x and θ.

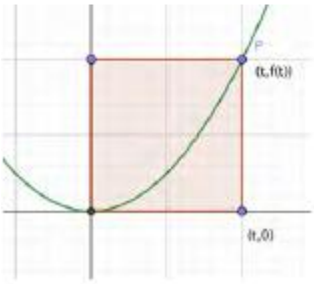
If we have a coordinate point (a, b) rotated θ radians anti-clockwise about the origin, we can find the x, y coordinates of the new point by using the following parametric equations:

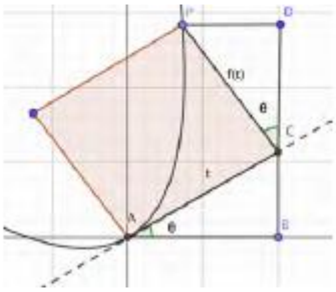
$$x = a\cos\theta - b\sin\theta$$
$$y = a\sin\theta + b\cos\theta$$

(b) Rotate the point $(1,1)$ anti-clockwise around the origin by $\frac{\pi}{6}, \frac{\pi}{4}$ and $\frac{\pi}{2}$ radians. Give your answers as coordinates.

(c) By squaring the equations, obtain an equation in terms of x and y only. What is the geometric significance of this equation?

We can rotate any curve by any angle θ by means of this method. Starting with a function $f(t)$, draw a rectangle centred at the origin through the point $P(t, f(t))$. Rotate the curve and the rectangle by θ radians anti-clockwise from the horizontal. Referring to the diagram, we note that $\angle ABC$ and $\angle CDP$ are right angles.





(d) Explain why $\angle CAB = \angle PCD$

(e) Show that P has coordinates $(t\cos\theta - f(t)\sin\theta\,, t\sin\theta + f(t)\cos\theta)$.

(f) Hence, show that $y\cos\theta - x\sin\theta = f(y\sin\theta + x\cos\theta)$.

(g) Hence, find the equation obtained when $y = 2x + 3$ is rotated by $\frac{\pi}{4}$ radians anti-clockwise around (0,0).

[adapted from Paper 3 Exploration questions by Andrew Chambers]

Answers

1. (e) $1 + x^2 + x^4 + x^6 + x^8 + x^{10}$.

2 (a)(i) $e^x - e^{4x}$ (a)(ii) $e^x - e^{(N+1)x}$ (iii) $x \le 0$ (iv) e^x

3 (a) $\frac{1}{3}\ln\left|x^3+1\right|+C$ (b)(i) $\ln|x+1|+C$ (ii) $\frac{2}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C$

(c) $\frac{1}{3}\ln|x+1|+\frac{1}{6}\ln\left|x^2-x+1\right|+\frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right)+C$

4. (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{8}+\frac{1}{4}$ (d)(ii) $\frac{3\pi}{32}+\frac{1}{4}$ (e) $\frac{3\pi}{32}+\frac{1}{4}$

5 (d) N = 250730 and onwards

6 (b) (i) $x = k\pi$ (ii) $\frac{\pi^2}{6}$ (iii) $\frac{3}{4}$ (c) (ii) 1.20

7 (a) $\ln x, \frac{1}{x}, -\frac{1}{x^2}$ (b) $-1+\frac{(x-1)^2}{2}-\frac{(x-1)^3}{6}$

8. (a) (iii) $-\frac{1}{2}$ (iv) $-\frac{1}{2}$ (b) (i) $\theta = \frac{k\pi}{7}, k \in \mathbb{Z}$ (iv) $\frac{1}{64}$

9. (b) (i) $2\sqrt{y^2+1}$ (d)(iii) $\frac{1}{\cosh^2 x}$

10. (b) 3.12 (c) π = 4p (d) 3.14 (e) $\pi = 2\sqrt{3}p$

11. (g) if previous outcome is same, generate 1, if previous outcome is different, generate 2
(h) $\frac{423}{512}$

12. (a) $y = x\tan\theta$ (b) $\left(\frac{-1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}\right)$; $\left(0, \sqrt{2}\right)$; $(-1,1)$
(c) $x^2 + y^2 = a^2 + b^2$. Circle centred at (0,0), radius $\sqrt{a^2+b^2}$. (g) $y = -3x - 3\sqrt{2}$.

Paper 3 Specimen Questions (from IB)

1. This question asks you to investigate compactness, a quantity that measures how compact an enclosed region is. The compactness, *C*, of an enclosed region can be defined by $C = \frac{4A}{\pi d^2}$, where *A* is the area of the region and *d* is the maximum distance between any two points in the region.

(a) Verify that $C = 1$ for a circular region. [2]

Consider a rectangle whose side lengths are in the ratio $x:1$.

(b) Show that $C = \frac{4x}{\pi\left(1+x^2\right)}$. [3]

(c) Show that $\frac{dC}{dx} = \frac{4\left(1-x^2\right)}{\pi\left(1+x^2\right)^2}$ [2]

(d) **(i)** Hence find the value of x which gives maximum compactness for a rectangle.

(ii) Identify the geometrical significance of this result. [3]

Consider an equilateral triangle of side length x units,

(e) Find an expression, in terms of x, for the area of this equilateral triangle. [2]

(f) Hence find the exact value of C for an equilateral triangle. [3]

Consider a regular polygon of n sides constructed such that its vertices lie on the circumference of a circle of radius r units.

(g) If $n > 2$ and even, show that $C = \frac{n}{2\pi}\sin\frac{2\pi}{n}$. [4]

(h) Illustrate, with the aid of a diagram, that $d \neq 2r$ when n is odd. [1]

If $n > 1$ and odd, it can be shown that $C = \frac{n\sin\frac{2\pi}{n}}{\pi\left(1+\cos\frac{\pi}{n}\right)}$,

(i) Find the regular polygon with the least number of sides for which the compactness is more than 0.995. [5]

(j) **(i)** For n even, use l'Hopital's rule, or otherwise, to show that $\lim_{n\to\infty} C = 1$.

(ii) Using your answer to part (i), show that $\lim_{n\to\infty} C = 1$, for $n > 1$ and odd.

(iii) Interpret geometrically the significance of parts **(i)** and **(ii)**. [6]

2. A **Gaussian integer** is a complex number, $z,$ such that $z = a + bi$ where $a, b \in$. In this question, you are asked to investigate certain divisibility properties of Gaussian integers.

Consider two Gaussian integers $\alpha = 3 + 4i$ and $\beta = 1 - 2i$ such that $\gamma = \alpha\beta$ for some Gaussian integer γ

(a) Find γ . [2]

Now consider two Gaussian integers $\alpha = 3 + 4i$ and $\gamma = 11 + 2i$.

(b) Determine whether $\dfrac{\gamma}{\alpha}$ is a Gaussian integer. [3]

The norm of a complex number z , denoted by $N(z)$, is defined by $N(z) = |z|^2$. For example, if $z = 2 + 3i$ then $N(2 + 3i) = 2^2 + 3^2 = 13$.

(c) On an Argand diagram, plot and label all Gaussian integers that have a norm less than 3.[2]

(d) Given that $\alpha = a + bi$ where $a, b \in \mathbb{Z}$, show that $N(\alpha) = a^2 + b^2$ [1]

A **Gaussian prime** is a Gaussian integer, z , that **cannot** be expressed in the form $z = \alpha\beta$ where α, β are Gaussian integers with $N(\alpha), N(\beta) > 1$.

(e) By expressing the positive integer $n = c^2 + d^2$ as the product of two Gaussian integers, each of norm $c^2 + d^2$, show that n is not a Gaussian prime. [3]

The positive integer 2 is a prime number, however it is not a Gaussian prime.

(f) Verify that 2 is not a Gaussian prime. [2]

(g) Write down another prime number of the form $c^2 + d^2$ that is not a Gaussian prime and express it as a product of two Gaussian integers. [2]

Let α, β be Gaussian integers.

(h) Show that $N(\alpha\beta) = N(\alpha)N(\beta)$. [6]

The result from part (h) provides a way of determining whether a Gaussian integer is a Gaussian prime.

(i) Hence show that $1 + 4i$ is a Gaussian prime. [3]

(j) Use proof by contradiction to prove that a prime number, p , that is not of the form $a^2 + b^2$ is a Gaussian prime. [6]

3. This question will investigate power series, as an extension to the Binomial Theorem for negative and fractional indices.

A power series in x is defined as a function of the form $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ where the $a_i \in \mathbb{R}$. It can be considered as an infinite polynomial.

(a) Expand $(1+x)^5$ using the Binomial Theorem. [2]

This is an example of a power series, but is only a finite power series, since only a finite number of the a_i are non-zero.

Consider the power series $1 - x + x^2 - x^3 + x^4 - \cdots$.

(b) By considering the ratio of consecutive terms, explain why this series is equal to $(1+x)^{-1}$ and state the values of x for which this equality is true. [4]

(c) Differentiate the equation obtained for part **(b)** and hence, find the first four terms in a power series for $(1+x)^{-2}$. [2]

(d) Repeat this process to find the first four terms in a power series for $(1+x)^{-3}$. [2]

(e) Hence, by recognizing the pattern, deduce the first four terms in a power series for $(1+x)^{-n}, n \in \mathbb{Z}^+$. [3]

We will now attempt to generalize further.

(f) Suppose $(1+x)^q, q \in \mathbb{Q}$ can be written as the power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$ By substituting $x = 0$, find the value of a_0. [1]

(g) By differentiating both sides of the expression and then substituting $x = 0$, find the value of a_1. [2]

(h) Repeat this procedure to find a_2 and a_3. [4]

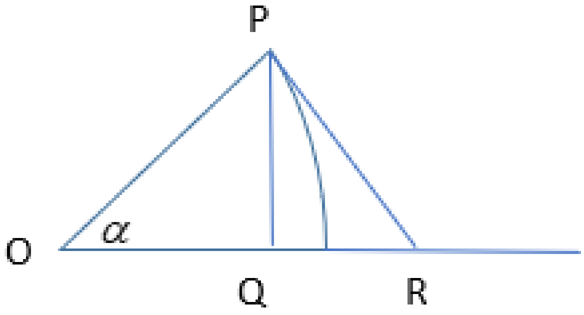
(i) Hence, write down the first four terms in what is called the Extended Binomial Theorem for $(1+x)^q, q \in \mathbb{Q}$. [1]

(j) Write down the power series for $\dfrac{1}{1+x^2}$. [2]

(k) Hence, using integration, find the power series for $\arctan x$, giving the first four non-zero terms. [4]

4. This question will explore connections between complex numbers and regular polygons.

This diagram below shows a sector of a circle of radius 1, with the angle subtended at the centre O being α, $0<\alpha<\frac{\pi}{2}$. A perpendicular is drawn from point P to intersect the x-axis at Q. The tangent to the circle at P intersects the x-axis at R.



(a) By considering the area of the two triangles and the area of the sector, show that $\cos\alpha\sin\alpha<\alpha<\frac{\sin\alpha}{\cos\alpha}$. [5]

(b) Hence, show that $\lim_{\alpha\to 0}\frac{\alpha}{\sin\alpha}=1$. [2]

(c) Let $z^n=1, z\in\mathbb{C}, n\in\mathbb{N}, n\geq 5$. Working in modulus/argument form, find the n solutions to this equation. [8]

(d) Represent these n solutions on an Argand diagram. Let their positions be denoted by $P_0, P_1, P_2, \ldots, P_{n-1}$ placed in order in an anticlockwise direction round the circle, starting on the positive x-axis. Show the positions of P_0, P_1, P_2 and P_{n-1}. [1]

(e) Show that the length of the line segment P_0P_1 is $2\sin\frac{\pi}{n}$. [4]

(f) Hence, write down the total length of the perimeter of the regular n sided polygon $P_0P_1P_2\ldots P_{n-1}P_0$. [1]

(g) Use part **(b)** to find the limit of this perimeter as $n\to\infty$. [2]

(h) Find the total area of this n sided polygon. [3]

(i) Using part **(b)**, find the limit of this area as $n\to\infty$. [2]

5. This question will investigate methods for finding definite integrals of powers of trigonometrical functions.

Let $I_n = \int_0^{\frac{\pi}{2}} sin^n\, x\, dx, n \in \mathbb{N}$.

(a) Find the exact values of I_0, I_1 and I_2. [6]

(b) Use integration by parts to show that $I_n = \frac{n-1}{n} I_{n-2}, n \geq 2$. [5]

(c) Explain where the condition $n \geq 2$ was used in your proof. [1]

(d) Hence, find the exact values of I_3 and I_4. [2]

Let $J_n = \int_0^{\frac{\pi}{2}} cos^n\, x\, dx, n \in \mathbb{N}$.

(e) Use the substitution $x = \frac{\pi}{2} - u$ to show that $J_n = I_n$. [4]

(f) Hence, find the exact values of J_5 and J_6. [2]

Let $T_n = \int_0^{\frac{\pi}{4}} tan^n\, x\, dx, n \in \mathbb{N}$.

(g) Find the exact values of T_0 and T_1. [3]

(h) Use the fact that $\tan^2 x = \sec^2 x - 1$ to show that $T_n = \frac{1}{n-1} - T_{n-2}, n \geq 2$. [3]

(i) Explain where the condition $n \geq 2$ was used in your proof. [1]

(j) Hence, find the exact values of T_2 and T_3. [2]

6. This question investigates the sum of sine and cosine functions.

(a) Sketch the graph of $y = 3\sin x + 4\cos x$, for $-2\pi \leq x \leq 2\pi$. [1]

(b) Write down the amplitude of this graph. [1]

(c) Write down the period of this graph. [1]

The expression $3\sin x + 4\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A, B \in \mathbb{R}^+$ and $C, D \in \mathbb{R}$ and $-\pi < C \leq \pi$.

(d) Use your answers from part (a) to write down the value of *A*, *B* and *D*. [1]

(e) Find the value of *C*. [2]

(f) Find $\arctan\frac{3}{4}$, giving the answer to 3 significant figures. [1]

(g) Comment on your answer to part (c)(i). [1]

The expression $5\sin x + 12\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $A, B \in \mathbb{R}^+$ and $C, D \in \mathbb{R}$ and $-\pi < C \leq \pi$.

(h) By considering the graph of $y = 5\sin x + 12\cos x$, find the value of *A*, *B*, *C* and *D*. [5]

In general, the expression $a\sin x + b\cos x$ can be written in the form $A\cos(Bx + C) + D$, where $a, b, A, B \in \mathbb{R}^+$ and $C, D \in \mathbb{R}$ and $-\pi < C \leq \pi$.

(i) Conjecture an expression, in terms of *a* and *b*, for

(a) A

(b) B

(c) C

(d) D [4]

The expression $a\sin x + b\cos x$ can also be written in the form

$$\sqrt{a^2+b^2}\left(\frac{a}{\sqrt{a^2+b^2}}\sin x + \frac{b}{\sqrt{a^2+b^2}}\cos x\right).$$

Let $\dfrac{a}{\sqrt{a^2+b^2}} = \sin\theta.$

(j) Show that $\dfrac{b}{\sqrt{a^2+b^2}} = \cos\theta.$ [2]

(k) Show that $\dfrac{a}{b} = \tan\theta.$ [1]

(l) Hence, prove your conjecture in part **(i)**. [6]

7. This question investigates some applications of differential equations to modelling population growth.

One model for population growth is to assume that the rate of change of the population is proportional to the population, i.e. $\dfrac{dP}{dt} = kP,$ where $k \in \mathbb{R}$, t is the time (in years) and P is the population.

(a) Show that the general solution of this differential equation is $P = Ae^{kt}$, where $A \in \mathbb{R}$. [5]

The initial population is 1000. Given that $k = 0.003,$ use your answer from part **(a)** to find

(b) the population after 10 years. [2]

(c) the number of years it will take for the population to triple. [2]

(d) $\lim\limits_{t\to\infty} P.$ [1]

Consider now the situation where *k* is not a constant, but a function of time. Given that $k = 0.003 + 0.002t,$ find

(e) the solution of the differential equation, giving your answer in the form $P = f(t).$ [5]

(f) the number of years it will take for the population to triple. [4]

Another model for population growth assumes

- there is a maximum value for the population, *L*.
- that *k* is not a constant, but is proportional to $\left(1 - \dfrac{P}{L}\right).$

(g) Show that $\dfrac{dP}{dt} = \dfrac{m}{L}P\left(L-P\right)$ where $m \in \mathbb{R}$. [2]

(h) Solve the differential equation $\frac{dP}{dt}=\frac{m}{L}P\left(L-P\right)$, giving your answer in the form $P=g\left(t\right)$. [10]

(i) Given that the initial population is 1000, L = 10000 and m = 0.003, find the number of years it will take for the population to triple. [4]

Answers:

1. (d) (i) $x = 1$ (ii) The most compact rectangle is a square. (e) $A=\frac{\sqrt{3}x^2}{4}$ (f) $C=\frac{\sqrt{3}}{\pi}$ (i) 29

(j) (iii) As $n\to\infty$, the polygon becomes a circle with compactness = 1.

2. (a) $11-2i$ (b) not a Gaussian integer.
3. (a) $1+5x+10x^2+10x^3+5x^4+x^5$ (b) It is an Infinite GP with $a=1, r=-x.\ |x|<1$
(c) $1-2x+3x^2-4x^3$ (d) $1-3x+6x^2-10x^3$
(e) $(1+x)^{-n}=1-nx+\frac{n(n+1)}{2!}x^2-\frac{n(n+1)(n+2)}{3!}x^3$
(f) 1 (g) q (h) $a_2=\frac{q(q-1)}{2!}, a_3=\frac{q(q-1)(q-2)}{3!}$
(i) $(1+x)^q=1+qx+\frac{q(q-1)}{2!}x^2+\frac{q(q-1)(q-2)}{3!}x^3$ (j) $1-x^2+x^4-x^6+\cdots$
(i) $x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\cdots$
4. (c) $z=\frac{cis2k\pi}{n}, 0\le k\le n-1$ (f) $2n\sin\frac{\pi}{n}$ (g) 2π (h) $\frac{n}{2}\sin\frac{2\pi}{n}$ (i) π
5. (a) $I_0=\frac{\pi}{2}, I_1=1, I_2=\frac{\pi}{4}$ (d) $I_3=\frac{2}{3}, I_4=\frac{3\pi}{16}$ (f) $J_5=\frac{8}{15},\ J_6=\frac{5\pi}{32}$
(g) $T_0=\frac{\pi}{4}, T_1=\ln\sqrt{2}$ (j) $T_2=1-\frac{\pi}{4}, T_3=\frac{1}{2}-\ln\sqrt{2}$
6. (b) 5, (c) 2π (d) $A=5, B=1, D=0$ (e) $C=-0.644$ (f) 0.644
(h) $A=13, B=1, C=-0.395, D=0$ (i) $A=\sqrt{a^2+b^2}; B=1, C=-\arctan\frac{a}{b}\ D=0$
7. (b) 1030 (c) 366 (d) ∞ (e) $P=1000e^{0.003t+0.001t^2}$ (f) 31.7 (i) 450 years

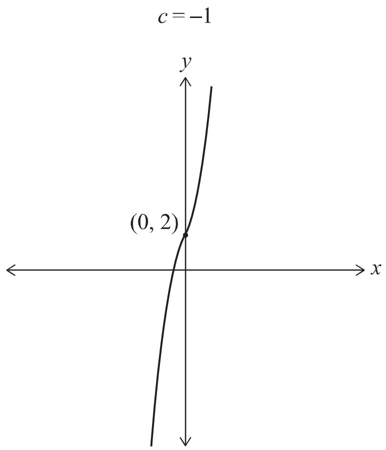
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

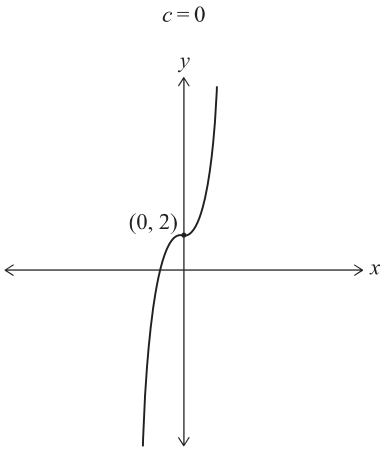
1. [Maximum mark: 27]

This question asks you to explore the behaviour and key features of cubic polynomials of the form $x^3 - 3cx + d$.

Consider the function $f(x) = x^3 - 3cx + 2$ for $x \in \mathbb{R}$ and where c is a parameter, $c \in \mathbb{R}$.

The graphs of $y = f(x)$ for $c = -1$ and $c = 0$ are shown in the following diagrams.





(a) On separate axes, sketch the graph of $y = f(x)$ showing the value of the y-intercept and the coordinates of any points with zero gradient, for

(i) $c = 1$; [3]

(ii) $c = 2$. [3]

(b) Write down an expression for $f'(x)$. [1]

(This question continues on the following page)

(Question 1 continued)

(c) Hence, or otherwise, find the set of values of c such that the graph of $y = f(x)$ has

(i) a point of inflexion with zero gradient; [1]

(ii) one local maximum point and one local minimum point; [2]

(iii) no points where the gradient is equal to zero. [1]

(d) Given that the graph of $y = f(x)$ has one local maximum point and one local minimum point, show that

(i) the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$; [3]

(ii) the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$. [1]

(e) Hence, for $c > 0$, find the set of values of c such that the graph of $y = f(x)$ has

(i) exactly one x-axis intercept; [2]

(ii) exactly two x-axis intercepts; [2]

(iii) exactly three x-axis intercepts. [2]

Consider the function $g(x) = x^3 - 3cx + d$ for $x \in \mathbb{R}$ and where $c,\ d \in \mathbb{R}$.

(f) Find all conditions on c and d such that the graph of $y = g(x)$ has exactly one x-axis intercept, explaining your reasoning. [6]

2. [Maximum mark: 28]

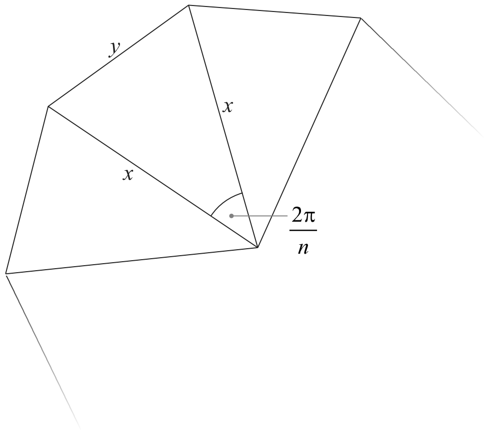
This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.

For each polygon in this question, let the numerical value of its area be A and let the numerical value of its perimeter be P.

(a) Find the side length, s, where $s > 0$, of a square such that $A = P$. [3]

An n-sided regular polygon can be divided into n congruent isosceles triangles. Let x be the length of each of the two equal sides of one such isosceles triangle and let y be the length of the third side. The included angle between the two equal sides has magnitude $\frac{2\pi}{n}$.

Part of such an n-sided regular polygon is shown in the following diagram.



(b) Write down, in terms of x and n, an expression for the area, A_T, of one of these isosceles triangles. [1]

(c) Show that $y = 2x\sin\frac{\pi}{n}$. [2]

Consider a n-sided regular polygon such that $A = P$.

(d) Use the results from parts (b) and (c) to show that $A = P = 4n\tan\frac{\pi}{n}$. [7]

(This question continues on the following page)

(Question 2 continued)

The Maclaurin series for $\tan x$ is $x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$

(e) (i) Use the Maclaurin series for $\tan x$ to find $\lim_{n \to \infty}\left(4n\tan\frac{\pi}{n}\right)$. [3]

(ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths a, b and $\sqrt{a^2+b^2}$, where $a \geq b$, such that $A = P$.

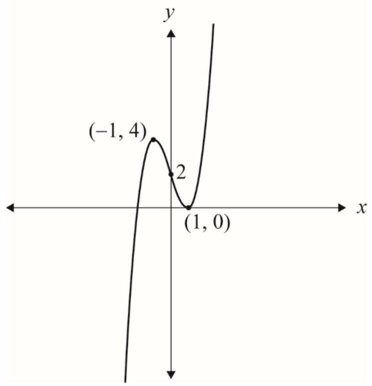
(f) Show that $a = \frac{8}{b-4} + 4$. [7]

(g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which $a,\ b,\ A,\ P \in \mathbb{Z}$. [3]

(ii) Determine the area and perimeter of these two right-angled triangles. [1]

References:

1. (a) (i)



$c = 1$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

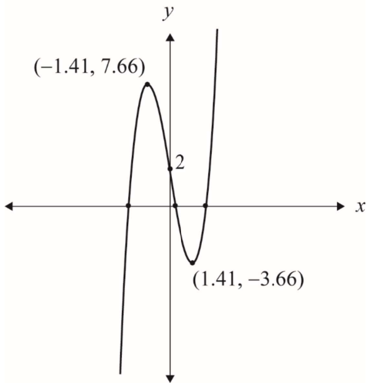
local minimum point correctly labelled ***A1***

[3 marks]

continued…

Question 1 continued

(ii)



$c = 2$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

local minimum point correctly labelled ***A1***

Note: Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

[3 marks]

continued…

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$ ***A1***

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$ ***A1***

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ ***A1***

[2 marks]

(iii) $c < 0$ ***A1***

Note: The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x) = 0$ for x ***(M1)***

$x = \pm\sqrt{c}$ ***(A1)***

Note: Award ***(A1)*** if either $x = -\sqrt{c}$ or $x = \sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ ***A1***

$f\left(-\sqrt{c}\right) = -c^{\frac{3}{2}} + 3c^{\frac{3}{2}} + 2 \; \left(= -c\sqrt{c} + 3c\sqrt{c} + 2\right)$

the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$ ***AG***

[3 marks]

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ ***A1***

$f\left(\sqrt{c}\right) = c^{\frac{3}{2}} - 3c^{\frac{3}{2}} + 2 \; \left(= c\sqrt{c} - 3c\sqrt{c} + 2\right)$

the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$ ***AG***

[1 mark]

continued…

Question 1 continued

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

EITHER

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

OR

the minimum point is above the $x-$ axis ***R1***

Note: Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$0 < c < 1$ ***A1***

Note: Condone $c < 1$. The ***A1*** is independent of the ***R1***.

[2 marks]

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

OR

evidence from the graph in part(a)(i) ***(M1)***

THEN

$c = 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

OR

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

THEN

$c > 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(f) case 1:

$c \leq 0$ (independent of the value of d) ***A1***

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) ***R1***

OR

$\Rightarrow g'(x) \geq 0$ for $x \in$ ~ ***R1***

OR

the graph of $y = f(x)$ has no local maximum or local minimum points, hence any vertical translation of this graph ($y = g(x)$) will also have no local maximum or local minimum points ***R1***

THEN

therefore there is only one x-axis intercept ***AG***

Note: Award at most ***A0R1*** if only $c < 0$ is considered.

continued…

Question 1 continued

case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

Note: Award ***(A1)*** for a correct y-coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

EITHER

considers both points above the x-axis or both points below the x-axis

OR

considers either the local minimum point only above the x-axis OR the local maximum point only below the x-axis

THEN

$d > 2c^{\frac{3}{2}}$ (both points above the x-axis) ***A1***

$d < -2c^{\frac{3}{2}}$ (both points below the x-axis) ***A1***

Note: Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

2. (a) $A = s^2$ and $P = 4s$ ***(A1)***

$A = P \Rightarrow s^2 = 4s$ ***(M1)***

$s(s-4) = 0$

$\Rightarrow s = 4 (s > 0)$ ***A1***

Note: Award ***A1M1A0*** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2}x^2 \sin\frac{2\pi}{n}$ ***A1***

Note: Award ***A1*** for a correct alternative form expressed in terms of x and n only.

For example, using Pythagoras' theorem, $A_T = x\sin\frac{\pi}{n}\sqrt{x^2 - x^2\sin^2\frac{\pi}{n}}$ or $A_T = 2\left(\frac{1}{2}\left(x\sin\frac{\pi}{n}\right)\left(x\cos\frac{\pi}{n}\right)\right)$ or $A_T = x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

[1 mark]

continued…

Question 2 continued

(c) **METHOD 1**

uses $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ ***(M1)***

$\frac{\frac{y}{2}}{x} = \sin\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x\cos\frac{\pi}{n}$ ***(M1)***

$\left(\frac{y}{2}\right)^2 + \left(x\cos\frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2\left(1 - \cos^2\frac{\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

continued…

Question 2 continued

METHOD 3

uses the cosine rule **(M1)**

$y^2 = 2x^2 - 2x^2\cos\frac{2\pi}{n} \quad \left(= 2x^2\left(1 - \cos\frac{2\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ **A1**

$y = 2x\sin\frac{\pi}{n}$ **AG**

[2 marks]

METHOD 4

uses the sine rule **(M1)**

$\frac{y}{\sin\frac{2\pi}{n}} = \frac{x}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$

$y\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ **A1**

$y = 2x\sin\frac{\pi}{n}$ **AG**

[2 marks]

continued…

Question 2 continued

(d) $A = P \Rightarrow nA_T = ny$ **(M1)**

Note: Award ***M1*** for equating correct expressions for A and P.

$\frac{1}{2}nx^2\sin\frac{2\pi}{n} = 2nx\sin\frac{\pi}{n} \quad \left(nx^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2nx\sin\frac{\pi}{n}\right)$

$\frac{1}{2}x^2\sin\frac{2\pi}{n} = 2x\sin\frac{\pi}{n} \quad \left(x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\right)$ **A1**

uses $\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ (seen anywhere in part (d) or in part (b)) **(M1)**

$x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}$

attempts to either factorise or divide their expression **(M1)**

$x\sin\frac{\pi}{n}\left(x\cos\frac{\pi}{n} - 2\right) = 0$

$x = \frac{2}{\cos\frac{\pi}{n}}, \left(x\sin\frac{\pi}{n} \neq 0\right)$(or equivalent) **A1**

continued…

Question 2 continued

EITHER

substitutes $x=\dfrac{2}{\cos\frac{\pi}{n}}$ (or equivalent) into $P=ny$ ***(M1)***

$$P=2n\left(\frac{2}{\cos\frac{\pi}{n}}\right)\left(\sin\frac{\pi}{n}\right)$$ ***A1***

Note: Other approaches are possible. For example, award ***A1*** for $P=2nx\cos\frac{\pi}{n}\tan\frac{\pi}{n}$ and ***M1*** for substituting $x=\dfrac{2}{\cos\frac{\pi}{n}}$ into P.

OR

substitutes $x=\dfrac{2}{\cos\frac{\pi}{n}}$ (or equivalent) into $A=nA_T$ ***(M1)***

$$A=\frac{1}{2}n\left(\frac{2}{\cos\frac{\pi}{n}}\right)^2\left(\sin\frac{2\pi}{n}\right)$$

$$A=\frac{1}{2}n\left(\frac{2}{\cos\frac{\pi}{n}}\right)^2\left(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)$$ ***A1***

THEN

$$A=P=4n\tan\frac{\pi}{n}$$ ***AG***

[7 marks]

continued…

Question 2 continued

(e) (i) attempts to use the Maclaurin series for $\tan x$ with $x=\frac{\pi}{n}$ ***(M1)***

$$\tan\frac{\pi}{n}=\frac{\pi}{n}+\frac{\left(\frac{\pi}{n}\right)^3}{3}+\frac{2\left(\frac{\pi}{n}\right)^5}{15}(+\ldots)$$

$$4n\tan\frac{\pi}{n}=4n\left(\frac{\pi}{n}+\frac{\pi^3}{3n^3}+\frac{2\pi^5}{15n^5}(+\ldots)\right)$$ (or equivalent) ***A1***

$$=4\left(\pi+\frac{\pi^3}{3n^2}+\frac{2\pi^5}{15n^4}+\ldots\right)$$

$$\Rightarrow\lim_{n\to\infty}\left(4n\tan\frac{\pi}{n}\right)=4\pi$$ ***A1***

Note: Award a maximum of ***M1A1A0*** if $\lim\limits_{n\to\infty}$ is not stated anywhere.

[3 marks]

(ii) (as $n\to\infty$, $P\to4\pi$ and $A\to4\pi$)

the polygon becomes a circle of radius 2 ***R1***

Note: Award ***R1*** for alternative responses such as:
the polygon becomes a circle of area 4π OR
the polygon becomes a circle of perimeter 4π OR
the polygon becomes a circle with $A=P=4\pi$.
Award ***R0*** for polygon becomes a circle.

[1 mark]

(f) $A=\frac{1}{2}ab$ and $P=a+b+\sqrt{a^2+b^2}$ ***(A1)(A1)***

equates their expressions for A and P ***M1***

$A=P\Rightarrow a+b+\sqrt{a^2+b^2}=\frac{1}{2}ab$

$\sqrt{a^2+b^2}=\frac{1}{2}ab-(a+b)$ ***M1***

Note: Award ***M1*** for isolating $\sqrt{a^2+b^2}$ or $\pm 2\sqrt{a^2+b^2}$. This step may be seen later.

$a^2+b^2=\left(\frac{1}{2}ab-(a+b)\right)^2$

$a^2+b^2=\frac{1}{4}a^2b^2-2\left(\frac{1}{2}ab\right)(a+b)+(a+b)^2$ ***M1***

$\left(=\frac{1}{4}a^2b^2-a^2b-ab^2+a^2+2ab+b^2\right)$

Note: Award ***M1*** for attempting to expand their RHS of either $a^2+b^2=\ldots$
or $4\left(a^2+b^2\right)=\ldots$.

EITHER

$ab\left(\frac{1}{4}ab-a-b+2\right)=0\ (ab\neq 0)$ ***A1***

$\frac{1}{4}ab-a-b+2=0$

$ab-4a=4b-8$

OR

$\frac{1}{4}a^2b^2-a^2b-ab^2+2ab=0$

$a\left(\frac{1}{4}b^2-b\right)+\left(2b-b^2\right)=0\ \left(a\left(b^2-4b\right)+\left(8b-4b^2\right)=0\right)$ ***A1***

$a=\frac{4b^2-8b}{b^2-4b}$

continued…

Question 2 continued

THEN

$\Rightarrow a=\frac{4b-8}{b-4}$ ***A1***

$a=\frac{4b-16+8}{b-4}$

$a=\frac{8}{b-4}+4$ ***AG***

Note: Award a maximum of ***A1A1M1M1M0A0A0*** for attempting to verify.
For example, verifying that $A=P=\frac{16}{b-4}+2b+4$ gains 4 of the 7 marks.

[7 marks]

continued…

Question 2 continued

(g) (i) using an appropriate method ***(M1)***

eg substituting values for b or using divisibility properties

$(5,12,13)$ and $(6,8,10)$ ***A1A1***

Note: Award ***A1A0*** for either one set of three correct side lengths or two sets of two correct side lengths.

[3 marks]

(ii) $A = P = 30$ and $A = P = 24$ ***A1***

Note: Do not award ***A1FT***.

[1 mark]

Total [28 marks]

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 31]

This question asks you to explore the behaviour and some key features of the function $f_n(x) = x^n(a-x)^n$, where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$.

In parts (a) and (b), **only** consider the case where $a = 2$.

Consider $f_1(x) = x(2-x)$.

(a) Sketch the graph of $y = f_1(x)$, stating the values of any axes intercepts and the coordinates of any local maximum or minimum points. [3]

Consider $f_n(x) = x^n(2-x)^n$, where $n \in \mathbb{Z}^+$, $n > 1$.

(b) Use your graphic display calculator to explore the graph of $y = f_n(x)$ for

- the odd values $n = 3$ and $n = 5$;
- the even values $n = 2$ and $n = 4$.

Hence, copy and complete the following table. [6]

	Number of local maximum points	**Number of local minimum points**	**Number of points of inflexion with zero gradient**
$n = 3$ and $n = 5$			
$n = 2$ and $n = 4$			

Now consider $f_n(x) = x^n(a-x)^n$ where $a \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, $n > 1$.

(c) Show that $f_n'(x) = nx^{n-1}(a-2x)(a-x)^{n-1}$. [5]

(d) State the three solutions to the equation $f_n'(x) = 0$. [2]

(e) Show that the point $\left(\frac{a}{2}, f_n\left(\frac{a}{2}\right)\right)$ on the graph of $y = f_n(x)$ is always above the horizontal axis. [3]

(This question continues on the following page)

(Question 1 continued)

(f) Hence, or otherwise, show that $f_n'\left(\frac{a}{4}\right) > 0$, for $n \in \mathbb{Z}^+$. [2]

(g) By using the result from part (f) and considering the sign of $f_n'(-1)$, show that the point $(0, 0)$ on the graph of $y = f_n(x)$ is

(i) a local minimum point for even values of n, where $n > 1$ and $a \in \mathbb{R}^+$; [3]

(ii) a point of inflexion with zero gradient for odd values of n, where $n > 1$ and $a \in \mathbb{R}^+$. [2]

Consider the graph of $y = x^n(a-x)^n - k$, where $n \in \mathbb{Z}^+$, $a \in \mathbb{R}^+$ and $k \in \mathbb{R}$.

(h) State the conditions on n and k such that the equation $x^n(a-x)^n = k$ has four solutions for x. [5]

2. [Maximum mark: 24]

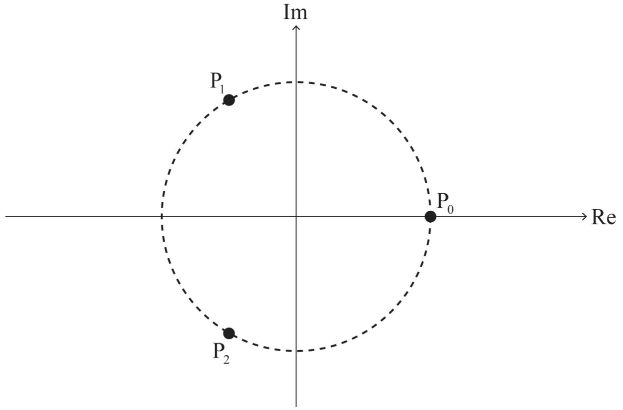
This question asks you to investigate and prove a geometric property involving the roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ for integers n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}, where $\omega = e^{\frac{2\pi i}{n}}$. Each root can be represented by a point P_0, P_1, P_2, ..., P_{n-1}, respectively, on an Argand diagram.

For example, the roots of the equation $z^2 = 1$ where $z \in \mathbb{C}$ are 1 and ω. On an Argand diagram, the root 1 can be represented by a point P_0 and the root ω can be represented by a point P_1.

Consider the case where $n = 3$.

The roots of the equation $z^3 = 1$ where $z \in \mathbb{C}$ are 1, ω and ω^2. On the following Argand diagram, the points P_0, P_1 and P_2 lie on a circle of radius 1 unit with centre $O(0, 0)$.



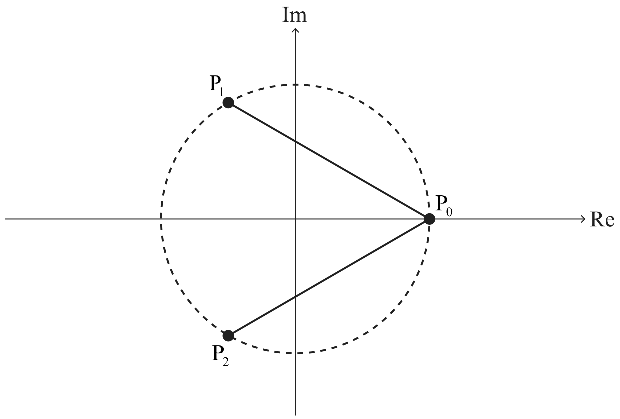
(a) (i) Show that $(\omega - 1)(\omega^2 + \omega + 1) = \omega^3 - 1$. [2]

(ii) Hence, deduce that $\omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

Line segments $[P_0P_1]$ and $[P_0P_2]$ are added to the Argand diagram in part (a) and are shown on the following Argand diagram.



P_0P_1 is the length of $[P_0P_1]$ and P_0P_2 is the length of $[P_0P_2]$.

(b) Show that $P_0P_1 \times P_0P_2 = 3$. [3]

Consider the case where $n = 4$.

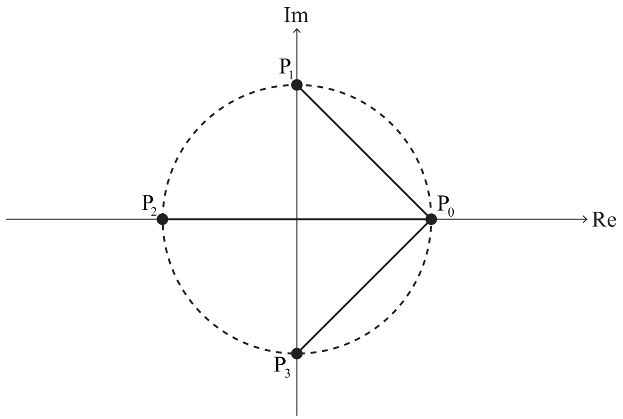
The roots of the equation $z^4 = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2 and ω^3.

(c) By factorizing $z^4 - 1$, or otherwise, deduce that $\omega^3 + \omega^2 + \omega + 1 = 0$. [2]

(This question continues on the following page)

(Question 2 continued)

On the following Argand diagram, the points P_0, P_1, P_2 and P_3 lie on a circle of radius 1 unit with centre $\mathrm{O}(0, 0)$. $[\mathrm{P}_0\mathrm{P}_1]$, $[\mathrm{P}_0\mathrm{P}_2]$ and $[\mathrm{P}_0\mathrm{P}_3]$ are line segments.



(d) Show that $\mathrm{P}_0\mathrm{P}_1 \times \mathrm{P}_0\mathrm{P}_2 \times \mathrm{P}_0\mathrm{P}_3 = 4$. [4]

For the case where $n = 5$, the equation $z^5 = 1$ where $z \in \mathbb{C}$ has roots 1, ω, ω^2, ω^3 and ω^4.

It can be shown that $\mathrm{P}_0\mathrm{P}_1 \times \mathrm{P}_0\mathrm{P}_2 \times \mathrm{P}_0\mathrm{P}_3 \times \mathrm{P}_0\mathrm{P}_4 = 5$.

Now consider the general case for integer values of n, where $n \geq 2$.

The roots of the equation $z^n = 1$ where $z \in \mathbb{C}$ are 1, ω, ω^2, ..., ω^{n-1}. On an Argand diagram, these roots can be represented by the points P_0, P_1, P_2, ..., P_{n-1} respectively where $[\mathrm{P}_0\mathrm{P}_1]$, $[\mathrm{P}_0\mathrm{P}_2]$, ..., $[\mathrm{P}_0\mathrm{P}_{n-1}]$ are line segments. The roots lie on a circle of radius 1 unit with centre $\mathrm{O}(0, 0)$.

(e) Suggest a value for $\mathrm{P}_0\mathrm{P}_1 \times \mathrm{P}_0\mathrm{P}_2 \times ... \times \mathrm{P}_0\mathrm{P}_{n-1}$. [1]

$\mathrm{P}_0\mathrm{P}_1$ can be expressed as $|1 - \omega|$.

(f) (i) Write down expressions for $\mathrm{P}_0\mathrm{P}_2$ and $\mathrm{P}_0\mathrm{P}_3$ in terms of ω. [2]

(ii) Hence, write down an expression for $\mathrm{P}_0\mathrm{P}_{n-1}$ in terms of n and ω. [1]

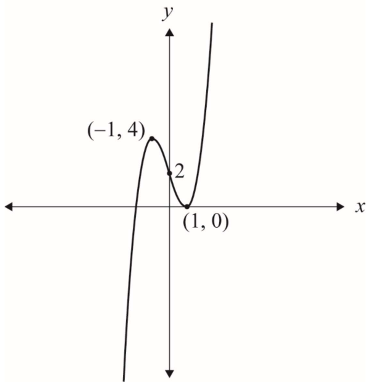
Consider $z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \ldots + z + 1)$ where $z \in \mathbb{C}$.

(g) (i) Express $z^{n-1} + z^{n-2} + \ldots + z + 1$ as a product of linear factors over the set $\mathbb{C}$. [3]

(ii) Hence, using the part (g)(i) and part (f) results, or otherwise, prove your suggested result to part (e). [4]

References:

1. (a) (i)



$c = 1$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

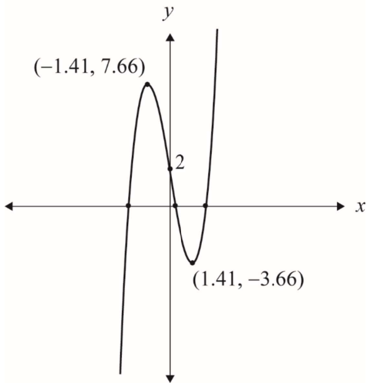
local minimum point correctly labelled ***A1***

[3 marks]

continued…

Question 1 continued

(ii)



$c = 2$: positive cubic with correct *y*-intercept labelled ***A1***

local maximum point correctly labelled ***A1***

local minimum point correctly labelled ***A1***

Note: Accept the following exact answers:

Local maximum point coordinates $\left(-\sqrt{2}, 2+4\sqrt{2}\right)$.

Local minimum point coordinates $\left(\sqrt{2}, 2-4\sqrt{2}\right)$.

[3 marks]

continued…

Question 1 continued

(b) $f'(x) = 3x^2 - 3c$ **A1**

Note: Accept $3x^2 - 3c$ (an expression).

[1 mark]

(c) (i) $c = 0$ **A1**

[1 mark]

(ii) considers the number of solutions to their $f'(x) = 0$ ***(M1)***

$3x^2 - 3c = 0$

$c > 0$ **A1**

[2 marks]

(iii) $c < 0$ **A1**

Note: The ***(M1)*** in part (c)(ii) can be awarded for work shown in either (ii) or (iii).

[1 mark]

(d) attempts to solve their $f'(x) = 0$ for x ***(M1)***

$x = \pm\sqrt{c}$ ***(A1)***

Note: Award ***(A1)*** if either $x = -\sqrt{c}$ or $x = \sqrt{c}$ is subsequently considered.
Award the above ***(M1)(A1)*** if this work is seen in part (c).

(i) correctly evaluates $f\left(-\sqrt{c}\right)$ **A1**

$f\left(-\sqrt{c}\right) = -c^{\frac{3}{2}} + 3c^{\frac{3}{2}} + 2 \ \left(= -c\sqrt{c} + 3c\sqrt{c} + 2\right)$

the y-coordinate of the local maximum point is $2c^{\frac{3}{2}} + 2$ **AG**

[3 marks]

(ii) correctly evaluates $f\left(\sqrt{c}\right)$ **A1**

$f\left(\sqrt{c}\right) = c^{\frac{3}{2}} - 3c^{\frac{3}{2}} + 2 \ \left(= c\sqrt{c} - 3c\sqrt{c} + 2\right)$

the y-coordinate of the local minimum point is $-2c^{\frac{3}{2}} + 2$ **AG**

[1 mark]

continued…

Question 1 continued

(e) (i) the graph of $y = f(x)$ will have one $x-$ axis intercept if

EITHER

$-2c^{\frac{3}{2}} + 2 > 0$ (or equivalent reasoning) ***R1***

OR

the minimum point is above the $x-$ axis ***R1***

Note: Award ***R1*** for a rigorous approach that does not (only) refer to sketched graphs.

THEN

$0 < c < 1$ ***A1***

Note: Condone $c < 1$. The ***A1*** is independent of the ***R1***.

[2 marks]

(ii) the graph of $y = f(x)$ will have two $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 = 0$ (or equivalent reasoning) ***(M1)***

OR

evidence from the graph in part(a)(i) ***(M1)***

THEN

$c = 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(iii) the graph of $y = f(x)$ will have three $x-$ axis intercepts if

EITHER

$-2c^{\frac{3}{2}} + 2 < 0$ (or equivalent reasoning) ***(M1)***

OR

reasoning from the results in both parts (e)(i) and (e)(ii) ***(M1)***

THEN

$c > 1$ ***A1***

[2 marks]

continued…

Question 1 continued

(f) case 1:

$c \le 0$ (independent of the value of d) **A1**

EITHER

$g'(x) = 0$ does not have two solutions (has no solutions or 1 solution) **R1**

OR

$\Rightarrow g'(x) \ge 0$ for $x \in$ ~ **R1**

OR

the graph of $y = f(x)$ has no local maximum or local minimum points, hence any vertical translation of this graph ($y = g(x)$) will also have no local maximum or local minimum points **R1**

THEN

therefore there is only one x-axis intercept **AG**

Note: Award at most ***A0R1*** if only $c < 0$ is considered.

continued…

Question 1 continued

case 2

$c > 0$

$\left(-\sqrt{c}, 2c^{\frac{3}{2}} + d\right)$ is a local maximum point and $\left(\sqrt{c}, -2c^{\frac{3}{2}} + d\right)$ is a local minimum point ***(A1)***

Note: Award ***(A1)*** for a correct y-coordinate seen for either the maximum or the minimum.

considers the positions of the local maximum point and/or the local minimum point ***(M1)***

EITHER

considers both points above the x-axis or both points below the x-axis

OR

considers either the local minimum point only above the x-axis OR the local maximum point only below the x-axis

THEN

$d > 2c^{\frac{3}{2}}$ (both points above the x-axis) **A1**

$d < -2c^{\frac{3}{2}}$ (both points below the x-axis) **A1**

Note: Award at most ***(A1)(M1)A0A0*** for case 2 if $c > 0$ is not clearly stated.

[6 marks]

Total [27 marks]

2. (a) $A = s^2$ and $P = 4s$ ***(A1)***

$A = P \Rightarrow s^2 = 4s$ ***(M1)***

$s(s-4) = 0$

$\Rightarrow s = 4\,(s > 0)$ ***A1***

Note: Award ***A1M1A0*** if both $s = 4$ and $s = 0$ are stated as final answers.

[3 marks]

(b) $A_T = \frac{1}{2}x^2 \sin\frac{2\pi}{n}$ ***A1***

Note: Award ***A1*** for a correct alternative form expressed in terms of x and n only.

For example, using Pythagoras' theorem, $A_T = x\sin\frac{\pi}{n}\sqrt{x^2 - x^2\sin^2\frac{\pi}{n}}$ or $A_T = 2\left(\frac{1}{2}\left(x\sin\frac{\pi}{n}\right)\left(x\cos\frac{\pi}{n}\right)\right)$ or $A_T = x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$.

[1 mark]

continued…

Question 2 continued

(c) **METHOD 1**

uses $\sin\theta = \frac{\text{opp}}{\text{hyp}}$ ***(M1)***

$\frac{\frac{y}{2}}{x} = \sin\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

METHOD 2

uses Pythagoras' theorem $\left(\frac{y}{2}\right)^2 + h^2 = x^2$ and $h = x\cos\frac{\pi}{n}$ ***(M1)***

$\left(\frac{y}{2}\right)^2 + \left(x\cos\frac{\pi}{n}\right)^2 = x^2 \quad \left(y^2 = 4x^2\left(1 - \cos^2\frac{\pi}{n}\right)\right)$

$= 4x^2\sin^2\frac{\pi}{n}$ ***A1***

$y = 2x\sin\frac{\pi}{n}$ ***AG***

[2 marks]

continued…

Question 2 continued

METHOD 3

uses the cosine rule **(M1)**

$y^2 = 2x^2 - 2x^2 \cos\frac{2\pi}{n} \quad \left(= 2x^2\left(1 - \cos\frac{2\pi}{n}\right)\right)$

$= 4x^2 \sin^2 \frac{\pi}{n}$ **A1**

$y = 2x \sin\frac{\pi}{n}$ **AG**

[2 marks]

METHOD 4

uses the sine rule **(M1)**

$\frac{y}{\sin\frac{2\pi}{n}} = \frac{x}{\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)}$

$y\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ **A1**

$y = 2x\sin\frac{\pi}{n}$ **AG**

[2 marks]

continued…

Question 2 continued

(d) $A = P \Rightarrow nA_T = ny$ **(M1)**

Note: Award ***M1*** for equating correct expressions for A and P.

$\frac{1}{2}nx^2\sin\frac{2\pi}{n} = 2nx\sin\frac{\pi}{n} \quad \left(nx^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2nx\sin\frac{\pi}{n}\right)$

$\frac{1}{2}x^2\sin\frac{2\pi}{n} = 2x\sin\frac{\pi}{n} \quad \left(x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}\right)$ **A1**

uses $\sin\frac{2\pi}{n} = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$ (seen anywhere in part (d) or in part (b)) **(M1)**

$x^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = 2x\sin\frac{\pi}{n}$

attempts to either factorise or divide their expression **(M1)**

$x\sin\frac{\pi}{n}\left(x\cos\frac{\pi}{n} - 2\right) = 0$

$x = \frac{2}{\cos\frac{\pi}{n}}, \left(x\sin\frac{\pi}{n} \neq 0\right)$(or equivalent) **A1**

continued…

Question 2 continued

EITHER

substitutes $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ (or equivalent) into $P = ny$ ***(M1)***

$P = 2n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)\left(\sin\dfrac{\pi}{n}\right)$ ***A1***

Note: Other approaches are possible. For example, award ***A1*** for $P = 2nx\cos\dfrac{\pi}{n}\tan\dfrac{\pi}{n}$

and ***M1*** for substituting $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ into P.

OR

substitutes $x = \dfrac{2}{\cos\dfrac{\pi}{n}}$ (or equivalent) into $A = nA_T$ ***(M1)***

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)^2\left(\sin\dfrac{2\pi}{n}\right)$

$A = \dfrac{1}{2}n\left(\dfrac{2}{\cos\dfrac{\pi}{n}}\right)^2\left(2\sin\dfrac{\pi}{n}\cos\dfrac{\pi}{n}\right)$ ***A1***

THEN

$A = P = 4n\tan\dfrac{\pi}{n}$ ***AG***

[7 marks]

continued…

Question 2 continued

(e) (i) attempts to use the Maclaurin series for $\tan x$ with $x = \dfrac{\pi}{n}$ ***(M1)***

$$\tan\frac{\pi}{n} = \frac{\pi}{n} + \frac{\left(\frac{\pi}{n}\right)^3}{3} + \frac{2\left(\frac{\pi}{n}\right)^5}{15}(+...)$$

$4n\tan\dfrac{\pi}{n} = 4n\left(\dfrac{\pi}{n} + \dfrac{\pi^3}{3n^3} + \dfrac{2\pi^5}{15n^5}(+...)\right)$ (or equivalent) ***A1***

$$= 4\left(\pi + \frac{\pi^3}{3n^2} + \frac{2\pi^5}{15n^4} + ...\right)$$

$\Rightarrow \lim\limits_{n\to\infty}\left(4n\tan\dfrac{\pi}{n}\right) = 4\pi$ ***A1***

Note: Award a maximum of ***M1A1A0*** if $\lim\limits_{n\to\infty}$ is not stated anywhere.

[3 marks]

(ii) (as $n \to \infty$, $P \to 4\pi$ and $A \to 4\pi$)

the polygon becomes a circle of radius 2 ***R1***

Note: Award ***R1*** for alternative responses such as:
the polygon becomes a circle of area 4π OR
the polygon becomes a circle of perimeter 4π OR
the polygon becomes a circle with $A = P = 4\pi$.
Award ***R0*** for polygon becomes a circle.

[1 mark]

(f) $A=\frac{1}{2}ab$ and $P=a+b+\sqrt{a^2+b^2}$ ***(A1)(A1)***

equates their expressions for A and P ***M1***

$A=P\Rightarrow a+b+\sqrt{a^2+b^2}=\frac{1}{2}ab$

$\sqrt{a^2+b^2}=\frac{1}{2}ab-(a+b)$ ***M1***

Note: Award ***M1*** for isolating $\sqrt{a^2+b^2}$ or $\pm 2\sqrt{a^2+b^2}$. This step may be seen later.

$a^2+b^2=\left(\frac{1}{2}ab-(a+b)\right)^2$

$a^2+b^2=\frac{1}{4}a^2b^2-2\left(\frac{1}{2}ab\right)(a+b)+(a+b)^2$ ***M1***

$\left(=\frac{1}{4}a^2b^2-a^2b-ab^2+a^2+2ab+b^2\right)$

Note: Award ***M1*** for attempting to expand their RHS of either $a^2+b^2=\ldots$
or $4\left(a^2+b^2\right)=\ldots$.

EITHER

$ab\left(\frac{1}{4}ab-a-b+2\right)=0\ (ab\neq 0)$ ***A1***

$\frac{1}{4}ab-a-b+2=0$

$ab-4a=4b-8$

OR

$\frac{1}{4}a^2b^2-a^2b-ab^2+2ab=0$

$a\left(\frac{1}{4}b^2-b\right)+\left(2b-b^2\right)=0\ \left(a\left(b^2-4b\right)+\left(8b-4b^2\right)=0\right)$ ***A1***

$a=\frac{4b^2-8b}{b^2-4b}$

continued…

Question 2 continued

THEN

$\Rightarrow a=\frac{4b-8}{b-4}$ ***A1***

$a=\frac{4b-16+8}{b-4}$

$a=\frac{8}{b-4}+4$ ***AG***

Note: Award a maximum of ***A1A1M1M1M0A0A0*** for attempting to verify.
For example, verifying that $A=P=\frac{16}{b-4}+2b+4$ gains 4 of the 7 marks.

[7 marks]

continued…

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question you will explore some of the properties of special functions f and g and their relationship with the trigonometric functions, sine and cosine.

Functions f and g are defined as $f(z)=\dfrac{\mathrm{e}^z+\mathrm{e}^{-z}}{2}$ and $g(z)=\dfrac{\mathrm{e}^z-\mathrm{e}^{-z}}{2}$, where $z\in\mathbb{C}$.

Consider t and u, such that $t,\ u\in\mathbb{R}$.

(a) Verify that $u=f(t)$ satisfies the differential equation $\dfrac{\mathrm{d}^2u}{\mathrm{d}t^2}=u$. [2]

(b) Show that $\left(f(t)\right)^2+\left(g(t)\right)^2=f(2t)$. [3]

(c) Using $\mathrm{e}^{\mathrm{i}u}=\cos u+\mathrm{i}\sin u$, find expressions, in terms of $\sin u$ and $\cos u$, for

  (i) $f(\mathrm{i}u)$; [3]

  (ii) $g(\mathrm{i}u)$. [2]

(d) Hence find, and simplify, an expression for $\left(f(\mathrm{i}u)\right)^2+\left(g(\mathrm{i}u)\right)^2$. [2]

(e) Show that $\left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2$. [4]

The functions $\cos x$ and $\sin x$ are known as circular functions as the general point $(\cos\theta,\sin\theta)$ defines points on the unit circle with equation $x^2+y^2=1$.

The functions $f(x)$ and $g(x)$ are known as hyperbolic functions, as the general point $\left(f(\theta),g(\theta)\right)$ defines points on a curve known as a hyperbola with equation $x^2-y^2=1$. This hyperbola has two asymptotes.

(f) Sketch the graph of $x^2-y^2=1$, stating the coordinates of any axis intercepts and the equation of each asymptote. [4]

The hyperbola with equation $x^2-y^2=1$ can be rotated to coincide with the curve defined by $xy=k,\ k\in\mathbb{R}$.

(g) Find the possible values of k. [5]

2. [Maximum mark: 30]

In this question you will be exploring the strategies required to solve a system of linear differential equations.

Consider the system of linear differential equations of the form:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=x-y \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t}=ax+y,$$

where $x,y,t\in\mathbb{R}^+$ and a is a parameter.

First consider the case where $a=0$.

(a) (i) By solving the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t}=y$, show that $y=A\mathrm{e}^t$ where A is a constant. [3]

  (ii) Show that $\dfrac{\mathrm{d}x}{\mathrm{d}t}-x=-A\mathrm{e}^t$. [1]

  (iii) Solve the differential equation in part (a)(ii) to find x as a function of t. [4]

Now consider the case where $a=-1$.

(b) (i) By differentiating $\dfrac{\mathrm{d}y}{\mathrm{d}t}=-x+y$ with respect to t, show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}=2\dfrac{\mathrm{d}y}{\mathrm{d}t}$. [3]

  (ii) By substituting $Y=\dfrac{\mathrm{d}y}{\mathrm{d}t}$, show that $Y=B\mathrm{e}^{2t}$ where B is a constant. [3]

  (iii) Hence find y as a function of t. [2]

  (iv) Hence show that $x=-\dfrac{B}{2}\mathrm{e}^{2t}+C$, where C is a constant. [3]

Now consider the case where $a=-4$.

(c) (i) Show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}-2\dfrac{\mathrm{d}y}{\mathrm{d}t}-3y=0$. [3]

  From previous cases, we might conjecture that a solution to this differential equation is $y=F\mathrm{e}^{\lambda t}$, $\lambda\in\mathbb{R}$ and F is a constant.

  (ii) Find the two values for λ that satisfy $\dfrac{\mathrm{d}^2y}{\mathrm{d}t^2}-2\dfrac{\mathrm{d}y}{\mathrm{d}t}-3y=0$. [4]

  Let the two values found in part (c)(ii) be λ_1 and λ_2.

  (iii) Verify that $y=F\mathrm{e}^{\lambda_1 t}+G\mathrm{e}^{\lambda_2 t}$ is a solution to the differential equation in (c)(i), where G is a constant. [4]

References:

1. (a) $f'(t)=\frac{\mathrm{e}^t-\mathrm{e}^{-t}}{2}$ ***A1***

$f''(t)=\frac{\mathrm{e}^t+\mathrm{e}^{-t}}{2}$ ***A1***

$=f(t)$ ***AG***

[2 marks]

(b) **METHOD 1**

$\left(f(t)\right)^2+\left(g(t)\right)^2$

substituting f and g ***M1***

$=\frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2+(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$

$=\frac{(\mathrm{e}^t)^2+2+(\mathrm{e}^{-t})^2+(\mathrm{e}^t)^2-2+(\mathrm{e}^{-t})^2}{4}$ ***(M1)***

$=\frac{(\mathrm{e}^t)^2+(\mathrm{e}^{-t})^2}{2}\ \left(=\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}}{2}\right)$ ***A1***

$=f(2t)$ ***AG***

METHOD 2

$f(2t)=\frac{\mathrm{e}^{2t}+\mathrm{e}^{-2t}}{2}$

$=\frac{(\mathrm{e}^t)^2+(\mathrm{e}^{-t})^2}{2}$ ***M1***

$=\frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2+(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$ ***M1A1***

$=\left(f(t)\right)^2+\left(g(t)\right)^2$ ***AG***

[3 marks]

Note: Accept combinations of METHODS 1 & 2 that meet at equivalent expressions.

(c) (i) substituting $\mathrm{e}^{\mathrm{i}u}=\cos u+\mathrm{i}\sin u$ into the expression for f ***(M1)***

obtaining $\mathrm{e}^{-\mathrm{i}u}=\cos u-\mathrm{i}\sin u$ ***(A1)***

$f(\mathrm{i}u)=\frac{\cos u+\mathrm{i}\sin u+\cos u-\mathrm{i}\sin u}{2}$

Note: The ***M1*** can be awarded for the use of sine and cosine being odd and even respectively.

$=\frac{2\cos u}{2}$

$=\cos u$ ***A1***

[3 marks]

(ii) $g(\mathrm{i}u)=\frac{\cos u+\mathrm{i}\sin u-\cos u+\mathrm{i}\sin u}{2}$

substituting and attempt to simplify ***(M1)***

$=\frac{2\mathrm{i}\sin u}{2}$

$=\mathrm{i}\sin u$ ***A1***

[2 marks]

(d) **METHOD 1**

$\left(f(\mathrm{i}u)\right)^2+\left(g(\mathrm{i}u)\right)^2$

substituting expressions found in part (c) ***(M1)***

$=\cos^2 u-\sin^2 u\ (=\cos 2u)$ ***A1***

METHOD 2

$f(2\mathrm{i}u)=\frac{\mathrm{e}^{2\mathrm{i}u}+\mathrm{e}^{-2\mathrm{i}u}}{2}$

$=\frac{\cos 2u+\mathrm{i}\sin 2u+\cos 2u-\mathrm{i}\sin 2u}{2}$ ***M1***

$=\cos 2u$ ***A1***

Note: Accept equivalent final answers that have been simplified removing all imaginary parts eg $2\cos^2 u-1$ etc

[2 marks]

(e) $\left(f(t)\right)^2-\left(g(t)\right)^2=\frac{(\mathrm{e}^t+\mathrm{e}^{-t})^2-(\mathrm{e}^t-\mathrm{e}^{-t})^2}{4}$ **M1**

$=\frac{(\mathrm{e}^{2t}+\mathrm{e}^{-2t}+2)-(\mathrm{e}^{2t}+\mathrm{e}^{-2t}-2)}{4}$ **A1**

$=\frac{4}{4}=1$ **A1**

Note: Award ***A1*** for a value of 1 obtained from either LHS or RHS of given expression.

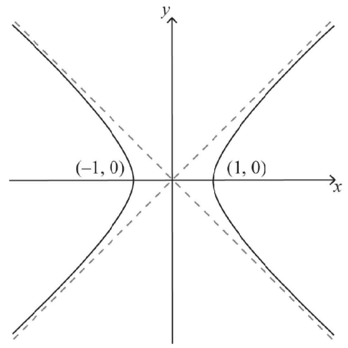
$\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2=\cos^2 u+\sin^2 u$ **M1**

$=1$ $\left(\text{hence}\left(f(t)\right)^2-\left(g(t)\right)^2=\left(f(\mathrm{i}u)\right)^2-\left(g(\mathrm{i}u)\right)^2\right)$ **AG**

Note: Award full marks for showing that $\left(f(z)\right)^2-\left(g(z)\right)^2=1,\ \forall z\in\mathbb{C}$.

[4 marks]

(f)



A1A1A1A1

Note: Award ***A1*** for correct curves in the upper quadrants, ***A1*** for correct curves in the lower quadrants **,** ***A1*** for correct x-intercepts of $(-1,\ 0)$ and $(1,\ 0)$ (condone $x=-1$ and 1), ***A1*** for $y=x$ and $y=-x$.

[4 marks]

(g) attempt to rotate by 45° in either direction ***(M1)***

Note: Evidence of an attempt to relate to a sketch of $xy=k$ would be sufficient for this ***(M1).***

attempting to rotate a particular point, *eg* $(1,\ 0)$ ***(M1)***

$(1,\ 0)$ rotates to $\left(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}}\right)$ (or similar) ***(A1)***

hence $k=\pm\frac{1}{2}$ ***A1A1***

[5 marks]

Total [25 marks]

2. (a) (i) **METHOD 1**

$\frac{\mathrm{d}y}{\mathrm{d}t} = y$

$\int \frac{\mathrm{d}y}{y} = \int \mathrm{d}t$ ***(M1)***

$\ln y = t + c$ **OR** $\ln|y| = t + c$ ***A1A1***

Note: Award ***A1*** for $\ln y$ and ***A1*** for t and c.

$y = A\mathrm{e}^t$ ***AG***

METHOD 2

rearranging to $\frac{\mathrm{d}y}{\mathrm{d}t} - y = 0$ AND multiplying by integrating factor e^{-t} ***M1***

$y\mathrm{e}^{-t} = A$ ***A1A1***

$y = A\mathrm{e}^t$ ***AG***

[3 marks]

(ii) substituting $y = A\mathrm{e}^t$ into differential equation in x ***M1***

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - A\mathrm{e}^t$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -A\mathrm{e}^t$ ***AG***

[1 mark]

(iii) integrating factor (IF) is $\mathrm{e}^{\int -1\mathrm{d}t}$ ***(M1)***

$= \mathrm{e}^{-t}$ ***(A1)***

$\mathrm{e}^{-t}\frac{\mathrm{d}x}{\mathrm{d}t} - x\mathrm{e}^{-t} = -A$

$x\mathrm{e}^{-t} = -At + D$ ***(A1)***

$x = (-At + D)\mathrm{e}^t$ ***A1***

Note: The first constant must be A, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[4 marks]

(b) (i) $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

EITHER

$= -x + y + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***(M1)***

$= \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ ***A1***

OR

$= -x + y + (-x + y)$ ***(M1)***

$= 2(-x + y)$ ***A1***

THEN

$= 2\frac{\mathrm{d}y}{\mathrm{d}t}$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}Y}{\mathrm{d}t} = 2Y$ ***A1***

$\int \frac{\mathrm{d}Y}{Y} = \int 2\mathrm{d}t$ ***M1***

$\ln|Y| = 2t + c$ OR $\ln Y = 2t + c$ ***A1***

$Y = B\mathrm{e}^{2t}$ ***AG***

[3 marks]

(iii) $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$

$y = \int B\mathrm{e}^{2t}\,\mathrm{d}t$ ***M1***

$y = \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

Note: The first constant must be B, and the second can be any constant for the final ***A1*** to be awarded. Accept a change of constant applied at the end.

[2 marks]

(iv) **METHOD 1**

substituting $\frac{\mathrm{d}y}{\mathrm{d}t} = B\mathrm{e}^{2t}$ and their (iii) into $\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y$ ***M1(M1)***

$B\mathrm{e}^{2t} = -x + \frac{B}{2}\mathrm{e}^{2t} + C$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

Note: Follow through from incorrect part (iii) cannot be awarded if it does not lead to the ***AG.***

METHOD 2

$\frac{\mathrm{d}x}{\mathrm{d}t} = x - \frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}x}{\mathrm{d}t} - x = -\frac{B}{2}\mathrm{e}^{2t} - C$

$\frac{\mathrm{d}(x\mathrm{e}^{-t})}{\mathrm{d}t} = -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}$ ***M1***

$x\mathrm{e}^{-t} = \int -\frac{B}{2}\mathrm{e}^{t} - C\mathrm{e}^{-t}\ \mathrm{d}t$

$x\mathrm{e}^{-t} = -\frac{B}{2}\mathrm{e}^{t} + C\mathrm{e}^{-t} + D$ ***A1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C + D\mathrm{e}^{t}$

$\frac{\mathrm{d}y}{\mathrm{d}t} = -x + y \Rightarrow B\mathrm{e}^{2t} = \frac{B}{2}\mathrm{e}^{2t} - C - D\mathrm{e}^{t} + \frac{B}{2}\mathrm{e}^{2t} + C \Rightarrow D = 0$ ***M1***

$x = -\frac{B}{2}\mathrm{e}^{2t} + C$ ***AG***

[3 marks]

(c) (i) $\frac{\mathrm{d}y}{\mathrm{d}t} = -4x + y$

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\mathrm{d}y}{\mathrm{d}t}$ seen anywhere ***M1***

METHOD 1

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = -4(x - y) + \frac{\mathrm{d}y}{\mathrm{d}t}$

attempt to eliminate x ***M1***

$= -4\left(\frac{1}{4}\left(y - \frac{\mathrm{d}y}{\mathrm{d}t}\right) - y\right) + \frac{\mathrm{d}y}{\mathrm{d}t}$

$= 2\frac{\mathrm{d}y}{\mathrm{d}t} + 3y$ ***A1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = 0$ ***AG***

METHOD 2

rewriting LHS in terms of x and y ***M1***

$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} - 3y = (-8x + 5y) - 2(-4x + y) - 3y$ ***A1***

$= 0$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}t} = F\lambda\mathrm{e}^{\lambda t}$, $\frac{\mathrm{d}^2y}{\mathrm{d}t^2} = F\lambda^2\mathrm{e}^{\lambda t}$ ***(A1)***

$F\lambda^2\mathrm{e}^{\lambda t} - 2F\lambda\mathrm{e}^{\lambda t} - 3F\mathrm{e}^{\lambda t} = 0$ ***(M1)***

$\lambda^2 - 2\lambda - 3 = 0$ (since $\mathrm{e}^{\lambda t} \neq 0$) ***A1***

λ_1 and λ_2 are 3 and -1 (either order) ***A1***

[4 marks]

(iii) **METHOD 1**

$y = Fe^{3t} + Ge^{-t}$

$\frac{dy}{dt} = 3Fe^{3t} - Ge^{-t}$, $\frac{d^2y}{dt^2} = 9Fe^{3t} + Ge^{-t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 9Fe^{3t} + Ge^{-t} - 2\left(3Fe^{3t} - Ge^{-t}\right) - 3\left(Fe^{3t} + Ge^{-t}\right)$ ***M1***

$= 9Fe^{3t} + Ge^{-t} - 6Fe^{3t} + 2Ge^{-t} - 3Fe^{3t} - 3Ge^{-t}$ ***A1***

$= 0$ ***AG***

METHOD 2

$y = Fe^{\lambda_1 t} + Ge^{\lambda_2 t}$

$\frac{dy}{dt} = F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}$, $\frac{d^2y}{dt^2} = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t}$ ***(A1)(A1)***

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = F\lambda_1^2 e^{\lambda_1 t} + G\lambda_2^2 e^{\lambda_2 t} - 2\left(F\lambda_1 e^{\lambda_1 t} + G\lambda_2 e^{\lambda_2 t}\right) - 3\left(Fe^{\lambda_1 t} + Ge^{\lambda_2 t}\right)$ ***M1***

$= Fe^{\lambda_1 t}\left(\lambda^2 - 2\lambda - 3\right) + Ge^{\lambda_2 t}\left(\lambda^2 - 2\lambda - 3\right)$ ***A1***

$= 0$ ***AG***

[4 marks]

Total [30 marks]

Mathematics: analysis and approaches
Higher level
Paper 3

Thursday 12 May 2022 (morning)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

2222–7113

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

This question asks you to explore properties of a family of curves of the type $y^2 = x^3 + ax + b$ for various values of a and b, where $a, b \in \mathbb{N}$.

(a) On the same set of axes, sketch the following curves for $-2 \le x \le 2$ and $-2 \le y \le 2$, clearly indicating any points of intersection with the coordinate axes.

(i) $y^2 = x^3,\ x \ge 0$ [2]

(ii) $y^2 = x^3 + 1,\ x \ge -1$ [2]

(b) (i) Write down the coordinates of the two points of inflexion on the curve $y^2 = x^3 + 1$. [1]

(ii) By considering each curve from part (a), identify two key features that would distinguish one curve from the other. [1]

Now, consider curves of the form $y^2 = x^3 + b$, for $x \ge -\sqrt[3]{b}$, where $b \in \mathbb{Z}^+$.

(c) By varying the value of b, suggest two key features common to these curves. [2]

Next, consider the curve $y^2 = x^3 + x,\ x \ge 0$.

(d) (i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2 + 1}{2\sqrt{x^3 + x}}$, for $x > 0$. [3]

(ii) Hence deduce that the curve $y^2 = x^3 + x$ has no local minimum or maximum points. [1]

The curve $y^2 = x^3 + x$ has two points of inflexion. Due to the symmetry of the curve these points have the same x-coordinate.

(e) Find the value of this x-coordinate, giving your answer in the form $x = \sqrt{\frac{p\sqrt{3} + q}{r}}$, where $p,\ q,\ r \in \mathbb{Z}$. [7]

(This question continues on the following page)

(Question 1 continued)

$\mathrm{P}(x, y)$ is defined to be a rational point on a curve if x and y are rational numbers.

The tangent to the curve $y^2 = x^3 + ax + b$ at a rational point P intersects the curve at another rational point Q.

Let C be the curve $y^2 = x^3 + 2$, for $x \geq -\sqrt[3]{2}$. The rational point $\mathrm{P}(-1, -1)$ lies on C.

(f) (i) Find the equation of the tangent to C at P. [2]

(ii) Hence, find the coordinates of the rational point Q where this tangent intersects C, expressing each coordinate as a fraction. [2]

(g) The point $\mathrm{S}(-1, 1)$ also lies on C. The line $[\mathrm{QS}]$ intersects C at a further point. Determine the coordinates of this point. [5]

Turn over

2. [Maximum mark: 27]

This question asks you to investigate conditions for the existence of complex roots of polynomial equations of degree 3 and 4.

The cubic equation $x^3 + px^2 + qx + r = 0$, where p, q, $r \in \mathbb{R}$, has roots α, β and γ.

(a) By expanding $(x - \alpha)(x - \beta)(x - \gamma)$ show that:

$$p = -(\alpha + \beta + \gamma)$$

$$q = \alpha\beta + \beta\gamma + \gamma\alpha$$

$$r = -\alpha\beta\gamma.$$

[3]

(b) (i) Show that $p^2 - 2q = \alpha^2 + \beta^2 + \gamma^2$. [3]

(ii) Hence show that $(\alpha - \beta)^2 + (\beta - \gamma)^2 + (\gamma - \alpha)^2 = 2p^2 - 6q$. [3]

(c) Given that $p^2 < 3q$, deduce that α, β and γ cannot all be real. [2]

Consider the equation $x^3 - 7x^2 + qx + 1 = 0$, where $q \in \mathbb{R}$.

(d) Using the result from part (c), show that when $q = 17$, this equation has at least one complex root. [2]

Noah believes that if $p^2 \geq 3q$ then α, β and γ are all real.

(e) (i) By varying the value of q in the equation $x^3 - 7x^2 + qx + 1 = 0$, determine the smallest positive integer value of q required to show that Noah is incorrect. [2]

(ii) Explain why the equation will have at least one real root for all values of q. [1]

(This question continues on the following page)

(Question 2 continued)

Now consider polynomial equations of degree 4.

The equation $x^4 + px^3 + qx^2 + rx + s = 0$, where p, q, r, $s \in \mathbb{R}$, has roots α, β, γ and δ.

In a similar way to the cubic equation, it can be shown that:

$$p = -(\alpha + \beta + \gamma + \delta)$$

$$q = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta$$

$$r = -(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)$$

$$s = \alpha\beta\gamma\delta.$$

(f) (i) Find an expression for $\alpha^2 + \beta^2 + \gamma^2 + \delta^2$ in terms of p and q. [3]

(ii) Hence state a condition in terms of p and q that would imply $x^4 + px^3 + qx^2 + rx + s = 0$ has at least one complex root. [1]

(g) Use your result from part (f)(ii) to show that the equation $x^4 - 2x^3 + 3x^2 - 4x + 5 = 0$ has at least one complex root. [1]

The equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$, has one integer root.

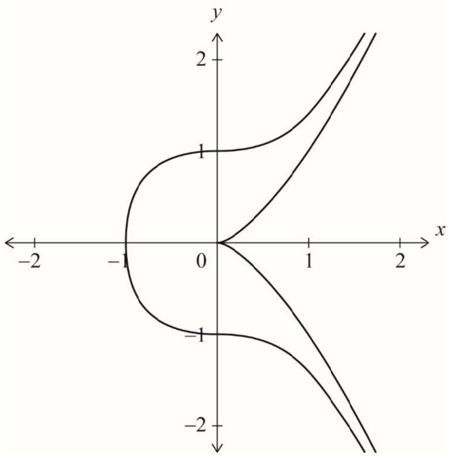
(h) (i) State what the result in part (f)(ii) tells us when considering this equation $x^4 - 9x^3 + 24x^2 + 22x - 12 = 0$. [1]

(ii) Write down the integer root of this equation. [1]

(iii) By writing $x^4 - 9x^3 + 24x^2 + 22x - 12$ as a product of one linear and one cubic factor, prove that the equation has at least one complex root. [4]

References:

1. (a) (i)



approximately symmetric about the x-axis graph of $y^2 = x^3$ ***A1***

including cusp/sharp point at (0, 0) ***A1***

[2 marks]

(ii) approximately symmetric about the x-axis graph of $y^2 = x^3 + 1$ with approximately correct gradient at axes intercepts ***A1***

some indication of position of intersections at $x = -1,\ y = \pm 1$ ***A1***

[2 marks]

Note: Final ***A1*** can be awarded if intersections are in approximate correct place with respect to the axes shown. Award ***A1A1A1A0*** if graphs 'merge' or 'cross' or are discontinuous at x-axis but are otherwise correct. Award ***A1A0A0A0*** if only one correct branch of both curves are seen.

Note: If they sketch graphs on separate axes, award a maximum of 2 marks for the 'best' response seen. This is likely to be ***A1A1A0A0.***

continued…

Question 1 continued

(b) (i) $(0, 1)$ and $(0, -1)$ ***A1***

[1 mark]

(ii) Any **two** from:

$y^2 = x^3$ has a cusp/sharp point, (the other does not)

graphs have different domains

$y^2 = x^3 + 1$ has points of inflexion, (the other does not)

graphs have different x-axis intercepts (one goes through the origin, and the other does not)

graphs have different y-axis intercepts ***A1***

Note: Follow through from their sketch in part (a)(i). In accordance with marking rules, mark their first two responses and ignore any subsequent.

[1 mark]

continued…

Question 1 continued

(c) Any **two** from:

as $x \to \infty, y \to \pm\infty$

as $x \to \infty$, $y^2 = x^3 + b$ is approximated by $y^2 = x^3$ (or similar)

they have x intercepts at $x = -\sqrt[3]{b}$

they have y intercepts at $y = (\pm)\sqrt{b}$

they all have the same range

$y = 0$ (or x-axis) is a line of symmetry

they all have the same line of symmetry $(y = 0)$

they have one x-axis intercept

they have two y-axis intercepts

they have two points of inflexion

at x-axis intercepts, curve is vertical/infinite gradient

there is no cusp/sharp point at x-axis intercepts ***A1A1***

Note: The last example is the only valid answer for things “not” present. Do not credit an answer of “they are all symmetrical” without some reference to the line of symmetry.

Note: Do not allow same/ similar shape or equivalent.

Note: In accordance with marking rules, mark their first two responses and ignore any subsequent.

[2 marks]

continued…

Question 1 continued

(d) (i) **METHOD 1**

attempt to differentiate implicitly ***M1***

$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2+1}{2y}$ OR $(\pm)2\sqrt{x^3+x}\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 1$ ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

METHOD 2

attempt to use chain rule $y = (\pm)\sqrt{x^3+x}$ ***M1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = (\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}(3x^2+1)$ ***A1A1***

Note: Award ***A1*** for $(\pm)\frac{1}{2}(x^3+x)^{-\frac{1}{2}}$, ***A1*** for $(3x^2+1)$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \pm\frac{3x^2+1}{2\sqrt{x^3+x}}$ ***AG***

[3 marks]

(ii) **EITHER**

local minima/maxima occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

$1 + 3x^2 = 0$ has no (real) solutions (or equivalent) ***R1***

OR

$\left(x^2 \geq 0 \Rightarrow\right)$ $3x^2 + 1 > 0$, so $\frac{\mathrm{d}y}{\mathrm{d}x} \neq 0$ ***R1***

THEN

so, no local minima/maxima exist ***AG***

[1 mark]

continued…

Question 1 continued

(e) **EITHER**

attempt to use quotient rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=(\pm)\frac{12x\sqrt{x+x^3}-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)}{4(x+x^3)}$$ ***A1A1***

Note: Award ***A1*** for correct $12x\sqrt{x+x^3}$ and correct denominator, ***A1*** for correct $-(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$.

Note: Future ***A*** marks may be awarded if the denominator is missing or incorrect.

stating or using $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ (may be seen anywhere) ***(M1)***

$$12x\sqrt{x+x^3}=(1+3x^2)(x+x^3)^{-\frac{1}{2}}(1+3x^2)$$

OR

attempt to use product rule to find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ ***M1***

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\frac{1}{2}\left(3x^2+1\right)\left(-\frac{1}{2}\right)\left(3x^2+1\right)\left(x^3+x\right)^{-\frac{3}{2}}+3x\left(x^3+x\right)^{-\frac{1}{2}}$$ ***A1A1***

Note: Award ***A1*** for correct first term, ***A1*** for correct second term.

setting $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ***(M1)***

continued…

Question 1 continued

OR

attempts implicit differentiation on $2y\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2+1$ ***M1***

$$2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=6x$$ ***A1***

recognizes that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=0$ ***(M1)***

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\pm\sqrt{3x}$$

$$(\pm)\frac{3x^2+1}{2\sqrt{x^3+x}}=(\pm)\sqrt{3x}$$ ***(A1)***

THEN

$12x(x+x^3)=(1+3x^2)^2$

$12x^2+12x^4=9x^4+6x^2+1$

$3x^4+6x^2-1=0$ ***A1***

attempt to use quadratic formula or equivalent ***(M1)***

$$x^2=\frac{-6\pm\sqrt{48}}{6}$$

$$(x>0\Rightarrow)x=\sqrt{\frac{2\sqrt{3}-3}{3}}\quad(p=2,\ q=-3,\ r=3)$$ ***A1***

Note: Accept any integer multiple of p, q and r (e.g. 4, –6 and 6).

[7 marks]

continued…

Question 1 continued

(f) (i) attempt to find tangent line through $(-1,-1)$ ***(M1)***

$y+1=-\frac{3}{2}(x+1)$ OR $y=-1.5x-2.5$ ***A1***

[2 marks]

(ii) attempt to solve simultaneously with $y^2=x^3+2$ ***(M1)***

> **Note:** The ***M1*** mark can be awarded for an unsupported correct answer in an incorrect format (e.g. $(4.25, \text{-}8.875)$).

obtain $\left(\frac{17}{4},-\frac{71}{8}\right)$ ***A1***

[2 marks]

continued…

Question 1 continued

(g) attempt to find equation of $[\text{QS}]$ ***(M1)***

$\frac{y-1}{x+1}=-\frac{79}{42}(=-1.88095\ldots)$ ***(A1)***

solve simultaneously with $y^2=x^3+2$ ***(M1)***

$x=0.28798\ldots\left(=\frac{127}{441}\right)$ ***A1***

$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$ ***A1***

$(0.288,-1.42)$

OR

attempt to find vector equation of $[\text{QS}]$ ***(M1)***

$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}-1\\1\end{pmatrix}+\lambda\begin{pmatrix}\frac{21}{4}\\-\frac{79}{8}\end{pmatrix}$ ***(A1)***

$x=-1+\frac{21}{4}\lambda$

$y=1-\frac{79}{8}\lambda$

attempt to solve $\left(1-\frac{79}{8}\lambda\right)^2=\left(-1+\frac{21}{4}\lambda\right)^3+2$ ***(M1)***

$\lambda=0.2453\ldots$

$x=0.28798\ldots\left(=\frac{127}{441}\right)$ ***A1***

$y=-1.4226\ldots\left(=\frac{13175}{9261}\right)$ ***A1***

$(0.288,-1.42)$

[5 marks]

[Total 28 marks]

2. (a) attempt to expand $(x-\alpha)(x-\beta)(x-\gamma)$ ***M1***

$=\left(x^2-(\alpha+\beta)x+\alpha\beta\right)(x-\gamma)$ **OR** $=(x-\alpha)\left(x^2-(\beta+\gamma)x+\beta\gamma\right)$ ***A1***

$\left(x^3+px^2+qx+r\right)=x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma$ ***A1***

comparing coefficients:

$p=-(\alpha+\beta+\gamma)$ ***AG***

$q=(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***AG***

$r=-\alpha\beta\gamma$ ***AG***

Note: For candidates who do not include the ***AG*** lines award full marks.

[3 marks]

(b) (i) $p^2-2q=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***(A1)***

attempt to expand $(\alpha+\beta+\gamma)^2$ ***(M1)***

$=\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\beta\gamma+\gamma\alpha)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ or equivalent ***A1***

$=\alpha^2+\beta^2+\gamma^2$ ***AG***

Note: Accept equivalent working from RHS to LHS.

[3 marks]

continued…

Question 2 continued

(ii) **EITHER**

attempt to expand $(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***(M1)***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$

$=2\left(p^2-2q\right)-2q$ or equivalent ***A1***

$=2p^2-6q$ ***AG***

OR

attempt to write $2p^2-6q$ in terms of α,β,γ ***(M1)***

$=2\left(p^2-2q\right)-2q$

$=2(\alpha^2+\beta^2+\gamma^2)-2(\alpha\beta+\beta\gamma+\gamma\alpha)$ ***A1***

$=\left(\alpha^2+\beta^2-2\alpha\beta\right)+\left(\beta^2+\gamma^2-2\beta\gamma\right)+\left(\gamma^2+\alpha^2-2\gamma\alpha\right)$ ***A1***

$=(\alpha-\beta)^2+(\beta-\gamma)^2+(\gamma-\alpha)^2$ ***AG***

Note: Accept equivalent working where LHS and RHS are expanded to identical expressions.

[3 marks]

continued…

Question 2 continued

(c) $p^2 < 3q \Rightarrow 2p^2 - 6q < 0$

$\Rightarrow (\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 < 0$ ***A1***

if all roots were real $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 \geq 0$ ***R1***

Note: Condone strict inequality in the ***R1*** line.
Note: Do not award ***A0R1.***

$\Rightarrow$ roots cannot all be real ***AG***

[2 marks]

(d) $p^2 = (-7)^2 = 49$ and $3q = 51$ ***A1***

so $p^2 < 3q$ $\Rightarrow$ the equation has at least one complex root ***R1***

Note: Allow equivalent comparisons; e.g. checking $2p^2 < 6q$

[2 marks]

(e) (i) use of GDC (*eg* graphs or tables) ***(M1)***

$q = 12$ ***A1***

[2 marks]

(ii) complex roots appear in conjugate pairs (so if complex roots occur the other root will be real OR all 3 roots will be real).
OR
a cubic curve always crosses the x-axis at at least one point. ***R1***

[1 mark]

continued…

Question 2 continued

(f) (i) attempt to expand $(\alpha+\beta+\gamma+\delta)^2$ ***(M1)***

$(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$ ***(A1)***

$\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

$\left(\Rightarrow \alpha^2+\beta^2+\gamma^2+\delta^2 =\right) p^2 - 2q$ ***A1***

[3 marks]

(ii) $p^2 < 2q$ **OR** $p^2 - 2q < 0$ ***A1***

Note: Allow ***FT*** on their result from part (f)(i).

[1 mark]

(g) $4 < 6$ OR $2^2 - 2\times 3 < 0$ ***R1***

hence there is at least one complex root. ***AG***

Note: Allow ***FT*** from part (f)(ii) for the ***R*** mark provided numerical reasoning is seen.

[1 mark]

continued…

Question 2 continued

(h) (i) $\left(p^2 > 2q\right)$ $\left(81 > 2\times 24\right)$ (so) nothing can be deduced ***R1***

Note: Do not allow ***FT*** for the ***R*** mark.

[1 mark]

(ii) -1 ***A1***

[1 mark]

(iii) attempt to express as a product of a linear and cubic factor ***M1***

$(x+1)(x^3-10x^2+34x-12)$ ***A1A1***

Note: Award ***A1*** for each factor. Award at most ***A1A0*** if not written as a product.

since for the cubic, $p^2 < 3q$ $(100 < 102)$ ***R1***

there is at least one complex root ***AG***

[4 marks]

[Total: 27 marks]

Mathematics: analysis and approaches
Higher level
Paper 3

Tuesday 8 November 2022 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

6 pages

8822–7103

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 28]

In this question you will investigate series of the form

$$\sum_{i=1}^{n} i^q = 1^q + 2^q + 3^q + \ldots + n^q \textbf{ where } n,\, q \in \mathbb{Z}^+$$

and use various methods to find polynomials, in terms of n, for such series.

When $q = 1$, the above series is arithmetic.

(a) Show that $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$. [1]

Consider the case when $q = 2$.

(b) The following table gives values of n^2 and $\sum_{i=1}^{n} i^2$ for $n = 1, 2, 3$.

n	n^2	$\sum_{i=1}^{n} i^2$
1	1	1
2	4	5
3	9	p

(i) Write down the value of p. [1]

(ii) The sum of the first n square numbers can be expressed as a cubic polynomial with three terms:

$$\sum_{i=1}^{n} i^2 = a_1 n + a_2 n^2 + a_3 n^3 \text{ where } a_1,\, a_2,\, a_3 \in \mathbb{Q}^+.$$

Hence, write down a system of three linear equations in a_1, a_2 and a_3. [3]

(iii) Hence, find the values of a_1, a_2 and a_3. [2]

(This question continues on the following page)

(Question 1 continued)

You will now consider a method that can be generalized for all values of q.

Consider the function $f(x) = 1 + x + x^2 + \ldots + x^n$, $n \in \mathbb{Z}^+$.

(c) Show that $x f'(x) = x + 2x^2 + 3x^3 + \ldots + nx^n$. [1]

Let $f_1(x) = x f'(x)$ and consider the following family of functions:

$$f_2(x) = x f_1'(x)$$

$$f_3(x) = x f_2'(x)$$

$$f_4(x) = x f_3'(x)$$

$$\ldots$$

$$f_q(x) = x f_{q-1}'(x)$$

(d) (i) Show that $f_2(x) = \sum_{i=1}^{n} i^2 x^i$. [2]

(ii) Prove by mathematical induction that $f_q(x) = \sum_{i=1}^{n} i^q x^i$, $q \in \mathbb{Z}^+$. [6]

(iii) Using sigma notation, write down an expression for $f_q(1)$. [1]

(e) By considering $f(x) = 1 + x + x^2 + \ldots + x^n$ as a geometric series, for $x \neq 1$, show that $f(x) = \dfrac{x^{n+1} - 1}{x - 1}$. [2]

(f) For $x \neq 1$, show that $f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$. [3]

(g) (i) Show that $\lim_{x \to 1} f_1(x)$ is in indeterminate form. [1]

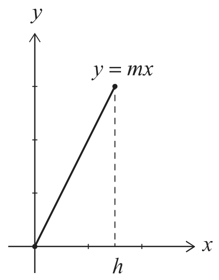
(ii) Hence, by applying l'Hôpital's rule, show that $\lim_{x \to 1} f_1(x) = \frac{1}{2} n(n+1)$. [5]

Turn over

2. [Maximum mark: 27]

In this question you will investigate curved surface areas and use calculus to derive key formulae used in geometry.

Consider the straight line from the origin, $y = mx$, where $0 \le x \le h$ and m, h are positive constants.



When this line is rotated through $360°$ about the x-axis, a cone is formed with a curved surface area A given by:

$$A = 2\pi \int_0^h y\sqrt{1 + m^2}\,\mathrm{d}x.$$

(a) Given that $m = 2$ and $h = 3$, show that $A = 18\sqrt{5}\,\pi$. [2]

(b) Now consider the general case where a cone is formed by rotating the line $y = mx$ where $0 \le x \le h$ through $360°$ about the x-axis.

(i) Deduce an expression for the radius of this cone r in terms of h and m. [1]

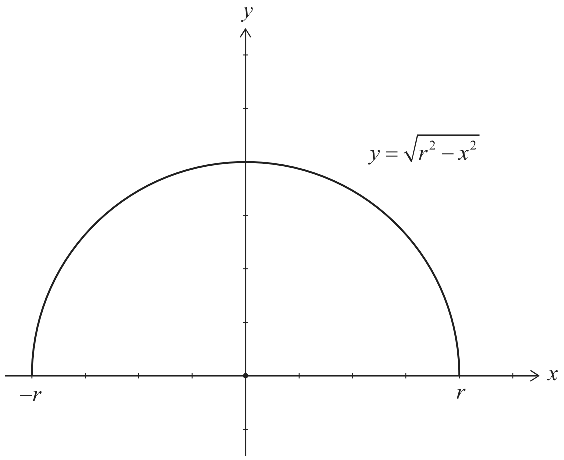
(ii) Deduce an expression for the slant height l in terms of h and m. [2]

(iii) Hence, by using the above integral, show that $A = \pi r l$. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the semi-circle, with radius r, defined by $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$.



(c) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$. [2]

A differentiable curve $y=f(x)$ is defined for $x_1 \le x \le x_2$ and $y \ge 0$. When any such curve is rotated through $360°$ about the x-axis, the surface formed has an area A given by:

$$A=2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x\ .$$

(d) A sphere is formed by rotating the semi-circle $y=\sqrt{r^2-x^2}$ where $-r \le x \le r$ through $360°$ about the x-axis. Show by integration that the surface area of this sphere is $4\pi r^2$. [4]

(This question continues on the following page)

(Question 2 continued)

(e) Let $f(x)=\sqrt{r^2-x^2}$ where $-r \le x \le r$.

The graph of $y=f(x)$ is transformed to the graph of $y=f(kx)$, $k>0$. This forms a different curve, called a semi-ellipse.

(i) Describe this geometric transformation. [2]

(ii) Write down the x-intercepts of the graph $y=f(kx)$ in terms of r and k. [1]

(iii) For $y=f(kx)$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x, r and k. [2]

(iv) The semi-ellipse $y=f(kx)$ is rotated $360°$ about the x-axis to form a solid called an ellipsoid.

Find an expression in terms of r and k for the surface area, A, of the ellipsoid.

Give your answer in the form $2\pi\int_{x_1}^{x_2}\sqrt{p(x)}\ \mathrm{d}x$, where $p(x)$ is a polynomial. [4]

(v) Planet Earth can be modelled as an ellipsoid. In this model:

- the ellipsoid has an axis of rotational symmetry running from the North Pole to the South Pole.
- the distance from the North Pole to the South Pole is $12\ 714\,\mathrm{km}$.
- the diameter of the equator is $12\ 756\,\mathrm{km}$.

By choosing suitable values for r and k, find the surface area of Earth in km^2 correct to 4 significant figures. Give your answer in the form $a \times 10^q$ where $1 \le a < 10$ and $q \in \mathbb{Z}^+$. [4]

References:

1. (a) **EITHER**

$S_n = \frac{n}{2}\left(2\times1+(n-1)\times1\right)$ ***A1***

OR

$u_1 = 1$ and either $u_n = n$ or $d = 1$ stated explicitly ***A1***

OR

$1+2+...+n$ (or equivalent) stated explicitly ***A1***

THEN

$S_n = \frac{n}{2}(1+n)$ ***AG***

Note: Award ***A0*** for a numerical verification.

[1 mark]

(b) (i) 14 ***A1***

[1 mark]

(ii) $a_1 + a_2 + a_3 = 1$ ***A1***

$2a_1 + 4a_2 + 8a_3 = 5$ ***A1***

$3a_1 + 9a_2 + 27a_3 = 14$ ***A1***

Note: For the third ***A*** mark, award ***A1FT*** for $3a_1 + 9a_2 + 27a_3 = p$ where p is their answer to part (b) (i).

[3 marks]

continued...

Question 1 continued

(iii) $a_1 = \frac{1}{6}$ $(= 0.166666... \approx 0.167)$, $a_2 = \frac{1}{2}$ $(= 0.5)$,

$a_3 = \frac{1}{3}$ $(= 0.333333... \approx 0.333)$ ***A2***

Note: Award ***A1*** if only two of a_1, a_2, a_3 are correct.
Only award ***FT*** if three linear equations, each in a_1, a_2 and a_3 are stated in part (b) (ii) or (iii).
Award ***A2FT*** for their a_1, a_2 and a_3.
Award ***A1FT*** for their a_1, a_2 and $a_3 = 0$.

[2 marks]

(c) $f'(x) = 1 + 2x + 3x^2 + ... + nx^{n-1}$ ***A1***

Note: Award ***A1*** for $f'(x) = \sum_{i=1}^{n} ix^{i-1}$.

$\Rightarrow x\,f'(x) = x + 2x^2 + 3x^3 + ... + nx^n$ ***AG***

[1 mark]

continued...

Question 1 continued

(d) (i) **METHOD 1**

$f_2(x) = xf_1'(x)$

$f_1'(x) = 1^2 + 2^2x + (3^2x^2) + ... + n^2x^{n-1}$ $\left(= 1 + 4x + (9x^2) + ... + n^2x^{n-1}\right)$ ***A1***

> **Note:** Award ***A1*** for
> $xf_1'(x) = x\left(1^2 + 2^2x + (3^2x^2) + ... + n^2x^{n-1}\right)$ $\left(= x\left(1 + 4x + (9x^2) + ... + n^2x^{n-1}\right)\right)$
> .

$xf_1'(x) = 1^2x + 2^2x^2 + (3^2x^3) + ... + n^2x^n$ $\left(= x + 4x^2 + (9x^3) + ... + n^2x^n\right)$ ***A1***

> **Note:** Award ***A1*** for $f_1'(x) = \sum_{i=1}^{n} i^2x^{i-1}$ and ***A1*** for $xf_1'(x) = x\sum_{i=1}^{n} i^2x^{i-1}$.
> The second ***A1*** is dependent on the first ***A1***.
> Award a maximum of ***A0A1*** if a general term is not considered.

$= \sum_{i=1}^{n} i^2x^i$ ***AG***

METHOD 2

$f_2(x) = x\frac{\mathrm{d}}{\mathrm{d}x}(xf'(x))$

$= x(f'(x) + xf''(x))$ $(= xf'(x) + x^2f''(x))$ ***A1***

$= x\sum_{i=1}^{n} ix^{i-1} + x^2\sum_{i=1}^{n} i(i-1)x^{i-2}$

$= \sum_{i=1}^{n} ix^i + \sum_{i=1}^{n} i(i-1)x^i$ $\left(= \sum_{i=1}^{n}(i + i^2 - i)x^i\right)$ ***A1***

$= \sum_{i=1}^{n} i^2x^i$ ***AG***

[2 marks]

continued...

Question 1 continued

(ii) consider $q = 1$

$f_1(x) = x + 2x^2 + ... + nx^n$ (reference to part (c)) and $f_1(x) = \sum_{i=1}^{n} ix^i$ ***R1***

assume true for $q = k$, ($f_k(x) = \sum_{i=1}^{n} i^kx^i$) ***M1***

> **Note:** Do not award ***M1*** for statements such as “let $q = k$” or “$q = k$ is true”.
> Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

consider $q = k + 1$

$f_{k+1}(x) = xf_k'(x)$ ***M1***

$= x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ **OR** $x(1 + 2^{k+1}x + 3^{k+1}x^2 + ... + n^{k+1}x^{n-1})$ ***A1***

> **Note:** Award the above ***M1*** if $f_{k+1}(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ or $xf_k'(x) = x\sum_{i=1}^{n} i^{k+1}x^{i-1}$ (or equivalent) is stated.

$= \sum_{i=1}^{n} i^{k+1}x^i$ **OR** $x + 2^{k+1}x^2 + 3^{k+1}x^3 + ... + n^{k+1}x^n$ ***A1***

since true for $q = 1$ and true for $q = k + 1$ if true for $q = k$, hence true for all q $(\in \mathbb{Z}^+)$ ***R1***

> **Note:** To obtain the final ***R1***, three of the previous five marks must have been awarded.

[6 marks]

(iii) $f_q(1) = 1^q + 2^q + 3^q + ... + n^q$

$= \sum_{i=1}^{n} i^q$ $\left(= \sum_{i=1}^{n} 1^i i^q\right)$ ***A1***

[1 mark]

continued...

Question 1 continued

(e) uses $S_n = \dfrac{u_1\left(r^n - 1\right)}{r-1}$ with $r = x$ and $u_1 = 1$ ***M1***

clear indication there are $\left(n+1\right)$ terms ***R1***

$f(x) = \dfrac{x^{n+1}-1}{x-1}$ ***AG***

[2 marks]

(f) **METHOD 1**

$f_1(x) = xf'(x)$

$= x\dfrac{(x-1)(n+1)x^n - 1\times\left(x^{n+1} - 1\right)}{(x-1)^2}$ ***M1A1***

Note: Award ***M1*** for attempting to use the quotient or the product rule to find $f'(x)$.

$= x\dfrac{(nx+x-n-1)x^n - \left(x^{n+1} - 1\right)}{(x-1)^2}\left(= x\dfrac{nx^{n+1} - nx^n - x^n + 1}{(x-1)^2}\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$= \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$ ***AG***

METHOD 2

attempts to form $(x-1)f_1(x)$ ***M1***

$(x-1)f_1(x) = nx^{n+1} - \left(x + x^2 + x^3 + \ldots + x^n\right)$

$f_1(x) = \dfrac{1}{x-1}\left(nx^{n+1} - \left(\dfrac{x^{n+1}-1}{x-1} - 1\right)\right)$ ***A1***

$f_1(x) = \dfrac{1}{x-1}\left(\dfrac{nx^{n+1}(x-1) - x^{n+1} + x}{x-1}\right)\left(= \dfrac{1}{x-1}\left(\dfrac{nx^{n+2} - nx^{n+1} - x^{n+1} + x}{x-1}\right)\right)$ ***A1***

Note: Award ***A1*** for any correct manipulation of the derivative that leads to the ***AG***.

$f_1(x) = \dfrac{nx^{n+2} - (n+1)x^{n+1} + x}{(x-1)^2}$ ***AG***

[3 marks]

(g) (i) $\lim\limits_{x\to 1} f_1(x) = \dfrac{n-(n+1)+1}{0}\left(= \dfrac{0}{0}\right)$ ***R1***

Note: Only award ***R1*** for sufficient simplification of the numerator, for example, as shown above.
Do not award ***R1*** if $\lim\limits_{x\to 1}$ is not referred to or stated.

[1 mark]

continued...

Question 1 continued

(ii) attempts to differentiate both the numerator and the denominator **M1**

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}$$ **A1**

> **Note:** Award ***A1*** for $\left(\left(\lim_{x\to 1}\right)\frac{n(n+2)x^{n+1}-n(n+1)x^n-(n+1)x^n+1}{2(x-1)}\right)$. This form can be used in subsequent work.

(l'Hôpital's rule applies again since)

$$\lim_{x\to 1}\frac{n(n+2)x^{n+1}-(n+1)^2x^n+1}{2(x-1)}=\frac{0}{0}$$ **R1**

> **Note:** Do not award ***R1*** if $\lim\limits_{x\to 1}$ is not referred to or stated.
> Subsequent marks are independent of this ***R*** mark.

attempts to differentiate both the numerator and the denominator **M1**

$$\lim_{x\to 1}\frac{n(n+2)(n+1)x^n-n(n+1)^2x^{n-1}}{2}$$

$$=\frac{n(n+2)(n+1)-n(n+1)^2}{2}\quad\left(=\frac{n^3+3n^2+2n-\left(n^3+2n^2+n\right)}{2}\right)$$

$$=\frac{n(n+1)\left((n+2)-(n+1)\right)}{2}\quad\left(=\frac{n^2+n}{2}\right)$$ **A1**

$$=\frac{1}{2}n(n+1)$$ **AG**

[5 marks]

Total [28 marks]

2. (a) **EITHER**

$$A=2\pi\int_0^3 2x\sqrt{1+2^2}\ \mathrm{d}x\ \left(=4\sqrt{5}\pi\int_0^3 x\,\mathrm{d}x\right)$$ ***(A1)***

$$=2\pi\sqrt{5}\ \left[x^2\right]_0^3\ \left(=2\pi\sqrt{5}\left(3^2-0^2\right)\right)$$ **A1**

OR

$$A=2\pi m\sqrt{1+m^2}\ \left[\frac{x^2}{2}\right]_0^h\ \left(=2\pi m\sqrt{1+m^2}\left(\frac{h^2}{2}\right)\right)$$ ***(A1)***

$$=2\pi(2)\sqrt{5}\ \left[\frac{x^2}{2}\right]_0^3\ \left(=2\pi(2)\sqrt{5}\left(\frac{3^2}{2}\right)\right)$$ **A1**

THEN

$$=18\sqrt{5}\,\pi$$ **AG**

[2 marks]

(b) (i) $r=mh$ **A1**

[1 mark]

(ii) $l=\sqrt{h^2+r^2}$ ***(M1)***

$l=\sqrt{h^2+h^2m^2}\ \left(=h\sqrt{1+m^2}\right)$ **A1**

[2 marks]

continued...

Question 2 continued

(iii) $A = 2\pi\int_0^h mx\sqrt{1+m^2}\ \mathrm{d}x$ ***(A1)***

$= 2\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$ ***(M1)***

Note: Award ***(M1)*** for $(c)\pi m\sqrt{1+m^2}\left[\frac{1}{2}x^2\right]_0^h$.
At least one of the above two lines needs to be seen.

$= \pi h^2 m\sqrt{1+m^2}\ \left(= \pi hm\times\sqrt{(h^2+h^2m^2)}\right)$ ***A1***

$= \pi rl$ ***AG***

Note: Award as above if $\frac{l}{h} = \sqrt{1+m^2}$ is used, for example.

[3 marks]

(c) **METHOD 1**

attempts to use the chain rule ***(M1)***

Note: Award ***(M1)*** for $\frac{\mathrm{d}y}{\mathrm{d}x} = (c)x\left(r^2-x^2\right)^{-\frac{1}{2}}$.

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}\left(r^2-x^2\right)^{-\frac{1}{2}}\ (-2x)\ \left(= -x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)$ ***A1***

METHOD 2

attempts implicit differentiation on $y^2 = r^2 - x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***A1***

[2 marks]

continued...

Question 2 continued

(d) **EITHER**

attempts to substitute $y = \sqrt{r^2-x^2}$ and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ into A ***(M1)***

$A = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\ \mathrm{d}x$

$= 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{1+\frac{x^2}{r^2-x^2}}\,\mathrm{d}x$ ***A1***

OR

attempts to substitute y and their $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y into A ***(M1)***

$A = 2\pi\int_{-r}^{r} y\sqrt{1+\left(-\frac{x}{y}\right)^2}\ \mathrm{d}x$

$= 2\pi\int_{-r}^{r} y\sqrt{1+\frac{x^2}{y^2}}\,\mathrm{d}x\ \left(= 2\pi\int_{-r}^{r}\sqrt{x^2+y^2}\ \mathrm{d}x\right)$ ***A1***

THEN

attempts to perform valid algebraic simplification to form a definite integral in terms of r only ***M1***

$= 2\pi\int_{-r}^{r} r\ \mathrm{d}x$

$= 2\pi r[x]_{-r}^{r}\ \left(= 2\pi r\left(r-(-r)\right)\right)$ ***A1***

$= 4\pi r^2$ ***AG***

Note: Award marks as above for $A = 4\pi\int_0^r\sqrt{r^2-x^2}\sqrt{1+\left(-x\left(r^2-x^2\right)^{-\frac{1}{2}}\right)^2}\ \mathrm{d}x$.

[4 marks]

continued...

Question 2 continued

(e) (i) **EITHER**

horizontal stretch **_A1_**

factor $\frac{1}{k}$ **_A1_**

OR

horizontal compression **_A1_**

factor k (invariant line y-axis) **_A1_**

Note: Award ***A1A1*** as above for correct alternative descriptions.
For example, dilation by a factor of $\frac{1}{k}$ from the y-axis.

[2 marks]

(ii) $\pm\frac{r}{k}$ **_A1_**

Note: Award ***A0*** for $\frac{r}{k}$ only and ***A0*** for $-\frac{r}{k}$ only.

[1 mark]

continued...

Question 2 continued

(iii) **METHOD 1**

attempts to use the chain rule ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(r^2-(kx)^2\right)^{-\frac{1}{2}}\times\left(-k^2 2x\right)\ \left(=-k^2x\left(r^2-(kx)^2\right)^{-\frac{1}{2}}\ \left(=\frac{-k^2x}{\sqrt{r^2-k^2x^2}}\right)\right)$ **_A1_**

METHOD 2

attempts implicit differentiation on $y^2=r^2-k^2x^2$ (or equivalent) ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{k^2x}{y}$

$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-k^2x}{\sqrt{r^2-k^2x^2}}$ **_A1_**

[2 marks]

continued...

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

2223–7108

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x)=x^n e^{-x}$.

Consider the family of functions $f_n(x)=x^n e^{-x}$, where $x \ge 0$ and $n \in \mathbb{Z}^+$.

When $n=1$, the function $f_1(x)=xe^{-x}$, where $x \ge 0$.

(a) Sketch the graph of $y=f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y=f_1(x)$, the x-axis and the line $x=b$, where $b>0$, is given by $\frac{e^b-b-1}{e^b}$. [6]

You may assume that the total area, A_n, of the region between the graph $y=f_n(x)$ and the x-axis can be written as $A_n=\int_0^\infty f_n(x)\,dx$ and is given by $\lim_{b\to\infty}\int_0^b f_n(x)\,dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b\to\infty}\frac{e^b-b-1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1. [1]

You are given that $A_2=2$ and $A_3=6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4; [2]

(ii) A_5. [1]

(e) Suggest an expression for A_n in terms of n, where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m, $\lim_{x\to\infty} x^m e^{-x}=0$. [8]

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2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers x_1, $x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f, of x_1 only. [2]

(b) (i) Find the value of x_1 for which the function is maximum. [1]

(ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, $x_1, x_2, \dots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \le \frac{(x_1 + x_2 + \dots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \dots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n.

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

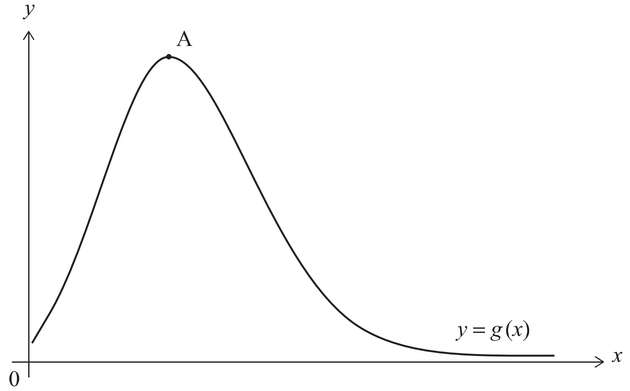
(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g, defined by $\ln(g(x)) = x\ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



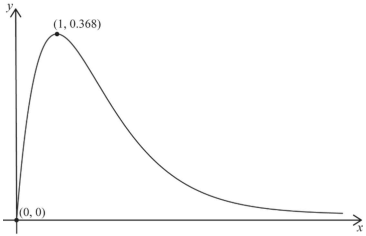
(h) Find, in terms of S, the x-coordinate of point A. [6]

(i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]

(j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \le a < 10$ and $k \in \mathbb{Z}^+$. [3]

References:

1. (a)



A1 for $(1,\ 0.368)$ or $\left(1,\ \frac{1}{\mathrm{e}}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through,the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued…

Question 1 continued

(b) $\int_0^b x\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

Note: Award ***(A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$= \left[-x\mathrm{e}^{-x}\right]_0^b + \int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$= \left[-x\mathrm{e}^{-x}\right]_0^b - \left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

Note: Award ***A1*** for correct second term.
Condone absence of limits to this point

attempt to substitute limits ***M1***

$= -b\mathrm{e}^{-b} - \mathrm{e}^{-b} + 1$ ***A1***

$= \frac{\mathrm{e}^b - b - 1}{\mathrm{e}^b}$ ***AG***

[6 marks]

(c) (i) $\lim_{b\to\infty} \frac{\mathrm{e}^b - b - 1}{\mathrm{e}^b} = \lim_{b\to\infty} \frac{\mathrm{e}^b - 1}{\mathrm{e}^b}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(= \lim_{b\to\infty} \frac{\mathrm{e}^b}{\mathrm{e}^b}\right) = 1$ ***A1***

[2 marks]

(ii) $\left(\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x = \right) 1$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) (i) correct integral ***(M1)***

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(ii) 120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$ ***A1***

[1 mark]

continued…

Question 1 continued

(f)

Note: Accept starting at $n = 0$, throughout this proof.

$n = 1$

$A_1 = 1$ and $1! = 1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n = 1$, and ***A1*** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered.

so true for $n = 1$

assume true for $n = k$, ($A_k = \int_0^\infty x^k e^{-x}\,dx = k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n = k + 1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^\infty x^{k+1} e^{-x}\,dx = \left[-x^{k+1}e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\,dx$ ***A1***

$(k+1)\int_0^\infty x^k e^{-x}\,dx$ simplified to $(k+1)k!$ seen ***A1***

$= 0 + (k+1)k!$

Note: Condone omission of the zero.

$= (k+1)!$ ***A1***

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

continued…

Question 2 continued

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + \ldots + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \le \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times \ldots \times x_n \le \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6\ \ (79.6262\ldots)$ ***A1***

[1 mark]

continued…

Question 2 continued

(f) considering $M_n(12)$ for higher values of n

$P(12)=81$ ***A1***

$n=4$
A1

Note: Award ***A0A0*** for $P(12)=82.6$ and $n=4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n=7$ ***A1***

$P(20)=\left(\frac{20}{7}\right)^7=1550\ (1554.260...)$ ***A1***

Note: Award ***(M1)A0A1*** for $n=7.36$ and $P(20)=1570$.

[3 marks]

continued…

Question 2 continued

(h) **EITHER**

$\ln(g(x))=x\left(\ln(S)-\ln x\right)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)}=\ln S-\ln x-x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)}=\ln\frac{S}{x}+\left(x\frac{x}{S}\times\frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g\left(x\right)=\mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'\left(x\right)=\mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x\times\frac{x}{S}\times\left(-Sx^{-2}\right)+\ln\left(\frac{S}{x}\right)\right]$

$=\mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right)-1\right]$ ***A1***

THEN

$g'(x)=\left(\ln\frac{S}{x}-1\right)g(x)$

$g(x)\neq 0$

$g'(x)=0\Rightarrow\ln\frac{S}{x}-1=0$ ***M1***

$x=\frac{S}{\mathrm{e}}\ (0.368S,\ 0.36789...S)$ ***A1***

[6 marks]

continued…

Mathematics: analysis and approaches
Higher level
Paper 3

9 May 2023

Zone A afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to examine the number and nature of intersection points of the graph of $y = \log_a x$ where $a \in \mathbb{R}^+$, $a \neq 1$ and the line $y = x$ for particular sets of values of a.

In this question you may either use the change of logarithm base formula $\log_a x = \frac{\ln x}{\ln a}$ or a graphic display calculator “logarithm to any base feature”.

The function f is defined by

$$f(x) = \log_a x \text{ where } x \in \mathbb{R}^+ \text{ and } a \in \mathbb{R}^+,\ a \neq 1.$$

(a) Consider the cases $a = 2$ and $a = 10$. On the same set of axes, sketch the following three graphs:

$$y = \log_2 x$$

$$y = \log_{10} x$$

$$y = x.$$

Clearly label each graph with its equation and state the value of any non-zero x-axis intercepts. [4]

(This question continues on the following page)

(Question 1 continued)

In parts (b) and (c), consider the case where $a = \mathrm{e}$. Note that $\ln x \equiv \log_{\mathrm{e}} x$.

(b) Use calculus to find the minimum value of the expression $x - \ln x$, justifying that this value is a local minimum. [5]

(c) Hence deduce that $x > \ln x$. [1]

(d) There exist values of a for which the graph of $y = \log_a x$ and the line $y = x$ do have intersection points. The following table gives three intervals for the value of a.

Interval	Number of intersection points
$0 < a < 1$	p
$1 < a < 1.4$	q
$1.5 < a < 2$	r

By investigating the graph of $y = \log_a x$ for different values of a, write down the values of p, q and r. [4]

In parts (e) and (f), consider $a \in \mathbb{R}^+$, $a \neq 1$.

For $1.4 \leq a \leq 1.5$, a value of a exists such that the line $y = x$ is a tangent to the graph of $y = \log_a x$ at a point P.

(e) Find the exact coordinates of P and the exact value of a. [8]

(f) Write down the exact set of values for a such that the graphs of $y = \log_a x$ and $y = x$ have

(i) two intersection points; [1]

(ii) no intersection points. [1]

Turn over

2. [Maximum mark: 31]

This question asks you to examine linear and quadratic functions constructed in systematic ways using arithmetic sequences.

Consider the function $L(x) = mx + c$ for $x \in \mathbb{R}$ where m, $c \in \mathbb{R}$ and m, $c \neq 0$.

Let $r \in \mathbb{R}$ be the root of $L(x) = 0$.

If m, r and c, in that order, are in arithmetic sequence then $L(x)$ is said to be an AS-linear function.

(a) Show that $L(x) = 2x - 1$ is an AS-linear function. [2]

Consider $L(x) = mx + c$.

(b) (i) Show that $r = -\dfrac{c}{m}$. [1]

(ii) Given that $L(x)$ is an AS-linear function, show that $L(x) = mx - \dfrac{m^2}{m+2}$. [4]

(iii) State any further restrictions on the value of m. [1]

There are only three **integer** sets of values of m, r and c, that form an AS-linear function. One of these is $L(x) = -x - 1$.

(c) Use part (b) to determine the other two AS-linear functions with integer values of m, r and c. [3]

Consider the function $Q(x) = ax^2 + bx + c$ for $x \in \mathbb{R}$ where $a \in \mathbb{R}$, $a \neq 0$ and b, $c \in \mathbb{R}$.

Let r_1, $r_2 \in \mathbb{R}$ be the roots of $Q(x) = 0$.

(d) Write down an expression for

(i) the sum of roots, $r_1 + r_2$, in terms of a and b. [1]

(ii) the product of roots, $r_1 r_2$, in terms of a and c. [1]

(This question continues on the following page)

(Question 2 continued)

If a, r_1, b, r_2 and c, in that order, are in arithmetic sequence, then $Q(x)$ is said to be an AS-quadratic function.

(e) Given that $Q(x)$ is an AS-quadratic function,

(i) write down an expression for $r_2 - r_1$ in terms of a and b; [1]

(ii) use your answers to parts (d)(i) and (e)(i) to show that $r_1 = \dfrac{a^2 - ab - b}{2a}$; [2]

(iii) use the result from part (e)(ii) to show that $b = 0$ or $a = -\dfrac{1}{2}$. [3]

Consider the case where $b = 0$.

(f) Determine the two AS-quadratic functions that satisfy this condition. [5]

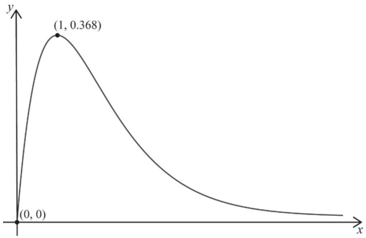
Now consider the case where $a = -\dfrac{1}{2}$.

(g) (i) Find an expression for r_1 in terms of b. [2]

(ii) Hence or otherwise, determine the exact values of b and c such that AS-quadratic functions are formed.

Give your answers in the form $\dfrac{-p \pm q\sqrt{s}}{2}$ where p, q, $s \in \mathbb{Z}^+$. [5]

1. (a)



A1 for $(1,\ 0.368)$ or $\left(1,\ \frac{1}{\mathrm{e}}\right)$ labelled at local maximum (accept correct coordinates written away from the graph)

A1 for graph clearly starting at, or passing through,the origin

A1 for correct domain

A1 for correct shape i.e.: single maximum, and asymptotic behaviour (equation not required) (or point of inflexion)

[4 marks]

continued…

Question 1 continued

(b) $\int_0^b x\mathrm{e}^{-x}\,\mathrm{d}x$ ***(A1)***

Note: Award ***(A1)*** for correct integrand and limits (which can be seen later in the question)

Use of integration by parts

$=\left[-x\mathrm{e}^{-x}\right]_0^b+\int_0^b \mathrm{e}^{-x}\,\mathrm{d}x$ ***A1A1***

Note: Award ***A1*** for each part (including the correct sign with each)

$=\left[-x\mathrm{e}^{-x}\right]_0^b-\left[\mathrm{e}^{-x}\right]_0^b$ ***A1***

Note: Award ***A1*** for correct second term.
Condone absence of limits to this point

attempt to substitute limits ***M1***

$=-b\mathrm{e}^{-b}-\mathrm{e}^{-b}+1$ ***A1***

$=\frac{\mathrm{e}^b-b-1}{\mathrm{e}^b}$ ***AG***

[6 marks]

(c) (i) $\lim_{b\to\infty}\frac{\mathrm{e}^b-b-1}{\mathrm{e}^b}=\lim_{b\to\infty}\frac{\mathrm{e}^b-1}{\mathrm{e}^b}$ ***A1***

Note: Award ***A1*** for correct quotient. Condone absence of limit.

$\left(=\lim_{b\to\infty}\frac{\mathrm{e}^b}{\mathrm{e}^b}\right)=1$ ***A1***

[2 marks]

(ii) $\left(\int_0^\infty x\mathrm{e}^{-x}\,\mathrm{d}x=\right)1$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) (i) correct integral ***(M1)***

Note: Award ***M1*** for correct integrand with limits from 0 to a larger number.

24 ***A1***

[2 marks]

(ii) 120 ***A1***

Note: The ***M1*** can be awarded if either part (d)(i) or part (d)(ii) is correct.

[1 mark]

(e) $A_n = n!$ ***A1***

[1 mark]

continued…

Question 1 continued

(f)

Note: Accept starting at $n = 0$, throughout this proof.

$n = 1$

$A_1 = 1$ and $1! = 1$ ***M1A1***

Note: Award ***M1*** for considering the case where $n = 1$, and ***A1*** if it is clear that both $A_1 = 1$ and $1! = 1$ have been considered.

so true for $n = 1$

assume true for $n = k$, ($A_k = \int_0^\infty x^k e^{-x}\,dx = k!$) ***M1***

Note: Award ***M0*** for statements such as "let $n = k$".

Note: Subsequent marks after this ***M1*** are independent of this mark and can be awarded.

when $n = k + 1$

attempt to integrate by parts ***M1***

Note: To obtain the ***M1***, a minimum of an expression +/- an integral must be seen.

$\int_0^\infty x^{k+1} e^{-x}\,dx = \left[-x^{k+1} e^{-x}\right]_0^\infty + (k+1)\int_0^\infty x^k e^{-x}\,dx$ ***A1***

$(k+1)\int_0^\infty x^k e^{-x}\,dx$ simplified to $(k+1)k!$ seen ***A1***

$= 0 + (k+1)k!$

Note: Condone omission of the zero.

$= (k+1)!$ ***A1***

Hence if true for $n = k$ then also true for $n = k + 1$. As true for $n = 1$ so true for $n \in \mathbb{Z}^+$. ***R1***

Note: Award the final ***R1*** mark provided at least four of the previous marks are gained.

[8 marks]

Total [25 marks]

2. (a) $x_2 = 12 - x_1$ ***(M1)***

$f(x) = x_1(12 - x_1)$ ***A1***

[2 marks]

(b) (i) $(x_1 =)\ 6$ ***A1***

Note: Award the ***A1*** if 6 seen in part (ii).

[1 mark]

(ii) $f(6)$ **OR** 6^2 **OR** graph with maximum at $(6, 36)$ ***M1***

$= 36$ ***AG***

[1 mark]

(c) $M_2(12) = \left(\frac{12}{2}\right)^2 = 36$ which is the maximum product (from (b)(ii)) ***A1***

Note: Both the 36 **AND** a link to part (b), which may be simply seeing the word "maximum" must be seen to award the ***A1***.

[1 mark]

continued…

Question 2 continued

(d) (i) let all x_i be labelled as x (or x_1 or x_n etc.) ***(M1)***

Note: Do not accept use of a specific number for x or n.

$(x^n)^{\frac{1}{n}} = x$ and $\frac{nx}{n} = x$ ***A1***

[2 marks]

(ii) $x_1 + x_2 + ... + x_n = S$ ***(A1)***

$(x_1 \times x_2 \times ... \times x_n)^{\frac{1}{n}} \le \frac{S}{n}$ ***M1***

Note: Award ***M1*** for use of the inequality, which may be seen as an equality.

$x_1 \times x_2 \times ... \times x_n \le \left(\frac{S}{n}\right)^n$ (as both sides are positive) ***M1***

LHS and RHS are equal when all values of x_i are equal (to $\frac{S}{n}$) ***R1***

$M_n(S) = \left(\frac{S}{n}\right)^n$ ***AG***

[4 marks]

(e) (i) $M_3(12) = 4^3 = 64$ ***A1***

[1 mark]

(ii) $M_4(12) = 3^4 = 81$ ***A1***

[1 mark]

(iii) $M_5(12) = 2.4^5 = 79.6\ \ (79.6262...)$ ***A1***

[1 mark]

continued…

Question 2 continued

(f) considering $M_n(12)$ for higher values of n

$P(12) = 81$ **A1**

$n = 4$
A1

Note: Award ***A0A0*** for $P(12) = 82.6$ and $n = 4.41$.

[2 marks]

(g) Consideration of graph or table of $\left(\frac{20}{n}\right)^n$ including values either side of 7 ***(M1)***

Maximum occurs when $n = 7$ ***A1***

$P(20) = \left(\frac{20}{7}\right)^7 = 1550\ (1554.260...)$ ***A1***

Note: Award ***(M1)A0A1*** for $n = 7.36$ and $P(20) = 1570$.

[3 marks]

continued…

Question 2 continued

(h) **EITHER**

$\ln(g(x)) = x(\ln(S) - \ln x)$ ***M1***

attempt to use implicit differentiation and product rule ***M1M1***

$\frac{g'(x)}{g(x)} = \ln S - \ln x - x\frac{1}{x}$ ***A1***

OR

attempt to use implicit differentiation, product rule and chain rule ***M1M1M1***

$\frac{g'(x)}{g(x)} = \ln\frac{S}{x} + \left(x\frac{x}{S} \times \frac{-S}{x^2}\right)$ ***A1***

OR

attempt to make equation explicit to $g(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}$ ***M1***

attempt to use product rule and chain rule ***M1M1***

$g'(x) = \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[x \times \frac{x}{S} \times \left(-Sx^{-2}\right) + \ln\left(\frac{S}{x}\right)\right]$

$= \mathrm{e}^{x\ln\left(\frac{S}{x}\right)}\left[\ln\left(\frac{S}{x}\right) - 1\right]$ ***A1***

THEN

$g'(x) = \left(\ln\frac{S}{x} - 1\right)g(x)$

$g(x) \neq 0$

$g'(x) = 0 \Rightarrow \ln\frac{S}{x} - 1 = 0$ ***M1***

$x = \frac{S}{\mathrm{e}}\ (0.368S,\ 0.36789...S)$ ***A1***

[6 marks]

continued…

Question 2 continued

(i) $\ln(g(x)) = x\ln\left(\frac{S}{x}\right) \Rightarrow \ln(g(x)) = \ln\left(\frac{S}{x}\right)^x$ ***M1***

$g(x) = \left(\frac{S}{x}\right)^x$ ***A1***

$= M_x(S)$ for $x \in \mathbb{Z}^+$ ***AG***

[2 marks]

(j) $\frac{100}{\mathrm{e}} = 36.8$ ***M1***

$\left(\frac{100}{36}\right)^{36} = 9.3996... \times 10^{15}$ **AND** $\left(\frac{100}{37}\right)^{37} = 9.47406... \times 10^{15}$ ***R1***

largest possible product is 9.47×10^{15} $(9.47406... \times 10^{15})$ ***A1***

Note: Award ***A1*** independently of the ***R1*** (but not independently of the ***M1***).

[3 marks]

Total [30 marks]

Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics: analysis and approaches
Higher level
Paper 3

31 October 2023

Zone A afternoon | **Zone B** afternoon | **Zone C** afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

8823–7103

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 24]

This question asks you to explore some properties of the family of curves $y = x^3 + ax^2 + b$ where $x \in \mathbb{R}$ and a, b are real parameters.

Consider the family of curves $y = x^3 + ax^2 + b$ for $x \in \mathbb{R}$, where $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

First consider the case where $a = 3$ and $b \in \mathbb{R}$.

(a) By systematically varying the value of b, or otherwise, find the two values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly two x-axis intercepts. [2]

(b) Write down the set of values of b such that the curve $y = x^3 + 3x^2 + b$ has exactly

(i) one x-axis intercept; [1]

(ii) three x-axis intercepts. [1]

Now consider the case where $a = -3$ and $b \in \mathbb{R}$.

(c) Write down the set of values of b such that the curve $y = x^3 - 3x^2 + b$ has exactly

(i) two x-axis intercepts; [1]

(ii) one x-axis intercept; [1]

(iii) three x-axis intercepts. [1]

(This question continues on the following page)

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(Question 1 continued)

For the following parts of this question, consider the curve $y = x^3 + ax^2 + b$ for $a \in \mathbb{R}$, $a \neq 0$ and $b \in \mathbb{R}$.

(d) Consider the case where the curve has exactly three x-axis intercepts. State whether each point of zero gradient is located above or below the x-axis. [1]

(e) Show that the curve has a point of zero gradient at $\mathrm{P}(0, b)$ and a point of zero gradient at $\mathrm{Q}\left(-\frac{2}{3}a, \frac{4}{27}a^3 + b\right)$. [5]

(f) Consider the points P and Q for $a > 0$ and $b > 0$.

(i) Find an expression for $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and hence determine whether each point is a local maximum or a local minimum. [3]

(ii) Determine whether each point is located above or below the x-axis. [1]

(g) Consider the points P and Q for $a < 0$ and $b > 0$.

(i) State whether P is a local maximum or a local minimum and whether it is above or below the x-axis. [1]

(ii) State the conditions on a and b that determine when Q is below the x-axis. [1]

(h) Prove that if $4a^3b + 27b^2 < 0$ then the curve, $y = x^3 + ax^2 + b$, has exactly three x-axis intercepts. [5]

Turn over

---

2. [Maximum mark: 31]

This question begins by asking you to examine families of curves that intersect every member of another family of curves at right-angles. You will then examine a family of curves that intersects every member of another family of curves at an acute angle, α.

(a) Consider a family of straight lines, L, with equation $y = mx$, where m is a parameter. Each member of L intersects every member of a family of curves, C, at right-angles.

Note: In parts (i), (ii) and (iii), you are not required to consider the case where $x = 0$.

(i) Write down an expression for the gradient of L in terms of x and y. [1]

(ii) Hence show that the gradient of C is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$. [1]

(iii) By solving the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$, show that the family of curves, C, has equation $x^2 + y^2 = k$ where k is a parameter. [2]

A family of curves has equation $y^2 = 4a^2 - 4ax$ where a is a positive real parameter.

A second family of curves has equation $y^2 = 4b^2 + 4bx$ where b is a positive real parameter.

(b) Consider the case where $a = 2$ and $b = 1$. On the same set of axes, sketch the curves $y^2 = 16 - 8x$ and $y^2 = 4 + 4x$. On your sketch, clearly label each curve and any x-intercepts.

Note: You are not required to find the coordinates of any points of intersection of the two curves. [3]

(c) By solving $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ simultaneously, show that these curves intersect at the points $\mathrm{M}\left(a - b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a - b, -2\sqrt{ab}\right)$. [6]

(d) At point M, show that the curves $y^2 = 4a^2 - 4ax$ and $y^2 = 4b^2 + 4bx$ intersect at right-angles. [5]

(This question continues on the following page)

1. (a) varies the value of b with $a = 3$ ***(M1)***

> **Note:** The ***(M1)*** in part (a) can also be awarded for a correct answer to either part (b)(i) or (b)(ii).
> Award ***(M1)*** for evidence that $b = 0$ case is considered/determined.

$b = -4,\ 0$ ***A1***

[2 marks]

(b) (i) $b < -4$ or $b > 0$ ***A1***

[1 mark]

(ii) $-4 < b < 0$ ***A1***

[1 mark]

continued…

Question 1 continued

(c) (i) $b = 0, 4$ ***A1***

[1 mark]

(ii) $b < 0$ or $b > 4$ ***A1***

[1 mark]

(iii) $0 < b < 4$ ***A1***

[1 mark]

continued…

Question 1 continued

(d) one point of zero gradient is located on either side (of the x- axis) (or equivalent) ***A1***

[1 mark]

(e) **METHOD 1**

$\frac{dy}{dx} = 3x^2 + 2ax$ ***(A1)***

attempts to solve their $\frac{dy}{dx} = 0$ for x ***M1***

$x(3x+2a)(=0)$ OR $x = \frac{-2a \pm \sqrt{4a^2}}{6}$ OR $x + \frac{a}{3} = \pm\frac{a}{3}$

$x = -\frac{2}{3}a,\ 0$ ***A1***

when $x = 0,\ y = b$ and so $\mathrm{P}(0,\ b)$ is a point of zero gradient ***AG***

Note: The following two marks are independent of the first three marks.

substitutes their expression for x in terms of a into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b\ \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $\mathrm{Q}\left(-\frac{2}{3}a,\ \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

METHOD 2

$\frac{dy}{dx} = 3x^2 + 2ax$ ***(A1)***

substitutes either $x = 0$ or $x = -\frac{2}{3}a$ into their $\frac{dy}{dx}$ ***M1***

when $x = 0,\ \frac{dy}{dx} = 0$ and $y = b$ so $\mathrm{P}(0,\ b)$ is a point of zero gradient ***AG***

$\frac{dy}{dx} = 3\left(-\frac{2}{3}a\right)^2 + 2a\left(-\frac{2}{3}a\right)$

$= \frac{4}{3}a^2 - \frac{4}{3}a^2\ (=0)\ \left(= 3\left(\frac{4}{9}a^2\right) - \frac{4}{3}a^2(=0), = \frac{12}{9}a^2 - \frac{4}{3}a^2(=0)\right)$ ***A1***

and so $\frac{dy}{dx} = 0$ when $x = -\frac{2}{3}a$ ***AG***

Note: The following two marks are independent of the first three marks.

substitutes $x = -\frac{2}{3}a$ into $y = x^3 + ax^2 + b$ ***(M1)***

$y = \left(-\frac{2}{3}a\right)^3 + a\left(-\frac{2}{3}a\right)^2 + b$

$y = -\frac{8}{27}a^3 + \frac{4}{9}a^3 + b\ \left(y = -\frac{8}{27}a^3 + \frac{12}{27}a^3 + b\right)$ ***A1***

so $\mathrm{Q}\left(-\frac{2}{3}a,\ \frac{4}{27}a^3 + b\right)$ is a point of zero gradient ***AG***

[5 marks]

continued…

Question 1 continued

(f) (i) $\frac{d^2y}{dx^2} = 6x + 2a$ ***A1***

when $x = 0$, $\frac{d^2y}{dx^2} = 2a$ $(a > 0)$ and so (P) is a (local) minimum (point) ***R1***

when $x = -\frac{2}{3}a$, $\frac{d^2y}{dx^2} = -2a$ $(a > 0)$ and so (Q) is a (local) maximum (point) ***R1***

[3 marks]

(ii) (P and Q are) both above (the x-axis) ***A1***

Note: Award ***A1*** if it is made clear that both points are above (the x-axis).
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(g) (i) (P) is a (local) maximum (point) and is above (the x-axis) ***A1***

[1 mark]

(ii) (Q is below the x-axis when) $\frac{4}{27}a^3 + b < 0$ ***A1***

Note: Award ***A1*** for an equivalent correct inequality, eg. $\frac{4}{27}a^3 < -b$.
Accept a labelled sketch that clearly shows this information.

[1 mark]

continued…

Question 1 continued

(h) **METHOD 1**

attempts to factorize $4a^3b+27b^2\,(<0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3+b\right)(<0)$ OR $b\left(4a^3+27b\right)(<0)$ ***A1***

$b>0$ and $\frac{4}{27}a^3+b<0$ or $b<0$ and $\frac{4}{27}a^3+b>0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3+b$ is less than zero (or equivalent).

when b and $\frac{4}{27}a^3+b$ have opposite sign, P and Q are located on either side (of the x- axis) (or equivalent) ***R1***

Note: Accept labelled sketches that clearly show this information.

P and Q are located on either side (of the x- axis) if (and only if) the curve has exactly three x- axis intercepts ***R1***

if $4a^3b+27b^2<0$, then the graph of $y=x^3+ax^2+b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

continued…

Question 1 continued

METHOD 2

attempts to factorize $4a^3b+27b^2\,(<0)$ ***(M1)***

$27b\left(\frac{4}{27}a^3+b\right)(<0)$ OR $b\left(4a^3+27b\right)(<0)$ ***A1***

either $b>0$ and $\frac{4}{27}a^3+b<0$ or $b<0$ and $\frac{4}{27}a^3+b>0$ ***A1***

Note: Only award this ***A1*** if both cases are stated.
Award ***A1*** for stating that exactly one of b and $\frac{4}{27}a^3+b$ is less than zero (or equivalent).

$b>0$ and $\frac{4}{27}a^3+b<0$, $\left(\Rightarrow 0<b<-\frac{4}{27}a^3\right)\Rightarrow a<0$ and hence three x- axis intercepts ***R1***

$b<0$ and $\frac{4}{27}a^3+b>0$, $\left(\Rightarrow -\frac{4}{27}a^3<b<0\right)\Rightarrow a>0$ and hence three x- axis intercepts ***R1***

Note: Accept labelled sketches that clearly show this information.

if $4a^3b+27b^2<0$, then the graph of $y=x^3+ax^2+b$ has exactly three x- axis intercepts ***AG***

Note: For proving the converse, award a maximum of 3 marks (likely to be similar steps but presented in reverse; 2nd ***A1*** line not necessary in reverse method).

[5 marks]

Total [24 marks]

2. (a) (i) $\frac{y}{x}$ ***A1***

[1 mark]

(ii) $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\left(\frac{y}{x}\right)} \left(= -\frac{1}{m}\right)$ ***A1***

> **Note:** Award ***A1*** for responses such as 'the gradient is the negative (opposite) reciprocal of $\frac{y}{x}$ or $\frac{y}{x} \times m = -1$ (or equivalent).
>
> Award ***A1*** for $\frac{y}{x} \times \left(-\frac{x}{y}\right) = -1$.
>
> Do not award ***FT*** from part (a) (i).

so $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y}$ ***AG***

[1 mark]

Question 2 continued

(iii) attempts to separate variables x and y ***(M1)***

$\int y \, \mathrm{d}y = -\int x \, \mathrm{d}x$

> **Note:** Award ***(M1)*** for $y \, \mathrm{d}y = -x \, \mathrm{d}x$.

$\frac{y^2}{2} = -\frac{x^2}{2} + c$ ***A1***

> **Note:** Award ***A1*** for $\frac{y^2}{2} + c_1 = -\frac{x^2}{2} + c_2$.
>
> Award ***A0*** for $\frac{y^2}{2} = -\frac{x^2}{2}$.

$\frac{x^2}{2} + \frac{y^2}{2} = c$

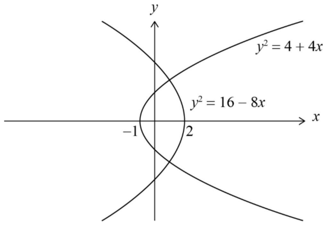
$\Rightarrow x^2 + y^2 = k$ (where $k = 2c$) ***AG***

[2 marks]

continued…

Question 2 continued

(b)



two parabolic shaped curves with approximately correct shape/position (e.g. two intersection points, in first and fourth quadrant) ***A1A1***

x-intercepts -1 and 2 ***A1***

[3 marks]

continued…

Question 2 continued

(c) at intersection, $4a^2 - 4ax = 4b^2 + 4bx$

$4a^2 - 4b^2 = 4ax + 4bx$ ($a^2 - b^2 = ax + bx$, $4a^2 - 4b^2 - 4ax - 4bx = 0$) ***A1***

attempts to factorize either the LHS or the RHS of the first two equations above (or equivalent) OR attempts to partially factorize the LHS side of $a^2 - b^2 - ax - bx = 0$ (or equivalent) ***(M1)***

$(a+b)(a-b) = (a+b)x$

Note: Accept alternative forms such as $4(a+b)(a-b) = 4(a+b)x$ or $(a+b)((a-b)-x) = 0$.

recognition that $a+b>0$ (or equivalent, eg. $a>0, b>0$) (allows division by $a+b$) ***R1***

Note: Subsequent marks are not dependent on this ***R1.***

$x = a - b$ ***A1***

Note: As $x = a - b$ is an ***AG***, only award the above ***A1*** if $a^2 - b^2 = (a+b)(a-b)$ has been used.

substitutes $x = a - b$ into either $y^2 = 4a^2 - 4ax$ or $y^2 = 4b^2 + 4bx$ and attempts to simplify ***(M1)***

$y^2 = 4a^2 - 4a(a-b)$ OR $y^2 = 4b^2 + 4b(a-b)$

$y^2 = 4a^2 - 4a^2 + 4ab \Rightarrow y = \pm 2\sqrt{ab}$ ***A1***

so $\mathrm{M}\left(a-b, 2\sqrt{ab}\right)$ and $\mathrm{N}\left(a-b, -2\sqrt{ab}\right)$ ***AG***

[6 marks]

continued…

Question 2 continued

(d) **METHOD 1**

attempts implicit differentiation on either curve ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4a}{2y}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4b}{2y}$ (or equivalent) ***A1***

substitutes $y=2\sqrt{ab}$ into either $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{4a}{2y}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4b}{2y}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\frac{a}{b}}$ $(=-\frac{a}{\sqrt{ab}})$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{b}{a}}$ $(=\frac{b}{\sqrt{ab}})$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{a}}=-1$ OR $-\frac{a}{\sqrt{ab}}\times\frac{b}{\sqrt{ab}}=-1$ (or equivalent) ***A1***

OR

eg. the negative (opposite) reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

continued…

Question 2 continued

METHOD 2

attempts chain rule differentiation on either $y=\sqrt{4a^2-4ax}$ or $y=\sqrt{4b^2+4bx}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ (or equivalent) and $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ (or equivalent) ***A1***

substitutes $x=a-b$ into either $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{\sqrt{4a^2-4ax}}$ or $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2b}{\sqrt{4b^2+4bx}}$ ***(M1)***

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sqrt{\frac{a}{b}}$ $(=-\frac{a}{\sqrt{ab}})$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=\sqrt{\frac{b}{a}}$ $(=\frac{b}{\sqrt{ab}})$ (or equivalent) ***A1***

EITHER

$-\sqrt{\frac{a}{b}}\times\sqrt{\frac{b}{a}}=-1$ OR $-\frac{a}{\sqrt{ab}}\times\frac{b}{\sqrt{ab}}=-1$ (or equivalent) ***A1***

OR

eg. the negative reciprocal of $-\sqrt{\frac{a}{b}}$ is $\sqrt{\frac{b}{a}}$ (or equivalent) ***A1***

OR

the product of the two gradients is -1 ***A1***

THEN

so at point M, the curves intersect at right angles ***AG***

[5 marks]

continued…

Question 2 continued

(e) (i) $g(x,\ y)=\dfrac{-\dfrac{x}{y}+\tan\dfrac{\pi}{4}}{1-\left(-\dfrac{x}{y}\right)\tan\dfrac{\pi}{4}}$ ***(A1)***

$g(x,\ y)=\dfrac{-\dfrac{x}{y}+1}{1+\dfrac{x}{y}}\ \left(=\dfrac{\dfrac{-x+y}{y}}{\dfrac{y+x}{y}}\right)$ ***A1***

so $g(x,\ y)=\dfrac{y-x}{y+x}$ ***AG***

[2 marks]

continued…

Question 2 continued

(ii) let $y=vx$ ***(M1)***

$\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$ ***(A1)***

$\left(v+x\dfrac{dv}{dx}=\right)\dfrac{vx-x}{vx+x}\ \left(=\dfrac{v-1}{v+1}\right)$ ***(A1)***

attempts to express $x\dfrac{dv}{dx}$ as a single fraction in v ***(M1)***

$x\dfrac{dv}{dx}=-\dfrac{v^2+1}{v+1}$ (or equivalent) ***(A1)***

attempts to separate variables x and v ***(M1)***

$\displaystyle\int\frac{v+1}{v^2+1}\,dv=-\int\frac{1}{x}\,dx$ (or equivalent)

$\dfrac{1}{2}\ln\left(v^2+1\right)+\arctan v=-\ln|x|(+d)$ (or equivalent) ***A1A1***

$\dfrac{1}{2}\ln\left(\dfrac{y^2}{x^2}+1\right)+\arctan\dfrac{y}{x}+\ln|x|=d$ (or equivalent) ***A1***

[9 marks]

continued…

Question 2 continued

(f) **METHOD 1**

$$g(x,\,y)=\frac{\frac{1}{\tan\alpha}f\left(x,y\right)+1}{\frac{1}{\tan\alpha}-f\left(x,y\right)}$$ ***M1***

EITHER

as $\alpha\to\frac{\pi}{2}$, $\frac{1}{\tan\alpha}\to 0$, (hence $g(x,\,y)\to-\frac{1}{f(x,\,y)}$) ***R1***

OR

as $\alpha\to\frac{\pi}{2}$, $\tan\alpha\to\infty$ and so $g(x,\,y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

THEN

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\,y)=-\frac{1}{f(x,\,y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

METHOD 2

uses either $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$ or $\frac{1}{\tan\alpha}=\frac{\cos\alpha}{\sin\alpha}$ to form $g(x,\,y)=\frac{\cos\alpha f\left(x,y\right)+\sin\alpha}{\cos\alpha-\sin\alpha f\left(x,y\right)}$ ***M1***

as $\alpha\to\frac{\pi}{2}$, $\cos\alpha\to 0$ and $\sin\alpha\to 1$ and so $g(x,\,y)\to\frac{0\times f\left(x,y\right)+1}{0-f\left(x,y\right)}$ ***R1***

$$\lim_{\alpha\to\frac{\pi}{2}}g(x,\,y)=-\frac{1}{f(x,\,y)}$$ ***AG***

Note: The ***R1*** is dependent on the ***M1***.

[2 marks]

Total [31 marks]

Topic 5a: Calculus – Differentiation & Its Applications

1. If $2x^2 - 3y^2 = 2$, find the two values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 5$.

(Total 4 marks)

2. Differentiate $y = \arccos(1 - 2x^2)$ with respect to x, and simplify your answer.

(Total 4 marks)

3. **Give exact answers in this part of the question.**

The temperature $g(t)$ at time t of a given point of a heated iron rod is given by

$$g(t) = \frac{\ln t}{\sqrt{t}}, \qquad \text{where } t > 0.$$

(a) Find the interval where $g'(t) > 0$. **(4)**

(b) Find the interval where $g''(t) > 0$ and the interval where $g''(t) < 0$. **(5)**

(c) Find the value of t where the graph of $g(t)$ has a point of inflexion. **(3)**

(d) Let t^* be a value of t for which $g'(t^*) = 0$ and $g''(t^*) < 0$. Find t^*. **(3)**

(e) Find the point where the normal to the graph of $g(t)$ at the point $(t^*, g(t^*))$ meets the t-axis. **(3)**

(Total 18 marks)

4. Let $f(x) = \ln|x^5 - 3x^2|$, $-0{\cdot}5 < x < 2$, $x \neq a$, $x \neq b$; (a, b are values of x for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros of $f(x)$. Show also the position of any asymptotes. **(2)**

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$). **(3)**

(b) Find the **exact** values of a and b. **(3)**

(c) Find $f'(x)$, and indicate clearly where $f'(x)$ is not defined. **(3)**

(d) Find the **exact** value of the x-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.) **(3)**

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the x-axis. (Do **not** evaluate the integral.) **(2)**

(Total 16 marks)

5. Differentiate from first principles $f(x) = \cos x$.

(Total 8 marks)

6. For the function $f : x \mapsto x^2 \, 1\text{n}\, x$, $x > 0$, find the function f', the derivative of f with respect to x.

(Total 3 marks)

7. For the function $f : x \mapsto \frac{1}{2}\sin 2x + \cos x$, find the possible values of $\sin x$ for which $f'(x) = 0$.

(Total 3 marks)

8. For what values of m is the line $y = mx + 5$ a tangent to the parabola $y = 4 - x^2$?

(Total 3 marks)

9. The tangent to the curve $y^2 - x^3$ at the point P(1, 1) meets the x-axis at Q and the y-axis at R. Find the ratio PQ : QR.

(Total 3 marks)

10. (a) Sketch and label the curves $y = x^2$ for $-2 \le x \le 2$, and $y = -\frac{1}{2}\ln x$ for $0 < x \le 2$. **(2)**

(b) Find the x-coordinate of P, the point of intersection of the two curves. **(2)**

(c) If the tangents to the curves at P meet the y-axis at Q and R, calculate the area of the triangle PQR. **(6)**

(d) Prove that the two tangents at the points where $x = a$, $a > 0$, on each curve are always perpendicular. **(4)**

(Total 14 marks)

11. (a) Let $y = \dfrac{a + b\sin x}{b + a\sin x}$, where $0 < a < b$.

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{(b^2 + a^2)\cos x}{(b + a\sin x)^2}$.

(4)

(ii) Find the maximum and minimum values of y.

(4)

(iii) Show that the graph of $y = \dfrac{a + b\sin x}{b + a\sin x}$, $0 < a < b$ cannot have a vertical asymptote.

(2)

(b) For the graph of $y = \dfrac{4 + 5\sin x}{5 + 4\sin x}$ for $0 \le x \le 2\pi$,

(i) write down the y-intercept;

(ii) find the x-intercepts m and n, (where $m < n$) correct to four significant figures;

(iii) sketch the graph.

(5)

(c) The area enclosed by the graph of $y = \dfrac{4 + 5\sin x}{5 + 4\sin x}$ and the x-axis from $x = 0$ to $x = n$ is denoted by A. Write down, but do **not** evaluate, an expression for the area A.

(2)
(Total 17 marks)

12. If $f(x) = \ln(2x - 1), x > \dfrac{1}{2}$, find

(a) $f'(x)$;

(b) the value of x where the gradient of $f(x)$ is equal to x.

(Total 3 marks)

13. Find the x-coordinate, between –2 and 0, of the point of inflexion on the graph of the function $f : x \mapsto x^2 e^x$. Give your answer to 3 decimal places.

(Total 3 marks)

14. Find the gradient of the tangent to the curve $3x^2 + 4y^2 = 7$ at the point where x = 1 and $y > 0$.

(Total 3 marks)

15. The function f is given by $f : x \mapsto e^{(1+\sin \pi x)}$, $x \ge 0$.

(a) Find $f'(x)$.

Let x_n be the value of x where the $(n + 1)^{\text{th}}$ maximum or minimum point occurs, $n \in \mathbb{N}$. (*ie* x_0 is the value of x where the first maximum or minimum occurs, x_1 is the value of x where the second maximum or minimum occurs, *etc*).

(b) Find x_n in terms of n.

(Total 3 marks)

16. Let $f(x) = x\left(\sqrt[3]{(x^2 - 1)^2}\right)$, $-1.4 \le x \le 1.4$

(a) **Sketch** the graph of $f(x)$. (An exact scale diagram is **not** required.)

On your graph indicate the approximate position of

(i) each zero;

(ii) each maximum point;

(iii) each minimum point.

(4)

(b) (i) Find $f'(x)$, clearly stating its domain.

(ii) Find the x-coordinates of the maximum and minimum points of $f(x)$, for $-1 < x < 1$.

(7)

(c) Find the x-coordinate of the point of inflexion of $f(x)$, where $x > 0$, giving your answer correct to **four** decimal places.

(2)
(Total 13 marks)

17. The line $y = 16x - 9$ is a tangent to the curve $y = 2x^3 + ax^2 + bx - 9$ at the point (1,7). Find the values of a and b.

(Total 3 marks)

18. Consider the function $y = \tan x - 8 \sin x$.

(a) Find $\frac{dy}{dx}$.

(b) Find the value of $\cos x$ for which $\frac{dy}{dx} = 0$.

(Total 3 marks)

19. Consider the tangent to the curve $y = x^3 + 4x^2 + x - 6$.

(a) Find the equation of this tangent at the point where $x = -1$.

(b) Find the coordinates of the point where this tangent meets the curve again.

(Total 3 marks)

20. Let $y = \sin(kx) - kx\cos(kx)$, where k is a constant.

Show that $\frac{dy}{dx} = k^2 x \sin(kx)$.

(Total 3 marks)

21. A curve has equation $xy^3 + 2x^2y = 3$. Find the equation of the tangent to this curve at the point (1, 1).

(Total 6 marks)

22. The function f is defined by

$$f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$$

(a) (i) Find an expression for $f'(x)$, simplifying your answer.

(ii) The tangents to the curve of $f(x)$ at points A and B are parallel to the x-axis. Find the coordinates of A and of B.

(5)

(b) (i) Sketch the graph of $y = f'(x)$.

(ii) Find the x-coordinates of the three points of inflexion on the graph of f.

(5)

(c) Find the range of

(i) f;

(ii) the composite function $f \circ f$.

(5)
(Total 15 marks)

23. Air is pumped into a spherical ball which expands at a rate of 8 cm^3 per second (8 cm^3 s^{-1}). Find the **exact** rate of increase of the radius of the ball when the radius is 2 cm.

(Total 6 marks)

24. A curve has equation $x^3 y^2 = 8$. Find the equation of the normal to the curve at the point (2, 1).

(Total 6 marks)

25. The function f is defined by $f(x) = \frac{x^2}{2^x}$, for $x > 0$.

(a) (i) Show that

$$f'(x) = \frac{2x - x^2 \ln 2}{2^x}$$

(ii) Obtain an expression for $f''(x)$, simplifying your answer as far as possible.

(5)

(b) (i) Find the **exact** value of x satisfying the equation $f'(x) = 0$

(ii) Show that this value gives a maximum value for $f(x)$.

(4)

(c) Find the x-coordinates of the two points of inflexion on the graph of f.

(3)
(Total 12 marks)

26. Consider the function $f(t) = 3\sec^2 t + 5t$.

(a) Find $f'(t)$.

(b) Find the **exact** values of

(i) $f(\pi)$;

(ii) $f'(\pi)$;

(Total 6 marks)

27. Consider the equation $2xy^2 = x^2y + 3$.

(a) Find y when $x = 1$ and $y < 0$.

(b) Find $\frac{dy}{dx}$ when $x = 1$ and $y < 0$.

(Total 6 marks)

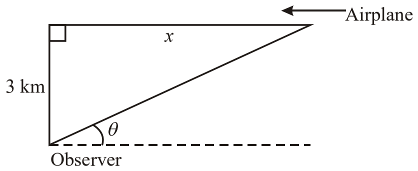
28. Let $y = e^{3x} \sin (\pi x)$.

(a) Find $\frac{dy}{dx}$.

(b) Find the smallest positive value of x for which $\frac{dy}{dx} = 0$.

(Total 6 marks)

29. An airplane is flying at a constant speed at a constant altitude of 3 km in a straight line that will take it directly over an observer at ground level. At a given instant the observer notes that the angle θ is $\frac{1}{3}\pi$ radians and is increasing at $\frac{1}{60}$ radians per second. Find the speed, in kilometres per hour, at which the airplane is moving towards the observer.



(Total 6 marks)

30. A curve has equation $f(x) = \frac{a}{b + e^{-cx}}$, $a \neq 0$, $b > 0$, $c > 0$.

(a) Show that $f''(x) = \frac{ac^2 e^{-cx}\left(e^{-cx} - b\right)}{\left(b + e^{-cx}\right)^3}$.

(4)

(b) Find the coordinates of the point on the curve where $f''(x) = 0$.

(2)

(c) Show that this is a point of inflexion.

(2)
(Total 8 marks)

31. The point P(1, p), where $p > 0$, lies on the curve $2x^2y + 3y^2 = 16$.

(a) Calculate the value of p.

(b) Calculate the gradient of the tangent to the curve at P.

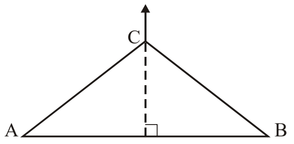
(Total 6 marks)

32. The function f is defined by $f : x \mapsto 3^x$.

Find the solution of the equation $f''(x) = 2$.

(Total 6 marks)

33. The following diagram shows an isosceles triangle ABC with AB = 10 cm and AC = BC. The vertex C is moving in a direction perpendicular to (AB) with speed 2 cm per second.



Calculate the rate of increase of the angle $\hat{CAB}$ at the moment the triangle is equilateral.

(Total 6 marks)

34. If $y = \ln (2x - 1)$, find $\frac{d^2y}{dx^2}$.

(Total 6 marks)

35. Find the equation of the normal to the curve $x^3 + y^3 - 9xy = 0$ at the point (2, 4).

(Total 6 marks)

36. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

(Total 6 marks)

37. Find the gradient of the normal to the curve $3x^2y + 2xy^2 = 2$ at the point (1, –2).

(Total 6 marks)

38. The function f is given by $f(x) = \frac{x^5 + 2}{x}$, $x \neq 0$. There is a point of inflexion on the graph of f at the point P. Find the coordinates of P.

(Total 6 marks)

39. An experiment is carried out in which the number n of bacteria in a liquid, is given by the formula $n = 650\,e^{kt}$, where t is the time in minutes after the beginning of the experiment and k is a constant. The number of bacteria doubles every 20 minutes. Find

(a) the **exact** value of k;

(b) the rate at which the number of bacteria is increasing when $t = 90$.

(Total 6 marks)

40. Let $f(x) = \dfrac{x^2 + 5x + 5}{x + 2}$, $x \neq -2$.

(a) Find $f'(x)$.

(b) Solve $f'(x) > 2$.

(Total 6 marks)

41. The function f is defined by $f(x) = e^{px}(x + 1)$, here $p \in \mathbb{R}$.

(a) (i) Show that $f'(x) = e^{px}(p(x + 1) + 1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}e^{px}(p(x + 1) + n),\ n \in \mathbb{Z}^+.$$

(7)

(b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f. Find the **exact** value of the x-coordinate of

(i) the minimum point;

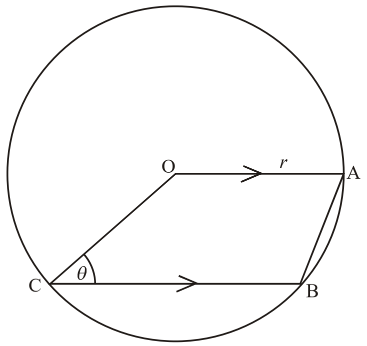
(ii) the point of inflexion.

(4)

(c) Let $p = \dfrac{1}{2}$. Let R be the region enclosed by the curve, the x-axis and the lines $x = -2$ and $x = 2$. Find the area of R.

(2)
(Total 13 marks)

42. The diagram shows a trapezium OABC in which OA is parallel to CB. O is the centre of a circle radius r cm. A, B and C are on its circumference. Angle $O\hat{C}B = \theta$.



Let T denote the area of the trapezium OABC.

(a) Show that $T = \dfrac{r^2}{2}(\sin\theta + \sin 2\theta)$.

(4)

For a **fixed** value of r, the value of T varies as the value of θ varies.

(b) Show that T takes its maximum value when θ satisfies the equation $4\cos^2\theta + \cos\theta - 2 = 0$, and verify that this value of T is a maximum.

(5)

(c) Given that the perimeter of the trapezium is 75 cm, find the maximum value of T.

(6)
(Total 15 marks)

43. Let f be a cubic polynomial function. Given that $f(0) = 2$, $f'(0) = -3$, $f(1) = f'(1)$ and $f''(-1) = 6$, find $f(x)$.

(Total 6 marks)

44. (a) Write down the term in x^r in the expansion of $(x + h)^n$, where $0 \leq r \leq n$, $n \in \mathbb{Z}^+$.

(1)

(b) Hence differentiate x^n, $n \in \mathbb{Z}^+$, from first principles.

(5)

(c) Starting from the result $x^n \times x^{-n} = 1$, deduce the derivative of x^{-n}, $n \in \mathbb{Z}^+$.

(4)
(Total 10 marks)

45. Let $f(x) = \cos^3(4x + 1)$, $0 \leq x \leq 1$.

(a) Find $f'(x)$.

(b) Find the **exact** values of the three roots of $f'(x) = 0$.

(Total 6 marks)

46. Given that $3^{x+y} = x^3 + 3y$, find $\dfrac{dy}{dx}$.

(Total 6 marks)

47. Let f be the function defined for $x > -\dfrac{1}{3}$ by $f(x) = \ln(3x + 1)$.

(a) Find $f'(x)$.

(b) Find the equation of the normal to the curve $y = f(x)$ at the point where $x = 2$.
Give your answer in the form $y = ax + b$ where $a, b \in \mathbb{R}$.

(Total 6 marks)

48. Let $y = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $\dfrac{dy}{d\theta} = \mathrm{i}y$.

[You may assume that for the purposes of differentiation and integration, i may be treated in the same way as a real constant.]

(3)

(b) **Hence** show, using integration, that $y = e^{\mathrm{i}\theta}$.

(5)

(c) Use this result to deduce de Moivre's theorem.

(2)

(d) (i) Given that $\dfrac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \neq 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

(ii) Hence deduce the value of $\lim\limits_{\theta\to 0}\dfrac{\sin 6\theta}{\sin\theta}$.

(10)
(Total 20 marks)

49. Let $f(x) = x\ln x - x$, $x > 0$.

(a) Find $f'(x)$.

(b) Using integration by parts find $\int (\ln x)^2\, dx$.

(Total 6 marks)

50. Let $y = x\arcsin x$, $x \in\,]-1, 1[$. Show that $\dfrac{d^2y}{dx^2} = \dfrac{2-x^2}{\left(1-x^2\right)^{\frac{3}{2}}}$.

(Total 6 marks)

51. Given that $e^{xy} - y^2\ln x = e$ for $x \geq 1$, find $\dfrac{dy}{dx}$ at the point (1, 1).

(Total 6 marks)

52. The following table shows the values of two functions f and g and their first derivatives when $x = 1$ and $x = 0$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	4	1	–4	5
1	–2	3	–1	2

(a) Find the derivative of $\dfrac{3f(x)}{g(x)-1}$ when $x = 0$.

(b) Find the derivative of $f(g(x) + 2x)$ when $x = 1$.

(Total 6 marks)

53. The function f is defined by $f(x) = \dfrac{2x}{x^2+6}$ for $x \geq b$ where $b \in \mathbb{R}$.

(a) Show that $f'(x) = \dfrac{12-2x^2}{(x^2+6)^2}$.

(b) Hence find the smallest **exact** value of b for which the inverse function f^{-1} exists. Justify your answer.

(Total 6 marks)

54. Consider the curve with equation $x^2 + xy + y^2 = 3$.

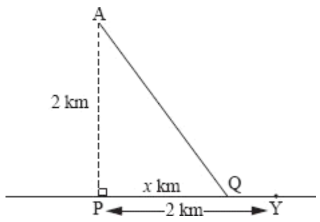
(a) Find in terms of k, the gradient of the curve at the point $(-1, k)$.

(5)

(b) Given that the tangent to the curve is parallel to the x-axis at this point, find the value of k.

(1)
(Total 6 marks)

55. André wants to get from point A located in the sea to point Y located on a straight stretch of beach. P is the point on the beach nearest to A such that AP = 2 km and PY = 2 km. He does this by swimming in a straight line to a point Q located on the beach and then running to Y.



When André swims he covers 1 km in $5\sqrt{5}$ minutes. When he runs he covers 1 km in 5 minutes.

(a) If PQ = x km, $0 \le x \le 2$, find an expression for the time T minutes taken by André to reach point Y.

(4)

(b) Show that $\frac{dT}{dx} = \frac{5\sqrt{5}\,x}{\sqrt{x^2+4}} - 5.$

(3)

(c) (i) Solve $\frac{dT}{dx} = 0.$

(ii) Use the value of x found in **part (c) (i)** to determine the time, T minutes, taken for André to reach point Y.

(iii) Show that $\frac{d^2T}{dx^2} = \frac{20\sqrt{5}}{\left(x^2+4\right)^{\frac{3}{2}}}$ and **hence** show that the time found in **part (c) (ii)** is a minimum.

(11)
(Total 18 marks)

56. Find the gradient of the tangent to the curve $x^3\,y^2 = \cos(\pi y)$ at the point $(-1, 1)$.

(Total 12 marks)

57. A particle moves along a straight line. When it is a distance s from a fixed point, where $s > 1$, the velocity v is given by $v = \frac{(3s+2)}{(2s-1)}$. Find the acceleration when $s = 2$.

(Total 4 marks)

58. The acceleration, $a(t)$ m s^{-2}, of a fast train during the first 80 seconds of motion is given by

$$a(t) = -\frac{1}{20}t + 2$$

where t is the time in seconds. If the train starts from rest at $t = 0$, find the distance travelled by the train in the first minute.

(Total 4 marks)

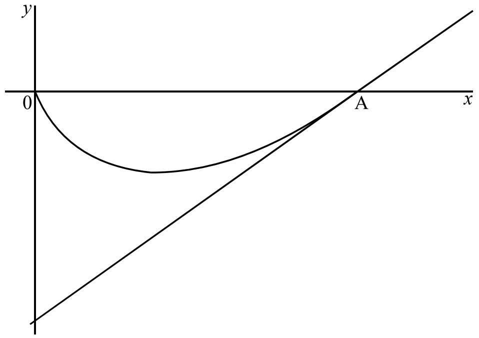
59. Consider the function $f_k(x) = \begin{cases} x\ln x - kx, & x > 0 \\ 0, & x = 0 \end{cases}$, where $k \in \mathbb{N}$

(a) Find the derivative of $f_k(x)$, $x > 0$.

(2)

(b) Find the interval over which $f_0(x)$ is increasing.
The graph of the function $f_k(x)$ is shown below.

(2)



(c) (i) Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.

(ii) One x-intercept is at (0, 0). Find the coordinates of the other x-intercept.

(4)

(d) Find the area enclosed by the curve and the x-axis.

(5)

(e) Find the equation of the tangent to the curve at A.

(2)

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis.

(2)

(g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence.

(3)
(Total 20 marks)

60. The velocity, v, of an object, at a time t, is given by $v = ke^{-\frac{t}{2}}$, where t is in seconds and v is in m s^{-1}. Find the distance travelled between $t = 0$ and $t = a$.

(Total 3 marks)

61. Find the coordinates of the point which is nearest to the origin on the line

$$L: x = 1 - \lambda, y = 2 - 3\lambda, z = 2.$$

(Total 3 marks)

62. A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = \sin x$, where $0 \le x \le \pi$.

(a) Write down an expression for the area of the rectangle.

(b) Find the maximum area of the rectangle.

(Total 3 marks)

63. Let $f : x \mapsto e^{\sin x}$.

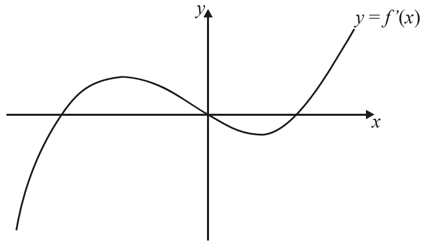
(a) Find $f'(x)$.

There is a point of inflexion on the graph of f, for $0 < x < 1$.

(b) Write down, but do not solve, an equation in terms of x, that would allow you to find the value of x at this point of inflexion.

(Total 3 marks)

64. The diagram shows the graph of $y = f'(x)$.



Indicate, and label clearly, **on the graph**

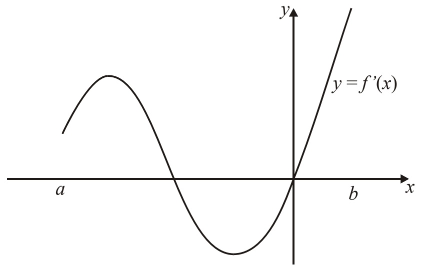
(a) the points where $y = f(x)$ has minimum points;

(b) the points where $y = f(x)$ has maximum points;

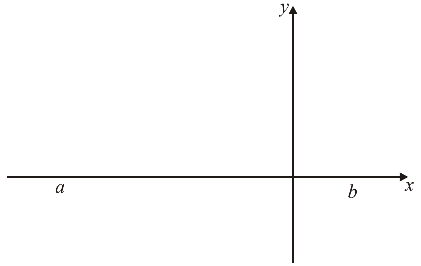
(c) the points where $y = f(x)$ has points of inflexion.

(Total 3 marks)

65. The diagram shows a sketch of the graph of $y = f'(x)$ for $a \le x \le b$.



On the grid below, which has the same scale on the x-axis, draw a sketch of the graph of $y = f(x)$ for $a \le x \le b$, given that $f(0) = 0$ and $f(x) \ge 0$ for all x. On your graph you should clearly indicate any minimum or maximum points, or points of inflexion.



(Total 3 marks)

66. An astronaut on the moon throws a ball vertically upwards. The height, s metres, of the ball, after t seconds, is given by the equation $s = 40t + 0.5at^2$, where a is a constant. If the ball reaches its maximum height when $t = 25$, find the value of a.

(Total 3 marks)

67. A point $P(x, x^2)$ lies on the curve $y = x^2$. Calculate the minimum distance from the point $A\left(2, -\frac{1}{2}\right)$ to the point P.

(Total 3 marks)

68. A particle is moving along a straight line so that t seconds after passing through a fixed point O on the line, its velocity $v(t)$ m s^{-1} is given by

$v(t) = t\sin\left(\frac{\pi}{3}t\right)$.

(a) Find the values of t for which $v(t) = 0$, given that $0 \le t \le 6$.

(3)

(b) (i) Write down a mathematical expression for the **total** distance travelled by the particle in the first six seconds after passing through O.

(ii) Find this distance.

(4)
(Total 7 marks)

69. A particle is projected along a straight line path. After t seconds, its velocity v metres per second is given by $v = \frac{1}{2+t^2}$.

(a) Find the distance travelled in the first second.

(b) Find an expression for the acceleration at time t.

(Total 6 marks)

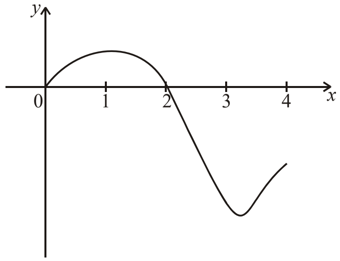
70. A rectangle is drawn so that its lower vertices are on the x-axis and its upper vertices are on the curve $y = e^{-x^2}$. The area of this rectangle is denoted by A.

(a) Write down an expression for A in terms of x.

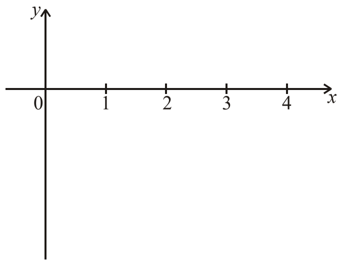
(b) Find the maximum value of A.

(Total 6 marks)

71. The diagram below shows the graph of $y_1 = f(x)$, $0 \le x \le 4$



On the axes below, sketch the graph of $y_2 = \int_0^x f(t)\mathrm{d}t$, marking clearly the points of inflexion.



(Total 6 marks)

72. A particle moves in a straight line with velocity v, in metres per second, at time t seconds, given by

$$v(t) = 6t^2 - 6t,\ t \ge 0$$

Calculate the total distance travelled by the particle in the first two seconds of motion.

(Total 6 marks)

73. Find the x-coordinate of the point of inflexion on the graph of $y = xe^x$, $-3 \le x \le 1$.

(Total 6 marks)

74. The point B(a, b) is on the curve $f(x) = x^2$ such that B is the point which is closest to A(6, 0). Calculate the value of a.

(Total 6 marks)

75. (a) On the same axes sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$$f(x) = 4 - (1 - x)^2, \text{ for } -2 \le x \le 4,$$

$$g(x) = \ln(x + 3) - 2, \text{ for } -3 \le x \le 5.$$

(2)

(b) (i) Write down the equation of any vertical asymptotes.

(ii) State the x-intercept and y-intercept of $g(x)$.

(3)

(c) Find the values of x for which $f(x) = g(x)$.

(2)

(d) Let A be the region where $f(x) \ge g(x)$ and $x \ge 0$.

(i) On your graph shade the region A.

(ii) Write down an integral that represents the area of A.

(iii) Evaluate this integral.

(4)

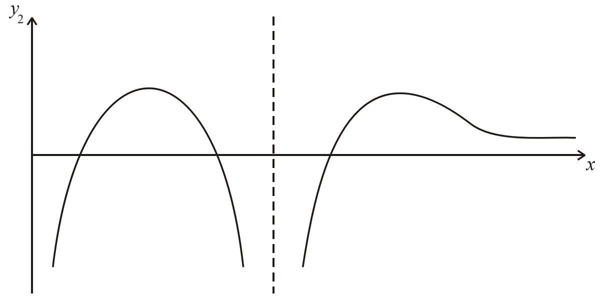
(e) In the region A find the maximum vertical distance between $f(x)$ and $g(x)$.

(3)
(Total 14 marks)

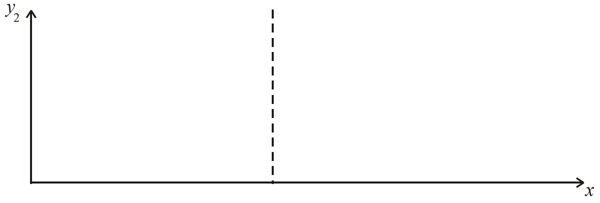
76. A particle moves in a straight line. Its velocity v m s^{-1} after t seconds is given by $v = e^{-\sqrt{t}} \sin t$. Find the total distance travelled in the time interval $[0, 2\pi]$.

(Total 6 marks)

77. The diagram below shows the graph of $y_1 = f(x)$.



On the axes below, sketch the graph of $y_2 = |f'(x)|$.



(Total 6 marks)

78. A closed cylindrical can has a volume of 500 cm^3. The height of the can is h cm and the radius of the base is r cm.

(a) Find an expression for the total surface area A of the can, in terms of r.

(b) Given that there is a minimum value of A for $r > 0$, find this value of r.

(Total 6 marks)

79. The displacement s metres of a moving body B from a fixed point O at time t seconds is given by

$$s = 50t - 10t^2 + 1000.$$

(a) Find the velocity of B in m s^{-1}.

(b) Find its maximum displacement from O.

(Total 6 marks)

80. The line L is given by the parametric equations $x = 1 - \lambda$, $y = 2 - 3\lambda$, $z = 2$. Find the coordinates of the point on L which is nearest to the origin.

(Total 6 marks)

81. The function f is defined on the domain $x \geq 1$ by $f(x) = \dfrac{\ln x}{x}$.

(a) (i) Show, by considering the first and second derivatives of f, that there is one maximum point on the graph of f.

(ii) State the **exact** coordinates of this point.

(9)

(iii) The graph of f has a point of inflexion at P. Find the x-coordinate of P.

(3)

Let R be the region enclosed by the graph of f, the x-axis and the line $x = 5$.

(c) Find the **exact** value of the area of R.

(6)

(d) The region R is rotated through an angle 2π about the x-axis. Find the volume of the solid of revolution generated.

(3)
(Total 21 marks)

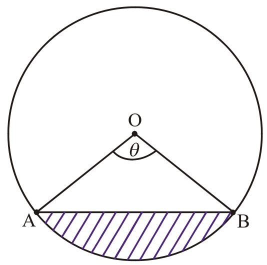
82. The normal to the curve $y = \dfrac{k}{x} + \ln x^2$, for $x \neq 0$, $k \in \mathbb{R}$, at the point where $x = 2$, has equation $3x + 2y = b$, where $b \in \mathbb{R}$. Find the **exact** value of k.

(Total 6 marks)

83. The curve $y = \dfrac{x^3}{3} - x^2 - 3x + 4$ has a local maximum point at P and a local minimum point at Q. Determine the equation of the straight line passing through P and Q, in the form $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$.

(Total 6 marks)

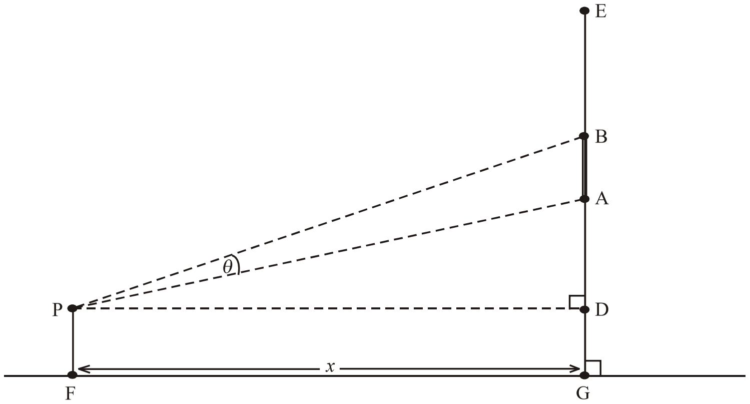
84. The following diagram shows the points A and B on the circumference of a circle, centre O, and radius 4 cm, where $\hat{AOB} = \theta$. Points A and B are moving on the circumference so that θ is increasing at a constant rate.



Given that the rate of change of the length of the minor arc AB is numerically equal to the rate of change of the area of the shaded segment, find the acute value of θ.

(Total 6 marks)

85. A man PF is standing on horizontal ground at F at a distance x from the bottom of a vertical wall GE. He looks at the picture AB on the wall. The angle BPA is θ.



Let DA = a, DB = b, where angle $\hat{PDE}$ is a right angle. Find the value of x for which tan θ is a maximum, giving your answer in terms of a and b. Justify that this value of x does give a maximum value of tan θ.

(Total 9 marks)

86. Let $f(x) = 3x^2 - x + 4$. Find the values of m for which the line $y = mx + 1$ is a tangent to the graph of f.

(Total 6 marks)

87. Particle A moves in a straight line, starting from O_A, such that its velocity in metres per second for $0 \le t \le 9$ is given by

$$v_A = -\frac{1}{2}t^2 + 3t + \frac{3}{2}.$$

Particle B moves in a straight line, starting from O_B, such that its velocity in metres per second for $0 \le t \le 9$ is given by

$$v_B = e^{0.2t}.$$

(a) Find the maximum value of v_A, justifying that it is a maximum.

(5)

(b) Find the acceleration of B when $t = 4$.

(3)

The displacements of A and B from O_A and O_B respectively, at time t are s_A metres and s_B metres. When $t = 0$, $s_A = 0$, and $s_B = 5$.

(c) Find an expression for s_A and for s_B, giving your answers in terms of t.

(7)

(d) (i) Sketch the curves of s_A and s_B on the same diagram.

(ii) Find the values of t at which $s_A = s_B$.

(8)

(Total 23 marks)

88. The radius and height of a cylinder are both equal to x cm. The curved surface area of the cylinder is increasing at a constant rate of 10 cm^2/sec. When $x = 2$, find the rate of change of

(a) the radius of the cylinder;

(b) the volume of the cylinder.

(Total 6 marks)

89. The function f is defined by $f(x) = \frac{\ln x}{x^3}$, $x \ge 1$.

(a) Find $f'(x)$ and $f''(x)$, simplifying your answers.

(6)

(b) (i) Find the **exact** value of the x-coordinate of the maximum point and justify that this is a maximum.

(ii) Solve $f''(x) = 0$, and show that at this value of x, there is a point of inflexion on the graph of f.

(iii) Sketch the graph of f, indicating the maximum point and the point of inflexion.

(11)

The region enclosed by the x-axis, the graph of f and the line $x = 3$ is denoted by R.

(c) Find the volume of the solid of revolution obtained when R is rotated through 360° about the x-axis.

(3)

(d) Show that the area of R is $\frac{1}{18}$ (4 – ln 3).

(6)

(Total 26 marks)

90. The volume of a solid is given by

$$V = \frac{4}{3}\pi r^3 + \pi r^2 h.$$

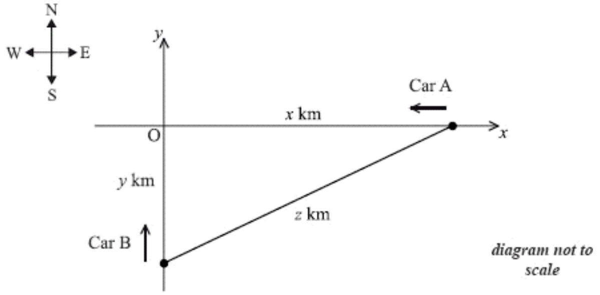
At the time when the radius is 3 cm, the volume is 1π cm^3, the radius is changing at a rate of 2 cm/min and the volume is changing at a rate of 204π cm^3/min. Find the rate of change of the height at this time.

(Total 6 marks)

91. A particle moves in a straight line. At time t seconds, its displacement from a fixed point O is s metres, and its velocity, v metres per second, is given by $v = 3t^2 - 4t + 2$, $t \geq 0$. When $t = 0$, $s = -3$. Find the value of t when the particle is at O.

(Total 6 marks)

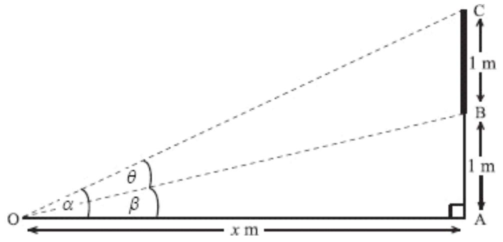
92. Car A is travelling on a straight east-west road in a westerly direction at 60 km h^{-1}. Car B is travelling on a straight north-south road in a northerly direction at 70 km h^{-1}. The roads intersect at the point O. When Car A is x km east of O, and Car B is y km south of O, the distance between the cars is z km.



Find the rate of change of z when Car A is 0.8 km east of O and Car B is 0.6 km south of O.

(Total 6 marks)

93. A television screen, BC, of height one metre, is built into a wall. The bottom of the television screen at B is one metre above an observer's eye level. The angles of elevation ($\text{A}\hat{\text{O}}\text{C}$, $\text{A}\hat{\text{O}}\text{B}$) from the observer's eye at O to the top and bottom of the television screen are α and β radians respectively. The horizontal distance from the observer's eye to the wall containing the television screen is x metres. The observer's angle of vision ($\text{B}\hat{\text{O}}\text{C}$) is θ radians, as shown below.



(a) (i) Show that $\theta = \arctan\frac{2}{x} - \arctan\frac{1}{x}$.

(ii) Hence, or otherwise, find the **exact** value of x for which θ is a maximum and justify that this value of x gives the maximum value of θ.

(iii) Find the maximum value of θ.

(17)

(b) Find where the observer should stand so that the angle of vision is 15°.

(5)

(Total 22 marks)

94. Given that $y = e^{-x^2}$ find

(a) $\frac{d^2 y}{dx^2}$;

(3)

(b) the exact values of the x-coordinates of the points of inflexion on the graph of $y = e^{-x^2}$, justifying that they are points of inflexion.

(3)

(Total 6 marks)

95. (a) A curve is defined by the implicit equation $2xy + 6x^2 - 3y^2 = 6$.

Show that the tangent at the point A with coordinates $\left(1, \frac{2}{3}\right)$ has gradient $\frac{20}{3}$.

(6)

(b) The line x =1 cuts the curve at point A, with coordinates $\left(1, \frac{2}{3}\right)$, and at point B.

Find, in the form $\boldsymbol{r} = \begin{pmatrix} a \\ b \end{pmatrix} + s\begin{pmatrix} c \\ d \end{pmatrix}$

(i) the equation of the tangent at A;

(ii) the equation of the normal at B.

(10)

(c) Find the acute angle between the tangent at A and the normal at B.

(4)
(Total 20 marks)

96. The acceleration in m s^{-2} of a particle moving in a straight line at time t seconds, $t > 0$, is given by the formula $a = -\frac{1}{(1+t)^2}$. When $t = 1$, the velocity is 8 m s^{-1}.

(a) Find the velocity when $t = 3$.

(6)

(b) Find the limit of the velocity as $t \to \infty$.

(1)

(c) Find the exact distance travelled between $t = 1$ and $t = 3$.

(4)
(Total 11 marks)

97. If $f(x) = x - 3x^{\frac{2}{3}}$, $x > 0$,

(a) find the x-coordinate of the point P where $f'(x) = 0$;

(2)

(b) determine whether P is a maximum or minimum point.

(3)
(Total 5 marks)

98. The function f is defined by $f(x) = x\,e^{2x}$.

It can be shown that $f^{(n)}(x) = (2^n x + n\,2^{n-1})\,e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(a) By considering $f^{(n)}(x)$ for $n = 1$ and $n = 2$, show that there is one minimum point P on the graph of f, and find the coordinates of P.

(7)

(b) Show that f has a point of inflexion Q at $x = -1$.

(5)

(c) Determine the intervals on the domain of f where f is

(i) concave up;

(ii) concave down.

(2)

(d) Sketch f, clearly showing any intercepts, asymptotes and the points P and Q.

(4)

(e) Use mathematical induction to prove that $f^{(n)}(x) = (2^n x + n2^{n-1})\,e^{2x}$ for all $n \in \mathbb{Z}^+$, where $f^{(n)}(x)$ represents the n^{th} derivative of $f(x)$.

(9)
(Total 27 marks)

99. A gourmet chef is renowned for her spherical shaped soufflé. Once it is put in the oven, its volume increases at a rate proportional to its radius.

(a) Show that the radius r cm of the soufflé, at time t minutes after it has been put in the oven, satisfies the differential equation $\frac{dr}{dt} = \frac{k}{r}$, where k is a constant.

(5)

(b) Given that the radius of the soufflé is 8 cm when it goes in the oven, and 12 cm when it's cooked 30 minutes later, find, to the nearest cm, its radius after 15 minutes in the oven.

(8)
(Total 13 marks)

100. A normal to the graph of $y = \arctan(x - 1)$, for $x > 0$, has equation $y = -2x + c$, where $c \in \mathbb{R}$.

Find the value of c.

(Total 6 marks)

101. A family of cubic functions is defined as $f_k(x) = k^2x^3 - kx^2 + x$, $k \in \mathbb{Z}^+$.

(a) Express in terms of k

(i) $f'_k(x)$ and $f''_k(x)$;

(ii) the coordinates of the points of inflexion P_k on the graphs of f_k.

(6)

(b) Show that all P_k lie on a straight line and state its equation.

(2)

(c) Show that for all values of k, the tangents to the graphs of f_k at P_k are parallel, and find the equation of the tangent lines.

(5)
(Total 13 marks)

102. Consider the curve with equation $f(x) = e^{-2x^2}$ for $x < 0$.

Find the coordinates of the point of inflexion and justify that it is a point of inflexion.

(Total 7 marks)

Answers

1. $\frac{dy}{dx} = \pm\frac{5}{6}$

2. $\frac{dy}{dx} = \frac{4x}{(4x^2 - 4x^4)^{1/2}}$

3. (a) $0 < t < e^2$.

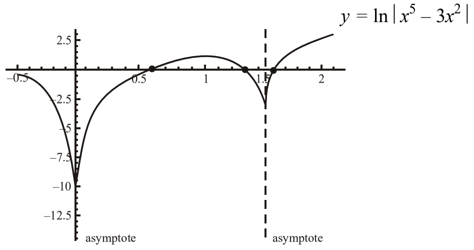
(b) $0 < t < e^{8/3}$.

(c) The required value of t is $e^{8/3}$.

(d) $t^* = e^2$

(e) $(e^2, 0)$.

4. (a) (i)



(ii) $f(x) = 0$ for $x = 0.599, 1.35, 1.51$

(b) $x = 0$ or $x = 3^{1/3}$

(c) $x = 0$ and $x = 3^{1/3}$

(d) $x = \left(\frac{6}{5}\right)^{\frac{1}{3}}$

(e) $A = \int_{0.599}^{1.35} f(x)dx$

6. $f' : x \mapsto 2x \ln x + x$

7. $\sin x = -1$ or $\frac{1}{2}$

8. $m = \pm 2$.

9. 2 : 1.

10. (b) 0.548

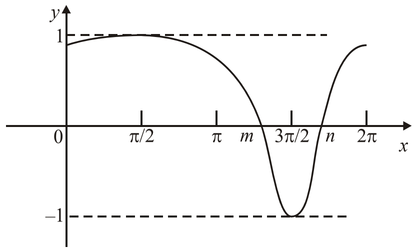
(c) 0.302

11. (a) (ii) maximum $y = 1$ and minimum $y = -1$.

(b) (i) y-intercept = 0.8

(ii) $x = 4.069, 5.356$.

(iii)



(c) Area = $\int_0^{5.356} \left|\frac{4 + 5\sin x}{5 + 4\sin x}\right| dx$

12. (a) $f'(x) = \frac{2}{2x - 1}$

(b) $x - 1.28$ (using a graphic display calculator or the quadratic formula)

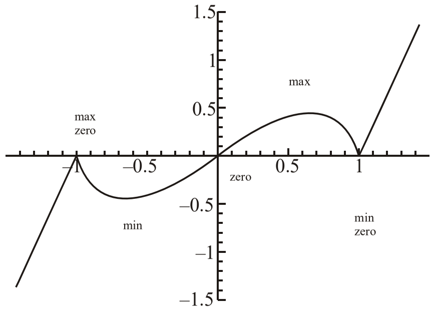
13. $x = -0.586$

14. $-\frac{3}{4}$

15. (a) $f'(x) = \pi \cos(\pi x)e^{(1+\sin \pi x)}$

(b) $x_n = \frac{2n + 1}{2}$

16. (a) $f(x) = x\left(\sqrt[3]{(x^2 - 1)^2}\right)$



(b) (i) $-1.4 \le x \le 1.4, x \ne \pm 1$

(ii) $x = -\sqrt{\frac{3}{7}}$ (or –0.655).

(c) $x = \pm 1.1339$

17. $a = -4$ and $b = 18$.

18. (a) $\frac{dy}{dx} = \sec^2 x - 8\cos x$

(b) $\cos x = \frac{1}{2}$

19. (a) $y = -4x - 8$.

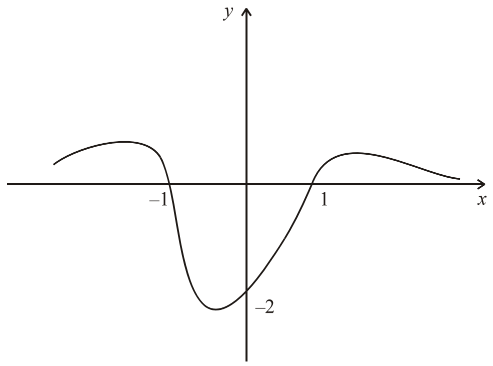
(b) (–2, 0).

21. $x + y = 2$

22. (a) (i) $\frac{2(x^2-1)}{(x^2+x+1)^2}$

(ii) $A\left(1, \frac{1}{3}\right)$ B(–1, 3) $\left(\text{or } A(-1, 3)\ B\left(1, \frac{1}{3}\right)\right)$

(b) (i)



(ii) $x = 1.53, -0.347, 1.88$

(c) (i) Range of f is $\left[\frac{1}{3}, 3\right]$.

(ii) $\left[\frac{1}{3}, \frac{7}{13}\right]$.

23. $\frac{1}{2\pi}$

24. $y - 1 = \frac{4}{3}(x - 2)$

25. (a) (ii) $\frac{x^2(\ln 2)^2 - 4x\ln 2 + 2}{2^x}$

(b) (i) $x = \frac{2}{\ln 2}$

(c) = 0.845, 4.93)

26. (a) $6\sec^2 t \tan t + 5 (= 6\tan^2 t + 6\tan t + 5)$

(b) $3 + 5\pi$; 5

27. (a) $y = -1$

(b) $\frac{dy}{dx}=\frac{4}{5}$

28. (a) $\frac{dy}{dx}=3e^{3x}\sin(\pi x)+\pi e^{3x}\cos(\pi x)$

(b) $x = 0.7426\ldots$ (0.743 to 3 sf)

29. The airplane is moving towards him at 240 km h^{-1}

30. (b) $\left(-\frac{1}{c}\ln b,\frac{a}{2b}\right)$

31. (a) $p=2$

(b) $-\frac{4}{7}$

32. 0.460

33. $\frac{d\theta}{dt}=\frac{1}{10}$

34. $\Rightarrow \frac{d^2y}{dx^2}=\frac{-4}{(2x-1)^2}$ or $-4(2x-1)^{-2}$

35. $y=-1.25x+6.5$

36. (a) $10\cos\left(5x-\frac{\pi}{2}\right)$

(b) $f(x)=-\frac{2}{5}\cos\left(5x-\frac{\pi}{2}\right)+\frac{7}{5}$

37. Gradient of normal is $\frac{5}{4}$.

38. $(-0.803,-2.08)\left(\text{or}\left(-\frac{1}{\sqrt[5]{3}},-\frac{5}{3}\sqrt[5]{3}\right)\right)$

39. (a) $k=\frac{\ln 2}{20}$

(b) 510

40. (a) $f'(x)=1+\frac{1}{(x+2)^2}$

(b) $-3<x<-1$

41. (b) (i) $\Rightarrow x=-\frac{1+\sqrt{3}}{\sqrt{3}}\left(=-\frac{\sqrt{3}+3}{3}\right)$

(ii) $\Rightarrow x=-\frac{2+\sqrt{3}}{\sqrt{3}}\left(=-\frac{2\sqrt{3}+3}{3}\right)$

(c) 8.08

42. (b) there is a **maximum** (when $\theta=0.9359$)

(c) 296 cm^2

43. $f(x)=-\frac{1}{5}x^3+\frac{12}{5}x^2-3x+2$ $\left(\text{Accept } a=-\frac{1}{5},b=\frac{12}{5},c=-3,d=2\right)$

44. (a) r^{th} term $=\binom{n}{n-r}x^r h^{n-r}\left(=\frac{n!}{r!(n-r)!}x^r h^{n-r}\right)$

45. (a) $f'(x)=-12\cos^2(4x+1)(\sin(4x+1))$

(b) $x=\frac{\pi}{8}-\frac{1}{4},x=\frac{3\pi}{8}-\frac{1}{4}$ **or** $x=\frac{\pi-1}{4}$

46. $\frac{dy}{dx}=\frac{3x^2-(\ln 3)3^{x+y}}{(\ln 3)3^{x+y}-3}$

47. (a) $f'(x)=\frac{1}{3x+1}\times 3\left(=\frac{3}{3x+1}\right)$

(b) $y=-\frac{7}{3}x+\frac{14}{3}+\ln 7$

48. (d) (ii) 6

49. (a) $\ln x$

(b) $x(\ln x)^2-2x\ln x+2x+C$

51. $\frac{dy}{dx}=\frac{1-e}{e}$

52. (a) -3

(b) 12

53. (b) $b = \sqrt{6}$

54. (a) $\frac{dy}{dx} = \frac{2-k}{2k-1}$

(b) $k = 2$

55. (a) $5\sqrt{5}\sqrt{(x^2+4)} + 5(2-x)$ (mins)

(c) (i) $x = 1$

(ii) 30 (mins)

56. $\frac{dy}{dx} = \frac{3}{2}$

57. $a = \frac{-56}{27}$

58. 1800 m.

59. (a) $f_k'(x) = \ln x + 1 - k$

(b) $x > \frac{1}{e}$

(c) (i) $x = e^{k-1}$

(ii) $(e^k, 0)$

(d) $\frac{e^{2k}}{4}$

(e) $y = x - e^k$.

60. $-2k(e^{-a/2} - 1)$

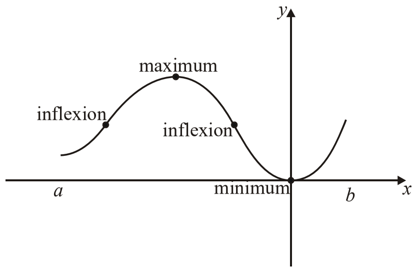
61. $\left(\frac{3}{10}, \frac{-1}{10}, 2\right)$.

62. (a) Area = $(\pi - 2x)\sin x$.

(b) Maximum Area = 1.12 units^2

63. (a) $f'(x) = \cos x \times e^{\sin x}$

65.



66. –1.6

67. The minimum distance is 1.63

68. (a) $v(t) = t\sin\left(\frac{\pi}{3}t\right) = 0$

(b) (ii) 11.5 m.

69. (a) 0.435

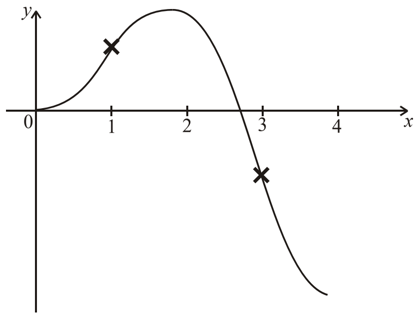
(b) $\frac{-2t}{(2+t^2)^2}$

70. (a) $2x e^{-x^2}$

(b) $\sqrt{2}e^{-\frac{1}{2}}$ (or 0.858)

\71.

✕ denotes pts of inflexion



72. 6 m.

73. **x** = –2

74. $a = 1.33$

75. (b) (i) $x = -3$ is the vertical asymptote.

(ii) x-intercept: $x = 4.39$
y-intercept: $y = -0.901$

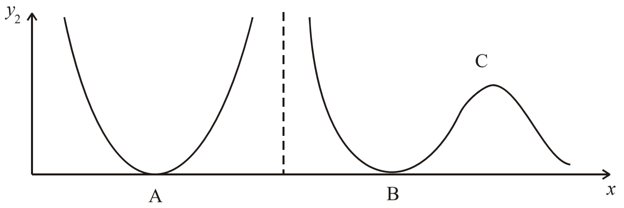
(c) $x = -1.34$ or $x = 3.05$

(d) (ii) Area of $A = \int_0^{3.05} \left(4 - (1-x)^2\right) - (\ln(x+3) - 2)\mathrm{d}x$

(e) The maximum value is 4.63.

76. Distance travelled = 0.852

77.



78. (a) $S = 2\pi r^2 + \dfrac{1000}{r}$

(b) $\Rightarrow r = \sqrt[3]{\dfrac{250}{\pi}}$ (or $r = 4.30$)

79. (a) $50 - 20t$

(b) $s = 1062.5$ m

80. $\left(\dfrac{3}{10}, \dfrac{-1}{10}, 2\right)$.

81. (a) (ii) Exact coordinates $x = \mathrm{e}$, $y = \dfrac{1}{\mathrm{e}}$

(b) $x = \mathrm{e}^{\frac{3}{2}}(4.48)$

(c) $\dfrac{1}{2}(\ln 5)^2$ (= 1.30)

(d) 1.38

82. 4/3

83. $8x + 3y - 9 = 0$

84. $\theta = \dfrac{\pi}{3}$ (Accept 60°)

85. $x = \sqrt{ab}$ (maximum)

86. $m = 5$, $m = -7$

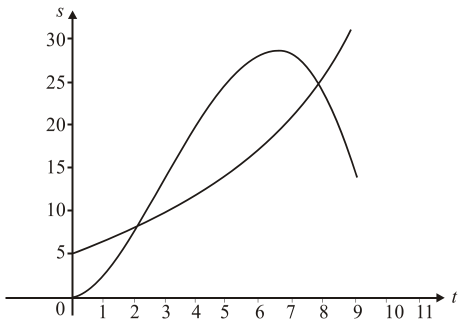
87. (a) $v_A = 6$ (m s^{-1})

(b) $a = 0.445$ (m s^{-2})

(c) $s_A = -\dfrac{1}{6}t^3 + \dfrac{3}{2}t^2 + \dfrac{3}{2}t$

$s_B = 5\mathrm{e}^{0.2t}$

(d) (i)



(ii) $t = 1.95$ and $t = 7.81$

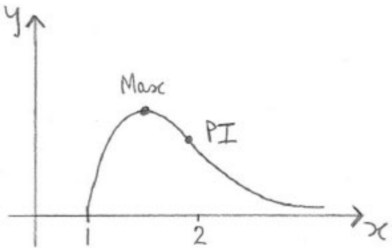
88. (a) $\frac{5}{4\pi}$ (0.398)

(b) 15

89. (a) $\frac{-7+12\ln x}{x^5}$

(b) (i) $x = e^{\frac{1}{3}}$

(iii)



(c) V= 0.0458

90. $\frac{dh}{dt}$ = 8 (cm/min)

91. T = 1.81 (sec)

92. Rate is –90 (km h^{-1})

93. (a) (ii) $x = \sqrt{2}$ (as $x > 0$)

(iii) $\theta = \arctan \frac{\sqrt{2}}{4}$ $(=0.340 \text{ radians})$.

(b) x = 0.649, 3.08 (m)

94. (a) $\frac{d^2y}{dx^2} = -2e^{-x^2} + 4x^2e^{-x^2}$

95. (b) (i) $\boldsymbol{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + s\begin{pmatrix}3\\20\end{pmatrix}$

(ii) $r = \begin{pmatrix}1\\0\end{pmatrix} + t\begin{pmatrix}6\\1\end{pmatrix}$

(c) θ = 1.26 rads or 72.0°

96. (a) $v = \frac{1}{4} + \frac{15}{2} = \frac{31}{4}\,\text{m s}^{-1}$

(b) 15/2

(c) (ln 2 + 15) metres

97. (a) 8

(b) at x = 8, $f(x)$ has a minimum.

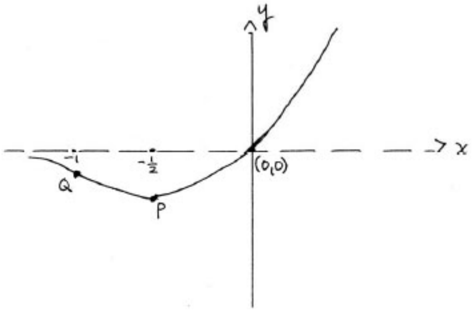
98. (a) $P\left(-\frac{1}{2}, -\frac{1}{2e}\right)$, minimum

(b) $x = -1$

(c) (i) $f(x)$ is concave up for $x > -1$.

(ii) $f(x)$ is concave down for $x < -1$.

(d)



99. (b) r ≈ 10 cm

100. $c = 4 + \frac{\pi}{4}$

101. (a) (i) $f_k'(x) = 3k^2x^2 - 2kx + 1$

$f_k''(x) = 6k^2x - 2k$

(ii) $\left(\frac{1}{3k}, \frac{7}{27k}\right)$

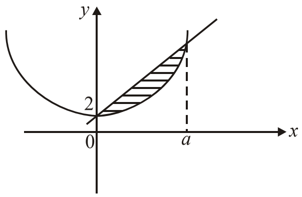
(c) $y = \frac{2}{3}x + \frac{1}{27k}$

102. $x = -0.5$.

Topic 5b: Calculus – Integration & Its Applications

1. The area of the enclosed region shown in the diagram is defined by

$y \geq x^2 + 2$, $y \leq ax + 2$, where $a > 0$.



This region is rotated 360° about the x-axis to form a solid of revolution. Find, in terms of a, the volume of this solid of revolution.

(Total 4 marks)

2. Using the substitution $u = \frac{1}{2}x + 1$, or otherwise, find the integral

$$\int x\sqrt{\frac{1}{2}x+1}\ \mathrm{d}x.$$

(Total 4 marks)

3. When air is released from an inflated balloon it is found that the rate of decrease of the volume of the balloon is proportional to the volume of the balloon. This can be represented by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv$, where v is the volume, t is the time and k is the constant of proportionality.

(a) If the initial volume of the balloon is v_0, find an expression, in terms of k, for the volume of the balloon at time t.

(b) Find an expression, in terms of k, for the time when the volume is $\frac{v_0}{2}$.

(Total 4 marks)

4. Consider the function $f : x \mapsto x - x^2$ for $-1 \leq x \leq k$, where $1 < k \leq 3$.

(a) Sketch the graph of the function f.
(3)

(b) Find the total finite area enclosed by the graph of f, the x-axis and the line $x = k$.
(4)
(Total 7 marks)

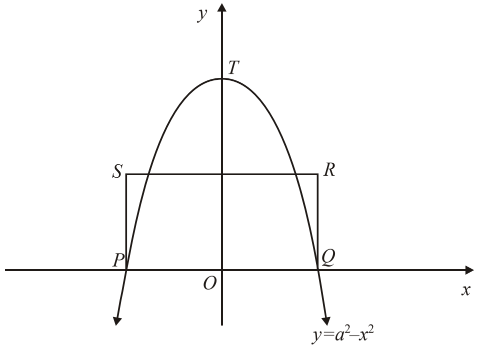
5. The area between the graph of $y = \mathrm{e}^x$ and the x-axis from $x = 0$ to $x = k$ ($k > 0$) is rotated through 360° about the x-axis. Find, in terms of k and e, the volume of the solid generated.

(Total 4 marks)

6. Find the real number $k > 1$ for which $\int_1^k \left(1+\frac{1}{x^2}\right)\mathrm{d}x = \frac{3}{2}$.

(Total 4 marks)

7. In the diagram, PTQ is an arc of the parabola $y = a^2 - x^2$, where a is a positive constant, and $PQRS$ is a rectangle. The area of the rectangle $PQRS$ is equal to the area between the arc PTQ of the parabola and the x-axis.



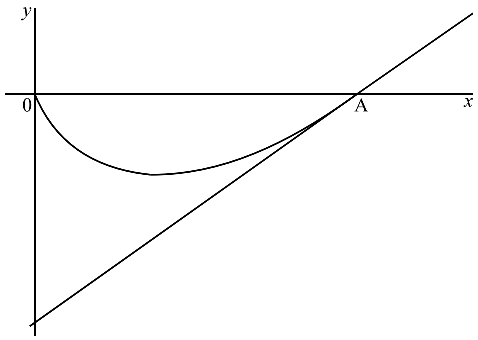
Find, in terms of a, the dimensions of the rectangle.

(Total 4 marks)

8. Consider the function $f_k(x) = \begin{cases} x\ln x - kx, & x > 0 \\ 0, & x = 0 \end{cases}$, where k ∈ ℕ

(a) Find the derivative of $f_k(x)$, $x > 0$.
(2)

(b) Find the interval over which $f_0(x)$ is increasing.
The graph of the function $f_k(x)$ is shown below.
(2)



(c) (i) Show that the stationary point of $f_k(x)$ is at $x = e^{k-1}$.

(ii) One x-intercept is at (0, 0). Find the coordinates of the other x-intercept.

(4)

(d) Find the area enclosed by the curve and the x-axis.

(5)

(e) Find the equation of the tangent to the curve at A.

(2)

(f) Show that the area of the triangular region created by the tangent and the coordinate axes is twice the area enclosed by the curve and the x-axis.

(2)

(g) Show that the x-intercepts of $f_k(x)$ for consecutive values of k form a geometric sequence.

(3)
(Total 20 marks)

9. Find the values of $a > 0$, such that $\int_a^{a^2} \frac{1}{1+x^2}\,dx = 0.22$.

(Total 3 marks)

10. Let $f(x) = \ln|x^5 - 3x^2|$, $-0{\cdot}5 < x < 2$, $x \neq a$, $x \neq b$; (a, b are values of x for which $f(x)$ is not defined).

(a) (i) Sketch the graph of $f(x)$, indicating on your sketch the number of zeros of $f(x)$. Show also the position of any asymptotes.

(2)

(ii) Find all the zeros of $f(x)$, (that is, solve $f(x) = 0$).

(3)

(b) Find the **exact** values of a and b.

(3)

(c) Find $f(x)$, and indicate clearly where $f'(x)$ is not defined.

(3)

(d) Find the **exact** value of the x-coordinate of the local maximum of $f(x)$, for $0 < x < 1.5$. (You may assume that there is no point of inflexion.)

(3)

(e) **Write down** the definite integral that represents the area of the region **enclosed** by $f(x)$ and the x-axis. (Do **not** evaluate the integral.)

(2)
(Total 16 marks)

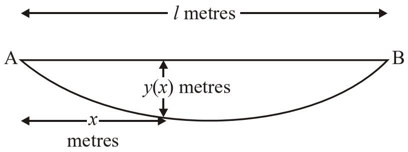
11. Calculate the area bounded by the graph of $y = x\sin(x^2)$ and the x-axis, between $x = 0$ and the smallest positive x-intercept.

(Total 3 marks)

12. Solve the differential equation $xy\frac{dy}{dx} = 1 + y^2$, given that $y = 0$ when $x = 2$.

(Total 3 marks)

13. A uniform rod of length l metres is placed with its ends on two supports A, B at the same horizontal level.



If $y(x)$ metres is the amount of sag (*ie* the distance below [AB]) at a distance x metres from support A, then it is known that

$$\frac{d^2y}{dx^2} = \frac{1}{125l^3}\left(x^2 - lx\right).$$

(a) (i) Let $z = \frac{1}{125l^3}\left(\frac{x^3}{3} - \frac{lx^2}{2}\right) + \frac{1}{1500}$. Show that $\frac{dz}{dx} = \frac{1}{125l^3}\left(x^2 - lx\right)$.

(ii) Given that $\frac{dw}{dx} = z$ and $w(0) = 0$, find $w(x)$.

(iii) Show that w satisfies $\frac{d^2w}{dx^2} = \frac{1}{125l^3}(x^2 - lx)$, and that $w(l) = w(0) = 0$.

(8)

(b) Find the sag at the centre of a rod of length 2.4 metres.

(2)
(Total 10 marks)

14. Find $\int \ln x \, dx$.

(Total 3 marks)

15. The equation of motion of a particle with mass m, subjected to a force kx can be written as $kx = mv\frac{dv}{dx}$, where x is the displacement and v is the velocity. When $x = 0$, $v = v_0$. dx Find v, in terms of v_0, k and m, when $x = 2$.

(Total 3 marks)

16. Find the value of a such that $\int_0^a \cos^2 x \, dx = 0.740$. Give your answer to 3 decimal places.

(Total 3 marks)

17. Find the area of the region enclosed by the graphs of $y = \sin x$ and $y = x^2 - 2x + 1.5$, where $0 \le x \le \pi$.

(Total 3 marks)

18. (a) Sketch and label the graphs of $f(x) = e^{-x^2}$ and $g(x) = e^{x^2} - 1$ for $0 \le x \le 1$, and shade the region A which is bounded by the graphs and the y-axis.

(3)

(b) Let the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = g(x)$ be p.

Without finding the value of p, show that

$$\frac{p}{2} < \text{area of region } A < p.$$

(4)

(c) Find the value of p correct to four decimal places.

(2)

(d) Express the area of region A as a definite integral and calculate its value.

(3)
(Total 12 marks)

19. Let $f(t) = t^{\frac{1}{3}}\left(1 - \frac{1}{2t^{\frac{5}{3}}}\right)$. Find $\int f(t) \, dt$.

(Total 3 marks)

20. Let $f : x \mapsto \frac{\sin x}{x}$, $\pi \le x \le 3\pi$. Find the area enclosed by the graph of f and the x-axis.

(Total 3 marks)

21. Find the general solution of the differential equation $\frac{dx}{dr} = kx(5 - x)$ where $0 < x < 5$, and k is a constant.

(Total 3 marks)

22. Let $f(x) = x \cos 3x$.

(a) Use integration by parts to show that

$$\int f(x) \, dx = \frac{1}{3}x \sin 3x + \frac{1}{9}\cos 3x + c.$$

(3)

(b) Use your answer to part (a) to calculate the **exact** area enclosed by $f(x)$ and the x-axis in each of the following cases. **Give your answers in terms of π.**

(i) $\frac{\pi}{6} \le x \le \frac{3\pi}{6}$

(ii) $\frac{3\pi}{6} \le x \le \frac{5\pi}{6}$

(iii) $\frac{5\pi}{6} \le x \le \frac{7\pi}{6}$.

(4)

(c) Given that the above areas are the first three terms of an arithmetic sequence, find an expression for the total area enclosed by $f(x)$ and the x-axis for $\frac{\pi}{6} \le x \le \frac{(2n+1)\pi}{6}$, where $n \in \mathbb{Z}^+$. **Give your answers in terms of *n* and π.**

(4)
(Total 11 marks)

23. A sample of radioactive material decays at a rate which is proportional to the amount of material present in the sample. Find the half-life of the material if 50 grams decay to 48 grams in 10 years.

(Total 3 marks)

24. Find the area enclosed by the curves $y = \frac{2}{1+x^2}$ and $y = e^{\frac{x}{3}}$, given that $-3 \le x \le 3$.

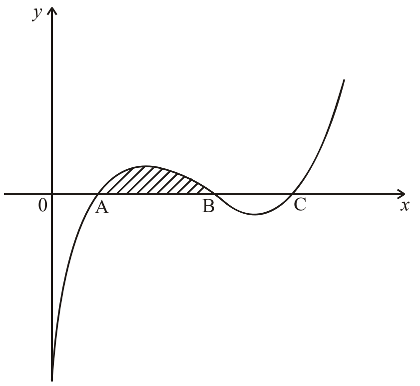
(Total 3 marks)

25. (a) Use integration by parts to find $\int x^2 \ln x \, dx$.

(b) Evaluate $\int_1^2 x^2 \ln x dx$

(Total 6 marks)

26. The figure below shows part of the curve $y = x^3 - 7x^2 + 14x - 7$. The curve crosses the x-axis at the points A, B and C.



(a) Find the x-coordinate of A.

(b) Find the x-coordinate of B.

(c) Find the area of the shaded region.

(Total 6 marks)

27. Find $\int (\theta \cos \theta - \theta) d\theta$.

(Total 6 marks)

28. The tangent to the curve $y = f(x)$ at the point $P(x, y)$ meets the x-axis at $Q(x - 1, 0)$. The curve meets the y-axis at $R(0, 2)$. Find the equation of the curve.

(Total 6 marks)

29. (a) On the same axes sketch the graphs of the functions, $f(x)$ and $g(x)$, where

$f(x) = 4 - (1 - x)^2$, for $-2 \leq x \leq 4$,

$g(x) = \ln(x + 3) - 2$, for $-3 \leq x \leq 5$.

(2)

(b) (i) Write down the equation of any vertical asymptotes.

(ii) State the x-intercept and y-intercept of $g(x)$.

(3)

(c) Find the values of x for which $f(x) = g(x)$.

(2)

(d) Let A be the region where $f(x) \geq g(x)$ and $x \geq 0$.

(i) On your graph shade the region A.

(ii) Write down an integral that represents the area of A.

(iii) Evaluate this integral.

(4)

(e) In the region A find the maximum vertical distance between $f(x)$ and $g(x)$.

(3)
(Total 14 marks)

30. Using the substitution $y = 2 - x$, or otherwise, find $\int \left(\frac{x}{2 - x}\right)^2 dx$.

(Total 6 marks)

31. The function f with domain $\left[0, \frac{\pi}{2}\right]$ is defined by $f(x) = \cos x + \sqrt{3} \sin x$.

This function may also be expressed in the form $R \cos(x - \alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(a) Find the **exact** value of R and of α.

(3)

(b) (i) Find the range of the function f.

(ii) State, giving a reason, whether or not the inverse function of f exists.

(5)

(c) Find the **exact** value of x satisfying the equation $f(x) = \sqrt{2}$.

(3)

(d) Using the result

$$\int \sec x dx = \ln |\sec x + \tan x| + C, \text{ where } C \text{ is a constant,}$$

show that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{f(x)} = \frac{1}{2} \ln(3 + 2\sqrt{3}).$$

(5)
(Total 16 marks)

32. Calculate the area enclosed by the curves $y = \ln x$ and $y = e^x - e$, $x > 0$.

(Total 6 marks)

33. Given that $\frac{dy}{dx} = e^x - 2x$ and $y = 3$ when $x = 0$, find an expression for y in terms of x.

(Total 6 marks)

34. (a) Find $\int_0^m \frac{dx}{2x+3}$, giving your answer in terms of m.

(b) Given that $\int_0^m \frac{dx}{2x+3} = 1$, calculate the value of m.

(Total 6 marks)

35. Find $\int \frac{\ln x}{\sqrt{x}} dx$.

(Total 6 marks)

36. The temperature T °C of an object in a room, after t minutes, satisfies the differential equation

$$\frac{dT}{dt} = k(T - 22), \text{ where } k \text{ is a constant.}$$

(a) Solve this equation to show that $T = Ae^{kt} + 22$, where A is a constant.

(3)

(b) When $t = 0$, $T = 100$, and when $t = 15$, $T = 70$.

(i) Use this information to find the value of A and of k.

(ii) Hence find the value of t when $T = 40$.

(7)
(Total 10 marks)

37. Find the total area of the two regions enclosed by the curve $y = x^3 - 3x^2 - 9x + 27$ and the line $y = x + 3$.

(Total 6 marks)

38. Using the substitution $2x = \sin \theta$, or otherwise, find $\int \left(\sqrt{1-4x^2}\right)dx$.

(Total 6 marks)

39. Consider the complex number $z = \cos\theta + i \sin\theta$.

(a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2 \cos n\theta.$$

(2)

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4 \cos 2\theta + 3).$$

(4)

(c) Let $g(a) = \int_0^a \cos^4 \theta d\theta$.

(i) Find $g(a)$.

(ii) Solve $g(a) = 1$

(5)
(Total 11 marks)

40. Consider the differential equation $\frac{dy}{d\theta} = \frac{y}{e^{2\theta} + 1}$.

(a) Use the substitution $x = e^{\theta}$ to show that

$$\int \frac{dy}{y} = \int \frac{dx}{x(x^2+1)}.$$

(3)

(b) Find $\int \frac{dx}{x(x^2+1)}$.

(4)

(c) Hence find y in terms of θ, if $y = \sqrt{2}$ when $\theta = 0$.

(4)
(Total 11 marks)

41. The function f' is given by $f'(x) = 2\sin\left(5x - \frac{\pi}{2}\right)$.

(a) Write down $f''(x)$.

(b) Given that $f\left(\frac{\pi}{2}\right) = 1$, find $f(x)$.

(Total 6 marks)

42. Use the substitution $u = x + 2$ to find $\int \frac{x^3}{(x+2)^2} dx$.

(Total 6 marks)

43. Solve the differential equation $x\frac{dy}{dx} - y^2 = 1$, given that $y = 0$ when $x = 2$. Give your answer in the form $y = f(x)$.

(Total 6 marks)

***44.** (a) Express as partial fractions $\frac{2x+4}{(x^2+4)(x-2)}$.

(b) Hence or otherwise, find $\int \frac{2x+4}{(x^2+4)(x-2)}\,\mathrm{d}x$.

(Total 6 marks)

45. The function f is defined by $f(x) = \mathrm{e}^{px}(x+1)$, here $p \in \mathbb{R}$.

(a) (i) Show that $f'(x) = \mathrm{e}^{px}(p(x+1)+1)$.

(ii) Let $f^{(n)}(x)$ denote the result of differentiating $f(x)$ with respect to x, n times. Use mathematical induction to prove that

$$f^{(n)}(x) = p^{n-1}\mathrm{e}^{px}(p(x+1)+n),\ n \in \mathbb{Z}^+.$$

(7)

(b) When $p = \sqrt{3}$, there is a minimum point and a point of inflexion on the graph of f. Find the **exact** value of the x-coordinate of

(i) the minimum point;

(ii) the point of inflexion.

(4)

(c) Let $p = \frac{1}{2}$. Let R be the region enclosed by the curve, the x-axis and the lines $x = -2$ and $x = 2$. Find the area of R.

(2)
(Total 13 marks)

46. Find $\int \mathrm{e}^{x} \cos x\,\mathrm{d}x$.

(Total 6 marks)

47. (a) Given that $\frac{x^2}{(1+x)(1+x^2)} \equiv \frac{a}{(1+x)} + \frac{bx+c}{(1+x^2)}$, calculate the value of a, of b and of c.

(5)

(b) (i) Hence, find $I = \int \frac{x^2}{(1+x)(1+x^2)}\mathrm{d}x$.

(ii) If $I = \frac{\pi}{4}$ when $x = 1$, calculate the value of the constant of integration giving your answer in the form $p + q \ln r$ where $p, q, r \in \mathbb{R}$

(7)
(Total 12 marks)

48. Let $f(x) = 2^{0.5x}$ and $g(x) = 3^{-0.5x} + \frac{5}{3}$. Let R be the region completely enclosed by the graphs of f and g, and the y-axis. Find the area of R.

(Total 6 marks)

49. Find $\int \mathrm{e}^{2x} \sin x\,\mathrm{d}x$.

(Total 6 marks)

50. Solve the differential equation

$$(x+2)^2\frac{\mathrm{d}y}{\mathrm{d}x} = 4xy \quad (x > -2)$$

given that $y = 1$ when $x = -1$.

(Total 6 marks)

51. The region enclosed by the curves $y^2 = kx$ and $x^2 = ky$, where $k > 0$, is denoted by R. Given that the area of R is 12, find the value of k.

(Total 6 marks)

42. The function f is defined by $f(x) = \frac{\ln x}{x^3}$, $x \geq 1$.

(a) Find $f'(x)$ and $f''(x)$, simplifying your answers.

(6)

(b) (i) Find the **exact** value of the x-coordinate of the maximum point and justify that this is a maximum.

(ii) Solve $f''(x) = 0$, and show that at this value of x, there is a point of inflexion on the graph of f.

(iii) Sketch the graph of f, indicating the maximum point and the point of inflexion.

(11)

The region enclosed by the x-axis, the graph of f and the line $x = 3$ is denoted by R.

(c) Find the volume of the solid of revolution obtained when R is rotated through 360° about the x-axis.

(3)

(d) Show that the area of R is $\frac{1}{18}(4 - \ln 3)$.

(6)
(Total 26 marks)

53. Let $y = \cos\theta + \mathrm{i}\sin\theta$.

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \mathrm{i}y$.

[You may assume that for the purposes of differentiation and integration, i may be treated in the same way as a real constant.]

(3)

(b) **Hence** show, using integration, that $y = \mathrm{e}^{\mathrm{i}\theta}$.

(5)

(c) Use this result to deduce de Moivre's theorem. (assume (b) is true)

(2)

(d) (i) Given that $\frac{\sin 6\theta}{\sin\theta} = a\cos^5\theta + b\cos^3\theta + c\cos\theta$, where $\sin\theta \neq 0$, use de Moivre's theorem with $n = 6$ to find the values of the constants a, b and c.

(ii) Hence deduce the value of $\lim\limits_{\theta\to 0}\frac{\sin 6\theta}{\sin\theta}$.

(10)
(Total 20 marks)

54. Let $f(x) = x\ln x - x, x > 0$.

(a) Find $f'(x)$.

(b) Using integration by parts find $\int(\ln x)^2\,\mathrm{d}x$.

(Total 6 marks)

55. The function f is defined as $f(x) = \sin x\ln x$ for $x \in [0.5, 3.5]$.

(a) Write down the x-intercepts.

(b) The area above the x-axis is A and the **total** area below the x-axis is B. If $A = kB$, find k.

(Total 6 marks)

56. Solve the differential equation $(x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} - xy = 0$ where $x > 0, y > 0$, given that $y = 1$ when $x = 1$.

(Total 6 marks)

57. Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$ given that $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$.

(Total 6 marks)

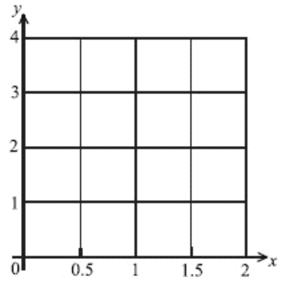
58. The graph of $y = \sin(3x)$ for $0 \le x \le \frac{\pi}{4}$ is is rotated through 2π radians about the x-axis.

Find the **exact** volume of the solid of revolution formed.

(Total 6 marks)

59. For $x \ge \frac{1}{2}$, let $f(x) = x^2\ln(x + 1)$ and $g(x) = \sqrt{2x - 1}$.

(a) Sketch the graphs of f and g on the grid below.



(b) Let A be the region completely enclosed by the graphs of f and g.
Find the area of A.

(Total 6 marks)

60. Find $\int_0^{\ln 3}\frac{e^x}{e^{2x} + 9}\,\mathrm{d}x$, expressing your answer in **exact** form.

(Total 6 marks)

61. (a) Using the formula for $\cos(A + B)$ prove that $\cos^2\theta = \frac{\cos 2\theta + 1}{2}$.

(3)

(b) Hence, find $\int\cos^2 x\,\mathrm{d}x$.

(4)

Let $f(x) = \cos x$ and $g(x) = \sec x$ for $x \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$.

Let R be the region enclosed by the two functions.

(c) Find the **exact** values of the x-coordinates of the points of intersection.

(4)

(d) Sketch the functions f and g and clearly shade the region R.

(3)

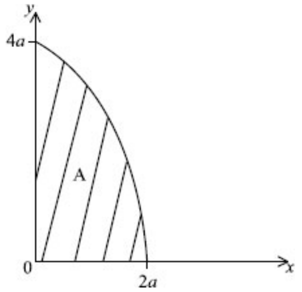
The region R is rotated through 2π about the x-axis to generate a solid.

(e) (i) Write down an integral which represents the volume of this solid.

(ii) Hence find the **exact** value of the volume.

(10)
(Total 24 marks)

63. The diagram below shows the shaded region A which is bounded by the axes and part of the curve $y^2 = 8a(2a - x)$, $a > 0$. Find in terms of a the volume of the solid formed when A is rotated through 360° around the x-axis.



(Total 6 marks)

64. Find $\int_0^a \arcsin x\,dx, 0 < a < 1$.

(Total 6 marks)

65. Solve the differential equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, given that $y = \sqrt{3}$ when $x = \frac{\sqrt{3}}{3}$.

Give your answer in the form $y = \frac{ax+\sqrt{a}}{a-x\sqrt{a}}$ where $a \in \mathbb{Z}^+$.

(Total 6 marks)

66. Find the area between the curves $y = 2 + x - x^2$ and $y = 2 - 3x + x^2$.

(Total 7 marks)

67. The region bounded by the curve $y = \frac{ln(x)}{x}$ and the lines $x = 1, x = e, y = 0$ is rotated

through 2π radians about the x-axis.

Find the volume of the solid generated.

(Total 12 marks)

68. Show that $\int_0^{\frac{\pi}{6}} x \, sin \, 2x \, dx = \frac{\sqrt{3}}{8} - \frac{\pi}{24}$.

(Total 6 marks)

69. By using an appropriate substitution find

$$\int \frac{\tan(\ln y)}{y} dy, y > 0.$$

(Total 6 marks)

70. The curve $y = e^{-x} - x + 1$ intersects the x-axis at P.

(a) Find the x-coordinate of P.

(2)

(b) Find the area of the region completely enclosed by the curve and the coordinate axes.

(3)
(Total 5 marks)

71. A particle moves in a straight line in a positive direction from a fixed point O.

The velocity v m s^{-1}, at time t seconds, where $t \geq 0$, satisfies the differential equation

$$\frac{dv}{dt} = \frac{-v(1+v^2)}{50}.$$

The particle starts from O with an initial velocity of 10 m s^{-1}.

(a) (i) Express as a definite integral, the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(ii) **Hence** calculate the time taken for the particle's velocity to decrease from 10 m s^{-1} to 5 m s^{-1}.

(5)

(b) (i) Show that, when $v > 0$, the motion of this particle can also be described by the differential equation $\frac{dv}{dx} = \frac{-(1+v^2)}{50}$ where x metres is the displacement from O.

(ii) Given that $v = 10$ when $x = 0$, solve the differential equation expressing x in terms of v.

(iii) **Hence** show that $v = \frac{10 - \tan\frac{x}{50}}{1 + 10\tan\frac{x}{50}}$.

(14)
(Total 19 marks)

72. (a) Using l'Hopital's Rule, show that $\lim_{x\to\infty} xe^{-x} = 0$.

(2)

(b) Determine $\int_0^a xe^{-x}\,dx$.

(5)

(c) Show that the integral $\int_0^\infty xe^{-x}\,dx$ is convergent and find its value.

(2)
(Total 9 marks)

Answers

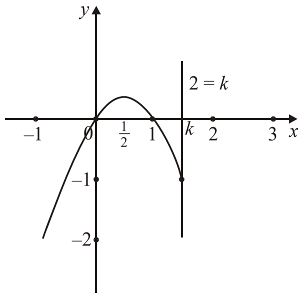
1. $\frac{2a^3\pi}{15}(a^2+5)$

2. $\frac{8}{15}\left(\frac{1}{2}x+1\right)^{3/2}\left(\frac{3}{2}x-2\right)+C$

3. (a) $v = v_0e^{-kt}$

(b) $t = \frac{\ln 2}{k}$

4. (a)



(b) $\frac{k^3}{3}-\frac{k^2}{2}+\frac{1}{6}$

5. $\frac{\pi}{2}(e^{2k}-1)$

6. $k = 2$

7. $2a \times \frac{2}{3}a^2.$

8. (a) $f_k'(x) = \ln x + 1 - k$

(b) $x > \frac{1}{e}$

(c) (i) $x = e^{k-1}$

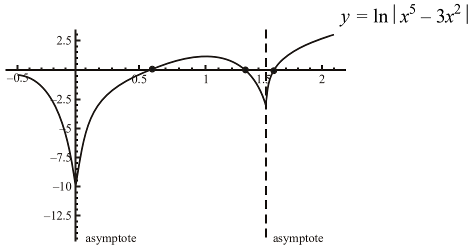
(ii) $(e^k, 0)$

(d) $\frac{e^{2k}}{4}$

(e) $y = x - e^k.$

9. $a = 2.04$ or $a = 2.62$

10. (a) (i)



(ii) $x = 0.599, 1.35, 1.51$

(b) $x = 0$ or $x = 3^{1/3}$

(c) $x = 0$ and $x = 3^{1/3}$

(d) $x = \left(\frac{6}{5}\right)^{\frac{1}{3}}$

(e) $A = \int_{0.599}^{1.35} f(x)\,dx$

11. l

12. $x^2 - 4y^2 = 4.$

13. (a) (ii) $w(x) = \frac{1}{125l^3}\left(\frac{x^4}{12}-\frac{lx^3}{6}\right)+\frac{x}{1500}$

(iii) 0

(b) 0.0005 m.

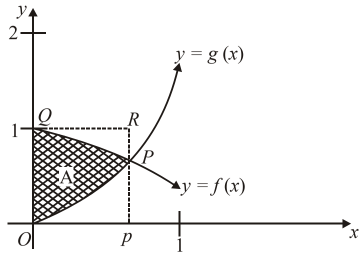
14. $x\ln x - x + C$

15. $v = \sqrt{v_0^2 + \frac{4k}{m}}$

16. $a = 1.047$

17. 0.271 units^2

18. (a)



(c) $p = 0.6937$

(d) 0.467

19. $\frac{3}{4}t^{\frac{4}{3}} + \frac{3}{2}t^{-\frac{1}{3}} + C$

20. 0.690 units^2

21. $\left(\frac{x}{5-x}\right)^{\frac{1}{5}} = Ae^{kt}$ **or** $\left(\frac{x}{5-x}\right) = Ae^{5kt}$

22. (b) (i) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{\pi}{6}}^{\frac{3\pi}{6}}\right| = \frac{2\pi}{9}$

(ii) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{3\pi}{6}}^{\frac{5\pi}{6}}\right| = \frac{4\pi}{9}$

(iii) Area = $\left|\left[\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}\right| = \frac{6\pi}{9}$

(c) $\frac{n\pi}{9}(n+1)$

23. half-life = 170 years

24. 1.22.

25. (a) $\frac{x^3}{3}\ln x - \frac{x^3}{9}$

(b) 1.07

26. (a) $x = 0.753$

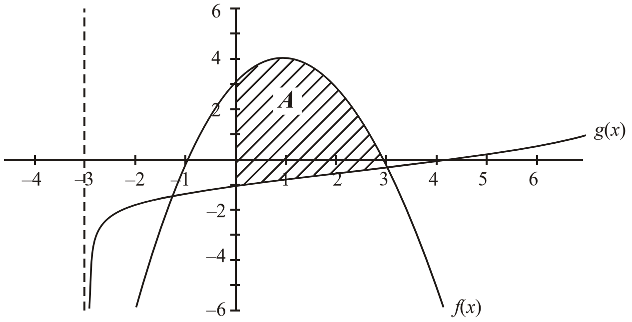
(b) $x = 2.45$

(c) 1.78

27. $\theta\sin\theta + \cos\theta - \frac{\theta^2}{2} + c$

28. $y = 2e^x$

29. (a)



(b) (i) $x = -3$ is the vertical asymptote.

(ii) x-intercept: $x = 4.39$ ($= e^2 - 3$)
y-intercept: $y = -0.901$ ($= \ln 3 - 2$)

(c) $x = -1.34$ or $x = 3.05$

(d) (ii) Area of $A = \int_0^{3.05}\left(4-(1-x)^2\right) - (\ln(x+3) - 2)dx$

(iii) 10.6

(e) The maximum value is 4.63.

30. $\frac{4}{2-x} + 4\ln|2-x| - (2-x) + c$

31. (a) $R = 2, \alpha = \frac{\pi}{3}$

(b) (i) [1, 2]

(c) $x = \frac{\pi}{12}$

(d) $\frac{1}{2}\ \ln(3+2\sqrt{3})$.

32. 0.201

33. $y = e^x - x^2 + 2$

34. (a) $\frac{1}{2}\ln\left|\frac{2m+3}{3}\right| \left(\text{or } \frac{1}{2}\ln|2m+3| - \frac{1}{2}\ln 3\right)$

(b) $m = \frac{3}{2}(e^2 - 1)$ (= 9.58)

35. $2x^{\frac{1}{2}}\ln x - 4x^{\frac{1}{2}} + C$

36. (b) (i) $k = \frac{1}{15}\ln\frac{48}{78}$ (= –0.0324)

(ii) $t = -\frac{1}{0.0324}\ln\frac{18}{78}$ (= 45.3)

37. 101.75

38. $\int\left(\sqrt{1-4x^2}\right)dx = \frac{1}{4}\left[2x\sqrt{1-4x^2} + \arcsin 2x\right] + C$

39. (c) (i) $g(a) = \frac{1}{8}\left(\frac{1}{4}\sin 4a + 2\sin 2a + 3a\right)$

(ii) $a = 2.96$

40. (b) $= \ln x - \frac{1}{2}\ln(x^2+1) + C$

(c) $y = \frac{2e^{\theta}}{\sqrt{e^{2\theta}+1}}$

41. (a) $10\cos\left(5x - \frac{\pi}{2}\right)$

(b) $f(x) = -\frac{2}{5}\cos\left(5x - \frac{\pi}{2}\right) + \frac{7}{5}$

42. $= \frac{(x+2)^2}{2} - 6(x+2) + 12\ln|x+2| + \frac{8}{x+2} + c$

43. $y = \tan\left(\ln\frac{x}{2}\right)$

44. (a) $\left(-\frac{x}{x^2+4} + \frac{1}{x-2}\right)$

(b) $\left(= \ln\frac{A(x-2)}{\sqrt{x^2+4}}\right)$

45. (b) (i) $\Rightarrow x = -\frac{1+\sqrt{3}}{\sqrt{3}} \left(= -\frac{\sqrt{3}+3}{3}\right)$

(ii) $\Rightarrow x = -\frac{2+\sqrt{3}}{\sqrt{3}} \left(= -\frac{2\sqrt{3}+3}{3}\right)$

(c) 8.08

46. $\int e^x \cos x\, dx = \frac{e^x}{2}(\cos x + \sin x) + k$

47. (a) $a = \frac{1}{2} \Rightarrow b = \frac{1}{2}, c = -\frac{1}{2}$.

(b) (i) $\frac{1}{2}\ln|1+x| + \frac{1}{4}\ln|1+x^2| - \frac{1}{2}\arctan x + k$

(ii) $\frac{3\pi}{8} - \frac{3}{4}\ln 2 = k \quad \left(\textit{accept } p = \frac{3\pi}{8}, q = -\frac{3}{4}, r = 2\right)$

48. Area = 1.66

49. $\int e^{2x}\sin x\, dx = \frac{e^{2x}}{5}(2\sin x - \cos x) + C$

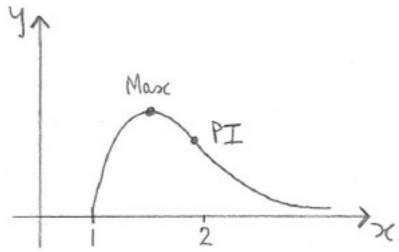
50. $\ln y = 4\ln(x+2) + \frac{8}{x+2} - 8$

51. $k = 6$

52. (a) $f'(x) = \frac{x^3 \times \frac{1}{x} - 3x^2 \ln x}{x^6}$; $f''(x) = \frac{-\frac{3}{x} \times x^4 - 4x^3(1 - 3\ln x)}{x^8} = \frac{-7 + 12\ln x}{x^5}$

(b) (i) $x = e^{\frac{1}{3}}$

(iii)



(c) 0.0458

53. (d) (ii) 6

54. (a) $\ln x$

(b) $x(\ln x)^2 - 2x\ln x + 2x + C$

55. (a) x-intercepts are $x = 1$ and $x = \pi$

(b) $k = 5.17$

56. $y=\sqrt{\dfrac{x^2+1}{2}}$

57. $y=-\dfrac{1}{x^2-1}$ $\left(=\dfrac{1}{1-x^2}\right)$

58. $\dfrac{\pi}{2}\left(\dfrac{\pi}{4}+\dfrac{1}{6}\right)$ $\left(=\dfrac{\pi^2}{8}+\dfrac{\pi}{12}\right)$

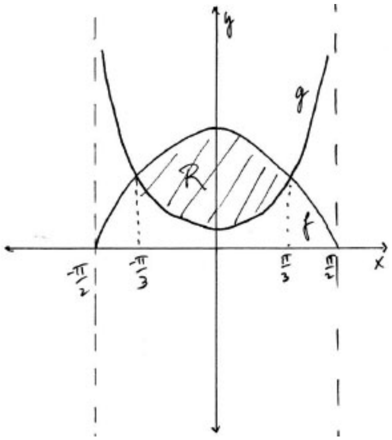
59. (b) $A = 0.201$

60. $\dfrac{1}{3}\left(\arctan 1-\arctan\dfrac{1}{3}\right)$ $\left(=\dfrac{\pi}{12}-\dfrac{1}{3}\arctan\dfrac{1}{3},\dfrac{1}{3}\arctan\dfrac{1}{2}\right)$

61. (b) $\int\cos^2 x\mathrm{d}x=\dfrac{1}{4}\sin 2x+\dfrac{1}{2}x+C$

(c) $x=\pm\dfrac{\pi}{3}$

(d)



(e) (i) $\pi\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(16\cos^2 x-\sec^2 x\right)\mathrm{d}x$

(ii) $2\pi\left(\sqrt{3}+\dfrac{8\pi}{3}\right)$

63. $16\pi a^3$

64. $a\arcsin a+\sqrt{1-a^2}-1$

65. $y=\dfrac{3x+\sqrt{3}}{3-x\sqrt{3}}$

66. $\dfrac{8}{3}\left(=2\dfrac{2}{3}\right)$

67. $2\pi-\dfrac{5\pi}{\mathrm{e}}$

69. $\int\dfrac{\tan(\ln y)}{y}\mathrm{d}y=\ln|\sec(\ln y)|+c$

70. (a) $x = 1.28$

(b) 1.18

71. (a) (i) $t = -50\int_{10}^{5} \frac{1}{v\left(1+v^2\right)} \mathrm{d}v \quad \left(= 50\int_{5}^{10} \frac{1}{v\left(1+v^2\right)} \mathrm{d}v\right)$

(ii) $t = 0.732\,(\text{sec}) \quad \left(= 25\ln\frac{104}{101}(\text{sec})\right)$

(b) (ii) $x = 50(\arctan 10 - \arctan v)$

72. (b) $1 - ae^{-a} - e^{-a}$

(c) 1

Topic 5c : Calculus – Maclaurin's Series, l'Hopital's Rule, and Differential Equations

1. Consider the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}=y+\sqrt{x^2-y^2},\ x>0,\ x^2>y^2.$$

(a) Show that this is a homogeneous differential equation. [1]

(b) Find the general solution, giving your answer in the form $y=f(x)$. [7]

2. (a) The function f is defined by $f(x)=\arcsin(2x)$, where $-\frac{1}{2}\leq x\leq\frac{1}{2}$.

By finding a suitable number of derivatives of f, find the first two non-zero terms in the Maclaurin series for f. [8]

(b) Hence or otherwise, find $\lim\limits_{x\to 0}\dfrac{\arcsin(2x)-2x}{(2x)^3}$. [3]

3. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4x^2+y^2-xy}{x^2}$, with $y=2$ when $x=1$.

(a) Use Euler's method, with step length $h=0.1$, to find an approximate value of y when $x=1.4$. [5]

(b) (i) Express m^2-2m+4 in the form $(m-a)^2+b$, where $a,\ b\in\mathbb{Z}$. [1]

(ii) Solve the differential equation, for $x>0$, giving your answer in the form $y=f(x)$. [10]

4. (a) A simple model to predict the population of the world is set up as follows. At time t years the population of the world is x, which can be assumed to be a continuous variable. The rate of increase of x due to births is $0.056x$ and the rate of decrease of x due to deaths is $0.035x$.

Show that $\frac{\mathrm{d}x}{\mathrm{d}t}=0.021x$. [1]

(b) Find a prediction for the number of years it will take for the population of the world to double. [6]

5. Using L'Hôpital's rule, find $\lim\limits_{x\to 0}\left(\frac{\tan 3x-3\tan x}{\sin 3x-3\sin x}\right)$. [9]

6. (a) Consider the differential equation $2xy\frac{\mathrm{d}y}{\mathrm{d}x}=y^2-x^2$, where $x>0$.

Solve the differential equation and show that a general solution is $x^2+y^2=cx$ where c is a positive constant. [11]

(b) Prove that there are two horizontal tangents to the general solution curve and state their equations, in terms of c. [5]

7. (a) Use L'Hôpital's rule to determine the value of

$$\lim_{x\to 0}\left(\frac{\mathrm{e}^{-3x^2}+3\cos(2x)-4}{3x^2}\right)$$

[5]

(b) Hence find $\lim\limits_{x\to 0}\left(\frac{\int_0^x\left(\mathrm{e}^{-3t^2}+3\cos(2t)-4\right)\mathrm{d}t}{\int_0^x 3t^2\,\mathrm{d}t}\right)$. [3]

8. (a) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x}=1+\frac{y}{x}$, where $x\neq 0$.

Given that $y(1)=1$, use Euler's method with step length $h=0.25$ to find an approximation for $y(2)$. Give your answer to two significant figures. [4]

(b) Solve the equation $\frac{\mathrm{d}y}{\mathrm{d}x}=1+\frac{y}{x}$ for $y(1)=1$. [6]

(c) Find the percentage error when $y(2)$ is approximated by the final rounded value found in part (a). Give your answer to two significant figures. [3]

9. A differential equation is given by $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{y}{x}$, where $x>0$ and $y>0$.

(a) Solve this differential equation by separating the variables, giving your answer in the form $y=f(x)$. [3]

(b) Solve the same differential equation by using the standard homogeneous substitution $y=vx$. [4]

(c) Solve the same differential equation by the use of an integrating factor. [5]

(d) If $y=20$ when $x=2$, find y when $x=5$. [1]

10. The function f is defined by $f(x) = (\arcsin x)^2,\ -1 \le x \le 1$.

The function f satisfies the equation $(1-x^2)f''(x) - xf'(x) - 2 = 0$.

(a) Show that $f'(0) = 0$. [2]

(b) By differentiating the above equation twice, show that

$$(1-x^2)f^{(4)}(x) - 5xf^{(3)}(x) - 4f''(x) = 0$$

where $f^{(3)}(x)$ and $f^{(4)}(x)$ denote the 3rd and 4th derivative of $f(x)$ respectively. [4]

(c) Hence show that the Maclaurin series for $f(x)$ up to and including the term in x^4 is

$x^2 + \frac{1}{3}x^4$. [3]

(d) Use this series approximation for $f(x)$ with $x = \frac{1}{2}$ to find an approximate value for π^2. [2]

11. Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} - y = x^p + 1$ where $x \in \mathbb{R},\ x \neq 0$ and p is a positive integer, $p > 1$.

(a) Solve the differential equation given that $y = -1$ when $x = 1$. Give your answer in the form $y = f(x)$. [8]

(b) (i) Show that the x-coordinate(s) of the points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ satisfy the equation $x^{p-1} = \frac{1}{p}$. [2]

(ii) Deduce the set of values for p such that there are two points on the curve $y = f(x)$ where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. Give a reason for your answer. [2]

12. (a) Find the first three terms of the Maclaurin series for $\ln(1+\mathrm{e}^x)$. [6]

(b) Hence, or otherwise, determine the value of $\lim\limits_{x\to 0}\frac{2\ln(1+\mathrm{e}^x)-x-\ln 4}{x^2}$. [4]

13. Use l'Hôpital's rule to determine the value of [7]

$$\lim_{x\to 0}\frac{\sin^2 x}{x\ln(1+x)}.$$

14. Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\frac{2x}{1+x^2}\right)y = x^2$, given that $y = 2$ when $x = 0$.

(a) Show that $1 + x^2$ is an integrating factor for this differential equation. [5]

(b) Hence solve this differential equation. Give the answer in the form $y = f(x)$. [6]

15. The curves $y = f(x)$ and $y = g(x)$ both pass through the point $(1,\ 0)$ and are defined by the differential equations $\frac{\mathrm{d}y}{\mathrm{d}x} = x - y^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x} = y - x^2$ respectively.

(a) Show that the tangent to the curve $y = f(x)$ at the point $(1,\ 0)$ is normal to the curve $y = g(x)$ at the point $(1,\ 0)$. [2]

(b) Find $g(x)$. [6]

(c) Use Euler's method with steps of 0.2 to estimate $f(2)$ to 5 decimal places. [5]

16. The function f is defined by $f(x) = \mathrm{e}^{-x}\cos x + x - 1$.

By finding a suitable number of derivatives of f, determine the first non-zero term in its Maclaurin series. [7]

17. (a) Show that $y = \frac{1}{x}\int f(x)\mathrm{d}x$ is a solution of the differential equation

$x\frac{\mathrm{d}y}{\mathrm{d}x} + y = f(x),\ x > 0.$ [3]

(b) Hence solve $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = x^{-\frac{1}{2}},\ x > 0$, given that $y = 2$ when $x = 4$. [5]

Answers:

1. (b) $y = x\sin(\ln x + C)$	2(a) $f(x) = 2x + \frac{8x^3}{6} + \cdots$ (b) $\frac{1}{6}$
3(a) $y(1.4) \approx 5.34$ (c) $(m-1)^2 + 3$ (b)(i) $a = 1, b = 3$ (b)(ii) $y = x\left(\sqrt{3}\tan\left(\sqrt{3}\ln x + \frac{\pi}{6}\right) + 1\right)$	4(b) 33 (accept 34)
5. -2	7(a) -3 (b) -3
8(a) 3.3 (b) $y = x\ln\lvert x\rvert + x$ (c) 2.5%	9(a)-(c) $y = kx$ (d) 50
10(d) 9.75 $\left(\simeq \frac{39}{4}\right)$	11(a) $y = \frac{1}{p-1}(x^p - x) - 1$ (b)(ii) p is odd
12(a) $\ln 2 + \frac{1}{2}x + \frac{1}{8}x^2 + \cdots$ (b) $\frac{1}{4}$	13. 1
14(b) $y = \frac{1}{1+x^2}\left(\frac{x^3}{3} + \frac{x^5}{5} + 2\right)$	15(b) $g(x) = x^2 + 2x + 2 - 5\mathrm{e}^{x-1}$ (c) 1.10033
16. $\frac{2x^3}{3!}$ $\left(= \frac{x^3}{3}\right)$	17(b) $y = \frac{1}{x}\left(2x^{\frac{1}{2}} + 4\right)$

Past IB Questions

Topic 1: Number and Algebra – Counting Principles (Permutations & Combinations)

1. Mr Blue, Mr Black, Mr Green, Mrs White, Mrs Yellow and Mrs Red sit around a circular table for a meeting. Mr Black and Mrs White must not sit together.

Calculate the number of different ways these six people can sit at the table without Mr Black and Mrs White sitting together.

(Total 3 marks)

2. In how many ways can six different coins be divided between two students so that each student receives at least one coin?

(Total 3 marks)

3. How many four-digit numbers are there which contain at least one digit 3?

(Total 3 marks)

4. A committee of four children is chosen from eight children. The two oldest children cannot both be chosen. Find the number of ways the committee may be chosen.

(Total 6 marks)

5. There are 30 students in a class, of which 18 are girls and 12 are boys. Four students are selected at random to form a committee. Calculate the probability that the committee contains

(a) two girls and two boys;

(b) students all of the same gender.

(Total 6 marks)

6. A team of five students is to be chosen at random to take part in a debate. The team is to be chosen from a group of eight medical students and three law students. Find the probability that

(a) only medical students are chosen;

(b) all three law students are chosen.

(Total 6 marks)

7. There are 25 disks in a bag. Some of them are black and the rest are white. Two are simultaneously selected at random. Given that the probability of selecting two disks of the same colour is equal to the probability of selecting two disks of different colour, how many black disks are there in the bag?

(Total 6 marks)

8. There are 10 seats in a row in a waiting room. There are six people in the room.

(a) In how many different ways can they be seated?

(b) In the group of six people, there are three sisters who must sit next to each other. In how many different ways can the group be seated?

(Total 6 marks)

9. Twelve people travel in three cars, with four people in each car. Each car is driven by its owner. Find the number of ways in which the remaining nine people may be allocated to the cars. (The arrangement of people within a particular car is not relevant).

(Total 6 marks)

Answers:

1. 72.
2. 62.
3. 3168.
4. 55.
5. (a) 0.368; (b) 0.130.
6. (a) $\frac{4}{33}$; (b) $\frac{2}{33}$.
7. 10 or 15.
8. (a) 151200; (b) 10080.
9. 1680.

Past IB Questions

Topic 4: Statistics & Probability – Manipulation & Presentation of Statistical Data

1. A sample of 70 batteries was tested to see how long they last. The results were:

Time (hours)	Number of batteries (frequency)
$0 \le t < 10$	2
$10 \le t < 20$	4
$20 \le t < 30$	8
$30 \le t < 40$	9
$40 \le t < 50$	12
$50 \le t < 60$	13
$60 \le t < 70$	8
$70 \le t < 80$	7
$80 \le t < 90$	6
$90 \le t \le 100$	1
Total	**70**

Find

(a) the sample standard deviation;

(b) an unbiased estimate of the standard deviation of the population from which this sample is taken.

(Total 3 marks)

2. A machine fills bottles with orange juice. A sample of six bottles is taken at random. The bottles contain the following amounts (in ml) of orange juice: 753, 748, 749, 752, 750, 751.

Find

(a) the sample standard deviation;

(b) an unbiased estimate of the population standard deviation from which this sample is taken.

(Total 3 marks)

3. A machine produces packets of sugar. The weights in grams of thirty packets chosen at random are shown below.

Weight (g)	29.6	29.7	29.8	29.9	30.0	30.1	30.2	30.3
Frequency	2	3	4	5	7	5	3	1

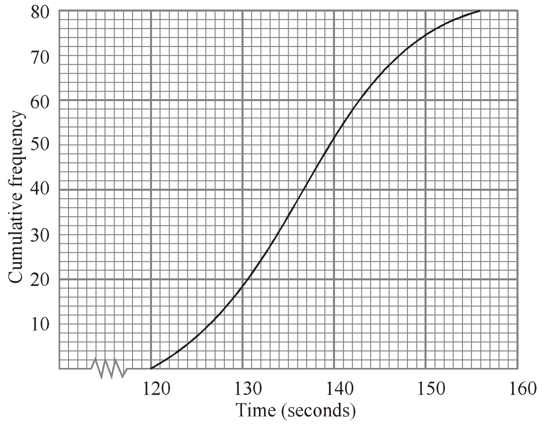
Find unbiased estimates of

(a) the mean of the population from which this sample is taken;

(b) the variance of the population from which this sample is taken.

(Total 3 marks)

4. The 80 applicants for a Sports Science course were required to run 800 metres and their times were recorded. The results were used to produce the following cumulative frequency graph.



Estimate

(a) the median;

(b) the interquartile range.

(Total 6 marks)

5. Consider the six numbers, 2, 3, 6, 9, a and b. The mean of the numbers is 6 and the variance is 10. Find the value of a and of b, if $a < b$.

(Total 6 marks)

6. A teacher drives to school. She records the time taken on each of 20 randomly chosen days. She finds that

$$\sum_{i=1}^{20} x_i = 626 \text{ and } \sum_{i=1}^{20} x_i^{\,2} = 19780.8, \text{ where } x_i \text{ denotes the time, in minutes, taken on}$$

the i^{th} day.

Calculate an unbiased estimate of

(a) the mean time taken to drive to school;

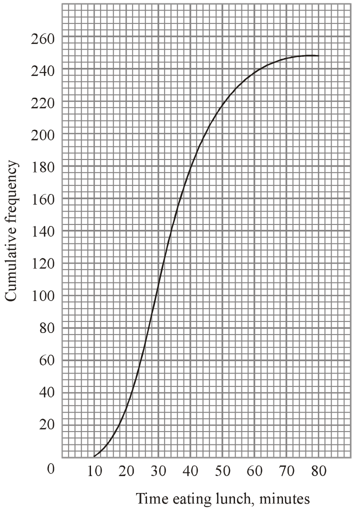
(b) the variance of the time taken to drive to school.

(Total 6 marks)

7. The cumulative frequency curve below indicates the amount of time 250 students spend eating lunch.

(a) Estimate the number of students who spend between 20 and 40 minutes eating lunch.

(b) If 20% of the students spend more than x minutes eating lunch, estimate the value of x.

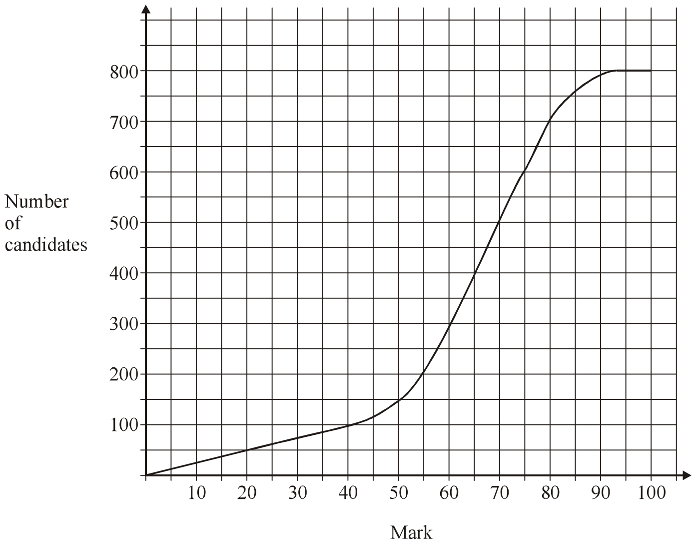


(Total 6 marks)

8. A fair six-sided die, with sides numbered 1, 1, 2, 3, 4, 5 is thrown. Find the mean and variance of the score.

(Total 6 marks)

9. A test marked out of 100 is written by 800 students. The cumulative frequency graph for the marks is given below.



(a) Write down the number of students who scored 40 marks or less on the test.

(b) The middle 50% of test results lie between marks a and b, where $a < b$. Find a and b.

(Total 6 marks)

10. The table below shows the probability distribution of a discrete random variable X.

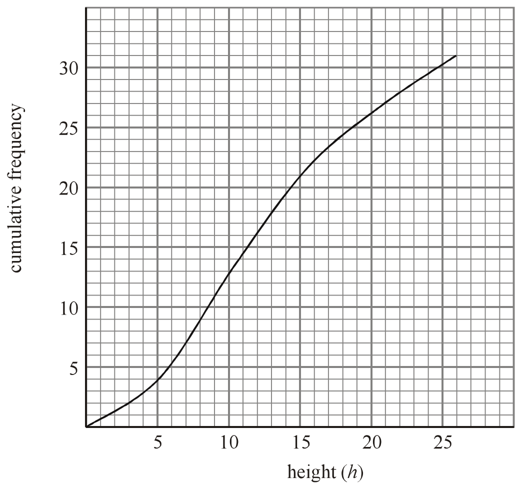
x	0	1	2	3
$P(X = x)$	0.2	a	b	0.25

(a) Given that $E(X) = 1.55$, find the value of a and of b.

(b) Calculate Var(X).

(Total 6 marks)

11. The following is the cumulative frequency diagram for the heights of 30 plants given in centimetres.



(a) Use the diagram to estimate the median height.

(b) Complete the following frequency table.

Height (h)	Frequency
$0 \le h < 5$	4
$5 \le h < 10$	9
$10 \le h < 15$	
$15 \le h < 20$	
$20 \le h < 25$	

(c) Hence estimate the mean height.

(Total 6 marks)

12. In a sample of 50 boxes of light bulbs, the number of defective light bulbs per box is shown below.

Number of defective light bulbs per box	0	1	2	3	4	5	6
Number of boxes	7	3	15	11	6	5	3

(a) Calculate the median number of defective light bulbs per box.

(b) Calculate the mean number of defective light bulbs per box.

(Total 6 marks)

13. A sample of discrete data is drawn from a population and given as

66, 72, 65, 70, 69, 73, 65, 71, 75.

Find

(a) the interquartile range;

(2)

(b) an estimate for the mean of the population;

(2)

(c) an unbiased estimate of the variance of the population.

(2)

(Total 6 marks)

14. Consider the data set $\{k-2, k, k+1, k+4\}$, where $k \in \mathbb{R}$.

(a) Find the mean of this data set in terms of k.

(3)

Each number in the above data set is now **decreased** by 3.

(b) Find the mean of this **new** data set in terms of k.

(2)

(Total 5 marks)

Answers

1. (a) $s_n = 21.4$; (b) $s_{n-1} = 21.6$

2. (a) $s_n = 1.71$; (b) $s_{n-1} = \sqrt{\frac{n}{n-1}} s_n = 1.87$

3. (a) 29.9 (b) 0.0336
4. (a) Median = 135;
   (b) $Q_1 = 130$, $Q_2 = 141$, IQ range = 11
5. $a = 5, b = 11$
6. (a) 31.3 (b) 9.84
7. (a) 28 spent less than 20 minutes
   184 spent less than 40 minutes
   156 spent between 20 and 40 minutes
(b) $x = 44$ minutes

8. $\text{Var}(X) = \frac{20}{9}$

9. (a) 100 students score 40 marks or fewer
(b) $a = 55, b = 75$.
10 (a) $a = 0.3$ and $b = 0.25$ (b) 1.15

11. (a) Median = 11 (c) 11.8
(b)

Height	Frequency
$0 \le h < 5$	4
$5 \le h < 10$	9
$10 \le h < 15$	**8**
$15 \le h < 20$	**5**
$20 \le h < 25$	**4**

12. (a) 2.5 (b) 2.66
13. (a) 7 or 6 (b) 69.6 (c) 13.0
14. (a) $\bar{x} = \frac{4k+3}{4} \left(= k + \frac{3}{4}\right)$

(b) $\bar{x} = \frac{4k+3}{4} - 3 \left(= \frac{4k-9}{4}, k - \frac{9}{4}\right)$

Anglo-Chinese School (Independent)

Year 6 IBDP Mathematics HL

Past IB Questions

Topic 4: Statistics & Probability (Core) – Probability

1. A bag contains 2 red balls, 3 blue balls and 4 green balls. A ball is chosen at random from the bag and is not replaced. A second ball is chosen. Find the probability of choosing one green ball and one blue ball in any order. **(Total 4 marks)**

2. In a bilingual school there is a class of 21 pupils. In this class, 15 of the pupils speak Spanish as their first language and 12 of these 15 pupils are Argentine. The other 6 pupils in the class speak English as their first language and 3 of these 6 pupils are Argentine.

A pupil is selected at random from the class and is found to be Argentine. Find the probability that the pupil speaks Spanish as his/her first language.

(Total 4 marks)

3. A new blood test has been shown to be effective in the early detection of a disease. The probability that the blood test correctly identifies someone with this disease is 0.99, and the probability that the blood test correctly identifies someone without that disease is 0.95. The incidence of this disease in the general population is 0.0001.

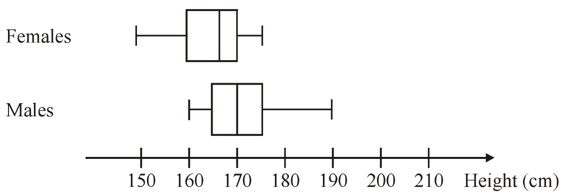
A doctor administered the blood test to a patient and the test result indicated that this patient had the disease. What is the probability that the patient has the disease?

(Total 6 marks)

4. The local Football Association consists of ten teams. Team A has a 40% chance of winning any game against a higher-ranked team, and a 75% chance of winning any game against a lower-ranked team. If A is currently in fourth position, find the probability that A wins its next game.

(Total 4 marks)

5. The box-and-whisker plots shown represent the heights of female students and the heights of male students at a certain school.



(a) What percentage of female students are shorter than any male students?

(b) What percentage of male students are shorter than some female students?

(c) From the diagram, **estimate** the mean height of the male students. **(Total 3 marks)**

6. Given that events A and B are independent with $P(A \cap B) = 0.3$ and $P(A \cap B') = 0.3$, find $P(A \cup B)$.

(Total 3 marks)

7. A girl walks to school every day. If it is not raining, the probability that she is late is $\frac{1}{5}$.

If it is raining, the probability that she is late is $\frac{2}{3}$. The probability that it rains on a particular day is $\frac{1}{4}$.

On one particular day the girl is late. Find the probability that it was raining on that day.

(Total 3 marks)

8. Given that $P(X) = \frac{2}{3}$, $P(Y|X) = \frac{2}{5}$ and $P(Y|X') = \frac{1}{4}$, find

(a) $P(Y')$;

(b) $P(X' \cup Y')$.

(Total 3 marks)

9. The probability that a man leaves his umbrella in any shop he visits is $\frac{1}{3}$. After visiting two shops in succession, he finds he has left his umbrella in one of them. What is the probability that he left his umbrella in the second shop?

(Total 3 marks)

10. The probability that it rains during a summer's day in a certain town is 0.2. In this town, the probability that the daily maximum temperature exceeds 25°C is 0.3 when it rains and 0.6 when it does not rain. Given that the maximum daily temperature exceeded 25°C on a particular summer's day, find the probability that it rained on that day.

(Total 6 marks)

11. Jack and Jill play a game, by throwing a die in turn. If the die shows a 1, 2, 3 or 4, the player who threw the die wins the game. If the die shows a 5 or 6, the other player has the next throw. Jack plays first and the game continues until there is a winner.

(a) Write down the probability that Jack wins on his first throw. **(1)**

(b) Calculate the probability that Jill wins on her first throw. **(2)**

(c) Calculate the probability that Jack wins the game. **(3)**

(Total 6 marks)

12. Bag A contains 2 red and 3 green balls.

(a) Two balls are chosen at random from the bag without replacement. Find the probability that 2 red balls are chosen.

(2)

Bag B contains 4 red and *n* green balls.

(b) Two balls are chosen without replacement from this bag. If the probability that two red balls are chosen is $\frac{2}{15}$, show that $n = 6$.

(4)

A standard die with six faces is rolled. If a 1 or 6 is obtained, two balls are chosen from bag A, otherwise two balls are chosen from bag B.

(c) Calculate the probability that two red balls are chosen.

(4)

(d) Given that two red balls are chosen, find the probability that a 1 or a 6 was obtained on the die.

(3)

(Total 13 marks)

13. Given that $(A \cup B)' = \varnothing$, $P(A'|B) = \frac{1}{3}$ and $P(A) = \frac{6}{7}$, find $P(B)$.

(Total 6 marks)

14. Bag 1 contains 4 red cubes and 5 blue cubes. Bag 2 contains 7 red cubes and 2 blue cubes.
Two cubes are drawn at random, the first from Bag 1 and the second from Bag 2.

(a) Find the probability that the cubes are of the same colour.

(b) Given that the cubes selected are of different colours, find the probability that the red cube was selected from Bag 1.

(Total 6 marks)

15. Only two international airlines fly daily into an airport. UN Air has 70 flights a day and IS Air has 65 flights a day. Passengers flying with UN Air have an 18% probability of losing their luggage and passengers flying with IS Air have a 23% probability of losing their luggage. You overhear someone in the airport complain about her luggage being lost.

Find the probability that she travelled with IS Air.

(Total 6 marks)

Answers

1. $\frac{1}{3}$

2. $\frac{4}{5}$

3. 0.00198

4. $\frac{19}{30}$

5. (a) Required percentage = 25%; (b) Required percentage = 75% ; (c) 172 cm ± 1 cm

6. 0.8

7. $\frac{10}{19}$

8. (a) $\frac{13}{20}$; (b) $\frac{11}{15}$

9. 2/5

10. $\frac{1}{9}$ (or 0.111)

11. (a) $\frac{2}{3}$ (or 0.667); (b) $\frac{2}{9}$ (or 0.222); (c) $\frac{3}{4}$

12. (a) $\frac{1}{10}$; (c) $\frac{11}{90}$; (d) $\frac{3}{11}$

13. $P(B) = \frac{3}{7}$

14. (a) 0.469

(b) 0.186

15. 0.543

Past IB Questions

Topic 4: Statistics & Probability – Discrete Random Variables

1. A discrete random variable X has its probability distribution given by

$$P(X = x) = k(x + 1), \text{ where } x \text{ is } 0, 1, 2, 3, 4.$$

(a) Show that $k = \frac{1}{15}$

(b) Find $E(X)$.

(Total 6 marks)

2. In a game a player pays an entrance fee of $\$n$. He then selects one number from 1, 2, 3, 4, 5, 6 and rolls three standard dice.

If his chosen number appears on all three dice he wins four times his entrance fee.

If his number appears on exactly two of the dice he wins three times the entrance fee.

If his number appears on exactly one die he wins twice the entrance fee.

If his number does not appear on any of the dice he wins nothing.

(a) Copy and complete the probability table below.

Profit ($)	$-n$	n	$2n$	$3n$
Probability		$\frac{75}{216}$		

(4)

(b) Show that the player's expected profit is $\$\left(-\frac{17n}{216}\right)$.

(2)

(c) What should the entrance fee be so that the player's expected loss per game is 34 cents?

(2)
(Total 8 marks)

3. In a game, the probability of a player scoring with a shot is $\frac{1}{4}$. Let X be the number of shots the player takes to score, including the scoring shot. (You can assume that each shot is independent of the others.)

(a) Find $P(X = 3)$.

(2)

(b) Find the probability that the player will have at least three misses before scoring **twice**.

(6)

(c) Prove that the expected value of X is 4.

(You may use the result $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3$......)

(5)
(Total 13 marks)

4. A biased die with four faces is used in a game. A player pays 10 counters to roll the die. The table below shows the possible scores on the die, the probability of each score and the number of counters the player receives in return for each score.

Score	1	2	3	4
Probability	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
Number of counters player receives	4	5	15	n

Find the value of n in order for the player to get an expected return of 9 counters per roll.

(Total 4 marks)

5. In a game a player rolls a biased tetrahedral (four-faced) die. The probability of each possible score is shown below.

Score	1	2	3	4
Probability	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	x

Find the probability of a total score of six after two rolls.

(Total 3 marks)

6. The probability distribution of a discrete random variable X is given by

$$P(X = x) = k\left(\frac{2}{3}\right)^x, \text{ for } x = 0, 1, 2, \ldots\ldots$$

Find the value of k.

(Total 3 marks)

7. Two children, Alan and Belle, each throw two fair cubical dice simultaneously. The score for each child is the sum of the two numbers shown on their respective dice.

(a) (i) Calculate the probability that Alan obtains a score of 9.

(ii) Calculate the probability that Alan and Belle both obtain a score of 9.

(2)

(b) (i) Calculate the probability that Alan and Belle obtain the same score,

(ii) Deduce the probability that Alan's score exceeds Belle's score.

(4)

(c) Let X denote the largest number shown on the four dice.

(i) Show that for $\mathrm{P}(X \le x) = \left(\frac{x}{6}\right)^4$, for $x = 1, 2, \ldots 6$

(ii) Copy and complete the following probability distribution table.

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$				$\frac{671}{1296}$

(iii) Calculate $\mathrm{E}(X)$.

(7)
(Total 13 marks)

8. The probability distribution of a discrete random variable X is defined by

$$\mathrm{P}(X = x) = cx(5 - x), x = 1, 2, 3, 4.$$

(a) Find the value of c.

(3)

(b) Find $\mathrm{E}(X)$.

(3)

(Total 6 marks)

9. Consider the 10 data items $x_1, x_2, \ldots x_{10}$. Given that $\sum_{i=1}^{10} x_i^{\ 2} = 1341$ and the standard deviation is 6.9, find the value of $\overline{x}$.

(Total 6 marks)

Answers

1. (b) 8/3

2. (a)

Profit	$-n$	n	$2n$	$3n$
Probability	$\mathbf{\frac{215}{216}}$	$\frac{75}{216}$	$\mathbf{\frac{15}{216}}$	$\mathbf{\frac{1}{216}}$

(c) $n = 4.32$

3. (a) 9/64; (b) $\frac{189}{256}$ **4.** $n = 30$ **5.** $\frac{1}{4}$ **6**. $k = \frac{1}{3}$

7. (a) (i) $\frac{1}{9}$; (ii) 1/81 (b)(i) $\frac{73}{648}$; (ii) $\frac{575}{1296}$

(c) (ii)

x	1	2	3	4	5	6
$\mathrm{P}(X = x)$	$\frac{1}{1296}$	$\frac{15}{1296}$	$\mathbf{\frac{65}{1296}}$	$\mathbf{\frac{175}{1296}}$	$\mathbf{\frac{369}{1296}}$	$\frac{671}{1296}$

(iii) $\frac{6797}{1296}$

8. (a) $c = \frac{1}{20}\ (= 0.05)$

(b) 2.5

9. $\overline{x} = \pm 9.3$

Anglo-Chinese School (Independent)

Year 6 IBDP Mathematics HL

Past IB Questions

Topic 4 : Statistics & Probability – Binomial Distribution

1. The continuous random variable X has probability density function $f(x)$ where

$$f_k(x) = \begin{cases} \mathrm{e} - k\mathrm{e}^{kx}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) Show that $k = 1$. **(3)**

(b) What is the probability that the random variable X has a value that lies between $\frac{1}{4}$ and $\frac{1}{2}$? Give your answer in terms of e. **(2)**

(c) Find the mean and variance of the distribution. Give your answers *exactly*, in terms of e. **(6)**

The random variable X above represents the lifetime, in years, of a certain type of battery.

(d) Find the probability that a battery lasts more than six months. **(2)**

A calculator is fitted with three of these batteries. Each battery fails independently of the other two. Find the probability that at the end of six months

(e) none of the batteries has failed; **(2)**

(f) exactly one of the batteries has failed. **(2)**
(Total 17 marks)

2. A satellite relies on solar cells for its power and will operate provided that at least one of the cells is working. Cells fail independently of each other, and the probability that an individual cell fails within one year is 0.8.

(a) For a satellite with ten solar cells, find the probability that all ten cells fail within one year. **(2)**

(b) For a satellite with ten solar cells, find the probability that the satellite is still operating at the end of one year. **(2)**

(c) For a satellite with n solar cells, write down the probability that the satellite is still operating at the end of one year. Hence, find the smallest number of solar cells required so that the probability of the satellite still operating at the end of one year is at least 0.95. **(5)**
(Total 9 marks)

3. In a school, $\frac{1}{3}$ of the students travel to school by bus. Five students are chosen at random. Find the probability that exactly 3 of them travel to school by bus. **(Total 3 marks)**

4. X is a binomial random variable, where the number of trials is 5 and the probability of success of each trial is p. Find the values of p if $\mathrm{P}(X = 4) = 0.12$. **(Total 3 marks)**

5. A coin is biased so that when it is tossed the probability of obtaining heads is $\frac{2}{3}$. The coin is tossed 1800 times. Let X be the number of heads obtained. Find

(a) the mean of X;

(b) the standard deviation of X. **(Total 3 marks)**

6. When John throws a stone at a target, the probability that he hits the target is 0.4. He throws a stone 6 times.

(a) Find the probability that he hits the target **exactly** 4 times.

(b) Find the probability that he hits the target for the first time on his third throw.
(Total 6 marks)

7. When a boy plays a game at a fair, the probability that he wins a prize is 0.25. He plays the game 10 times. Let X denote the total number of prizes that he wins. Assuming that the games are independent, find

(a) $\mathrm{E}(X)$

(b) $\mathrm{P}(X \le 2)$. **(Total 6 marks)**

8. On a television channel the news is shown at the same time each day. The probability that Alice watches the news on a given day is 0.4. Calculate the probability that on five consecutive days, she watches the news on at most three days. **(Total 6 marks)**

9. Andrew shoots 20 arrows at a target. He has a probability of 0.3 of hitting the target. All shots are independent of each other. Let X denote the number of arrows hitting the target.

(a) Find the mean and standard deviation of X. **(5)**

(b) Find

(i) $\mathrm{P}(X = 5)$;

(ii) $\mathrm{P}(4 \le X \le 8)$. **(6)**

Bill also shoots arrows at a target, with probability of 0.3 of hitting the target. All shots are independent of each other.

(c) Calculate the probability that Bill hits the target for the first time on his third shot. **(3)**

(d) Calculate the minimum number of shots required for the probability of at least one shot hitting the target to exceed 0.99. **(5)**
(Total 19 marks)

10. A bag contains a very large number of ribbons. One quarter of the ribbons are yellow and the rest are blue. Ten ribbons are selected at random from the bag.

(a) Find the expected number of yellow ribbons selected.
(2)

(b) Find the probability that exactly six of these ribbons are yellow.
(2)

(c) Find the probability that at least two of these ribbons are yellow.
(3)

(d) Find the most likely number of yellow ribbons selected.
(4)

(e) What assumption have you made about the probability of selecting a yellow ribbon?
(1)
(Total 12 marks)

11. In an experiment, a trial is repeated n times. The trials are independent and the probability p of success in each trial is constant. Let X be the number of successes in the n trials. The mean of X is 0.4 and the standard deviation is 0.6.

(a) Find p.

(b) Find n.
(Total 6 marks)

12. A biology test consists of seven multiple choice questions. Each question has five possible answers, only one of which is correct. At least four correct answers are required to pass the test. Juan does not know the answer to any of the questions so, for each question, he selects the answer at random.

(a) Find the probability that Juan answers exactly four questions correctly.

(b) Find the probability that Juan passes the biology test.
(Total 6 marks)

13. Over a one month period, Ava and Sven play a total of n games of tennis.

The probability that Ava wins any game is 0.4. The result of each game played is independent of any other game played.

Let X denote the number of games won by Ava over a one month period.

(a) Find an expression for $P(X = 2)$ in terms of n.
(3)

(b) If the probability that Ava wins two games is 0.121 correct to three decimal places, find the value of n.
(3)
(Total 6 marks)

Answers

1. (b) $\frac{e}{4} - \sqrt{e} + \sqrt[4]{e}$; (c) $1 + \frac{e}{3} - \frac{e^2}{4}$; (d) $\sqrt{e} - \frac{e}{2}$ or 0.290 ; (e) $\left(\sqrt{e} - \frac{e}{2}\right)^3$ or 0.0243

(f) 0.179

2. (a) 0.107; (b) 0.893; (c) 14 solar cells are needed.

3. $\frac{40}{243}$ or 0.165

4. $p = 0.459$ or 0.973

5. (a) $E(X) = np = 1200$; (b) 20

6. (a) 0.138 ; (b) 0.144

7. (a) 2.5; (b) 0.526

8. 0.913

9. (a) 6; 2.05

(b) (i) 0.179

(ii) 0.780

(c) 0.147

(d) $n = 13$

10. (a) 2.5; (b) 0.0162; (c) 0.756; (d) 2

11. (a) $p = 0.1$

(b) $n = 4$

12. (a) 0.0287; (b) 0.0333

13. (a) $P(X = 2) = \binom{n}{r}(0.4)^2(0.6)^{n-2}$ $\left(= \frac{n(n-1)}{2}(0.4)^2(0.6)^{n-2}\right)$; (b) $n = 10$

Past IB Questions

Topic 4: Statistics & Probability – Continuous Random Variables

1. A business man spends X hours on the telephone during the day. The probability density function of X is given by

$$f(x) = \begin{cases} \frac{1}{12}(8x - x^3), \text{ for } 0 \le x \le 2 \\ 0, \quad \text{otherwise.} \end{cases}$$

(a) (i) Write down an integral whose value is E(X).

(ii) Hence evaluate E(X).
(3)

(b) (i) Show that the median, m, of X satisfies the equation $m^4 - 16m^2 + 24 = 0$.

(ii) Hence evaluate m.
(5)

(c) Evaluate the mode of X.
(3)
(Total 11 marks)

2. The lifetime of a particular component of a solar cell is Y years, where Y is a continuous random variable with probability density function

$$f(y) = \begin{cases} 0 & \text{when } y < 0 \\ 0.5\text{e}^{-y/2} & \text{when } y \ge 0. \end{cases}$$

(a) Find the probability, correct to four significant figures, that a given component fails within six months.
(3)

Each solar cell has three components which work independently and the cell will continue to run if at least two of the components continue to work.

(b) Find the probability that a solar cell fails within six months.
(4)
(Total 7 marks)

3. A continuous random variable X has probability density function

$$f(x) = \begin{cases} \dfrac{4}{\pi(1+x^2)}, & \text{for } 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

Find E(X).
(Total 3 marks)

4. The probability density function $f(x)$, of a continuous random variable X is defined by

$$f(x) = \begin{cases} \frac{1}{4}x(4 - x^2), \ 0 \le x \le 2 \\ 0, \text{ otherwise.} \end{cases}$$

Calculate the **median** value of X.
(Total 6 marks)

5. A continuous random variable X has probability density function given by

$$f(x) = k(2x - x^2), \quad \text{for } 0 \le x \le 2$$
$$f(x) = 0, \quad \text{elsewhere.}$$

(a) Find the value of k.

(b) Find P($0.25 \le x \le 0.5$).
(Total 6 marks)

6. The continuous random variable X has probability density function

$$f(x) = \frac{1}{6}x(1 + x^2) \text{ for } 0 \le x \le 2,$$
$$f(x) = 0 \text{ otherwise.}$$

(a) Sketch the graph of f for $0 \le x \le 2$.
(2)

(b) Write down the mode of X.
(1)

(c) Find the mean of X.
(4)

(d) Find the median of X.
(5)
(Total 12 marks)

7. The probability density function $f(x)$ of the continuous random variable X is defined on the interval $[0, a]$ by

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \le x \le 3. \\ \dfrac{27}{8x^2} & \text{for } 3 < x \le a. \end{cases}$$

Find the value of a.
(Total 6 marks)

8. The time, T minutes, required by candidates to answer a question in a mathematics examination has probability density function

$$f(t) = \begin{cases} \frac{1}{72}(12t - t^2 - 20), & \text{for } 4 \le t \le 10 \\ 0, & \end{cases}$$

(a) Find

(i) μ, the expected value of T;

(ii) σ^2, the variance of T.

(7)

(b) A candidate is chosen at random. Find the probability that the time taken by this candidate to answer the question lies in the interval $[\mu - \sigma, \mu]$.

(5)
(Total 12 marks)

9. The continuous random variable X has probability density function

$$f(x) = \begin{cases} \frac{x}{1+x^2}, & \textit{for } 0 \le x \le k \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the exact value of k.

(5)

(b) Find the mode of X.

(2)

(c) Calculate $P(1 \le X \le 2)$.

(3)
(Total 10 marks)

10. A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} e^x, & \text{for } 0 \le x \le \ln 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find the **exact** value of $E(X)$.

(Total 6 marks)

11. A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{8}{\pi(x^2+4)}, & 0 \le x \le 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) State the mode of X.

(b) Find the **exact** value of $E(X)$.

(Total 6 marks)

12. A continuous random variable X has probability density function defined by

$$f(x) = \begin{cases} \frac{c}{4+x^2}, & \text{for } -\frac{2}{\sqrt{3}} \le x \le 2\sqrt{3} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the **exact** value of the constant c in terms of π.

(5)

(b) Sketch the graph of $f(x)$ and hence state the mode of the distribution.

(3)

(c) Find the **exact** value of $E(X)$.

(4)
(Total 12 marks)

13. The random variable T has the probability density function

$$f(t) = \frac{\pi}{4} cos\left(\frac{\pi t}{2}\right), -1 \le t \le 1.$$

Find

(a) $P(T = 0)$;

(2)

(b) the interquartile range.

(5)

(Total 7 marks)

14. A continuous random variable X has probability density function

$$f(x) = \begin{cases} 12x^2(1-x), & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Find the probability that X lies between the mean and the mode.

(Total 6 marks)

Answers

1. (a) (i) $\frac{1}{12}\int_0^2 x(8x - x^3)dx$ (ii) $E(X) = 1.24$
(b) (ii) $m = 1.29$ (c) $x = 1.63$

2. (a) 0.2212. (b) 0.125. **3.** 0.441. **4.** $m = 1.08$
5. (a) $k = 3/4$ (b) 0.113 **6.** (b) Mode = 2 (c) $\frac{68}{45}$ (1.51); (d) $m = 1.61$
7. 54/11 **8.** (a) (i) 6.5 (ii) 2.15 (b) 0.321

9. (a) $k = \sqrt{e^2 - 1}$ (b) 1 (c) 0.458

10. $E(X) = 2 \ln 2 - 1$

11. (a) Mode = 0

(b) $E(X) = \dfrac{4}{\pi}(\ln 8 - \ln 4)\left(= \dfrac{4}{\pi}\ln\left(\dfrac{8}{4}\right), \dfrac{4}{\pi}\ln 2\right)$

12. (a) $c = \dfrac{4}{\pi}$

(b) The mode is zero.

(c) $E(X) = \dfrac{2}{\pi}\left(\ln 16 - \ln \dfrac{16}{3}\right)\left(= \dfrac{2}{\pi}\ln 3\right)$

13. (a) 0

(b) interquartile range is $\dfrac{1}{3} - \left(-\dfrac{1}{3}\right) = \dfrac{2}{3}$

14. $\int_{\frac{3}{5}}^{\frac{2}{3}} f(x)\mathrm{d}x = 0.117 \left(= \dfrac{1981}{16875}\right)$

Past IB Questions

Topic 4: Probability & Statistics – Normal Distribution

1. A factory has a machine designed to produce 1 kg bags of sugar. It is found that the average weight of sugar in the bags is 1.02 kg. Assuming that the weights of the bags are normally distributed, find the standard deviation if 1.7% of the bags weigh below 1 kg. Give your answer correct to the nearest 0.1 gram.

(Total 4 marks)

2. The random variable X is distributed normally with mean 30 and standard deviation 2. Find $p(27 \le X \le 34)$.

(Total 4 marks)

3. A machine is set to produce bags of salt, whose weights are distributed normally, with a mean of 110 g and standard deviation of 1.142 g. If the weight of a bag of salt is less than 108 g, the bag is rejected. With these settings, 4% of the bags are rejected.

The settings of the machine are altered and it is found that 7% of the bags are rejected.

(a) (i) If the mean has not changed, find the new standard deviation, **correct to three decimal places**.

(4)

The machine is adjusted to operate with this new value of the standard deviation.

(ii) Find the value, **correct to two decimal places**, at which the mean should be set so that only 4% of the bags are rejected.

(4)

(b) With the new settings from part (a), it is found that 80% of the bags of salt have a weight which lies between A g and B g, where A and B are symmetric about the mean. Find the values of A and B, giving your answers **correct to two decimal places**.

(4)
(Total 12 marks)

4. The diameters of discs produced by a machine are normally distributed with a mean of 10 cm and standard deviation of 0.1 cm. Find the probability of the machine producing a disc with a diameter smaller than 9.8 cm.

(Total 3 marks)

5. Z is the standardized normal random variable with mean 0 and variance 1. Find the value of a such that $P(|Z| \le a) = 0.75$.

(Total 3 marks)

6. The weights of a certain species of bird are normally distributed with mean 0.8 kg and standard deviation 0.12 kg. Find the probability that the weight of a randomly chosen bird of the species lies between 0.74 kg and 0.95 kg.

(Total 6 marks)

7. (a) At a building site the probability, P(A), that all materials arrive on time is 0.85. The probability, P(B), that the building will be completed on time is 0.60. The probability that the materials arrive on time and that the building is completed on time is 0.55.

(i) Show that events A and B are **not** independent.

(ii) All the materials arrive on time. Find the probability that the building will not be completed on time.

(5)

(b) There was a team of ten people working on the building, including three electricians and two plumbers. The architect called a meeting with five of the team, and randomly selected people to attend. Calculate the probability that **exactly two** electricians and **one** plumber were called to the meeting.

(2)

(c) The number of hours a week the people in the team work is normally distributed with a mean of 42 hours. 10% of the team work 48 hours or more a week. Find the probability that **both** plumbers work more than 40 hours in a given week.

(8)
(Total 15 marks)

8. The random variable X is normally distributed and

$P(X \le 10) = 0.670$
$P(X \le 12) = 0.937.$

Find E(X).

(Total 6 marks)

9. A random variable X is normally distributed with mean μ and standard deviation σ, such that
$P(X > 50.32) = 0.119$, and $P(X < 43.56) = 0.305$.

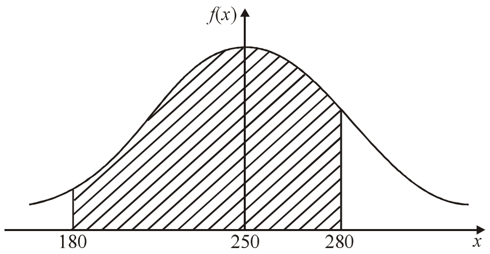
(a) Find μ and σ.

(5)

(b) Hence find $P(|X - \mu| < 5)$.

(2)
(Total 7 marks)

10. The following diagram shows the probability density function for the random variable X, which is normally distributed with mean 250 and standard deviation 50.



Find the probability represented by the shaded region.

(Total 6 marks)

11. Ian and Karl have been chosen to represent their countries in the Olympic discus throw. Assume that the distance thrown by each athlete is normally distributed. The mean distance thrown by Ian in the past year was 60.33 m with a standard deviation of 1.95 m.

(a) In the past year, 80% of Ian's throws have been longer than x metres. Find x, correct to **two** decimal places.

(3)

(b) In the past year, 80% of Karl's throws have been longer than 56.52 m. If the mean distance of his throws was 59.39 m, find the standard deviation of his throws, correct to **two** decimal places.

(3)

(c) This year, Karl's throws have a mean of 59.50 m and a standard deviation of 3.00 m. Ian's throws still have a mean of 60.33 m and standard deviation 1.95 m. In a competition an athlete must have at least one throw of 65 m or more in the first round to qualify for the final round. Each athlete is allowed three throws in the first round.

(i) Determine which of these two athletes is more likely to qualify for the final on their first throw.

(ii) Find the probability that **both** athletes qualify for the final.

(11)
(Total 17 marks)

12. The speeds of cars at a certain point on a straight road are normally distributed with mean μ and standard deviation σ. 15% of the cars travelled at speeds greater than 90 km h^{-1} and 12% of them at speeds less than 40 km h^{-1}. Find μ and σ.

(Total 6 marks)

13. A company buys 44% of its stock of bolts from manufacturer A and the rest from manufacturer B. The diameters of the bolts produced by each manufacturer follow a normal distribution with a standard deviation of 0.16 mm.

The mean diameter of the bolts produced by manufacturer A is 1.56 mm.
24.2% of the bolts produced by manufacturer B have a diameter less than 1.52 mm.

(a) Find the mean diameter of the bolts produced by manufacturer B.
(3)

A bolt is chosen at random from the company's stock.

(b) Show that the probability that the diameter is less than 1.52 mm is 0.312, to three significant figures.
(4)

(c) The diameter of the bolt is found to be less than 1.52 mm. Find the probability that the bolt was produced by manufacturer B.
(3)

(d) Manufacturer B makes 8000 bolts in one day. It makes a profit of \$1.50 on each bolt sold, on condition that its diameter measures between 1.52 mm and 1.83 mm. Bolts whose diameters measure less than 1.52 mm must be discarded at a loss of \$0.85 per bolt.
Bolts whose diameters measure over 1.83 mm are sold at a reduced profit of \$0.50 per bolt.

Find the expected profit for manufacturer B.
(6)
(Total 16 marks)

14. A random variable X is normally distributed with mean μ and variance σ^2. If $P(X > 6.2) = 0.9474$ and $P(X < 9.8) = 0.6368$, calculate the value of μ and of σ.
(Total 6 marks)

15. The weights in grams of bread loaves sold at a supermarket are normally distributed with mean 200 g. The weights of 88% of the loaves are less than 220 g.
Find the standard deviation.
(Total 6 marks)

16. A certain type of vegetable has a weight which follows a normal distribution with mean 450 grams and a standard deviation 50 grams.

(a) In a load of 2000 of these vegetables, calculate the expected number with a weight greater than 525 grams.

(b) Find the upper quartile of the distribution.
(Total 6 marks)

17. The lengths of a particular species of lizard are normally distributed with a mean length of 50 cm and a standard deviation of 4 cm. A lizard is chosen at random.

(a) Find the probability that its length is greater than 45 cm.

(b) Given that its length is greater than 45 cm, find the probability that its length is greater than 55 cm.
(Total 6 marks)

18. The time, T minutes, spent each day by students in Amy's school sending text messages may be modelled by a normal distribution.

30% of the students spend less than 10 minutes per day.
35% spend more than 15 minutes per day.

(a) Find the mean and standard deviation of T.
(6)

The number of text messages received by Amy during a fixed time interval may be modelled by a Poisson distribution with a mean of 6 messages per hour.

(b) Find the probability that Amy will receive exactly 8 messages between 16:00 and 18:00 on a random day.
(3)

(c) Given that Amy has received at least 10 messages between 16:00 and 18:00 on a random day, find the probability that she received 13 messages during that time.
(5)

(d) During a 5-day week, find the probability that there are exactly 3 days when Amy receives no messages between 17:45 and 18:00.
(4)
(Total 18 marks)

19. The times taken for buses travelling between two towns are normally distributed with a mean of 35 minutes and standard deviation of 7 minutes.

(a) Find the probability that a randomly chosen bus completes the journey in less than 40 minutes.
(2)

(b) 90% of buses complete the journey in less than t minutes. Find the value of t.
(5)

(c) A random sample of 10 buses had their travel time between the two towns recorded. Find the probability that exactly 6 of these buses complete the journey in less than 40 minutes.
(4)
(Total 11 marks)

20. A furniture manufacturer makes tables. A table leg is considered to be oversize if its width is greater than 10.5 cm and undersize if its width is less than 9.5 cm. From past experience it is found that 2% of the table legs that are made are oversize and that 4% of the table legs are undersize. The widths of the table legs are normally distributed with mean μ cm and standard deviation σ cm. Find the value of μ and of σ.

(Total 6 marks)

21. A company produces computer microchips, which have a life expectancy that follows a normal distribution with a mean of 90 months and a standard deviation of 3.7 months.

(a) If a microchip is guaranteed for 84 months find the probability that it will fail before the guarantee ends.

(2)

(b) The probability that a microchip does not fail before the end of the guarantee is required to be 99%. For how many months should it be guaranteed?

(2)

(c) A rival company produces microchips where the probablity that they will fail after 84 months is 0.88. Given that the life expectancy also follows a normal distribution with standard deviation 3.7 months, find the mean.

(2)
(Total 6 marks)

22. The distance travelled by students to attend Gauss College is modelled by a normal distribution with mean 6 km and standard deviation 1.5 km.

(a) (i) Find the probability that the distance travelled to Gauss College by a randomly selected student is between 4.8 km and 7.5 km.

(ii) 15% of students travel less than *d* km to attend Gauss College. Find the value of *d*.

(7)

At Euler College, the distance travelled by students to attend their school is modelled by a normal distribution with mean μ km and standard deviation σ km.

(b) If 10% of students travel more than 8 km and 5% of students travel less than 2 km, find the value of μ and of σ.

(6)

The number of telephone calls, *T*, received by Euler College each minute can be modelled by a Poisson distribution with a mean of 3.5.

(c) (i) Find the probability that at least three telephone calls are received by Euler College in **each** of two successive one-minute intervals.

(ii) Find the probability that Euler College receives 15 telephone calls during a randomly selected five-minute interval.

(8)
(Total 21 marks)

Answers

1. 9.4 g

2. 0.910

3. (a) (i) $\sigma = 1.355$.

(ii) $\mu = 110.37$

(b) $A = 108.63$; $B = 112.11$

4. 0.0228

5. $a = 1.15$

6. 0.586

7. (a) (ii) $\frac{6}{17}(= 0.353)$

(b) $\frac{60}{252}\left(= \frac{5}{21} = 0.238\right)$

(c) 0.443

8. E(*X*) = 9.19

9. (a) $\sigma = 4$; $\mu = 45.6$

(b) 0.789

10. 0.645

11. (a) $x = 58.69$ m

(b) $\sigma = 3.41$

(c) (i) Karl is more likely to qualify.

(ii) 0.00239

12. $\mu = 66.6$, σ=22.6

13. (a) 1.63

(c) 0.434

(d) $6605.28

14. $\sigma = 1.83$ and $\sigma = 9.16$

15. $\sigma = 17.0$

16. (a) 134

(b) 484

17. (a) 0.894

(b) 0.118

18. (a) $\mu = 12.9$, σ = 5.50

(b) 0.0655

(c) 0.139

(d) 0.0670

19. (a) 0.762

(b) 44.0

(c) 0.131

20. $\sigma = 0.263$; $\mu = 9.96$

21. (a) 0.0524

(b) $x = 81.4$

(c) mean is 88.3

22. (a) (i) 0.629

(ii) 4.45

(b) $\sigma = 2.05$ (km) **and** $\mu = 5.37$ (km)

(c) (i) 0.461

(ii) 0.0849

Mathematics: analysis and approaches
Higher level
Paper 1

Specimen paper

2 hours

Candidate session number

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

13 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Let A and B be events such that $\mathrm{P}(A) = 0.5$, $\mathrm{P}(B) = 0.4$ and $\mathrm{P}(A \cup B) = 0.6$.
Find $\mathrm{P}(A \mid B)$.

2. [Maximum mark: 5]

(a) Show that $(2n-1)^2+(2n+1)^2=8n^2+2$, where $n \in \mathbb{Z}$. [2]

(b) Hence, or otherwise, prove that the sum of the squares of any two consecutive odd integers is even. [3]

Turn over

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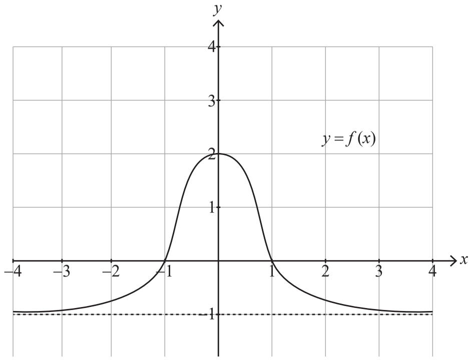
3. [Maximum mark: 5]

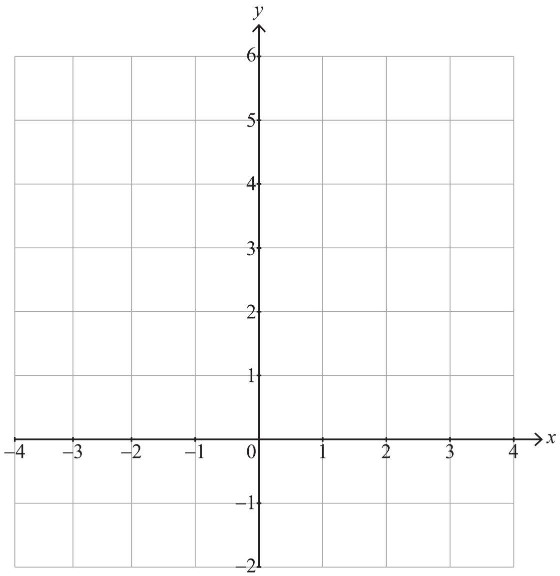
Let $f'(x)=\dfrac{8x}{\sqrt{2x^2+1}}$. Given that $f(0)=5$, find $f(x)$.

4. [Maximum mark: 5]

The following diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = -1$. The graph crosses the x-axis at $x = -1$ and $x = 1$, and the y-axis at $y = 2$.



On the following set of axes, sketch the graph of $y = [f(x)]^2 + 1$, clearly showing any asymptotes with their equations and the coordinates of any local maxima or minima.



Turn over

5. [Maximum mark: 5]

The functions f and g are defined such that $f(x) = \frac{x+3}{4}$ and $g(x) = 8x + 5$.

(a) Show that $(g \circ f)(x) = 2x + 11$. [2]

(b) Given that $(g \circ f)^{-1}(a) = 4$, find the value of a. [3]

6. [Maximum mark: 8]

(a) Show that $\log_9 (\cos 2x + 2) = \log_3 \sqrt{\cos 2x + 2}$. [3]

(b) Hence or otherwise solve $\log_3 (2\sin x) = \log_9 (\cos 2x + 2)$ for $0 < x < \frac{\pi}{2}$. [5]

Turn over

7. [Maximum mark: 7]

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} \frac{\pi x}{36} \sin\left(\frac{\pi x}{6}\right), & 0 \le x \le 6 \\ 0, & \text{otherwise} \end{cases}.$$

Find $\mathrm{P}(0 \le X \le 3)$.

8. [Maximum mark: 7]

The plane Π has the Cartesian equation $2x + y + 2z = 3$.

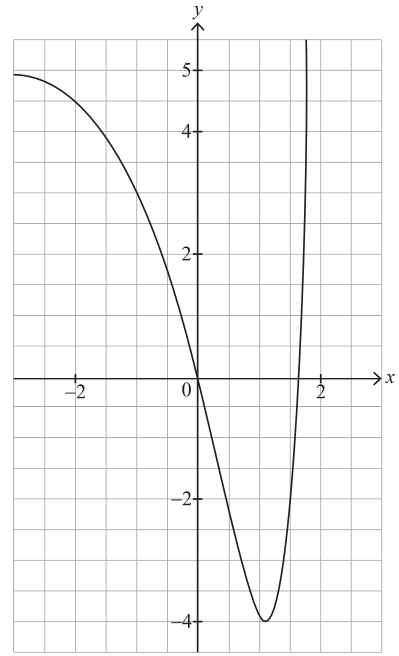
The line L has the vector equation $\boldsymbol{r} = \begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -2 \\ p \end{pmatrix}$, $\mu, p \in \mathbb{R}$. The acute angle between the line L and the plane Π is 30°.

Find the possible values of p.

Turn over

9. [Maximum mark: 8]

The function f is defined by $f(x) = \mathrm{e}^{2x} - 6\mathrm{e}^{x} + 5$, $x \in \mathbb{R}$, $x \leq a$. The graph of $y = f(x)$ is shown in the following diagram.



(a) Find the largest value of a such that f has an inverse function. [3]

(b) For this value of a, find an expression for $f^{-1}(x)$, stating its domain. [5]

(This question continues on the following page)

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 16]

Let $f(x) = \dfrac{\ln 5x}{kx}$ where $x > 0$, $k \in \mathbb{R}^+$.

(a) Show that $f'(x) = \dfrac{1 - \ln 5x}{kx^2}$. [3]

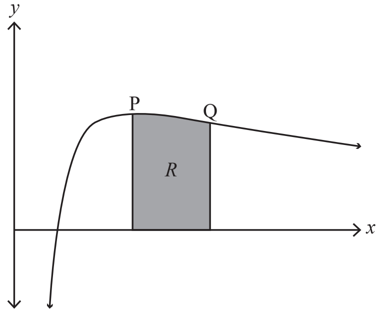
The graph of f has exactly one maximum point P.

(b) Find the x-coordinate of P. [3]

The second derivative of f is given by $f''(x) = \dfrac{2\ln 5x - 3}{kx^3}$. The graph of f has exactly one point of inflexion Q.

(c) Show that the x-coordinate of Q is $\dfrac{1}{5}e^{\frac{3}{2}}$. [3]

The region R is enclosed by the graph of f, the x-axis, and the vertical lines through the maximum point P and the point of inflexion Q.



(d) Given that the area of R is 3, find the value of k. [7]

Do **not** write solutions on this page.

11. [Maximum mark: 18]

(a) Express $-3 + \sqrt{3}i$ in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [5]

Let the roots of the equation $z^3 = -3 + \sqrt{3}i$ be u, v and w.

(b) Find u, v and w expressing your answers in the form $re^{i\theta}$, where $r > 0$ and $-\pi < \theta \le \pi$. [5]

On an Argand diagram, u, v and w are represented by the points U, V and W respectively.

(c) Find the area of triangle UVW. [4]

(d) By considering the sum of the roots u, v and w, show that $\cos\dfrac{5\pi}{18} + \cos\dfrac{7\pi}{18} + \cos\dfrac{17\pi}{18} = 0$. [4]

12. [Maximum mark: 21]

The function f is defined by $f(x) = e^{\sin x}$.

(a) Find the first two derivatives of $f(x)$ and hence find the Maclaurin series for $f(x)$ up to and including the x^2 term. [8]

(b) Show that the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero. [4]

(c) Using the Maclaurin series for $\arctan x$ and $e^{3x} - 1$, find the Maclaurin series for $\arctan(e^{3x} - 1)$ up to and including the x^3 term. [6]

(d) Hence, or otherwise, find $\lim\limits_{x \to 0} \dfrac{f(x) - 1}{\arctan(e^{3x} - 1)}$. [3]

Section A

1. attempt to substitute into $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ ***(M1)***

Note: Accept use of Venn diagram or other valid method.

$0.6 = 0.5 + 0.4 - \mathrm{P}(A \cap B)$ ***(A1)***

$\mathrm{P}(A \cap B) = 0.3$ (seen anywhere) ***A1***

attempt to substitute into $\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ ***(M1)***

$= \frac{0.3}{0.4}$

$\mathrm{P}(A \mid B) = 0.75\left(= \frac{3}{4}\right)$ ***A1***

Total [5 marks]

2. (a) attempting to expand the LHS ***(M1)***

$\mathrm{LHS} = (4n^2 - 4n + 1) + (4n^2 + 4n + 1)$ ***A1***

$= 8n^2 + 2 (= \mathrm{RHS})$ ***AG***

[2 marks]

(b) ***METHOD 1***

recognition that $2n-1$ and $2n+1$ represent two consecutive odd integers (for all odd integers n) ***R1***

$8n^2 + 2 = 2(4n^2 + 1)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

METHOD 2

recognition, *eg* that n and $n+2$ represent two consecutive odd integers (for $n \in \mathbb{Z}$) ***R1***

$n^2 + (n+2)^2 = 2(n^2 + 2n + 2)$ ***A1***

valid reason *eg* divisible by 2 (2 is a factor) ***R1***

so the sum of the squares of any two consecutive odd integers is even ***AG***

[3 marks]

Total [5 marks]

3. attempt to integrate ***(M1)***

$u = 2x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 4x$

$\int \frac{8x}{\sqrt{2x^2+1}}\,\mathrm{d}x = \int \frac{2}{\sqrt{u}}\,\mathrm{d}u$ ***(A1)***

EITHER

$= 4\sqrt{u}\,(+C)$ ***A1***

OR

$= 4\sqrt{2x^2+1}\,(+C)$ ***A1***

THEN

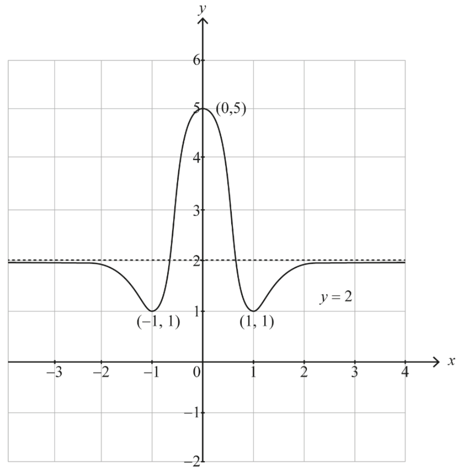
correct substitution into **their** integrated function (must have C) ***(M1)***

$5 = 4 + C \Rightarrow C = 1$

$f(x) = 4\sqrt{2x^2+1} + 1$ ***A1***

Total [5 marks]

4.



no y values below 1 ***A1***
horizontal asymptote at $y=2$ with curve approaching from below as $x \to \pm\infty$ ***A1***
$(\pm 1, 1)$ local minima ***A1***
$(0,5)$ local maximum ***A1***
smooth curve and smooth stationary points ***A1***
Total [5 marks]

5. (a) attempt to form composition ***M1***
correct substitution $g\left(\frac{x+3}{4}\right)=8\left(\frac{x+3}{4}\right)+5$ ***A1***
$(g \circ f)(x)=2x+11$ ***AG***
[2 marks]

(b) attempt to substitute 4 (seen anywhere) ***(M1)***
correct equation $a=2\times 4+11$ ***(A1)***
$a=19$ ***A1***
[3 marks]

Total [5 marks]

6. (a) attempting to use the change of base rule ***M1***
$\log_9(\cos 2x+2)=\frac{\log_3(\cos 2x+2)}{\log_3 9}$ ***A1***
$=\frac{1}{2}\log_3(\cos 2x+2)$ ***A1***
$=\log_3\sqrt{\cos 2x+2}$ ***AG***
[3 marks]

(b) $\log_3(2\sin x)=\log_3\sqrt{\cos 2x+2}$
$2\sin x=\sqrt{\cos 2x+2}$ ***M1***
$4\sin^2 x=\cos 2x+2$ (or equivalent) ***A1***
use of $\cos 2x=1-2\sin^2 x$ ***(M1)***
$6\sin^2 x=3$
$\sin x=(\pm)\frac{1}{\sqrt{2}}$ ***A1***
$x=\frac{\pi}{4}$ ***A1***

Note: Award ***A0*** if solutions other than $x=\frac{\pi}{4}$ are included.

[5 marks]

Total [8 marks]

7. attempting integration by parts, *eg*

$u = \frac{\pi x}{36}, \mathrm{d}u = \frac{\pi}{36}\mathrm{d}x, \mathrm{d}v = \sin\left(\frac{\pi x}{6}\right)\mathrm{d}x, v = -\frac{6}{\pi}\cos\left(\frac{\pi x}{6}\right)$ ***(M1)***

$\mathrm{P}(0 \le X \le 3) = \frac{\pi}{36}\left(\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 + \frac{6}{\pi}\int_0^3 \cos\left(\frac{\pi x}{6}\right)\mathrm{d}x\right)$ (or equivalent) ***A1A1***

Note: Award ***A1*** for a correct uv and ***A1*** for a correct $\int v \, \mathrm{d}u$.

attempting to substitute limits ***M1***

$\frac{\pi}{36}\left[-\frac{6x}{\pi}\cos\left(\frac{\pi x}{6}\right)\right]_0^3 = 0$ ***(A1)***

so $\mathrm{P}(0 \le X \le 3) = \frac{1}{\pi}\left[\sin\left(\frac{\pi x}{6}\right)\right]_0^3$ (or equivalent) ***A1***

$= \frac{1}{\pi}$ ***A1***

Total [7 marks]

8. recognition that the angle between the normal and the line is 60° (seen anywhere) ***R1***

attempt to use the formula for the scalar product ***M1***

$\cos 60° = \frac{\left|\begin{pmatrix}2\\1\\2\end{pmatrix} \bullet \begin{pmatrix}1\\-2\\p\end{pmatrix}\right|}{\sqrt{9} \times \sqrt{1+4+p^2}}$ ***A1***

$\frac{1}{2} = \frac{|2p|}{3\sqrt{5+p^2}}$ ***A1***

$3\sqrt{5+p^2} = 4|p|$

attempt to square both sides ***M1***

$9(5+p^2) = 16p^2 \Rightarrow 7p^2 = 45$

$p = \pm 3\sqrt{\frac{5}{7}}$ (or equivalent) ***A1A1***

Total [7 marks]

9. (a) attempt to differentiate and set equal to zero ***M1***

$f'(x) = 2\mathrm{e}^{2x} - 6\mathrm{e}^x = 2\mathrm{e}^x(\mathrm{e}^x - 3) = 0$ ***A1***

minimum at $x = \ln 3$

$a = \ln 3$ ***A1***

[3 marks]

(b) **Note:** Interchanging x and y can be done at any stage.

$y = (\mathrm{e}^x - 3)^2 - 4$ ***(M1)***

$\mathrm{e}^x - 3 = \pm\sqrt{y+4}$ ***A1***

as $x \le \ln 3$, $x = \ln\left(3 - \sqrt{y+4}\right)$ ***R1***

so $f^{-1}(x) = \ln\left(3 - \sqrt{x+4}\right)$ ***A1***

domain of f^{-1} is $x \in \mathbb{R}, -4 \le x < 5$ ***A1***

[5 marks]

Total [8 marks]

Section B

10. (a) attempt to use quotient rule ***(M1)***

correct substitution into quotient rule

$f'(x) = \dfrac{5kx\left(\dfrac{1}{5x}\right) - k\ln 5x}{(kx)^2}$ (or equivalent) ***A1***

$= \dfrac{k - k\ln 5x}{k^2x^2}, \left(k \in \mathbb{R}^+\right)$ ***A1***

$= \dfrac{1 - \ln 5x}{kx^2}$ ***AG***

[3 marks]

(b) $f'(x) = 0$ ***M1***

$\dfrac{1 - \ln 5x}{kx^2} = 0$

$\ln 5x = 1$ ***(A1)***

$x = \dfrac{\mathrm{e}}{5}$ ***A1***

[3 marks]

(c) $f''(x) = 0$ ***M1***

$\dfrac{2\ln 5x - 3}{kx^3} = 0$

$\ln 5x = \dfrac{3}{2}$ ***A1***

$5x = \mathrm{e}^{\frac{3}{2}}$ ***A1***

so the point of inflexion occurs at $x = \dfrac{1}{5}\mathrm{e}^{\frac{3}{2}}$ ***AG***

[3 marks]

continued…

Question 10 continued

(d) attempt to integrate ***(M1)***

$u = \ln 5x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = \dfrac{1}{x}$

$\displaystyle\int \frac{\ln 5x}{kx}\,\mathrm{d}x = \frac{1}{k}\int u\ \mathrm{d}u$ ***(A1)***

EITHER

$= \dfrac{u^2}{2k}$ ***A1***

so $\displaystyle\frac{1}{k}\int_1^{\frac{3}{2}} u\ \mathrm{d}u = \left[\frac{u^2}{2k}\right]_1^{\frac{3}{2}}$ ***A1***

OR

$= \dfrac{(\ln 5x)^2}{2k}$ ***A1***

so $\displaystyle\int_{\frac{\mathrm{e}}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}} \frac{\ln 5x}{kx}\,\mathrm{d}x = \left[\frac{(\ln 5x)^2}{2k}\right]_{\frac{\mathrm{e}}{5}}^{\frac{1}{5}\mathrm{e}^{\frac{3}{2}}}$ ***A1***

THEN

$= \dfrac{1}{2k}\left(\dfrac{9}{4} - 1\right)$

$= \dfrac{5}{8k}$ ***A1***

setting **their** expression for area equal to 3 ***M1***

$\dfrac{5}{8k} = 3$

$k = \dfrac{5}{24}$ ***A1***

[7 marks]

Total [16 marks]

11. (a) attempt to find modulus ***(M1)***

$r = 2\sqrt{3}\left(=\sqrt{12}\right)$ ***A1***

attempt to find argument in the correct quadrant ***(M1)***

$\theta = \pi + \arctan\left(-\frac{\sqrt{3}}{3}\right)$ ***A1***

$= \frac{5\pi}{6}$ ***A1***

$-3+\sqrt{3}\mathrm{i} = \sqrt{12}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\left(= 2\sqrt{3}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}\right)$

[5 marks]

(b) attempt to find a root using de Moivre's theorem ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{5\pi\mathrm{i}}{18}}$ ***A1***

attempt to find further two roots by adding and subtracting $\frac{2\pi}{3}$ to the argument ***M1***

$12^{\frac{1}{6}}\mathrm{e}^{-\frac{7\pi\mathrm{i}}{18}}$ ***A1***

$12^{\frac{1}{6}}\mathrm{e}^{\frac{17\pi\mathrm{i}}{18}}$ ***A1***

Note: Ignore labels for u, v and w at this stage.

[5 marks]

continued…

Question 11 continued

(c) ***METHOD 1***

attempting to find the total area of (congruent) triangles $\mathrm{UOV}, \mathrm{VOW}$ and UOW ***M1***

$\mathrm{Area} = 3\left(\frac{1}{2}\right)\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\sin\frac{2\pi}{3}$ ***A1A1***

Note: Award ***A1*** for $\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)$ and ***A1*** for $\sin\frac{2\pi}{3}$.

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

METHOD 2

$\mathrm{UV}^2 = \left(12^{\frac{1}{6}}\right)^2 + \left(12^{\frac{1}{6}}\right)^2 - 2\left(12^{\frac{1}{6}}\right)\left(12^{\frac{1}{6}}\right)\cos\frac{2\pi}{3}$ (or equivalent) ***A1***

$\mathrm{UV} = \sqrt{3}\left(12^{\frac{1}{6}}\right)$ (or equivalent) ***A1***

attempting to find the area of UVW using $\mathrm{Area} = \frac{1}{2}\times\mathrm{UV}\times\mathrm{VW}\times\sin\alpha$ for example ***M1***

$\mathrm{Area} = \frac{1}{2}\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\left(\sqrt{3}\times 12^{\frac{1}{6}}\right)\sin\frac{\pi}{3}$

$= \frac{3\sqrt{3}}{4}\left(12^{\frac{1}{3}}\right)$ (or equivalent) ***A1***

[4 marks]

(d) $u+v+w=0$ ***R1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\mathrm{i}\sin\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\mathrm{i}\sin\frac{5\pi}{18}+\cos\frac{17\pi}{18}+\mathrm{i}\sin\frac{17\pi}{18}\right)=0$ ***A1***

consideration of real parts ***M1***

$12^{\frac{1}{6}}\left(\cos\left(-\frac{7\pi}{18}\right)+\cos\frac{5\pi}{18}+\cos\frac{17\pi}{18}\right)=0$

$\cos\left(-\frac{7\pi}{18}\right)=\cos\frac{7\pi}{18}$ explicitly stated ***A1***

$\cos\frac{5\pi}{18}+\cos\frac{7\pi}{18}+\cos\frac{17\pi}{18}=0$ ***AG***

[4 marks]

Total [18 marks]

12. (a) attempting to use the chain rule to find the first derivative ***M1***

$f'(x)=(\cos x)\mathrm{e}^{\sin x}$ ***A1***

attempting to use the product rule to find the second derivative ***M1***

$f''(x)=\mathrm{e}^{\sin x}\left(\cos^2 x-\sin x\right)$ (or equivalent) ***A1***

attempting to find $f(0)$, $f'(0)$ and $f''(0)$ ***M1***

$f(0)=1;\ f'(0)=(\cos 0)\mathrm{e}^{\sin 0}=1;\ f''(0)=\mathrm{e}^{\sin 0}\left(\cos^2 0-\sin 0\right)=1$ ***A1***

substitution into the Maclaurin formula $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\ldots$ ***M1***

so the Maclaurin series for $f(x)$ up to and including the x^2 term is $1+x+\frac{x^2}{2}$ ***A1***

[8 marks]

(b) ***METHOD 1***

attempting to differentiate $f''(x)$ ***M1***

$f'''(x)=(\cos x)\mathrm{e}^{\sin x}\left(\cos^2 x-\sin x\right)-(\cos x)\mathrm{e}^{\sin x}(2\sin x+1)$ (or equivalent) ***A2***

substituting $x=0$ into **their** $f'''(x)$ ***M1***

$f'''(0)=1(1-0)-1(0+1)=0$

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

METHOD 2

substituting $\sin x$ into the Maclaurin series for e^x ***(M1)***

$\mathrm{e}^{\sin x}=1+\sin x+\frac{\sin^2 x}{2!}+\frac{\sin^3 x}{3!}+\ldots$

substituting Maclaurin series for $\sin x$ ***M1***

$$\mathrm{e}^{\sin x}=1+\left(x-\frac{x^3}{3!}+\ldots\right)+\frac{\left(x-\frac{x^3}{3!}+\ldots\right)^2}{2!}+\frac{\left(x-\frac{x^3}{3!}+\ldots\right)^3}{3!}+\ldots$$ ***A1***

coefficient of x^3 is $-\frac{1}{3!}+\frac{1}{3!}=0$ ***A1***

so the coefficient of x^3 in the Maclaurin series for $f(x)$ is zero ***AG***

[4 marks]

continued…

Question 12 continued

(c) substituting $3x$ into the Maclaurin series for e^x ***M1***

$\mathrm{e}^{3x}=1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots$ ***A1***

substituting $\left(\mathrm{e}^{3x}-1\right)$ into the Maclaurin series for $\arctan x$ ***M1***

$$\arctan\left(\mathrm{e}^{3x}-1\right)=\left(\mathrm{e}^{3x}-1\right)-\frac{\left(\mathrm{e}^{3x}-1\right)^3}{3}+\frac{\left(\mathrm{e}^{3x}-1\right)^5}{5}-\ldots$$

$$=\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots\right)-\frac{\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\ldots\right)^3}{3}+\ldots$$ ***A1***

selecting correct terms from above ***M1***

$$=\left(3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}\right)-\frac{(3x)^3}{3}$$

$$=3x+\frac{9x^2}{2}-\frac{9x^3}{2}$$ ***A1***

[6 marks]

(d) ***METHOD 1***

substitution of **their** series ***M1***

$$\lim_{x\to 0}\frac{x+\frac{x^2}{2}+\ldots}{3x+\frac{9x^2}{2}+\ldots}$$ ***A1***

$$=\lim_{x\to 0}\frac{1+\frac{x}{2}+\ldots}{3+\frac{9x}{2}+\ldots}$$

$$=\frac{1}{3}$$ ***A1***

METHOD 2

use of l'Hôpital's rule ***M1***

$$\lim_{x\to 0}\frac{(\cos x)\mathrm{e}^{\sin x}}{\frac{3\mathrm{e}^{3x}}{1+\left(\mathrm{e}^{3x}-1\right)^2}}$$ (or equivalent) ***A1***

$$=\frac{1}{3}$$ ***A1***

[3 marks]

Total [21 marks]

Mathematics: analysis and approaches
Higher level
Paper 2

Specimen

Candidate session number

2 hours

Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.

12 pages

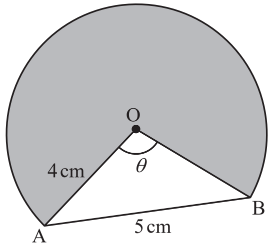
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 6]

The following diagram shows part of a circle with centre O and radius $4\,\mathrm{cm}$.



Chord AB has a length of $5\,\mathrm{cm}$ and $\mathrm{A\hat{O}B} = \theta$.

(a) Find the value of θ, giving your answer in radians. [3]

(b) Find the area of the shaded region. [3]

2. [Maximum mark: 6]

On 1st January 2020, Laurie invests $\$P$ in an account that pays a nominal annual interest rate of 5.5%, compounded **quarterly**.

The amount of money in Laurie's account **at the end of each year** follows a geometric sequence with common ratio, r.

(a) Find the value of r, giving your answer to four significant figures. [3]

Laurie makes no further deposits to or withdrawals from the account.

(b) Find the year in which the amount of money in Laurie's account will become double the amount she invested. [3]

Turn over

3. [Maximum mark: 6]

A six-sided biased die is weighted in such a way that the probability of obtaining a "six" is $\frac{7}{10}$.

The die is tossed five times. Find the probability of obtaining

(a) at most three "sixes". [3]

(b) the third "six" on the fifth toss. [3]

4. [Maximum mark: 7]

The following table below shows the marks scored by seven students on two different mathematics tests.

Test 1 (x)	15	23	25	30	34	34	40
Test 2 (y)	20	26	27	32	35	37	35

Let L_1 be the regression line of x on y. The equation of the line L_1 can be written in the form $x = ay + b$.

(a) Find the value of a and the value of b. [2]

Let L_2 be the regression line of y on x. The lines L_1 and L_2 pass through the same point with coordinates (p, q).

(b) Find the value of p and the value of q. [3]

(c) Jennifer was absent for the first test but scored 29 marks on the second test. Use an appropriate regression equation to estimate Jennifer's mark on the first test. [2]

5. [Maximum mark: 7]

The displacement, in centimetres, of a particle from an origin, O, at time t seconds, is given by $s(t) = t^2 \cos t + 2t \sin t,\ 0 \le t \le 5$.

(a) Find the maximum distance of the particle from O. [3]

(b) Find the acceleration of the particle at the instant it first changes direction. [4]

6. [Maximum mark: 6]

In a city, the number of passengers, X, who ride in a taxi has the following probability distribution.

x	1	2	3	4	5
$\mathrm{P}(X=x)$	0.60	0.30	0.03	0.05	0.02

After the opening of a new highway that charges a toll, a taxi company introduces a charge for passengers who use the highway. The charge is $2.40 per taxi plus $1.20 per passenger. Let T represent the amount, in dollars, that is charged by the taxi company per ride.

(a) Find $\mathrm{E}(T)$. [4]

(b) Given that $\mathrm{Var}(X) = 0.8419$, find $\mathrm{Var}(T)$. [2]

Turn over

7. [Maximum mark: 5]

Two ships, A and B, are observed from an origin O. Relative to O, their position vectors at time t hours after midday are given by

$$\boldsymbol{r}_{\mathrm{A}} = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + t\begin{pmatrix} 5 \\ 8 \end{pmatrix}$$

$$\boldsymbol{r}_{\mathrm{B}} = \begin{pmatrix} 7 \\ -3 \end{pmatrix} + t\begin{pmatrix} 0 \\ 12 \end{pmatrix}$$

where distances are measured in kilometres.

Find the minimum distance between the two ships.

8. [Maximum mark: 7]

The complex numbers w and z satisfy the equations

$$\frac{w}{z} = 2\mathrm{i}$$
$$z^* - 3w = 5 + 5\mathrm{i}.$$

Find w and z in the form $a + b\mathrm{i}$ where $a, b \in \mathbb{Z}$.

Turn over

9. [Maximum mark: 5]

Consider the graphs of $y = \frac{x^2}{x-3}$ and $y = m(x+3)$, $m \in \mathbb{R}$.

Find the set of values for m such that the two graphs have no intersection points.

Do **not** write solutions on this page.

Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

10. [Maximum mark: 15]

The length, $X\,\text{mm}$, of a certain species of seashell is normally distributed with mean 25 and variance, σ^2.

The probability that X is less than 24.15 is 0.1446.

(a) Find $\mathrm{P}(24.15 < X < 25)$. [2]

(b) (i) Find σ, the standard deviation of X.

(ii) Hence, find the probability that a seashell selected at random has a length greater than $26\,\text{mm}$. [5]

A random sample of 10 seashells is collected on a beach. Let Y represent the number of seashells with lengths greater than $26\,\text{mm}$.

(c) Find $\mathrm{E}(Y)$. [3]

(d) Find the probability that exactly three of these seashells have a length greater than $26\,\text{mm}$. [2]

A seashell selected at random has a length less than $26\,\text{mm}$.

(e) Find the probability that its length is between $24.15\,\text{mm}$ and $25\,\text{mm}$. [3]

Turn over

Do **not** write solutions on this page.

11. [Maximum mark: 21]

A large tank initially contains pure water. Water containing salt begins to flow into the tank The solution is kept uniform by stirring and leaves the tank through an outlet at its base. Let x grams represent the amount of salt in the tank and let t minutes represent the time since the salt water began flowing into the tank.

The rate of change of the amount of salt in the tank, $\frac{\mathrm{d}x}{\mathrm{d}t}$, is described by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = 10\mathrm{e}^{-\frac{t}{4}} - \frac{x}{t+1}$.

(a) Show that $t+1$ is an integrating factor for this differential equation. [2]

(b) Hence, by solving this differential equation, show that $x(t) = \frac{200 - 40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$. [8]

(c) Sketch the graph of x versus t for $0 \le t \le 60$ and hence find the maximum amount of salt in the tank and the value of t at which this occurs. [5]

(d) Find the value of t at which the amount of salt in the tank is decreasing most rapidly. [2]

The rate of change of the amount of salt leaving the tank is equal to $\frac{x}{t+1}$.

(e) Find the amount of salt that left the tank during the first 60 minutes. [4]

12. [Maximum mark: 19]

(a) Show that $\cot 2\theta = \frac{1 - \tan^2\theta}{2\tan\theta}$. [1]

(b) Verify that $x = \tan\theta$ and $x = -\cot\theta$ satisfy the equation $x^2 + (2\cot 2\theta)x - 1 = 0$. [7]

(c) Hence, or otherwise, show that the exact value of $\tan\frac{\pi}{12} = 2 - \sqrt{3}$. [5]

(d) Using the results from parts (b) and (c) find the exact value of $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$.

Give your answer in the form $a + b\sqrt{3}$ where $a, b \in \mathbb{Z}$. [6]

Section A

1. (a) ***METHOD 1***

attempt to use the cosine rule ***(M1)***

$\cos\theta = \frac{4^2 + 4^2 - 5^2}{2 \times 4 \times 4}$ (or equivalent) ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

METHOD 2

attempt to split triangle AOB into two congruent right triangles ***(M1)***

$\sin\left(\frac{\theta}{2}\right) = \frac{2.5}{4}$ ***A1***

$\theta = 1.35$ ***A1***

[3 marks]

(b) attempt to find the area of the shaded region ***(M1)***

$\frac{1}{2} \times 4 \times 4 \times (2\pi - 1.35...)$ ***A1***

$= 39.5\ (\text{cm}^2)$ ***A1***

[3 marks]

Total [6 marks]

2. (a) $\left(1 + \frac{5.5}{4 \times 100}\right)^4$ ***(M1)(A1)***

$= 1.056$ ***A1***

[3 marks]

continued…

Question 2 continued

(b) ***EITHER***

$2P = P \times \left(1 + \frac{5.5}{100 \times 4}\right)^{4n}$ **OR** $2P = P \times (\text{their } (a))^m$ ***(M1)(A1)***

Note: Award ***(M1)*** for substitution into loan payment formula. Award ***(A1)*** for correct substitution.

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 5.5$
$\text{P/Y} = 4$
$\text{C/Y} = 4$
$n = 50.756\ldots$ ***(M1)(A1)***

OR

$\text{PV} = \pm 1$
$\text{FV} = \mp 2$
$\text{I\%} = 100(\text{their } (a) - 1)$
$\text{P/Y} = 1$
$\text{C/Y} = 1$ ***(M1)(A1)***

THEN

$\Rightarrow 12.7$ years

Laurie will have double the amount she invested during 2032 ***A1***

[3 marks]

Total [6 marks]

3. (a) recognition of binomial ***(M1)***

$X \sim \text{B}(5, 0.7)$

attempt to find $\text{P}(X \le 3)$ ***M1***

$= 0.472 (= 0.47178)$ ***A1***

[3 marks]

(b) recognition of 2 sixes in 4 tosses ***(M1)***

$\text{P}(\text{3rd six on the 5th toss}) = \left[\binom{4}{2} \times (0.7)^2 \times (0.3)^2\right] \times 0.7 (= 0.2646 \times 0.7)$ ***A1***

$= 0.185 (= 0.18522)$ ***A1***

[3 marks]

Total [6 marks]

4. (a) $a = 1.29$ and $b = -10.4$ ***A1A1***

[2 marks]

(b) recognising both lines pass through the mean point ***(M1)***

$p = 28.7$, $q = 30.3$ ***A2***

[3 marks]

(c) substitution into **their** x on y equation ***(M1)***

$x = 1.29082(29) - 10.3793$

$x = 27.1$ ***A1***

Note: Accept 27.

[2 marks]

Total [7 marks]

5. (a) use of a graph to find the coordinates of the local minimum ***(M1)***

$s = -16.513...$ ***(A1)***

maximum distance is $16.5\,\text{cm}$ (to the left of O) ***A1***

[3 marks]

(b) attempt to find time when particle changes direction *eg* considering the first maximum on the graph of s or the first t – intercept on the graph of s'. ***(M1)***

$t = 1.51986...$ ***(A1)***

attempt to find the gradient of s' for **their** value of *t*, $s''(1.51986...)$ ***(M1)***

$= -8.92\ (\text{cm/s}^2)$ ***A1***

[4 marks]

Total [7 marks]

6. (a) ***METHOD 1***

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(X) = (1\times0.60) + (2\times0.30) + (3\times0.03) + (4\times0.05) + (5\times0.02)$

$\mathrm{E}(X) = 1.59(\$)$ ***(A1)***

use of $\mathrm{E}(1.20X + 2.40) = 1.20\mathrm{E}(X) + 2.40$ ***(M1)***

$\mathrm{E}(T) = 1.20(1.59) + 2.40$

$= 4.31(\$)$ ***A1***

METHOD 2

attempting to find the probability distribution for T ***(M1)***

t	3.60	4.80	6.00	7.20	8.40
$\mathrm{P}(T=t)$	0.60	0.30	0.03	0.05	0.02

(A1)

attempting to use the expected value formula ***(M1)***

$\mathrm{E}(T) = (3.60\times0.60) + (4.80\times0.30) + (6.00\times0.03) + (7.20\times0.05) + (8.40\times0.02)$

$= 4.31(\$)$ ***A1***

[4 marks]

(b) ***METHOD 1***

using $\mathrm{Var}(1.20X + 2.40) = (1.20)^2\,\mathrm{Var}(X)$ with $\mathrm{Var}(X) = 0.8419$ ***(M1)***

$\mathrm{Var}(T) = 1.21$ ***A1***

METHOD 2

finding the standard deviation for **their** probability distribution found in part (a) ***(M1)***

$\mathrm{Var}(T) = (1.101...)^2$

$= 1.21$ ***A1***

Note: Award ***M1A1*** for $\mathrm{Var}(T) = (1.093...)^2 = 1.20$.

[2 marks]

Total [6 marks]

7. attempting to find $\boldsymbol{r}_{\text{B}} - \boldsymbol{r}_{\text{A}}$ for example ***(M1)***

$\boldsymbol{r}_{\text{B}} - \boldsymbol{r}_{\text{A}} = \begin{pmatrix} 3 \\ -6 \end{pmatrix} + t\begin{pmatrix} -5 \\ 4 \end{pmatrix}$

attempting to find $\left|\boldsymbol{r}_{\text{B}} - \boldsymbol{r}_{\text{A}}\right|$ ***M1***

distance $d(t) = \sqrt{(3-5t)^2 + (4t-6)^2} \left(= \sqrt{41t^2 - 78t + 45}\right)$ ***A1***

using a graph to find the d – coordinate of the local minimum ***M1***

the minimum distance between the ships is $2.81\ (\text{km}) \left(= \frac{11\sqrt{41}}{41}\ (\text{km})\right)$ ***A1***

Total [5 marks]

8. substituting $w = 2\text{i}z$ into $z^* - 3w = 5 + 5\text{i}$ ***M1***

$z^* - 6\text{i}z = 5 + 5\text{i}$ ***A1***

let $z = x + y\text{i}$

comparing real and imaginary parts of $(x - y\text{i}) - 6\text{i}(x + y\text{i}) = 5 + 5\text{i}$ ***M1***

to obtain $x + 6y = 5$ and $-6x - y = 5$ ***A1***

attempting to solve for x and y ***M1***

$x = -1$ and $y = 1$ and so $z = -1 + \text{i}$ ***A1***

hence $w = -2 - 2\text{i}$ ***A1***

---

9. ***METHOD 1***

sketching the graph of $y = \frac{x^2}{x-3}$ ($y = x + 3 + \frac{9}{x-3}$) ***M1***

the (oblique) asymptote has a gradient equal to 1
and so the maximum value of m is 1 ***R1***

consideration of a straight line steeper than the horizontal line joining
$(-3, 0)$ and $(0, 0)$ ***M1***

so $m > 0$ ***R1***

hence $0 < m \leq 1$ ***A1***

METHOD 2

attempting to eliminate y to form a quadratic equation in x ***M1***

$x^2 = m(x^2 - 9)$

$\Rightarrow (m-1)x^2 - 9m = 0$ ***A1***

EITHER

attempting to solve $-4(m-1)(-9m) < 0$ for m ***M1***

OR

attempting to solve $x^2 < 0$ *ie* $\frac{9m}{m-1} < 0$ $(m \neq 1)$ for m ***M1***

THEN

$\Rightarrow 0 < m < 1$ ***A1***

a valid reason to explain why $m = 1$ gives no solutions *eg* if $m = 1$,
$(m-1)x^2 - 9m = 0 \Rightarrow -9 = 0$ and so $0 < m \leq 1$ ***R1***

Total [5 marks]

Section B

10. (a) attempt to use the symmetry of the normal curve ***(M1)***
eg diagram, $0.5-0.1446$

$\mathrm{P}(24.15<X<25)=0.3554$ ***A1***

[2 marks]

(b) (i) use of inverse normal to find z score ***(M1)***
$z=-1.0598$

correct substitution $\frac{24.15-25}{\sigma}=-1.0598$ ***(A1)***

$\sigma=0.802$ ***A1***

(ii) $\mathrm{P}(X>26)=0.106$ ***(M1)A1***

[5 marks]

(c) recognizing binomial probability ***(M1)***
$\mathrm{E}(Y)=10\times 0.10621$ ***(A1)***
$=1.06$ ***A1***

[3 marks]

(d) $\mathrm{P}(Y=3)$ ***(M1)***
$=0.0655$ ***A1***

[2 marks]

(e) recognizing conditional probability ***(M1)***
correct substitution ***A1***

$\frac{0.3554}{1-0.10621}$
$=0.398$ ***A1***

[3 marks]

Total [15 marks]

11. (a) ***METHOD 1***

using $I(t)=\mathrm{e}^{\int P(t)\mathrm{d}t}$ ***M1***
$\mathrm{e}^{\int \frac{1}{t+1}\mathrm{d}t}$
$=\mathrm{e}^{\ln(t+1)}$ ***A1***
$=t+1$ ***AG***

METHOD 2

attempting product rule differentiation on $\frac{\mathrm{d}}{\mathrm{d}t}\left(x\left(t+1\right)\right)$ ***M1***

$\frac{\mathrm{d}}{\mathrm{d}t}\left(x\left(t+1\right)\right)=\frac{\mathrm{d}x}{\mathrm{d}t}\left(t+1\right)+x$
$=\left(t+1\right)\left(\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{x}{t+1}\right)$ ***A1***
so $t+1$ is an integrating factor for this differential equation ***AG***

[2 marks]

continued…

Question 11 continued

(b) attempting to multiply through by $(t+1)$ and rearrange to give **(M1)**

$(t+1)\frac{\mathrm{d}x}{\mathrm{d}t}+x=10(t+1)\mathrm{e}^{-\frac{t}{4}}$ **A1**

$\frac{\mathrm{d}}{\mathrm{d}t}\left(x(t+1)\right)=10(t+1)\mathrm{e}^{-\frac{t}{4}}$

$x(t+1)=\int 10(t+1)\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$ **A1**

attempting to integrate the RHS by parts **M1**

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}+40\int \mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t$

$=-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+C$ **A1**

Note: Condone the absence of C .

EITHER

substituting $t=0, x=0 \Rightarrow C=200$ **M1**

$x=\frac{-40(t+1)\mathrm{e}^{-\frac{t}{4}}-160\mathrm{e}^{-\frac{t}{4}}+200}{t+1}$ **A1**

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ **M1**

OR

using $-40\mathrm{e}^{-\frac{t}{4}}$ as the highest common factor of $-40(t+1)\mathrm{e}^{-\frac{t}{4}}$ and $-160\mathrm{e}^{-\frac{t}{4}}$ giving

$x(t+1)=-40\mathrm{e}^{-\frac{t}{4}}(t+5)+C$ (or equivalent) **M1A1**

substituting $t=0,\ x=0 \Rightarrow C=200$ **M1**

THEN

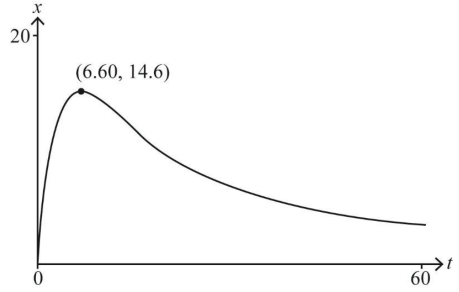
$x(t)=\frac{200-40\mathrm{e}^{-\frac{t}{4}}(t+5)}{t+1}$ ***AG***

[8 marks]

continued…

Question 11 continued

(c)



graph starts at the origin and has a local maximum (coordinates not required) ***A1***

sketched for $0 \le t \le 60$ ***A1***

correct concavity for $0 \le t \le 60$ ***A1***

maximum amount of salt is 14.6 (grams) at $t = 6.60$ (minutes) ***A1A1***

[5 marks]

(d) using an appropriate graph or equation (first or second derivative) ***M1***

amount of salt is decreasing most rapidly at $t=12.9$ (minutes) ***A1***

[2 marks]

(e) **EITHER**

attempting to form an integral representing the amount of salt that left the tank ***M1***

$\int_0^{60}\frac{x(t)}{t+1}\mathrm{d}t$

$\int_0^{60}\frac{200-40\mathrm{e}^{-\frac{t}{4}}(t+5)}{(t+1)^2}\mathrm{d}t$ ***A1***

OR

attempting to form an integral representing the amount of salt that entered the tank minus the amount of salt in the tank at $t=60$(minutes) ***M1***

amount of salt that left the tank is $\int_0^{60} 10\mathrm{e}^{-\frac{t}{4}}\,\mathrm{d}t - x(60)$ ***A1***

THEN

$=36.7$ (grams) ***A2***

[4 marks]

Total [21 marks]

12. (a) stating the relationship between cot and tan and stating the identity for $\tan 2\theta$ ***M1***

$\cot 2\theta = \frac{1}{\tan 2\theta}$ and $\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$

$\Rightarrow \cot 2\theta = \frac{1-\tan^2\theta}{2\tan\theta}$ ***AG***

[1 mark]

(b) ***METHOD 1***

attempting to substitute $\tan\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \tan^2\theta + 2\tan\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$\tan^2\theta + 1 - \tan^2\theta - 1 = 0 (= \text{RHS})$ ***A1***

so $x = \tan\theta$ satisfies the equation ***AG***

attempting to substitute $-\cot\theta$ for x and using the result from (a) ***M1***

$\text{LHS} = \cot^2\theta - 2\cot\theta\left(\frac{1-\tan^2\theta}{2\tan\theta}\right) - 1$ ***A1***

$= \frac{1}{\tan^2\theta} - \left(\frac{1-\tan^2\theta}{\tan^2\theta}\right) - 1$ ***A1***

$\frac{1}{\tan^2\theta} - \frac{1}{\tan^2\theta} + 1 - 1 = 0 (= \text{RHS})$ ***A1***

so $x = -\cot\theta$ satisfies the equation ***AG***

METHOD 2

let $\alpha = \tan\theta$ and $\beta = -\cot\theta$

attempting to find the sum of roots ***M1***

$\alpha + \beta = \tan\theta - \frac{1}{\tan\theta}$

$= \frac{\tan^2\theta - 1}{\tan\theta}$ ***A1***

$= -2\cot 2\theta$ (from part (a)) ***A1***

attempting to find the product of roots ***M1***

$\alpha\beta = \tan\theta \times (-\cot\theta)$ ***A1***

$= -1$ ***A1***

the coefficient of x and the constant term in the quadratic are $2\cot 2\theta$ and -1 respectively ***R1***

hence the two roots are $\alpha = \tan\theta$ and $\beta = -\cot\theta$ ***AG***

[7 marks]

continued…

Question 12 continued

(c) ***METHOD 1***

$x = \tan\frac{\pi}{12}$ and $x = -\cot\frac{\pi}{12}$ are roots of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$ ***R1***

Note: Award ***R1*** if only $x = \tan\frac{\pi}{12}$ is stated as a root of $x^2 + \left(2\cot\frac{\pi}{6}\right)x - 1 = 0$.

$x^2 + 2\sqrt{3}x - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$x = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ($-\cot\frac{\pi}{12} < 0$) ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

METHOD 2

attempting to substitute $\theta = \frac{\pi}{12}$ into the identity for $\tan 2\theta$ ***M1***

$\tan\frac{\pi}{6} = \frac{2\tan\frac{\pi}{12}}{1-\tan^2\frac{\pi}{12}}$

$\tan^2\frac{\pi}{12} + 2\sqrt{3}\tan\frac{\pi}{12} - 1 = 0$ ***A1***

attempting to solve **their** quadratic equation ***M1***

$\tan\frac{\pi}{12} = -\sqrt{3} \pm 2$ ***A1***

$\tan\frac{\pi}{12} > 0$ ***R1***

so $\tan\frac{\pi}{12} = 2 - \sqrt{3}$ ***AG***

[5 marks]

(d) $\tan\frac{\pi}{24} - \cot\frac{\pi}{24}$ is the sum of the roots of $x^2 + \left(2\cot\frac{\pi}{12}\right)x - 1 = 0$ ***R1***

$\tan\frac{\pi}{24} - \cot\frac{\pi}{24} = -2\cot\frac{\pi}{12}$ ***A1***

$= \frac{-2}{2-\sqrt{3}}$ ***A1***

attempting to rationalise **their** denominator ***(M1)***

$= -4 - 2\sqrt{3}$ ***A1A1***

[6 marks]

Total [19 marks]

Mathematics: analysis and approaches
Higher level
Paper 3

Specimen

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

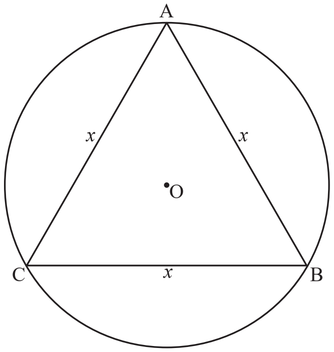
5 pages

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

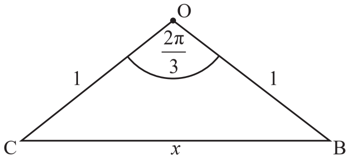
1. [Maximum mark: 30]

This question asks you to investigate regular n-sided polygons inscribed and circumscribed in a circle, and the perimeter of these as n tends to infinity, to make an approximation for π.

(a) Consider an equilateral triangle ABC of side length, x units, inscribed in a circle of radius 1 unit and centre O as shown in the following diagram.



The equilateral triangle ABC can be divided into three smaller isosceles triangles, each subtending an angle of $\frac{2\pi}{3}$ at O, as shown in the following diagram.



Using right-angled trigonometry or otherwise, show that the perimeter of the equilateral triangle ABC is equal to $3\sqrt{3}$ units. [3]

(b) Consider a square of side length, x units, inscribed in a circle of radius 1 unit.
By dividing the inscribed square into four isosceles triangles, find the exact perimeter of the inscribed square. [3]

(This question continues on the following page)

(Question 1 continued)

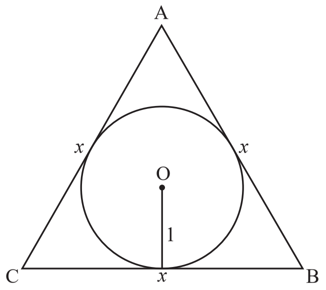
(c) Find the perimeter of a regular hexagon, of side length, x units, inscribed in a circle of radius 1 unit. [2]

Let $P_i(n)$ represent the perimeter of any n-sided regular polygon inscribed in a circle of radius 1 unit.

(d) Show that $P_i(n) = 2n\sin\left(\frac{\pi}{n}\right)$. [3]

(e) Use an appropriate Maclaurin series expansion to find $\lim_{n\to\infty} P_i(n)$ and interpret this result geometrically. [5]

Consider an equilateral triangle ABC of side length, x units, circumscribed about a circle of radius 1 unit and centre O as shown in the following diagram.



Let $P_c(n)$ represent the perimeter of any n-sided regular polygon circumscribed about a circle of radius 1 unit.

(f) Show that $P_c(n) = 2n\tan\left(\frac{\pi}{n}\right)$. [4]

Consider the function $P(x) = 2x\tan\left(\frac{\pi}{x}\right)$ where $x \in \mathbb{R}, x > 2$.

(g) (i) By writing $P(x)$ in the form $\dfrac{2\tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}}$, find $\lim_{x\to\infty} P(x)$.

(ii) Hence state the value of $\lim_{n\to\infty} P_c(n)$ for integers $n > 2$. [5]

(This question continues on the following page)

Turn over

(Question 1 continued)

(h) Use the results from part (d) and part (f) to determine an inequality for the value of π in terms of n. [2]

The inequality found in part (h) can be used to determine lower and upper bound approximations for the value of π.

(i) Determine the least value for n such that the lower bound and upper bound approximations are both within 0.005 of π. [3]

2. [Maximum mark: 25]

This question asks you to investigate some properties of the sequence of functions of the form $f_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$ and $n \in \mathbb{Z}^+$.

Important: When sketching graphs in this question, you are **not** required to find the coordinates of any axes intercepts or the coordinates of any stationary points unless requested.

(a) On the same set of axes, sketch the graphs of $y = f_1(x)$ and $y = f_3(x)$ for $-1 \le x \le 1$. [2]

(b) For odd values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for odd values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(c) On a new set of axes, sketch the graphs of $y = f_2(x)$ and $y = f_4(x)$ for $-1 \le x \le 1$. [2]

(d) For even values of $n > 2$, use your graphic display calculator to systematically vary the value of n. Hence suggest an expression for even values of n describing, in terms of n, the number of

(i) local maximum points;

(ii) local minimum points. [4]

(e) Solve the equation $f_n'(x) = 0$ and hence show that the stationary points on the graph of $y = f_n(x)$ occur at $x = \cos\frac{k\pi}{n}$ where $k \in \mathbb{Z}^+$ and $0 < k < n$. [4]

The sequence of functions, $f_n(x)$, defined above can be expressed as a sequence of polynomials of degree n.

(f) Use an appropriate trigonometric identity to show that $f_2(x) = 2x^2 - 1$. [2]

Consider $f_{n+1}(x) = \cos((n+1)\arccos x)$.

(g) Use an appropriate trigonometric identity to show that $f_{n+1}(x) = \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$. [2]

(h) Hence

(i) show that $f_{n+1}(x) + f_{n-1}(x) = 2xf_n(x)$, $n \in \mathbb{Z}^+$;

(ii) express $f_3(x)$ as a cubic polynomial. [5]

---

1. (a) ***METHOD 1***

consider right-angled triangle OCX where $\text{CX} = \frac{x}{2}$

$\sin\frac{\pi}{3} = \frac{\frac{x}{2}}{1}$ ***M1A1***

$\Rightarrow \frac{x}{2} = \frac{\sqrt{3}}{2} \Rightarrow x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

METHOD 2

eg use of the cosine rule $x^2 = 1^2 + 1^2 - 2(1)(1)\cos\frac{2\pi}{3}$ ***M1A1***

$x = \sqrt{3}$ ***A1***

$P_i = 3 \times x = 3\sqrt{3}$ ***AG***

Note: Accept use of sine rule.

[3 marks]

(b) $\sin\frac{\pi}{4} = \frac{1}{x}$ where x = side of square ***M1***

$x = \sqrt{2}$ ***A1***

$P_i = 4\sqrt{2}$ ***A1***

[3 marks]

(c) 6 equilateral triangles $\Rightarrow x = 1$ ***A1***

$P_i = 6$ ***A1***

[2 marks]

(d) in right-angled triangle $\sin\left(\frac{\pi}{n}\right) = \frac{\frac{x}{2}}{1}$ ***M1***

$\Rightarrow x = 2\sin\left(\frac{\pi}{n}\right)$ ***A1***

$P_i = n \times x$

$P_i = n \times 2\sin\left(\frac{\pi}{n}\right)$ ***M1***

$P_i = 2n\sin\left(\frac{\pi}{n}\right)$ ***AG***

[3 marks]

continued…

Question 1 continued

(e) consider $\lim_{n \to \infty} 2n\sin\left(\frac{\pi}{n}\right)$

use of $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ ***M1***

$2n\sin\left(\frac{\pi}{n}\right) = 2n\left(\frac{\pi}{n} - \frac{\pi^3}{6n^3} + \frac{\pi^5}{120n^5} - \dots\right)$ ***(A1)***

$= 2\left(\pi - \frac{\pi^3}{6n^2} + \frac{\pi^5}{120n^4} - \dots\right)$ ***A1***

$\Rightarrow \lim_{n \to \infty} 2n\sin\left(\frac{\pi}{n}\right) = 2\pi$ ***A1***

as $n \to \infty$ polygon becomes a circle of radius 1 and $P_i = 2\pi$ ***R1***

[5 marks]

(f) consider an n-sided polygon of side length x

$2n$ right-angled triangles with angle $\frac{2\pi}{2n} = \frac{\pi}{n}$ at centre ***M1A1***

opposite side $\frac{x}{2} = \tan\left(\frac{\pi}{n}\right) \Rightarrow x = 2\tan\left(\frac{\pi}{n}\right)$ ***M1A1***

Perimeter $P_c = 2n\tan\left(\frac{\pi}{n}\right)$ ***AG***

[4 marks]

(g) (i) $\lim_{x \to \infty} P(x) = \lim_{x \to \infty} \frac{2\tan\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$ ***R1***

attempt to use L'Hôpital's rule ***M1***

$\lim_{x \to \infty} P(x) = \lim_{x \to \infty} \frac{-\frac{2\pi}{x^2}\sec^2\left(\frac{\pi}{x}\right)}{-\frac{1}{x^2}}$ ***A1***

$\lim_{x \to \infty} P(x) = 2\pi$ ***A1***

(ii) $\lim_{n \to \infty} P_c(n) = 2\pi$ ***A1***

[5 marks]

continued…

Question 1 continued

(h) $P_i < 2\pi < P_c$

$2n\sin\left(\frac{\pi}{n}\right) < 2\pi < 2n\tan\left(\frac{\pi}{n}\right)$ ***M1***

$n\sin\left(\frac{\pi}{n}\right) < \pi < n\tan\left(\frac{\pi}{n}\right)$ ***A1***

[2 marks]

(i) attempt to find the lower bound and upper bound approximations within 0.005 of π ***(M1)***

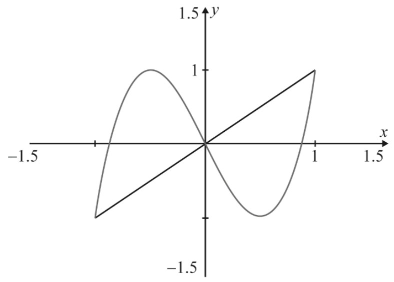
$n = 46$ ***A2***

[3 marks]

Total [30 marks]

2. (a) correct graph of $y = f_1(x)$ ***A1***

correct graph of $y = f_3(x)$ ***A1***



[2 marks]

(b) (i) graphical or tabular evidence that n has been systematically varied ***M1***

eg $n = 3$, 1 local maximum point and 1 local minimum point

$n = 5$, 2 local maximum points and 2 local minimum points

$n = 7$, 3 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-1}{2}$ local maximum points ***A1***

(ii) $\frac{n-1}{2}$ local minimum points ***A1***

Note: Allow follow through from an incorrect local maximum formula expression.

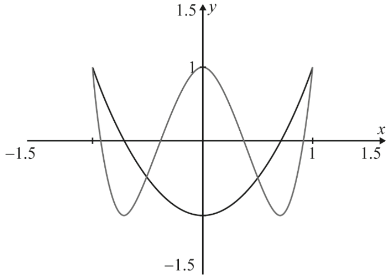
[4 marks]

continued…

Question 2 continued

(c) correct graph of $y = f_2(x)$ **A1**

correct graph of $y = f_4(x)$ **A1**



[2 marks]

(d) (i) graphical or tabular evidence that n has been systematically varied **M1**

eg $n = 2$, 0 local maximum point and 1 local minimum point
$n = 4$, 1 local maximum points and 2 local minimum points
$n = 6$, 2 local maximum points and 3 local minimum points ***(A1)***

$\frac{n-2}{2}$ local maximum points **A1**

(ii) $\frac{n}{2}$ local minimum points **A1**

[4 marks]

(e) $f_n(x) = \cos(n \arccos(x))$

$f'_n(x) = \frac{n \sin(n \arccos(x))}{\sqrt{1-x^2}}$ ***M1A1***

Note: Award ***M1*** for attempting to use the chain rule.

$f'_n(x) = 0 \Rightarrow n \sin(n \arccos(x)) = 0$ ***M1***

$n \arccos(x) = k\pi$ $(k \in \mathbb{Z}^+)$ ***A1***

leading to

$x = \cos\frac{k\pi}{n}$ ($k \in \mathbb{Z}^+$ and $0 < k < n$) ***AG***

[4 marks]

continued…

Question 2 continued

(f) $f_2(x) = \cos(2 \arccos x)$

$= 2(\cos(\arccos x))^2 - 1$ ***M1***

stating that $(\cos(\arccos x)) = x$ ***A1***

so $f_2(x) = 2x^2 - 1$ ***AG***

[2 marks]

(g) $f_{n+1}(x) = \cos((n+1)\arccos x)$

$= \cos(n \arccos x + \arccos x)$ ***A1***

use of $\cos(A+B) = \cos A \cos B - \sin A \sin B$ leading to ***M1***

$= \cos(n \arccos x)\cos(\arccos x) - \sin(n \arccos x)\sin(\arccos x)$ ***AG***

[2 marks]

(h) (i) $f_{n-1}(x) = \cos((n-1)\arccos x)$ ***A1***

$= \cos(n \arccos x)\cos(\arccos x) + \sin(n \arccos x)\sin(\arccos x)$ ***M1***

$f_{n+1}(x) + f_{n-1}(x) = 2\cos(n \arccos x)\cos(\arccos x)$ ***A1***

$= 2x f_n(x)$ ***AG***

(ii) $f_3(x) = 2x f_2(x) - f_1(x)$ ***(M1)***

$= 2x(2x^2 - 1) - x$

$= 4x^3 - 3x$ ***A1***

[5 marks]

Total [25 marks]