Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 6A 3D Vector Geometry (Lines)
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 6A 3D Vector Geometry (Lines)
Core Concept Notes

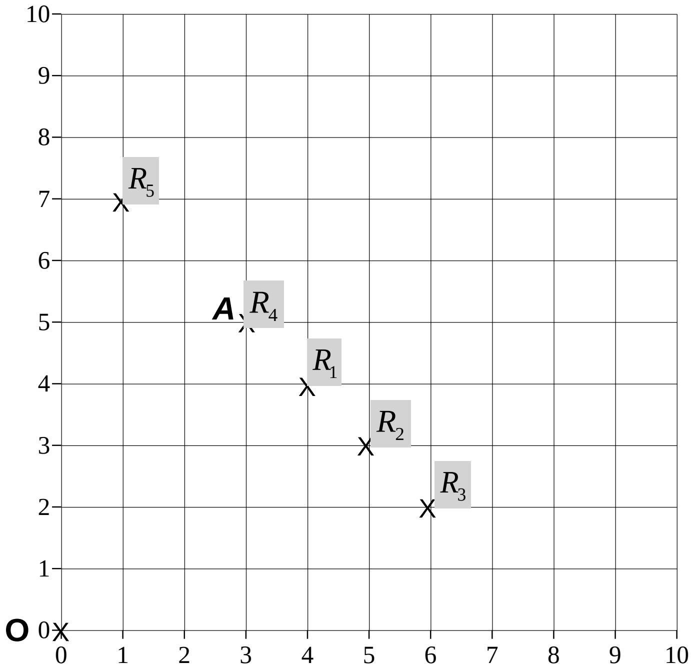
Success Criteria:

Surface Learning	**Deep Learning**	**Transfer Learning**
☐ Interpret and find equations of lines in the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$ (vector equation) or $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$ (cartesian equation) ☐ Convert the equations from one form to another ☐ Determine the relationship (i.e. intersecting, parallel or skew) between two lines ☐ Explain that two lines are coplanar if they are intersecting or parallel ☐ Find the angle between two lines	☐ Find the point of intersection of two lines if they intersect ☐ Find the length of projection of a vector onto a given line ☐ Find the foot of the perpendicular and perpendicular distance from a point to a line	☐ Find the reflection of a point in a line ☐ Interpret given information in contextual question

Pre Reading

Look through the following 2 examples BEFORE the first Independent Learning module on Chapter 6A in SLS.

1.



Using the grid, draw the following vectors, marking the points R_1, R_2, R_3, R_4 and R_5.

(a) $\overrightarrow{OR_1} = \overrightarrow{OA} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(b) $\overrightarrow{OR_2} = \overrightarrow{OA} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(c) $\overrightarrow{OR_3} = \overrightarrow{OA} + 3\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(d) $\overrightarrow{OR_4} = \overrightarrow{OA} + 0\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(e) $\overrightarrow{OR_5} = \overrightarrow{OA} - 2\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

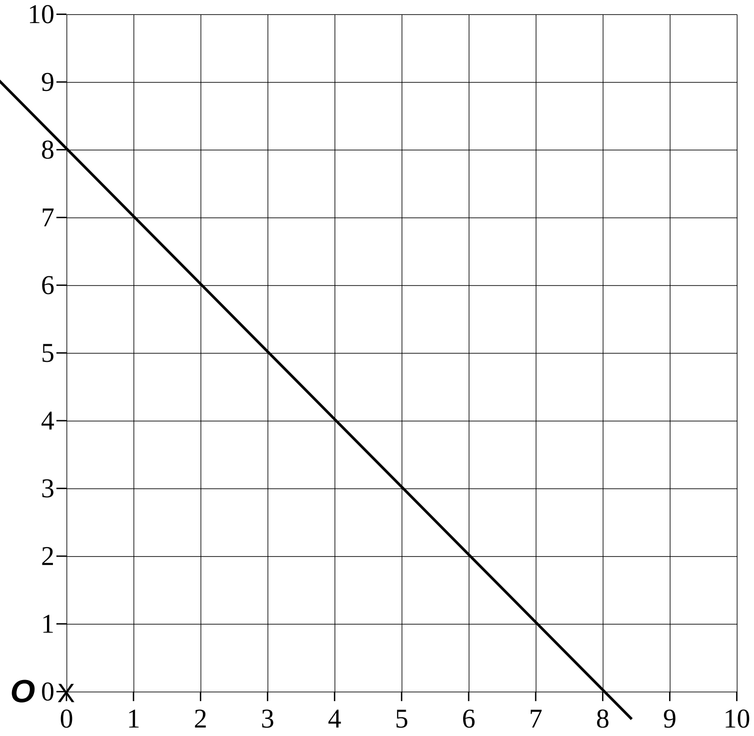
(i) Join the points R_1, R_2, R_3, R_4 and R_5. What do they form? (**A line segment)**

(ii) What is the geometrical representation of the following equation?

$$\overrightarrow{OR} = \overrightarrow{OA} + \lambda\begin{pmatrix} 1 \\ -1 \end{pmatrix},\ \lambda \in \mathbb{R}.$$

Position vector of a point on the line that passes through point *A* and is parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$**.**

2.



The position vector of any point from the line is given in the form

$$\mathbf{r} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda \begin{pmatrix} c \\ d \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}.$$

(i) State 2 possible values of $\begin{pmatrix} a \\ b \end{pmatrix}$.

Just provide the column vector of ANY point on the line. Possible values of $\begin{pmatrix} a \\ b \end{pmatrix}$ are $\begin{pmatrix} 0 \\ 8 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}$ etc

(ii) State 2 possible values of $\begin{pmatrix} c \\ d \end{pmatrix}$. What does $\frac{d}{c}$ represent?

Possible values of $\begin{pmatrix} c \\ d \end{pmatrix}$ are $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. $\frac{d}{c}$ represents the **gradient** of the straight line.

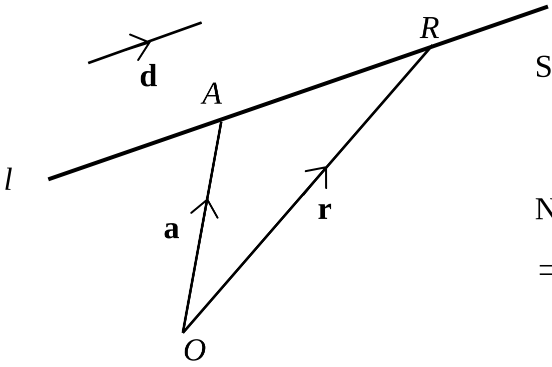
§1 Equation of Line

1.1 Vector Equation of a Line

In 3-dimensional space, a straight line is uniquely located in space if it has a known direction and passes through a known fixed point.

Consider a straight line *l* passing through a fixed point *A* with position vector **a** and which is parallel to a given vector **d**.

Let *R* be any point on the line, and **r** be the position vector of *R*.



Since line *l* is parallel to **d**,

$$\overrightarrow{AR} = \lambda\mathbf{d} \quad \text{for some } \lambda \in \mathbb{R}.$$

Now, $\overrightarrow{OR} = \overrightarrow{OA} + \overrightarrow{AR}$.

$\Rightarrow$ equation of line *l* : $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$.

Vector equation of the line *l* in parametric form which passes through the point with position vector **a** and parallel to the vector **d** is given by:

$$l : \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}, \quad \text{where} \quad \lambda \in \mathbb{R}$$

- $\mathbf{r}$: Position vector of **ANY point** on the line *l*.
- $\mathbf{a}$: Position vector of a **fixed point** on the line *l*.
- $\mathbf{d}$: **Direction** vector of the line (a vector that is parallel to the line *l*).
- λ is a real parameter

Note

(i) Each value of λ gives the position vector of a different point on the line.

(ii) If a point *R* lies on the line *l* with equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, then the position vector of the point *R* is given by:

$$\overrightarrow{OR} = \mathbf{a} + \lambda\mathbf{d} \text{ for } \textbf{some } \lambda \in \mathbb{R}.$$

(iii) An equation of this form $\mathbf{r} = \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$ represents a line passing through the origin.

(iv) Is the vector equation of a line unique? No. Why?

Example 1

Find a vector equation of the line passing through $A(1,-1,2)$ and parallel to the vector $3\mathbf{i}+\mathbf{k}$.

Solution:

A vector equation of the line is $l:\mathbf{r}=\begin{pmatrix}1\\-1\\2\end{pmatrix}+\lambda\begin{pmatrix}3\\0\\1\end{pmatrix}, \ \lambda\in\mathbb{R}$.

> The line passing through point A and parallel to vector $\underset{\sim}{b}$ has equation $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{b}, \ \lambda\in\mathbb{R}$

Discussion: What are the equations of the x-axis, y-axis and z-axis?

x-axis : $l_x:\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\0\\0\end{pmatrix}, \ \alpha\in\mathbb{R}$

> The x-axis passes through point $(0,0,0)$ and parallel to vector $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

y-axis : $l_y:\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\beta\begin{pmatrix}0\\1\\0\end{pmatrix}, \ \beta\in\mathbb{R}$

z-axis : $l_z:\mathbf{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\gamma\begin{pmatrix}0\\0\\1\end{pmatrix}, \ \gamma\in\mathbb{R}$

Example 2

Find a vector equation of the line through $A(5,2,7)$ and $B(-3,6,3)$. Determine whether the points $C(-7,8,1)$ and $D(1,2,4)$ lie on the line.

Solution:

$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$

$=\begin{pmatrix}-8\\4\\-4\end{pmatrix}=-4\begin{pmatrix}2\\-1\\1\end{pmatrix}$ $\Rightarrow$ a direction vector for the line is $\begin{pmatrix}2\\-1\\1\end{pmatrix}$

Therefore, a vector equation of the line *AB* is $\mathbf{r}=\begin{pmatrix}5\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix},\ \lambda\in\mathbb{R}$.

OR $\mathbf{r}=\begin{pmatrix}5\\2\\7\end{pmatrix}+\beta\begin{pmatrix}-8\\4\\-4\end{pmatrix},\ \beta\in\mathbb{R}$

OR $\mathbf{r}=\begin{pmatrix}-3\\6\\3\end{pmatrix}+s\begin{pmatrix}2\\-1\\1\end{pmatrix},\ s\in\mathbb{R}$

OR $\mathbf{r}=\begin{pmatrix}-3\\6\\3\end{pmatrix}+t\begin{pmatrix}-8\\4\\-4\end{pmatrix},\ t\in\mathbb{R}$

$$\begin{pmatrix}-7\\8\\1\end{pmatrix}=\begin{pmatrix}5\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}\Rightarrow\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-7\\8\\1\end{pmatrix}-\begin{pmatrix}5\\2\\7\end{pmatrix}$$

$$\Rightarrow\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-12\\6\\-6\end{pmatrix}$$

Since $\lambda=-6$ satisfies the equation, C lies on the line AB.

To determine if a point C lies on the line $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$, $\lambda\in\mathbb{R}$.
Check if there exists a real value of λ that satisfies the equation $\overrightarrow{OC}=\underset{\sim}{a}+\lambda\underset{\sim}{d}$.
If Yes, C lies on the line.
Otherwise, C does not lie on the line.

$$\begin{pmatrix}1\\2\\4\end{pmatrix}=\begin{pmatrix}5\\2\\7\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}\Rightarrow\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}1\\2\\4\end{pmatrix}-\begin{pmatrix}5\\2\\7\end{pmatrix}$$

$$\Rightarrow\lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-4\\0\\-3\end{pmatrix}\Rightarrow\begin{matrix}\lambda=-2\\\lambda=0\\\lambda=-3\end{matrix}$$

Since no consistent value of λ satisfies the equation, D does not lie on the line AB.

1.2 Parametric and Cartesian Forms

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, $\mathbf{a}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ and $\mathbf{d}=\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix}$.

Then from $\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$, we have:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}d_1\\d_2\\d_3\end{pmatrix},\lambda\in\mathbb{R} \text{ --- ②}$$

This is known as the ***vector equation (form)*** of the line.

From ② if we equate the **i**, **j** and **k** components, we have:

$$x=a_1+\lambda d_1,\quad y=a_2+\lambda d_2,\quad z=a_3+\lambda d_3,\ \lambda\in\mathbb{R} \text{ --- ③}$$

From ③ if we make λ the subject throughout, we obtain:

$$\frac{x-a_1}{d_1}=\frac{y-a_2}{d_2}=\frac{z-a_3}{d_3}\qquad(=\lambda)$$

This is known as the ***cartesian equation (form)*** of the line.

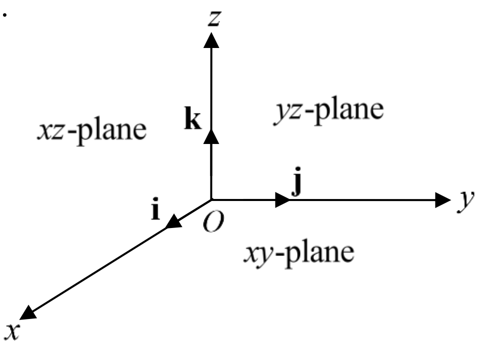
Special Cases: What happens when $d_1=0$ or/and $d_2=0$ or/and $d_3=0$?

Case 1: $d_1=0$, $d_2\neq 0$, $d_3\neq 0$

The vector equation of the line is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}0\\d_2\\d_3\end{pmatrix}$, $\lambda\in\mathbb{R}$.

The cartesian equation of the line is $x=a_1$, $\frac{y-a_2}{d_2}=\frac{z-a_3}{d_3}$.

This line is parallel to the yz -plane.



Case 2: $d_1=d_2=0$, $d_3\neq 0$

The vector equation of the line is $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\d_3\end{pmatrix}$, $\lambda\in\mathbb{R}$

The cartesian equation of the line is $x=a_1$, $y=a_2$, $z=a_3+\lambda d_3$.

This line is parallel to the z-axis.

Case 3: $d_1=d_2=d_3=0$. This gives you a point with position vector $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$, not a straight line!

Example 3

Write down the Cartesian equations of l_1 and l_2 where

$$l_1: \mathbf{r}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}, \lambda\in\mathbb{R} \quad \text{and} \quad l_2: \mathbf{r}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix}, \mu\in\mathbb{R}.$$

Solution:

Equate **i**, **j** and **k** components. Make λ the subject and equate.

(a) $l_1: \mathbf{r}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}, \lambda\in\mathbb{R}$

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix}$

$x=1+2\lambda \Rightarrow \lambda=\frac{x-1}{2}$

$y=1+3\lambda \Rightarrow \lambda=\frac{y-1}{3}$

$z=-4+\lambda \Rightarrow \lambda=z+4$

The Cartesian equation of l_1 is $\frac{x-1}{2}=\frac{y-1}{3}=z+4 \quad (\lambda\in\mathbb{R})$

(b) $l_2: \mathbf{r}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix}, \mu\in\mathbb{R}$

Note the **j** component of direction vector is 0.

Let $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\-4\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix}$

$x=1+2\mu \Rightarrow \mu=\frac{x-1}{2}$

$y=1$

$z=-4+\mu \Rightarrow \mu=z+4$

The Cartesian equation of l_2 is $\frac{x-1}{2}=z+4,\ y=1 \ (\mu\in\mathbb{R})$

Example 4

Find the vector equations of l_1 and l_2 where

$$l_1: \frac{x-1}{2}=\frac{y+2}{3}=z \quad \text{and} \quad l_2: x=3,\ 2y+1=\frac{z+1}{5}.$$

Solution:

Strategy: Introduce a parameter λ and make x, y, z the subject

Equate Cartesian equation to λ and make x, y, z the subject.
Rearrange to Vector Equation of line $\underset{\sim}{r}=\underset{\sim}{a}+\lambda\underset{\sim}{d},\ \lambda\in\mathbb{R}$.

$$l_1: \frac{x-1}{2}=\frac{y+2}{3}=z=\lambda \Rightarrow \begin{cases}x=1+2\lambda\\y=-2+3\lambda\\z=\lambda\end{cases}$$

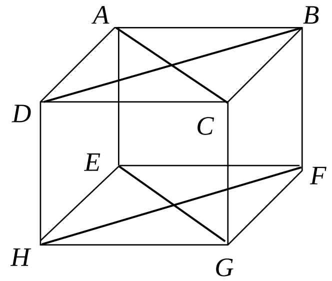
Since $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$, vector equation of l_1 is $\mathbf{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\\1\end{pmatrix},\ \lambda\in\mathbb{R}$.

$$l_2: x=3,\ 2y+1=\frac{z+1}{5}=\beta \Rightarrow \begin{cases}x=3\\y=\frac{-1+\beta}{2}\\z=-1+5\beta\end{cases}$$

$\therefore$ vector equation of l_2 is: $\mathbf{r}=\begin{pmatrix}3\\-\frac{1}{2}\\-1\end{pmatrix}+\beta\begin{pmatrix}0\\\frac{1}{2}\\5\end{pmatrix},\ \beta\in\mathbb{R}$

§2 Relationship Between Two Lines

Two lines in **3-dimensional space** can be



(i) Intersecting e.g. *AC*, *DB*

(ii) Parallel e.g. *AC*, *EG*

(iii) Non-intersecting and non-parallel (**skew lines**) e.g. *AC* and *HF*

Note that

- the intersecting lines in case **(i)** are **coplanar**, i.e. they are on the same plane.
- the parallel lines in case **(ii)** are **coplanar**, i.e. they are on the same plane.

Consider two lines whose equations are:

$$l_1 : \mathbf{r} = \mathbf{a_1} + \lambda\, \mathbf{d_1}\,,\ \lambda \in \mathbb{R}$$
$$l_2 : \mathbf{r} = \mathbf{a_2} + \mu\, \mathbf{d_2}\,,\ \mu \in \mathbb{R}$$

If two non-parallel lines l_1 and l_2 intersect at point P, then unique values of λ and μ can be found such that

$$\overrightarrow{OP} = \mathbf{a_1} + \lambda\, \mathbf{d_1} = \mathbf{a_2} + \mu\, \mathbf{d_2}.$$

Parallel Lines	2 lines are parallel if their direction vectors are parallel. $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{d}_1$ $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{d}_2$ are parallel if $\mathbf{d}_1 = k\mathbf{d}_2$ for some $k \in \mathbb{R}$. Furthermore, l_1 and l_2 are distinct if there does not exist a unique value of μ such that $\mathbf{a}_1 = \mathbf{a}_2 + \mu \mathbf{d}_2$.
Intersecting Lines	2 lines intersect at a point. $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{d}_1$ $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{d}_2$ intersect if there is a unique pair of λ and μ for which $\mathbf{a}_1 + \lambda \mathbf{d}_1 = \mathbf{a}_2 + \mu \mathbf{d}_2$.
Skew Lines	2 lines are skew lines if they are non-parallel and non-intersecting. $l_1 : \mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{d}_1$ $l_2 : \mathbf{r} = \mathbf{a}_2 + \mu \mathbf{d}_2$ are skew lines if **(i)** $\mathbf{d}_1$ is not parallel to $\mathbf{d}_2$ <u>AND</u> **(ii)** l_1 and l_2 do not intersect i.e. there does not exist unique values of λ and μ such that $\mathbf{a}_1 + \lambda \mathbf{d}_1 = \mathbf{a}_2 + \mu \mathbf{d}_2$.

Example 5

Determine whether the following pairs of lines are parallel, intersecting or skew.

(a) $l_1 : \mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + \lambda(3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k})$; $l_2 : \mathbf{r} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + \mu(-6\mathbf{i} + 4\mathbf{j} - 8\mathbf{k})$, $\lambda, \mu \in \mathbb{R}$

(b) $l_1 :\ \mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$, $\quad l_2 :\ \mathbf{r} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\mu \in \mathbb{R}$

(c) $l_1 : \frac{x+1}{-2} = \frac{y-1}{2} = \frac{z-3}{1}$; $l_2 : \frac{x-1}{3} = \frac{y-3}{1} = \frac{z-2}{6}$

Solution:

(a) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ and $l_2 : \mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$

Since $\begin{pmatrix} -6 \\ 4 \\ -8 \end{pmatrix} = -2 \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$ $\therefore$ l_1 and l_2 are parallel.

> 2 lines are parallel if their direction vectors are parallel.

$$\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix}$$

> Since point on l_2 does not satisfy equation of l_1, the 2 lines are distinct.

$\lambda \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow$ no solution for λ

$\therefore l_1$ & l_2 are not coincident.

(b) Since $\begin{pmatrix}1\\-1\\1\end{pmatrix}$ is not parallel to $\begin{pmatrix}2\\1\\3\end{pmatrix}$, l_1 and l_2 are not parallel.

Check if l_1 and l_2 intersect, i.e. check if scalars λ and μ can be found such that

$$\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}2\\4\\6\end{pmatrix}+\mu\begin{pmatrix}2\\1\\3\end{pmatrix}$$

Method 1: Using PlySmlt2 (SIMULT EQN SOLVER)

We can use GC to solve the system of linear equations that consists of 3 equations and 2 unknowns.

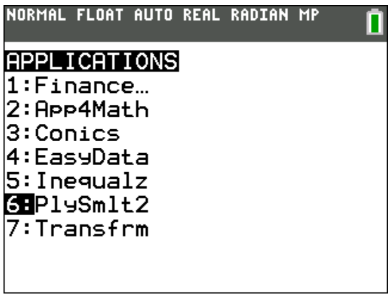
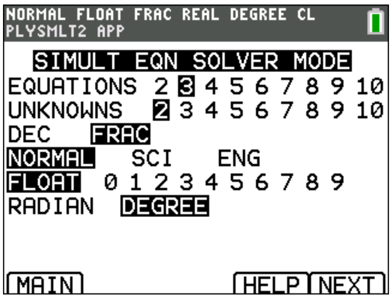
$$1+\lambda=2+2\mu \quad \text{— (1)}$$
$$-1-\lambda=4+\mu \quad \text{— (2)}$$
$$3+\lambda=6+3\mu \quad \text{— (3)}$$

Rearranging, we get
$$\lambda-2\mu=1$$
$$\lambda+\mu=-5$$
$$\lambda-3\mu=3$$

Using GC to solve (1), (2) and (3), $\lambda=-3,\ \mu=-2$

So l_1 and l_2 intersect at the point with position vector $\begin{pmatrix}1\\-1\\3\end{pmatrix}+(-3)\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-2\\2\\0\end{pmatrix}$.

Therefore, l_1 and l_2 are non-parallel but intersecting lines.

GC Keystrokes	**GC Screenshot**
Step 1: **(i)** Press **apps** and select **6:PlySmlt2**. **(ii)** Press **enter** to enter the main menu. **(iii)** Select **2: SIMULT EQN SOLVER**	
Step 2: **(i)** Select the required parameters. **(ii)** Press **graph** to go **NEXT**.	

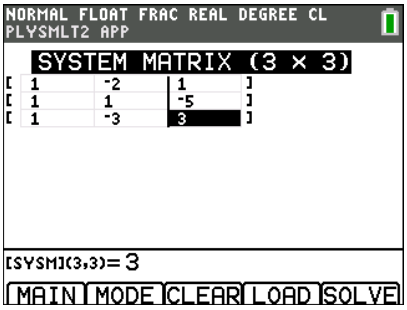
Step 3:

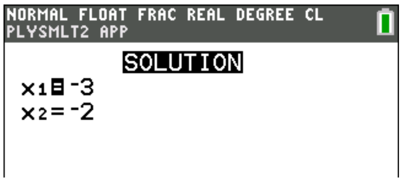
(i) Key in the entries in the matrix row by row. Enter $1,-2,1$ in the 1st row; $1,1,-5$ in the 2nd row and $1,-3,3$ in the 3rd row.

(ii) **Remember to press enter after keying in the last value.**

(iii) Press **graph** to **SOLVE**.

Solution: Using GC, $x_1=-3$ and $x_2=-2$.





Method 2

$$1+\lambda=2+2\mu \quad \text{— (1)}$$
$$-1-\lambda=4+\mu \quad \text{— (2)}$$
$$3+\lambda=6+3\mu \quad \text{— (3)}$$

Solving equations (1) and (2), $\lambda=-3, \mu=-2$

If l_1 and l_2 intersect, $\lambda=-3,\ \mu=-2$ must satisfy all 3 equations. So need to substitute into the 3rd unused equation to check.

Sub $\lambda=-3,\ \mu=-2$ into equation (3): $\text{LHS}=3+(-3)=0$

$\text{RHS}=6+3(-2)=0=\text{LHS}$

Thus $\lambda=-3,\ \mu=-2$ also satisfies equation (3).

So l_1 and l_2 intersect at the point with position vector $\begin{pmatrix}1\\-1\\3\end{pmatrix}+(-3)\begin{pmatrix}1\\-1\\1\end{pmatrix}=\begin{pmatrix}-2\\2\\0\end{pmatrix}$.

Therefore, l_1 and l_2 are non-parallel but intersecting lines.

(c) In vector form the lines are:

$$l_1 : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}, \lambda, \mu \in \mathbb{R}$$

Since $\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$ is not parallel to $\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix}$, l_1 and l_2 are not parallel

If l_1 and l_2 intersect, $\begin{pmatrix} -1-2\lambda \\ 1+2\lambda \\ 3+\lambda \end{pmatrix} = \begin{pmatrix} 1+3\mu \\ 3+\mu \\ 2+6\mu \end{pmatrix}$

$$\begin{aligned} \Rightarrow -1-2\lambda &= 1+3\mu \quad &\text{— (1)} \\ 1+2\lambda &= 3+\mu \quad &\text{— (2)} \\ 3+\lambda &= 2+6\mu \quad &\text{— (3)} \end{aligned}$$

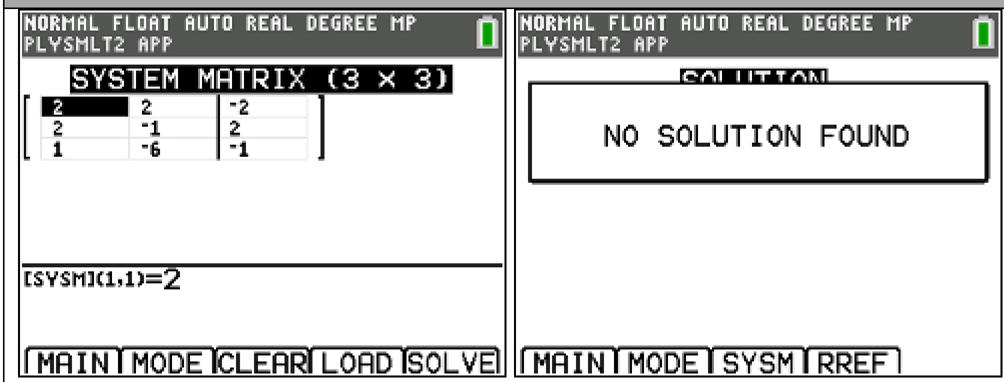
__**Method 1: Using PlySmlt2 (SIMULT EQN SOLVER)**__

Rearranging, $\begin{aligned} 2\lambda + 2\mu &= -2 \\ 2\lambda - \mu &= 2 \\ \lambda - 6\mu &= -1 \end{aligned}$

Using GC, there are no solution for λ and μ that satisfies all 3 equations. Hence l_1 and l_2 do not intersect.

Since l_1 and l_2 are non-parallel and non-intersecting, they are skew lines.

GC Screenshot



Method 2

$$-1-2\lambda=1+3\mu \quad \text{— (1)}$$
$$1+2\lambda=3+\mu \quad \text{— (2)}$$
$$3+\lambda=2+6\mu \quad \text{— (3)}$$

Solving equations (1) and (2), $\mu=-1,\ \lambda=\frac{1}{2}$.

To check if values obtained satisfy (or is consistent with) Equation (3):
Substituting $\mu=-1$ and $\lambda=\frac{1}{2}$ in (3),
LHS: $3+\frac{1}{2}=\frac{7}{2}$,
RHS: $2-6=-4\neq \text{LHS}$

Non-parallel and non-intersecting lines are skew lines.

$\therefore\ l_1$ and l_2 do not intersect.
Since l_1 and l_2 are non-parallel and non-intersecting, they are skew lines.

§3 Angle Between Two Lines

Recall from Vectors 1 that the angle θ between two **vectors** $\mathbf{a}$ and $\mathbf{b}$ is found by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

> Dot Product:
> $\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$, where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$.

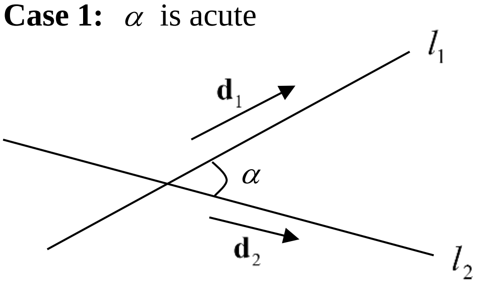
Consider two lines l_1 and l_2 whose vector equations are

$$\begin{aligned} l_1 &: \ \mathbf{r} = \mathbf{a_1} + \lambda\mathbf{d_1} \\ l_2 &: \ \mathbf{r} = \mathbf{a_2} + \mu\mathbf{d_2} \end{aligned} \quad \text{where } \lambda \in \mathbb{R}, \mu \in \mathbb{R}$$

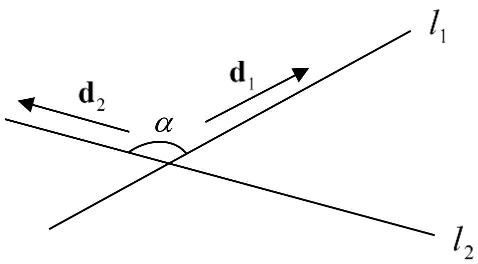
To find the angle between l_1 and l_2 , we first consider the angle α between their direction vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ using the scalar product formula

$$\cos\alpha = \frac{\mathbf{d_1}\bullet\mathbf{d}_2}{|\mathbf{d_1}||\mathbf{d}_2|}$$

Case 1: α is acute



Case 2: α is obtuse



By convention, we want to find the **acute** angle θ between l_1 and l_2.

Case 1: α is acute, $\mathbf{d_1}\bullet\mathbf{d}_2 > 0$ and $\cos\alpha > 0$, therefore $\theta = \alpha$

$$\cos\theta = \frac{\mathbf{d_1}\bullet\mathbf{d}_2}{|\mathbf{d_1}||\mathbf{d}_2|}$$

Case 2: α is obtuse, $\mathbf{d_1}\bullet\mathbf{d}_2 < 0$ and $\cos\alpha < 0$, therefore $\theta = 180° - \alpha$

$$\begin{aligned} \cos\theta &= \cos(180° - \alpha) \\ &= -\cos\alpha \\ &= \frac{-\mathbf{d_1}\bullet\mathbf{d}_2}{|\mathbf{d_1}||\mathbf{d}_2|} \text{ , where } -\mathbf{d_1}\bullet\mathbf{d}_2 > 0 \end{aligned}$$

Combining Case 1 and 2 from above, in conclusion, the **acute** angle θ between l_1 and l_2 can be found using the formula

$$\boxed{\cos\theta = \left|\frac{\mathbf{d_1}\bullet\mathbf{d}_2}{|\mathbf{d_1}||\mathbf{d}_2|}\right|}$$

Special Case: l_1 and l_2 are perpendicular ($\theta = 90°, \cos\theta = 0$) if and only if $\mathbf{d_1}\bullet\mathbf{d}_2 = 0$

> **Question:** Can we find the angle between 2 skew lines? Yes

Example 6

Find the acute angle between the lines, correct to the nearest 0.1°.

$$l_1 : \mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\1\\-2\end{pmatrix} \text{ and } l_2 : \mathbf{r} = \begin{pmatrix}3\\-1\\2\end{pmatrix} + \mu\begin{pmatrix}2\\0\\5\end{pmatrix}, \quad \lambda \in \mathbb{R}, \mu \in \mathbb{R}$$

Solution:

Let θ be the acute angle between l_1 and l_2.

$$\cos\theta = \frac{\left|\begin{pmatrix}3\\1\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\0\\5\end{pmatrix}\right|}{\sqrt{14}\sqrt{29}} = \frac{|-4|}{\sqrt{14}\sqrt{29}} = \frac{4}{\sqrt{14}\sqrt{29}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{4}{\sqrt{14}\sqrt{29}}\right) = 78.5°$$

For acute angle, cos θ is positive.

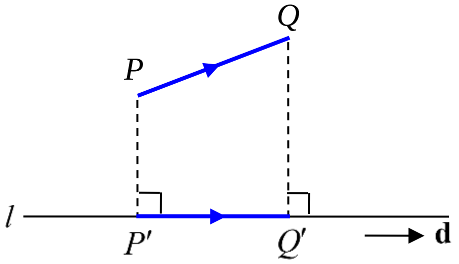
Note: Angle between the vectors $\begin{pmatrix}3\\1\\-2\end{pmatrix}$ and $\begin{pmatrix}2\\0\\5\end{pmatrix}$ is given by

$$\cos^{-1}\frac{\begin{pmatrix}3\\1\\-2\end{pmatrix} \bullet \begin{pmatrix}2\\0\\5\end{pmatrix}}{\sqrt{14}\sqrt{29}} = \cos^{-1}\left(\frac{-4}{\sqrt{14}\sqrt{29}}\right) = 101.5°$$

In short, always look out for key word like **<u>acute angle</u>** when finding angles between 2 lines or angles between 2 vectors.

§4 <u>Projection of a Vector onto a Line</u>

Consider the vector $\overrightarrow{PQ}$ and the line l with equation $l: \ \mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$, $\lambda \in \mathbb{R}$.



- The length of projection of $\overrightarrow{PQ}$ onto l = $P'Q'$ = $\left|\overrightarrow{P'Q'}\right|$ = $\left|\overrightarrow{PQ} \cdot \hat{\mathbf{d}}\right|$
- The projection vector of $\overrightarrow{PQ}$ onto l = $\overrightarrow{P'Q'}$ = $\left(\overrightarrow{PQ} \cdot \hat{\mathbf{d}}\right)\hat{\mathbf{d}}$

Example 7

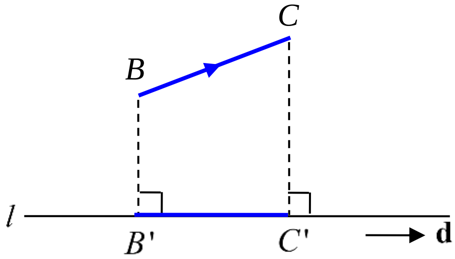
The line l passes through the point A with position vector $\begin{pmatrix}1\\2\\3\end{pmatrix}$ and is parallel to $\begin{pmatrix}1\\-2\\-2\end{pmatrix}$.

The points B and C have position vectors $\begin{pmatrix}1\\1\\7\end{pmatrix}$ and $\begin{pmatrix}3\\6\\4\end{pmatrix}$ respectively.

(i) Find the length of the projection of $\overrightarrow{BC}$ onto l.
(ii) Hence find the projection vector of $\overrightarrow{BC}$ onto l.

Solution:

(i) Let $\mathbf{d} = \begin{pmatrix}1\\-2\\-2\end{pmatrix}$ be the direction vector of line l.



So, the length of projection of $\overrightarrow{BC}$ onto l is $B'C'$

$$= \left|\overrightarrow{BC} \,.\, \hat{\mathbf{d}}\right| = \left|\frac{\overrightarrow{BC} \,.\, \mathbf{d}}{|\mathbf{d}|}\right|$$

$$= \left|\frac{\begin{pmatrix}2\\5\\-3\end{pmatrix} . \begin{pmatrix}1\\-2\\-2\end{pmatrix}}{\sqrt{1+4+4}}\right|$$

$$= \frac{|2-10+6|}{\sqrt{9}}$$

$$= \left|-\frac{2}{3}\right|$$

$$= \frac{2}{3} \text{ units}$$

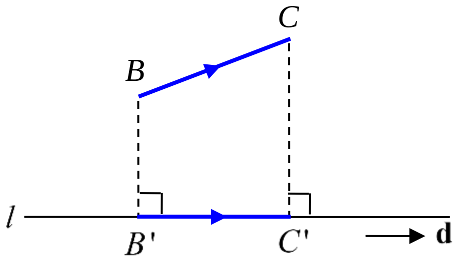
Length of projection is a length and thus it is a positive value.

(ii)
The projection vector of $\overrightarrow{BC}$ onto l is $\overrightarrow{B'C'}$

$$= \left(\overrightarrow{BC} \,.\, \hat{\mathbf{d}}\right)\hat{\mathbf{d}} = \left(\frac{\overrightarrow{BC} \,.\, \mathbf{d}}{|\mathbf{d}|}\right)\frac{\mathbf{d}}{|\mathbf{d}|}$$

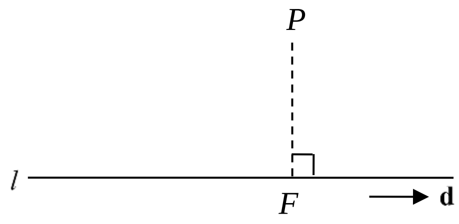
$$= -\frac{2}{3}\,\frac{\begin{pmatrix}1\\-2\\-2\end{pmatrix}}{3} = -\frac{2}{9}\begin{pmatrix}1\\-2\\-2\end{pmatrix}.$$

Projection Vector is a vector.



§5 Between Point and Line: Foot of the Perpendicular from a Point to a Line

Consider the line l with equation $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$, and the point P (not on l).



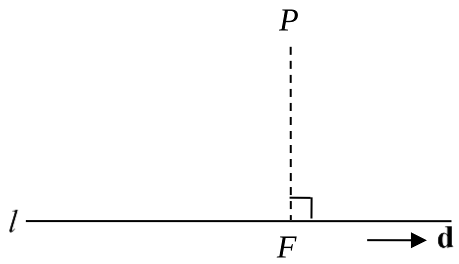
How do we find the position vector of F , the foot of the perpendicular from the point P to the line l?

Example 8

The line l has equation $l:\mathbf{r}=(6\mathbf{i}+2\mathbf{k})+\lambda(\mathbf{i}+2\mathbf{j})$, $\lambda\in\mathbb{R}$. Find the position vector of the foot of the perpendicular from the point $P(1,0,2)$ to l.

Solution:
Let F be the foot of the perpendicular from P to l.

Method 1



What is the aim of the question?
To find $\overrightarrow{OF}$.

What can we observe from the diagram?
$\overrightarrow{PF}\perp l$, so $\overrightarrow{PF}\cdot\mathbf{d}=0$

Since F lies on l then $\overrightarrow{OF}=\begin{pmatrix}6+\lambda\\2\lambda\\2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}6+\lambda\\2\lambda\\2\end{pmatrix}-\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}5+\lambda\\2\lambda\\0\end{pmatrix}$$

Since $\overrightarrow{PF}\perp l$, $\overrightarrow{PF}\cdot\mathbf{d}=0\Rightarrow\begin{pmatrix}5+\lambda\\2\lambda\\0\end{pmatrix}.\begin{pmatrix}1\\2\\0\end{pmatrix}=0\Rightarrow 5+\lambda+4\lambda=0\Rightarrow\lambda=-1$

Therefore, the position vector of the foot of the perpendicular from P to l is

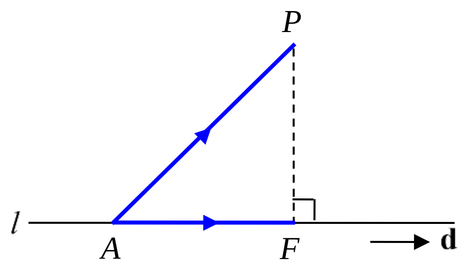
$$\overrightarrow{OF}=\begin{pmatrix}6-1\\2(-1)\\2\end{pmatrix}=\begin{pmatrix}5\\-2\\2\end{pmatrix}$$

<u>Method 2</u> (similar to finding projection vector)

Let A be the point with position vector $6\mathbf{i}+2\mathbf{k}$, which lies on l.

$$\overrightarrow{AP} = \overrightarrow{OP}-\overrightarrow{OA} = \begin{pmatrix}-5\\0\\0\end{pmatrix}$$

$\overrightarrow{AF}$ = projection vector of $\overrightarrow{AP}$ onto l



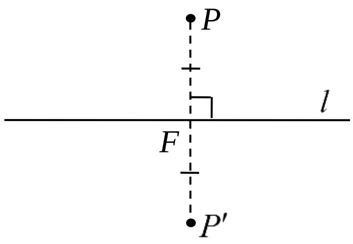
$$\overrightarrow{AF} = \left(\frac{\overrightarrow{AP}\bullet\mathbf{d}}{|\mathbf{d}|}\right)\frac{\mathbf{d}}{|\mathbf{d}|} = \frac{\begin{pmatrix}-5\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\2\\0\end{pmatrix}}{\sqrt{1+4+0}}\frac{\mathbf{d}}{|\mathbf{d}|} = -\sqrt{5}\,\frac{\begin{pmatrix}1\\2\\0\end{pmatrix}}{\sqrt{1+4+0}} = \begin{pmatrix}-1\\-2\\0\end{pmatrix}$$

Therefore, the position vector of the foot of the perpendicular from P to l is

$$\overrightarrow{OF} = \overrightarrow{OA}+\overrightarrow{AF} = \begin{pmatrix}6\\0\\2\end{pmatrix}+\begin{pmatrix}-1\\-2\\0\end{pmatrix} = \begin{pmatrix}5\\-2\\2\end{pmatrix}.$$

§6 <u>Between Point and Line: Reflection of a Point in a Line</u>

Consider the line l with equation $l:\mathbf{r}=\mathbf{a}+\lambda\mathbf{d}$, $\lambda\in\mathbb{R}$, and the point P (not on l).



How do we find the position vector of P', the reflection of P in the line l?

Note that if P' is the reflection of P in the line l, then

(i) PP' is perpendicular to l. So F is in fact the foot of the perpendicular from P to l.

(ii) P and P' are equidistant from F.

To find the reflection of P in l:

Step 1: Find the position vector of F, the foot of the perpendicular from P to l.

Step 2: **Using Ratio Theorem,** $\overrightarrow{OF}=\dfrac{\overrightarrow{OP}+\overrightarrow{OP'}}{2} \quad\Rightarrow\quad \overrightarrow{OP'}=2\overrightarrow{OF}-\overrightarrow{OP}$.

Example 9

The equation of a straight line l is $\mathbf{r}=(1+4\lambda)\mathbf{i}+3\lambda\,\mathbf{j}+2\mathbf{k}$, where λ is a parameter.

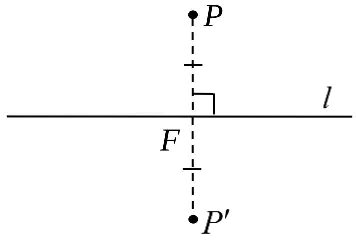
The point P has coordinates $(2,7,-1)$.

(i) Find the position vector of the foot of the perpendicular from P to l.

(ii) Find the position vector of the reflection of P in the line l.

Solution:

(i) Let F be the foot of the perpendicular from P to l.



$$l: \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}4\\3\\0\end{pmatrix}, \lambda\in\mathbb{R}$$

What is the aim of the question?

To find $\overrightarrow{OF}$.

What can we observe from the diagram?

$\overrightarrow{PF}\perp l$, so $\overrightarrow{PF}\cdot\mathbf{d}=0$

Foot of perpendicular F lies on the line l, i.e. F is a point on l.

Since F lies on l then $\overrightarrow{OF}=\begin{pmatrix}1+4\lambda\\3\lambda\\2\end{pmatrix}$ for some $\lambda\in\mathbb{R}$.

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}1+4\lambda\\3\lambda\\2\end{pmatrix}-\begin{pmatrix}2\\7\\-1\end{pmatrix}=\begin{pmatrix}4\lambda-1\\3\lambda-7\\3\end{pmatrix}$$

$$\overrightarrow{PF}\perp l \Rightarrow \begin{pmatrix}4\lambda-1\\3\lambda-7\\3\end{pmatrix}.\begin{pmatrix}4\\3\\0\end{pmatrix}=0\Rightarrow 16\lambda-4+9\lambda-21=0\Rightarrow\lambda=1$$

Therefore, the position vector of the foot of the perpendicular from P to l is

$$\overrightarrow{OF}=\begin{pmatrix}1+4(1)\\3(1)\\2\end{pmatrix}=\begin{pmatrix}5\\3\\2\end{pmatrix}.$$

(ii) Let P' be the reflection of P in the line l.

By Ratio Theorem,

$$\overrightarrow{OF}=\frac{\overrightarrow{OP}+\overrightarrow{OP'}}{2} \quad\Rightarrow\quad \overrightarrow{OP'} = 2\overrightarrow{OF}-\overrightarrow{OP} = 2\begin{pmatrix}5\\3\\2\end{pmatrix}-\begin{pmatrix}2\\7\\-1\end{pmatrix}=\begin{pmatrix}8\\-1\\5\end{pmatrix}$$

Therefore, the position vector of the reflection of P in the line l is $\overrightarrow{OP'}=\begin{pmatrix}8\\-1\\5\end{pmatrix}$.

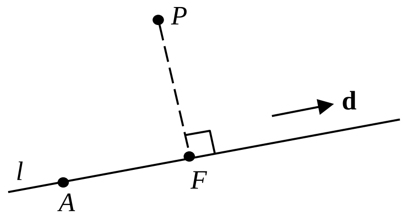
§7 Between Point and Line: Shortest Distance / Perpendicular Distance from a Point to a Line

❖ **If $\overrightarrow{OF}$ is known,**

Method 1

(1) First find $\overrightarrow{PF} = \overrightarrow{OF} - \overrightarrow{OP}$.

(2) Thus, shortest distance from P to l = $\left|\overrightarrow{PF}\right|$.



❖ **If $\overrightarrow{OF}$ is unknown,**

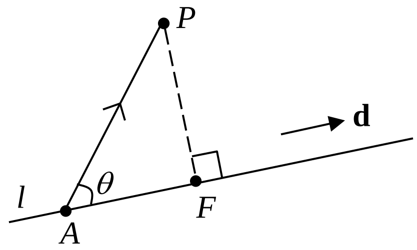
Method 2 (Use Cross Product)

(1) Use A, any point on l. (Consider the given fixed point on line with position vector $\mathbf{a}$).

Find $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$.

(2) Use Cross Product to find PF:

$$PF = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right|$$



❖ **If $\overrightarrow{OF}$ is unknown,**

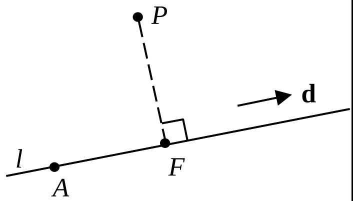
Method 3 (Find Length of Projection first and then use Pythagoras' Theorem)

(1) Use A, any point on l. (Consider the given fixed point on line with position vector $\mathbf{a}$).

Find $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA}$.

(2) Find AF, length of projection of $\overrightarrow{AP}$ onto l.

$$AF = \left|\overrightarrow{AP} \bullet \hat{\mathbf{d}}\right| = \left|\frac{\overrightarrow{AP} \bullet \mathbf{d}}{|\mathbf{d}|}\right|,$$ (Projection of $\overrightarrow{AP}$ onto line l)



(3) Use Pythagoras' Theorem to find: $PF = \sqrt{AP^2 - AF^2}$

Example 9 (modified)

The equation of a straight line l is $\mathbf{r}=(1+4\lambda)\mathbf{i}+3\lambda\,\mathbf{j}+2\mathbf{k}$, where λ is a parameter.

The point P has coordinates $(2,7,-1)$.

(i) Find the position vector of the foot of the perpendicular from P to l.

(ii) Hence or otherwise, find the shortest distance from P to l.

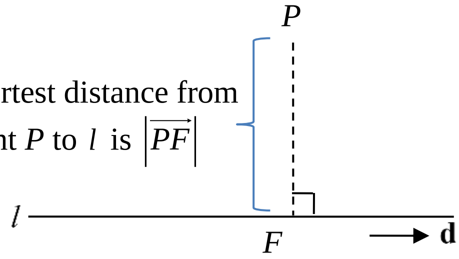
Solution:

(i) From the above Example 9, we have solved part **(i)** where $\lambda=1$ and the position vector of the foot of the perpendicular from P to l is $\overrightarrow{OF}=\begin{pmatrix}5\\3\\2\end{pmatrix}$

(ii) Using the result from part **(i)**, we can find the shortest distance from P to l which is $\left|\overrightarrow{PF}\right|$.

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}5\\3\\2\end{pmatrix}-\begin{pmatrix}2\\7\\-1\end{pmatrix}=\begin{pmatrix}3\\-4\\3\end{pmatrix}$$

$$\therefore\left|\overrightarrow{PF}\right|=\sqrt{(3)^2+(-4)^2+(3)^2}=\sqrt{34}$$



Hence, the shortest distance from P to l is $\sqrt{34}$ units.

Example 10

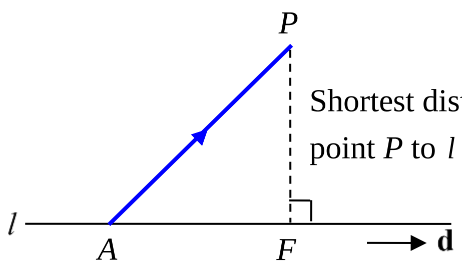
Find the perpendicular distance from $P(7,-2,4)$ to the line $\mathbf{r}=\begin{pmatrix}2\\3\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}$, $\lambda\in\mathbb{R}$.

Solution:

We will use **<u>Method 2</u> (Use Cross Product)** to find the shortest distance from P to line l.

Let point A on line l be $(2,3,1)$.

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}=\begin{pmatrix}7\\-2\\4\end{pmatrix}-\begin{pmatrix}2\\3\\1\end{pmatrix}=\begin{pmatrix}5\\-5\\3\end{pmatrix}$$



$$\left|\overrightarrow{PF}\right| = \left|\overrightarrow{AP} \times \hat{\mathbf{d}}\right|$$

$$= \frac{\left|\overrightarrow{AP} \times \mathbf{d}\right|}{\left|\mathbf{d}\right|}$$

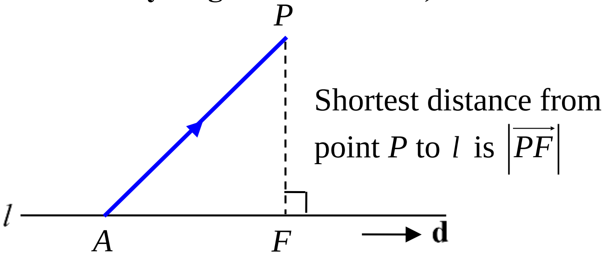
$$= \frac{\left|\begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right|}$$

$$= \frac{\left|\begin{pmatrix} 6 \\ 3 \\ -5 \end{pmatrix}\right|}{\sqrt{1^2 + (-2)^2}}$$

$$= \frac{\sqrt{6^2 + 3^2 + (-5)^2}}{\sqrt{5}}$$

$$= \frac{\sqrt{70}}{\sqrt{5}} = \sqrt{14}$$

<u>Method 3 (</u>Find Length of Projection first and then use Pythagoras' Theorem)



Let point A on line l be $(2,3,1)$.

$$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix}$$

$$\left|\overrightarrow{AP}\right| = \sqrt{59}$$

Length of projection of $\overrightarrow{AP}$ onto l is $\left|\overrightarrow{AF}\right|$.

$$\left|\overrightarrow{AF}\right| = \left|\overrightarrow{AP} \bullet \hat{\mathbf{d}}\right|$$

$$= \frac{\left|\overrightarrow{AP} \bullet \mathbf{d}\right|}{\left|\mathbf{d}\right|} = \frac{\left|\begin{pmatrix} 5 \\ -5 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right|}{\left|\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}\right|} = \frac{15}{\sqrt{1^2 + (-2)^2}} = \frac{15}{\sqrt{5}} = 3\sqrt{5}$$

Using Pythagoras' Theorem,

$$\left|\overrightarrow{PF}\right| = \sqrt{59 - \left(3\sqrt{5}\right)^2} = \sqrt{14}$$

H2 Mathematics (9758)
Chapter 6A 3D Vector Geometry (Lines)
Discussion Questions

Level 1

1 Find a vector equation, a cartesian equation and a set of parametric equations of the following lines:

(a) passing through the point with position vector $7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$ and parallel to $\mathbf{i}-3\mathbf{j}+\mathbf{k}$,

(b) passing through the points $(1,-2,1)$ and $(0,4,9)$,

(c) passing through the point $(3,0,2)$ and parallel to the line $x=\dfrac{y+4}{3}, z=1$.

2 For the following pairs of lines, determine whether they are parallel lines, intersecting lines or skew lines. Find the coordinates of the point of intersection for intersecting lines.

(a) $\dfrac{x-1}{3}=\dfrac{y-1}{-2}=z-1$, $\mathbf{r}=-2\mathbf{i}+3\mathbf{j}+\alpha(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$ where α is a real parameter.

(b) $\mathbf{r}=-2\mathbf{i}+3\mathbf{j}+\lambda(2\mathbf{i}+3\mathbf{j}-\mathbf{k})$, $\mathbf{r}=(-1-6\mu)\mathbf{i}+(3-9\mu)\mathbf{j}+(3\mu)\mathbf{k}$, where λ and μ are real parameters.

3 The lines l_1 and l_2 have equations $\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+s\begin{pmatrix}1\\2\\4\end{pmatrix}$ and $\mathbf{r}=\begin{pmatrix}1\\-1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\-1\end{pmatrix}$ respectively, where s and t are real parameters.

(i) Show that l_1 passes through the point $A(2,-1,-4)$, but that l_2 does not.

(ii) Find the acute angle between l_2 and the line joining $A(2,-1,-4)$ and $B(1,-1,1)$.

4 Given that point A has position vector $\begin{pmatrix}2\\0\\1\end{pmatrix}$ and point B has position vector $\begin{pmatrix}1\\0\\2\end{pmatrix}$, find the

(i) length of the projection of $\overrightarrow{AB}$ onto the z-axis,

(ii) projection vector of $\overrightarrow{AB}$ onto the z-axis.

5 Find the coordinates of the foot of perpendicular from the point $P(7, -2, 4)$ to the line $r=(2+\lambda)i+(3-2\lambda)\,j+k$, where λ is a real parameter.

Level 2

6 The equation of a straight line l is $\mathbf{r}=\begin{pmatrix}1\\2\\3\end{pmatrix}+t\begin{pmatrix}-1\\1\\1\end{pmatrix}$, where t is a parameter.

The point A on l is given by $t=0$, and the origin of the position vectors is O.

(a) Calculate the acute angle between OA and l, giving your answer correct to the nearest degree.

(b) Find the position vector of the point P on l such that OP is perpendicular to l.

(c) A point Q on l is such that the length of OQ is 5 units. Find the two possible position vectors of Q.

(d) The points R and S on l are given by $t=\lambda$ and $t=2\lambda$ respectively. Show that there is no value of λ for which OR and OS are perpendicular.

7 **N2015/I/7**

Referred to the origin O, points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Point C lies on OA, between O and A, such that $OC:CA=3:2$. Point D lies on OB, between O and B, such that $OD:DB=5:6$.

(i) Find the position vectors $\overrightarrow{OC}$ and $\overrightarrow{OD}$, giving your answers in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

(ii) Show that the vector equation of the line BC can be written as $\mathbf{r}=\frac{3}{5}\lambda\mathbf{a}+(1-\lambda)\mathbf{b}$, where λ is a parameter. Find in a similar form the vector equation of the line AD in terms of a parameter μ. [3]

(iii) Find, in terms of $\mathbf{a}$ and $\mathbf{b}$, the position vector of the point E where the lines BC and AD meet and find the ratio $AE:ED$. [5]

Level 3

8 Relative to the origin O, the point A has coordinates (4, 4, 7) and the line l has equation $r=-i+j+2k+\lambda(6i+j+k)$. Find the position vector of

(i) the foot of perpendicular from A to l,
(ii) the point A', the reflection of A in the line l.

Hence or otherwise, find the shortest distance from A to line l.

9 **2012/I/9 (modified)**

(i) Find a vector equation of the line through the points *A* and *B* with position vectors $7\mathbf{i}+8\mathbf{j}+9\mathbf{k}$ and $-\mathbf{i}-8\mathbf{j}+\mathbf{k}$ respectively. [3]

(ii) The perpendicular to this line from the point *C* with position vector $\mathbf{i}+8\mathbf{j}+3\mathbf{k}$ meets the line at the point *N*. Find the position vector of *N* and the ratio $AN:NB$. [5]

(iii) Find a Cartesian equation of the line which is a reflection of the line *AC* in the line *AB*. [4]

(iv) The point *D* has position vector $\mathbf{i}+8\mathbf{j}-2\mathbf{k}$. Find the length of projection of $\overrightarrow{CD}$ onto line *AB*. [4]

10 **2017(9758)/I/10**

Electrical engineers are installing electricity cables on a building site. Points (x, y, z) are defined relative to a main switching site at $(0,0,0)$, where units are metres. Cables are laid in straight lines and the widths of cables can be neglected.

An existing cable C starts at the main switching site and goes in the direction $\begin{pmatrix}3\\1\\-2\end{pmatrix}$.

A new cable is installed which passes through points $P(1,2,-1)$ and $Q(5,7,a)$.

(i) Find the value of a for which C and the new cable will meet. [4]

To ensure that the cables do not meet, the engineers use $a=-3$. The engineers wish to connect each of the points P and Q to a point R on C.

(ii) The engineers wish to reduce the length of cable required and believe in order to do this that angle PRQ should be 90°. Show that this is not possible. [4]

(iii) The engineers discover that the ground between P and R is difficult to drill through and now decide to make the length of PR as small as possible. Find the coordinates of R in this case and the exact minimum length. [5]

Extend: How can we find the exact minimum length without first finding the coordinates of R?

11 **9758 Specimen Paper/I/6**

(a) The non-zero vectors **a**, **b** and **c** are such that $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$. Given that $\mathbf{b}\neq-\mathbf{c}$, find a linear relationship between **a**, **b** and **c**. [3]

(b) The variable vector **v** satisfies the equation $\mathbf{v}\times(\mathbf{i}-3\mathbf{k})=2\mathbf{j}$. Find the set of vectors **v** and fully describe this set geometrically. [5]

Answer Key

Q1	**Vector Equation**	**Cartesian Equation**	**Parametric Equation**
(a)	$l_1: \mathbf{r} = \begin{pmatrix} 7 \\ 2 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$	$x-7 = \frac{y-2}{-3} = z+4$ **OR** $x-7 = \frac{2-y}{3} = z+4$	$x = 7+\lambda$ $y = 2-3\lambda$ $z = -4+\lambda,\ \lambda \in \mathbb{R}$
(b)	$l_2: \mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -6 \\ -8 \end{pmatrix}, \mu \in \mathbb{R}$	$x-1 = \frac{y+2}{-6} = \frac{z-1}{-8}$ **OR** $x-1 = -\frac{y+2}{6} = \frac{1-z}{8}$	$x = 1+\mu$ $y = -2-6\mu$ $z = 1-8\mu,\ \mu \in \mathbb{R}$
(c)	$l_5: \mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix}, \gamma \in \mathbb{R}$	$x-3 = \frac{y}{3},\ z = 2$	$x = 3+\gamma$ $y = 3\gamma$ $z = 2,\ \gamma \in \mathbb{R}$

2(a) l_1 and l_2 are intersecting lines and the coordinates of their point of intersection are $(-2,3,0)$.

2(b) l_3 and l_4 are distinct parallel lines.

3(ii) $55.9°$

4(i) 1, **(ii)** $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

5 $(5,-3,1)$

6(a) $52°$ **(b)** $\frac{1}{3}\begin{pmatrix} 7 \\ 2 \\ 5 \end{pmatrix}$ **(c)** $\overrightarrow{OQ} = \frac{1}{3}\begin{pmatrix} 14 \\ -5 \\ -2 \end{pmatrix}$ or $\overrightarrow{OQ} = \begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$

7(i) $\overrightarrow{OC} = \frac{3}{5}\mathbf{a}, \overrightarrow{OD} = \frac{5}{11}\mathbf{b}$ **(ii)** $\mathbf{r} = \frac{5}{11}\mu\mathbf{b} + (1-\mu)\mathbf{a},\ \ \mu \in \mathbb{R}$ **(iii)** $\overrightarrow{OE} = \frac{9}{20}\mathbf{a} + \frac{1}{4}\mathbf{b}$, $11:9$

8(i) $\begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix}$, **(ii)** $\begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix}$; $\sqrt{21}$

9(i) $\mathbf{r} = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix},\ \lambda \in \mathbb{R}$ **(ii)** $\begin{pmatrix} 5 \\ 4 \\ 7 \end{pmatrix}$, $1:3$ **(iii)** $x-7 = \frac{y-8}{-4} = z-9$ **(iv)** $\frac{5\sqrt{6}}{6}$

10(i) $a = -\frac{22}{5}$ **(iii)** $\left(\frac{3}{2}, \frac{1}{2}, -1\right)$, $\frac{\sqrt{10}}{2}$