APQ: Vectors II (The Scalar and Vector Products of Vectors)

1. MJC/2017/Prelim/I/Q4(a)

The points A and B relative to the origin O have position vectors $3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $-3\mathbf{i}+2\mathbf{j}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [2]

(ii) Using the result in part **(i)**, find the shortest distance from B to line OA. [2]

(i) Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta = \frac{\begin{pmatrix}3\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}-3\\2\\0\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}-3\\2\\0\end{pmatrix}\right|}$$

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{19}\sqrt{13}}\right) = 134.4^\circ \text{ (1 d.p)} = 2.35 \text{ radian (3 s.f)}$$

(ii) Let h be the shortest distance from B to line OA.

$$\sin 134.42^\circ = \frac{h}{|\mathbf{b}|}$$

$$\begin{aligned} h &= \sqrt{13}\sin 134.42^\circ \\ &= 2.5752 \\ &= 2.58 \text{ units (3 s.f)} \end{aligned}$$

2. PJC/2018/MYE/P2/Q1

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point M is on AB produced such that $AB : BM = 3:2$ and the point N is on OB such that $ON : NB = 1:2$. It is given that $|\mathbf{a}| = 3$, $\mathbf{b}$ is a unit vector and $\angle AOB = 60°$.

(i) Find $\overrightarrow{OM}$ in terms of **a** and **b**. [2]

(ii) Show that the area of triangle OMN can be written as $k|\mathbf{a} \times \mathbf{b}|$, where k is a constant to be found. Hence evaluate the exact area of triangle OMN. [4]

(iii) Find the length of projection OM onto ON. [3]

(i) Using Ratio Theorem, $\overrightarrow{OB} = \dfrac{2\overrightarrow{OA} + 3\overrightarrow{OM}}{5}$

$5\overrightarrow{OB} = 2\overrightarrow{OA} + 3\overrightarrow{OM}$

$3\overrightarrow{OM} = 5\overrightarrow{OB} - 2\overrightarrow{OA}$

$\overrightarrow{OM} = \dfrac{1}{3}\left(5\overrightarrow{OB} - 2\overrightarrow{OA}\right) = \dfrac{1}{3}(5\mathbf{b} - 2\mathbf{a})$

(ii) Area of triangle OMN

$= \dfrac{1}{2}\left|\overrightarrow{OM} \times \overrightarrow{ON}\right|$

$= \dfrac{1}{2}\left|\dfrac{1}{3}(5\mathbf{b} - 2\mathbf{a}) \times \dfrac{1}{3}\mathbf{b}\right|$

$= \dfrac{1}{9}|\mathbf{a} \times \mathbf{b}| \qquad , k = \dfrac{1}{9}$

$= \dfrac{1}{9}\left||\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}}\right|$

$= \dfrac{1}{9}(3)(1)\sin 60°$

$= \dfrac{\sqrt{3}}{6}$

(iii) Length of projection of OM on ON

$= \left|\overrightarrow{OM} \cdot \widehat{ON}\right|$

$= \left|\dfrac{\frac{1}{3}(5\mathbf{b} - 2\mathbf{a}) \cdot \frac{1}{3}\mathbf{b}}{\left|\overrightarrow{ON}\right|}\right|$

Note: $\overrightarrow{ON} = \dfrac{1}{3}\mathbf{b}$ and $\left|\overrightarrow{ON}\right| = \dfrac{1}{3}$

$= = \dfrac{1}{3}|5\mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b}|$

$= \dfrac{1}{3}\left|5|\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos 60°\right|$

$= \dfrac{1}{3}\left|5 - 2(3)(1)\dfrac{1}{2}\right|$

$= \dfrac{2}{3}$

3. CJC Mid Year 9758/2021/3 (modified)

The vectors **a** and **b** are such that $\mathbf{a}\times\mathbf{b}=2(q+3)\mathbf{i}+(p-2)\mathbf{j}+p(q+3)\mathbf{k}$ and $\mathbf{a}\bullet\mathbf{b}=-\frac{1}{8}$ where p and q are non-zero constants. It is given that **b** is a unit vector.

(a) If $p=2$ and $q=-3$, state the relationship between **a** and **b**. [1]

(b) (i) Show that $\left(4\mathbf{a}+\frac{1}{2}\mathbf{b}\right)\times\left(2\mathbf{a}-\frac{3}{2}\mathbf{b}\right)=-14(q+3)\mathbf{i}-7(p-2)\mathbf{j}-7p(q+3)\mathbf{k}$. [2]

(ii) If $p=1$ and $\left|\left(4\mathbf{a}+\frac{1}{2}\mathbf{b}\right)\times\left(2\mathbf{a}-\frac{3}{2}\mathbf{b}\right)\right|=112$, find the exact values of q. [2]

(iii) Given instead that $4\mathbf{a}+\frac{1}{2}\mathbf{b}$ and $2\mathbf{a}-\frac{3}{2}\mathbf{b}$ are perpendicular, find the value of $|\mathbf{a}|$. [3]

(a)	The vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ are parallel
(b)(i)	$\left(4\mathbf{a}+\frac{1}{2}\mathbf{b}\right)\times\left(2\mathbf{a}-\frac{3}{2}\mathbf{b}\right)$ $=8\mathbf{a}\times\mathbf{a}-6\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\frac{3}{4}\mathbf{b}\times\mathbf{b}$ $=\underset{\sim}{0}-6\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}-\underset{\sim}{0}$ $=-7\mathbf{a}\times\mathbf{b}$ $=-7\left[2(q+3)\mathbf{i}+(p-2)\mathbf{j}+p(q+3)\mathbf{k}\right]$ $=14(q+3)\mathbf{i}-7(p-2)\mathbf{j}-7p(q+3)\mathbf{k}$
(ii)	When $p=1$, $\left\|-7\left[2(q+3)\mathbf{i}-\mathbf{j}+2(q+3)\mathbf{k}\right]\right\|=112$ $7\sqrt{\left[2(q+3)\right]^2+(-1)^2+(q+3)^2}=112$ $\sqrt{4(q+3)^2+1+(q+3)^2}=16$ $5(q+3)^2+1=16^2$ $5(q+3)^2=255$ $(q+3)^2=51$ $q=-3\pm\sqrt{51}$
(iii)	$\left(4\mathbf{a}+\frac{1}{2}\mathbf{b}\right)\bullet\left(2\mathbf{a}-\frac{3}{2}\mathbf{b}\right)=0$

	$8\underset{\sim}{a}\cdot\underset{\sim}{a}-6\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{a}-\frac{3}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}=0$ $8\lvert\underset{\sim}{a}\rvert^2-5\underset{\sim}{a}\cdot\underset{\sim}{b}-\frac{3}{4}\lvert\underset{\sim}{b}\rvert^2=0$ $8\lvert\underset{\sim}{a}\rvert^2=\frac{3}{4}(1)^2+5\left(-\frac{1}{8}\right)$ $=\frac{1}{8}$ $\lvert\underset{\sim}{a}\rvert=\frac{1}{8}$

4. TJC Mid Year 9758/ 20201/ P1/ Q5 (modifed)

Referred to the origin *O*, the position vectors of points *A* and *B* are **a** and **b** respectively. Given **a** is a unit vector, $\lvert \mathbf{b} \rvert = 2$ and $\angle AOB = 120°$.

(i) Find $\lvert \mathbf{a}+3\mathbf{b}\rvert^2$. [3]

(ii) Find $\lvert(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})\rvert$ and state a geometrical interpretation of $\lvert(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})\rvert$.[3]

A point *E* lies on the line *AB* such that *OE* is perpendicular to *AB*.

(iii) Find the position vector of *E*, in terms of **a** and **b**. [3]

(i)	$\lvert\mathbf{a}+3\mathbf{b}\rvert^2=(\mathbf{a}+3\mathbf{b})\bullet(\mathbf{a}+3\mathbf{b})$ $=\mathbf{a}\bullet\mathbf{a}+3(\mathbf{a}\bullet\mathbf{b})+3(\mathbf{b}\bullet\mathbf{a})+9(\mathbf{b}\bullet\mathbf{b})$ $=\lvert\mathbf{a}\rvert^2+6(\mathbf{a}\bullet\mathbf{b})+9\lvert\mathbf{b}\rvert^2$ $=1+6\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos120°+9(2)^2$ $=1-6+36=31$ Need to see $\mathbf{a}\bullet\mathbf{b}=\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos120^\circ=-1$ Alternative: Use Cosine rule $\lvert\mathbf{a}+3\mathbf{b}\rvert^2=\lvert\mathbf{a}\rvert^2+\lvert3\mathbf{b}\rvert^2-2\lvert\mathbf{a}\rvert\lvert3\mathbf{b}\rvert\cos60°$ $=1+3^2(2^2)-6(1)(2)\left(\frac{1}{2}\right)=31$

(ii)	$\left\|(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})\right\| = \left\|3(\mathbf{a}\times\mathbf{a})-(\mathbf{a}\times\mathbf{b})+9(\mathbf{b}\times\mathbf{a})-3(\mathbf{b}\times\mathbf{b})\right\|$ $= \left\|-(\mathbf{a}\times\mathbf{b})-9(\mathbf{a}\times\mathbf{b})\right\|$ $= \left\|-10(\mathbf{a}\times\mathbf{b})\right\| = 10\left\|\mathbf{a}\times\mathbf{b}\right\| = 20\left(\frac{1}{2}\left\|\mathbf{a}\times\mathbf{b}\right\|\right)$ $= 10\|\mathbf{a}\|\|\mathbf{b}\|\sin 120°$ $= 10(1)(2)\left(\frac{\sqrt{3}}{2}\right) = 10\sqrt{3}$ $\left\|(\mathbf{a}+3\mathbf{b})\times(3\mathbf{a}-\mathbf{b})\right\|$ represents 20 times the area of the triangle *OAB* OR It is the 10 times the area of the parallelogram formed by *adjacent sides OA* and *OB*
(iii)	Let $\overrightarrow{OE} = \mathbf{a}+\lambda(\mathbf{b}-\mathbf{a}),\ \lambda \in \mathbb{R}$ since E lies on line *AB* OR by ratio thm since E lies on line $AB \Rightarrow$ *A*, *B* and *E* are collinear: $\overrightarrow{OE} = \lambda\mathbf{b}+(1-\lambda)\mathbf{a}$ Given $\overrightarrow{OE}$ is perpendicular to $\overrightarrow{AB}$ $\overrightarrow{OE}\bullet(\mathbf{b}-\mathbf{a}) = 0$ $\left[\lambda\mathbf{b}+(1-\lambda)\mathbf{a}\right]\bullet(\mathbf{b}-\mathbf{a}) = 0$ $\lambda(4)-\lambda(-1)+(1-\lambda)(-1)-(1-\lambda)(1) = 0$ $4\lambda+\lambda-2+2\lambda = 0$ $7\lambda = 2$ $\lambda = \frac{2}{7}$ $\overrightarrow{OE} = \frac{5}{7}\mathbf{a}+\frac{2}{7}\mathbf{b}$