Solutions (Differential Equations)

1	$(x+y)\frac{dy}{dx}=x^2+xy+x+1$ -------------(1) Given $z=x+y \Rightarrow \frac{dz}{dx}=1+\frac{dy}{dx} \Rightarrow \frac{dy}{dx}=\frac{dz}{dx}-1$ ----------(2) Sub (2) into (1) : $(z)\left(\frac{dz}{dx}-1\right)=x^2+x(z-x)+x+1$ $(z)\frac{dz}{dx}=z+xz+x+1$ $(z)\frac{dz}{dx}=z(x+1)+(x+1)$ $(z)\frac{dz}{dx}=(x+1)(z+1)$ $(shown)$ $\int\frac{z}{z+1}dz=\int(x+1)dx$ $\int 1-\frac{1}{z+1}dz=\frac{(x+1)^2}{2}+c$ $z-\ln\lvert z+1\rvert=\frac{1}{2}(x+1)^2+c$ $x+y-\ln\lvert x+y+1\rvert=\frac{1}{2}(x+1)^2+c$
	Speed = $\frac{dx}{dt}=k(8-x)$, $k=$ positive constant At t = 0, x = 0 , $\frac{dx}{dt}$ = 10 $\Rightarrow$ 10 = 8k $\Rightarrow$ $k=\frac{5}{4}$ $\Rightarrow \frac{dx}{dt}=\frac{5}{4}(8-x)$ (shown)
	Integrating wrt t , $\int\frac{1}{8-x}dx=\int\frac{5}{4}$ $-\ln(8-x)=\frac{5}{4}t+C$ $x=8-Ae^{-\frac{5}{4}t}$

	At $t=0$, $x=0$, $A=8 \Rightarrow$ $x=8-8e^{-\frac{5}{4}t}$ At $x=6$, $e^{-\frac{5}{4}t}=\frac{1}{4} \Rightarrow t=1.11$hrs
	$x=8-8e^{-\frac{5}{4}t}$ When $x=8$ (i.e. he completes his 8 km jog) , $t\to\infty$ Model predicts infinite amount of time to complete jog, therefore model is NOT suitable

2. Suggested solution

(a)(i)

To prove $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=k---(*)$

Consider

$$y=\frac{k\left(\ln x+\alpha\right)}{x}\Rightarrow xy=k\left(\ln x+\alpha\right)--(1)$$

Diff (1) wrt *x*,

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+y=k\left(\frac{1}{x}\right)\Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=k$$ **[shown]**

(a)(ii) $y=\frac{k\left(\ln x+\alpha\right)}{x}$

At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow xy=k$ **[from (*)]**

So $\frac{k}{x}=\frac{k\left(\ln x+\alpha\right)}{x}\Rightarrow \ln x=1-\alpha\Rightarrow x=\mathrm{e}^{1-\alpha}$

When $x=\mathrm{e}^{1-\alpha}$, $y=\frac{k}{x}=\frac{k}{\mathrm{e}^{1-\alpha}}=k\mathrm{e}^{\alpha-1}$**.**

Therefore, $\left(\mathrm{e}^{1-\alpha},\ k\mathrm{e}^{\alpha-1}\right)$ **is a stationary point of the curve** $y=\frac{k\left(\ln x+\alpha\right)}{x}$**.**

(b)(i) Given $y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\sqrt{x^2+y^2}$ **---(**)**

$v = x^2 + y^2 \Rightarrow \frac{dv}{dx} = 2x + 2y\frac{dy}{dx}$

Sub into (**):

$y\frac{dy}{dx} + x = \sqrt{x^2 + y^2} \Rightarrow \frac{1}{2}\frac{dv}{dx} = \sqrt{v} \Rightarrow \frac{dv}{dx} = 2\sqrt{v}$ **[shown]**

(b)(ii)

$\frac{dv}{dx} = 2\sqrt{v} \Rightarrow \frac{1}{2\sqrt{v}}\frac{dv}{dx} = 1 \Rightarrow \int \frac{1}{2\sqrt{v}}\ dv = \int 1\ dx$

$\Rightarrow \sqrt{v} = x + C$

Since $y = 0$ **when** $x = -2$**, we have** $v = 4$**.**

$\sqrt{4} = -2 + C \Rightarrow C = 4$

$\sqrt{v} = x + 4 \Rightarrow v = (x+4)^2$

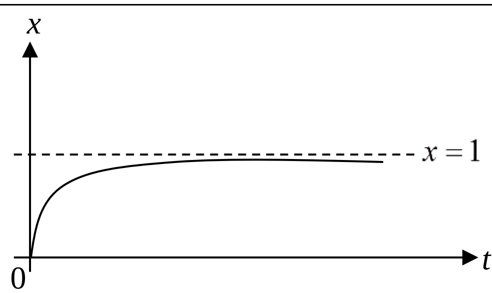
$\Rightarrow y^2 = (x+4)^2 - x^2 = 8x + 16$

Hence $f(x) = 8x + 16$**.**

3	(a)	Let x be the amount of radium at any time t. $\frac{dx}{dt} = -kx$ $\int \frac{1}{x} dx = -k\int dt$ $\ln x = -kt + c$ If m is the initial amount, then $\ln m = c$ $x = \frac{1}{2}m$ when t = 1600 $\ln \frac{m}{2} = -1600k + \ln m$ $k = \frac{\ln 2}{1600}$ $\ln x = -kt + c$

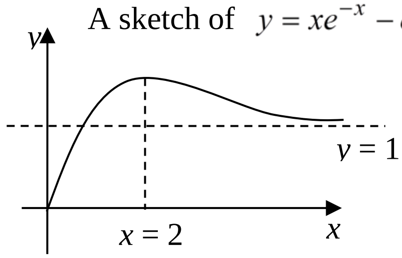
	When $x=\frac{99}{100}m$, $\ln\frac{99m}{100}=-\frac{\ln 2}{1600}t+\ln m$, $t = 23.2$ It will take 23 years to lose 1% of the initial mass. (b) $\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=\frac{1}{1+4t^2}=\frac{1}{4\left(\frac{1}{4}+t^2\right)}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{4}\int\frac{1}{\left(\frac{1}{4}+t^2\right)}\mathrm{d}t=\left(\frac{1}{4}\right)\left(\frac{1}{1/2}\right)\tan^{-1}\frac{t}{1/2}+c=\frac{1}{2}\tan^{-1}2t+c$ $\frac{\mathrm{d}x}{\mathrm{d}t}=0$ when $t=0\Rightarrow c=0$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{2}\tan^{-1}2t$ $x=\frac{1}{2}\int\tan^{-1}2t\ \mathrm{d}t$ $=\frac{1}{2}\left[t\tan^{-1}2t-\int\frac{2t}{1+4t^2}\mathrm{d}t\right]$ $=\frac{1}{2}\left[t\tan^{-1}2t-\frac{1}{4}\ln(1+4t^2)+d\right]$ $x = 0.5$ when $t = 0 \Rightarrow d = 1$ $\therefore\ x=\frac{1}{8}\left[4t\tan^{-1}2t-2\ln(1+4t^2)+4\right]$

4	Net rate of increase of fuel = rate of increase due to fuel pumped in − rate at which fuel is used up $\frac{\mathrm{d}x}{\mathrm{d}t}=2\left(\frac{3}{2}-x\right) - kx^2$, where $k>0$ When $x=1$, $\frac{\mathrm{d}x}{\mathrm{d}t}=0$, $k=1$. Hence $\frac{\mathrm{d}x}{\mathrm{d}t}=3-2x-x^2$.

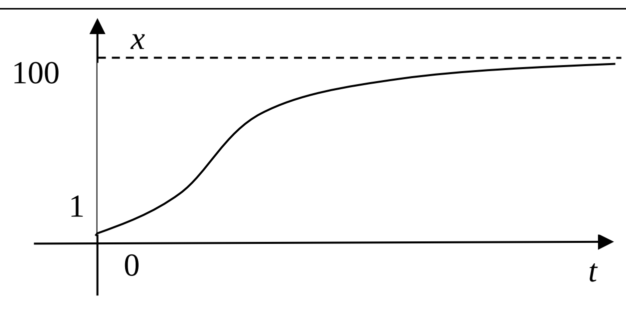
	$$\frac{dx}{dt} = -\left(x^2+2x-3\right)$$ $$\frac{1}{x^2+2x-3}\frac{dx}{dt} = -1$$ $$\int \frac{1}{x^2+2x-3}\,dx = -\int dt$$ $$\int \frac{1}{(x+1)-2^2}\,dx = -\int dt$$ $$\frac{1}{4}\ln\left\|\frac{x-1}{x+3}\right\| = -t+C$$ $$\left\|\frac{x-1}{x+3}\right\| = e^{-4t+4C}$$ $$\frac{x-1}{x+3} = Ae^{-4t} \quad \text{where } A=\pm e^{4C}$$ When $t=0$, $x=0$, $A=-\frac{1}{3}$, $$-3(x-1)=(x+3)e^{-4t}$$ $$x\left(3+e^{-4t}\right)=3-3e^{-4t}$$ $$x=\frac{3\left(1-e^{-4t}\right)}{3+e^{-4t}}$$ Alternatively: $$1-\frac{4}{x+3}=-\frac{1}{3}e^{-4t}$$ $$x=\frac{12}{3+e^{-4t}}-3$$
	

5	$\frac{dx}{dt}=\frac{a}{x}-bx$ where a and b are positive constants When $x=2$, $\frac{dx}{dt}=0 \Rightarrow \frac{a}{2}=b(2) \quad \Rightarrow \quad a=4b$ $\therefore \; \frac{dx}{dt}=\frac{4b}{x}-bx$ $= -b\left(\frac{x^2-4}{x}\right)$

	$= k\left(\dfrac{x^2-4}{x}\right)$, where k is a negative constant $\displaystyle\int \frac{x}{x^2-4}\ dx = \int k\ dt$ $\frac{1}{2}\ln\lvert x^2-4\rvert = kt + C$ Since $x > 2$, $x^2-4 = e^{2C}e^{2kt}$ $x^2 = Ae^{2kt}+4$ where $A = e^{2C}$ $x = \sqrt{\left(Ae^{2kt}+4\right)}$ (Reject $-\sqrt{\left(Ae^{2kt}+4\right)}$ since $x > 0$)

6	$\dfrac{dy}{dx} = (2-x)e^{-x} \Rightarrow y = \int (2-x)e^{-x}dx$ $\Rightarrow y = -(2-x)e^{-x} - \int e^{-x}dx + C$, where C is an arbitrary constant. $\Rightarrow y = (x-2)e^{-x} + e^{-x} + C$, $\Rightarrow y = xe^{-x} - e^{-x} + C$ When $x = 0$, $y = 0$. Therefore, $C = 1$. Therefore, $y = xe^{-x} - e^{-x} + 1$.
	A sketch of $y = xe^{-x} - e^{-x} + 1$  Max y occurs when $x = 2$. $\dfrac{dx}{dt} = e^{x} \Rightarrow \int e^{-x}dx = \int 1dt \Rightarrow e^{-x} = -t + C$. $\Rightarrow x = -\ln(-t + C)$ When $t = 0$, $x = 0$. Therefore $C = 1$. $\therefore$ $x = -\ln(1-t)$ When $x = 2$, $2 = -\ln(1-k)$ $\Rightarrow k = 1 - e^{-2}$

7	(a) Since $y = x$ and $\frac{dy}{dx} = 1$, $LHS = 1 = \frac{x^2 + x^2}{2x^2} = RHS$ (b) $y = ux$ $\frac{dy}{dx} = u + x\frac{du}{dx}$ $u + x\frac{du}{dx} = \frac{x^2 + u^2x^2}{2x(ux)}$ $u + x\frac{du}{dx} = \frac{1+u^2}{2u}$ $x\frac{du}{dx} = \frac{1+u^2}{2u} - u$ $x\frac{du}{dx} = \frac{1+u^2-2u^2}{2u}$ $x\frac{du}{dx} = \frac{1-u^2}{2u} (shown)$ $\int \frac{2u}{1-u^2}\, du = \int \frac{1}{x}\, dx$ $\int \frac{-2u}{1-u^2}\, du = -\int \frac{1}{x}\, dx$ $\ln(1-u^2) = -\ln x + C$ $1 - u^2 = \frac{A}{x}$ $u^2 = 1 - \frac{A}{x}$ $\frac{y^2}{x^2} = 1 - \frac{A}{x}$ $y^2 = x^2 - Ax$ (c) $\frac{d^2x}{dt^2} = 4ae^{-2t}$ $\frac{dx}{dt} = -2ae^{-2t} + C$ $x = ae^{-2t} + Ct + D$ Since entire population is wiped out by the disease eventually, as $t \to +\infty, x \to 0$. Hence, $C = 0$, $D = 0$. $\therefore x = ae^{-2t}$ a represents the initial population of the fish (in thousands).

8 (i)	$\frac{dx}{dt} = k(x)(100-x)$ At $t=0$, $x=1, \frac{dx}{dt}=1$ $1 = k(1)(99) \Rightarrow k = \frac{1}{99}$ therefore $\frac{dx}{dt} = \frac{1}{99}(100x - x^2)$
(ii)	$\frac{dx}{dt} = \frac{1}{99}(100x - x^2)$ $\int \frac{1}{x(100-x)} \mathrm{d}x = \int \frac{1}{99} \mathrm{d}t$ $\frac{1}{100}\int \frac{1}{x} + \frac{1}{100-x} \mathrm{d}x = \int \frac{1}{99} \mathrm{d}t$ $\ln x - \ln(100-x) = \frac{100}{99}t + C$ $\ln\left(\frac{x}{100-x}\right) = \frac{100}{99}t + C$ $\frac{x}{100-x} = e^{\frac{100}{99}t+C} = Ae^{\frac{100}{99}t}$ When $t=0$, $x=1$, $A = \frac{1}{99}$ $x = A100e^{\frac{100}{99}t} - xAe^{\frac{100}{99}t}$ $x = \frac{A100e^{\frac{100}{99}t}}{1+Ae^{\frac{100}{99}t}} = \frac{100e^{\frac{100}{99}t}}{99+e^{\frac{100}{99}t}}$
(iii)	
(iv)	Using GC, 5.64 years.
(v)	The farmers may be influenced by adoption of innovation from other sources, eg mass media, besides farmers. Or any other reasonable answer

9 (i)	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{4-9x^2}\right)=\dfrac{-18x}{2\sqrt{4-9x^2}}=\dfrac{-9x}{\sqrt{4-9x^2}}$
(ii)	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\dfrac{1}{\sqrt{4-9x^2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\displaystyle\int \dfrac{1}{\sqrt{4-9x^2}}\,\mathrm{d}x=\dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+c$ $y=\displaystyle\int \dfrac{1}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+c\,\mathrm{d}x$ $=\dfrac{x}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)-\displaystyle\int \dfrac{x}{3}\cdot\dfrac{1}{\sqrt{1-\dfrac{9x^2}{4}}}\left(\dfrac{3}{2}\right)\mathrm{d}x+cx+d$ $=\dfrac{x}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)-\displaystyle\int \dfrac{x}{\sqrt{4-9x^2}}\,\mathrm{d}x+cx+d$ $=\dfrac{x}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+\dfrac{1}{9}\sqrt{4-9x^2}+cx+d$ When $x = 0$, $y=\dfrac{2}{9}$. $\dfrac{2}{9}=\dfrac{2}{9}+d \quad \Rightarrow d = 0$ $y=\dfrac{x}{3}\sin^{-1}\left(\dfrac{3x}{2}\right)+\dfrac{1}{9}\sqrt{4-9x^2}+cx$

10	$\dfrac{d\theta}{dt}=k\left(\theta-25\right)$ $\displaystyle\int\dfrac{1}{\left(\theta-25\right)}\mathrm{d}\theta=k\int 1\mathrm{d}t$ $\Rightarrow \ln\lvert\theta-25\rvert=kt+C$ When $t=0$, $\theta=110\Rightarrow \ln\lvert 110-25\rvert=C$ When $t=5$, $\theta=80\Rightarrow \ln\lvert 80-25\rvert=5k+C$ $\therefore\ k=\dfrac{1}{5}\ln\dfrac{55}{85}$ When $\theta=45$,

	$\ln\lvert 45-25\rvert=\left(\frac{1}{5}\ln\frac{55}{85}\right)t+\ln 85 \Rightarrow t=16.62\,\text{min}\approx 17\,\text{min}$ The estimated time when coffee was brewed is 11.43AM

11(a)	$\frac{\mathrm{dy}}{\mathrm{d}x}=\frac{y}{x}-2x^2$ Let $y = ux$ $\frac{\mathrm{d}y}{\mathrm{d}x}=u+x\frac{\mathrm{d}u}{\mathrm{d}x}$ $u+x\frac{du}{dx}=u-2x^2$ $\frac{du}{dx}=-2x$ $\int du=\int -2x\,dx$ $u=-x^2+C$ $\frac{y}{x}=-x^2+C$ $y=-x^3+Cx$ when $y = 0$, $x = 1$ $\Rightarrow$ C = 1 $y=-x^3+x$
11(b)	$\frac{dx}{dt}=kx(1-x)$ When $\frac{dx}{dt}=\frac{1}{54}$, $x=\frac{1}{3}$, $k=\frac{1}{12}$ $\int\frac{1}{x(1-x)}\,dx=\frac{1}{12}\int dt$ $\int\frac{1}{x}-\frac{-1}{1-x}\,dx=\frac{1}{12}\int dt$ $\ln\lvert x\rvert-\ln\lvert 1-x\rvert=\frac{1}{12}t+C$ $\left\lvert\frac{x}{1-x}\right\rvert=Ae^{\frac{t}{12}}$ $\frac{x}{1-x}=Be^{\frac{t}{12}}$ when $t=0$, $x=\frac{1}{2}$, $B=1$ $\frac{x}{1-x}=e^{\frac{t}{12}}$

	$x=\dfrac{e^{\frac{t}{12}}}{1+e^{\frac{t}{12}}}$ (ii) when $t=12$, $x=\dfrac{e}{1+e}=0.731$

12	$w=\dfrac{y}{t^2}$ diff. w.r.t. t $\dfrac{dw}{dt}=\dfrac{t^2\dfrac{dy}{dt}-2yt}{t^4}$ $t\dfrac{dw}{dt}=w^2t^3+2wt-2w$ $t\left(\dfrac{t^2\dfrac{dy}{dt}-2yt}{t^4}\right)=\dfrac{y^2}{t^4}t^3+\dfrac{2y}{t^2}t-\dfrac{2y}{t^2}$ $t^3\dfrac{dy}{dt}-2t^2y=t^3y^2+2yt^3-2yt^2$ $\dfrac{dy}{dt}=y^2+2y$ $\int\dfrac{1}{y^2+2y}\,dy=\int 1\,dt$ $\int\dfrac{1}{y(y+2)}\,dy=\int 1\,dt$ $\int\dfrac{1}{2y}-\dfrac{1}{2(y+2)}\,dy=\int 1\,dt$ $\dfrac{1}{2}\ln\lvert y\rvert-\dfrac{1}{2}\ln\lvert y+2\rvert=t+c$

	$\frac{1}{2}\ln\left\|\frac{y}{y+2}\right\|=t+c$ $\ln\left\|\frac{y}{y+2}\right\|=2t+b$ $\frac{y}{y+2}=Ae^{2t}$ $y\left(1-Ae^{2t}\right)=2Ae^{2t}$ $y=\frac{2Ae^{2t}}{1-Ae^{2t}}$ $w=\frac{1}{t^2}\left(\frac{2Ae^{2t}}{1-Ae^{2t}}\right)$

13	$z=e^{2x}\frac{dy}{dx}$ $\frac{dz}{dx}=e^{2x}\frac{d^2y}{dx^2}+2e^{2x}\frac{dy}{dx}$ $=e^{2x}\left(\frac{d^2y}{dx^2}+2\frac{dy}{dx}\right)$ Given $\frac{d^2y}{dx^2}+2\frac{dy}{dx}=e^{1-4x}$ $e^{-2x}\frac{dz}{dx}=e^{1-4x}$ $\Rightarrow\frac{dz}{dx}=e^{1-4x}\cdot e^{2x}\ \Rightarrow\frac{dz}{dx}=e^{1-2x}$ (shown) Hence $\int dz=\int e^{1-2x}dx$ $z=-\frac{1}{2}e^{1-2x}+C$ $e^{2x}\frac{dy}{dx}=-\frac{1}{2}e^{1-2x}+C$ $\frac{dy}{dx}=-\frac{1}{2}e^{1-4x}+Ce^{-2x}\Rightarrow y=\frac{1}{8}e^{1-4x}-\frac{1}{2}Ce^{-2x}+D$

14 (a)	$y = xz ----(1)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = x\dfrac{\mathrm{d}z}{\mathrm{d}x} + z ----(2)$ Sub (1) and (2) into D.E.: $\left(\mathrm{e}^x + 1\right)\left(x\dfrac{\mathrm{d}z}{\mathrm{d}x} + \dfrac{y}{x} - \dfrac{y}{x}\right) = \dfrac{x^2}{xz}\left(\mathrm{e}^x - 1\right)$ $\left(\mathrm{e}^x + 1\right)\left(\dfrac{\mathrm{d}z}{\mathrm{d}x}\right) = \dfrac{\left(\mathrm{e}^x - 1\right)}{z}$ $z\dfrac{\mathrm{d}z}{\mathrm{d}x} = \dfrac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}$ $\int z \ \mathrm{d}z = \int \dfrac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1} \ \mathrm{d}x$ Method 1: $\int z \ \mathrm{d}z = \int \dfrac{\left(\mathrm{e}^{\frac{x}{2}}\right)\left(\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{-\frac{x}{2}}\right)}{\left(\mathrm{e}^{\frac{x}{2}}\right)\left(\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}\right)} \ \mathrm{d}x$ $\int z \ \mathrm{d}z = \int \dfrac{\left(\mathrm{e}^{\frac{x}{2}} - \mathrm{e}^{-\frac{x}{2}}\right)}{\left(\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}\right)} \ \mathrm{d}x$ $\dfrac{z^2}{2} = 2\ln\left\|\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}\right\| + C,$ where C is an arbitrary constant. $\left(\dfrac{y}{x}\right)^2 = 4\ln\left(\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}\right) + D,$ where $D = 2C$ $y^2 = 4x^2\ln\left(\mathrm{e}^{\frac{x}{2}} + \mathrm{e}^{-\frac{x}{2}}\right) + Dx^2$ Method 2:

$$\int z \, \mathrm{d}z = \int \left(\frac{\mathrm{e}^x}{\mathrm{e}^x + 1} - \frac{1}{\mathrm{e}^x + 1} \frac{\mathrm{e}^{-x}}{\mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = \ln\left|\mathrm{e}^x + 1\right| + C' - \int \left(\frac{1}{\mathrm{e}^x + 1} \frac{\mathrm{e}^{-x}}{\mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = \ln\left|\mathrm{e}^x + 1\right| + C' - \int \left(\frac{\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = \ln\left|\mathrm{e}^x + 1\right| + C' + \int \left(\frac{-\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = \ln\left|\mathrm{e}^x + 1\right| + \ln\left|1 + \mathrm{e}^{-x}\right| + C$$, where C is an arbitrary constant

$$z^2 = 2\ln\left(\mathrm{e}^x + 1\right) + 2\ln\left(1 + \mathrm{e}^{-x}\right) + D, \ D = 2C$$

$$\left(\frac{y}{x} \right)^2 = 2\ln\left(\mathrm{e}^x + 1\right) + 2\ln\left(1 + \mathrm{e}^{-x}\right) + D$$

$$y^2 = 2x^2 \ln\left(\mathrm{e}^x + 1\right) + 2x^2 \ln\left(1 + \mathrm{e}^{-x}\right) + Dx^2$$

Method 3:

$$\int z \, \mathrm{d}z = \int \left(1 - \frac{2}{\mathrm{e}^x + 1} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int \left(\frac{1}{\mathrm{e}^x + 1} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int \left(\frac{1}{\mathrm{e}^x + 1} \frac{\mathrm{e}^{-x}}{\mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' - 2\int \left(\frac{\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + C' + 2\int \left(\frac{-\mathrm{e}^{-x}}{1 + \mathrm{e}^{-x}} \right) \mathrm{d}x$$

$$\frac{z^2}{2} = x + 2\ln\left|1 + \mathrm{e}^{-x}\right| + C$$, where C is an arbitrary constant

$$z^2 = 2x + 4\ln\left(1 + \mathrm{e}^{-x}\right) + D, \ D = 2C$$

$$\left(\frac{y}{x} \right)^2 = 2x + 4\ln\left(1 + \mathrm{e}^{-x}\right) + D$$

$$y^2 = 2x^3 + 4x^2 \ln\left(1 + \mathrm{e}^{-x}\right) + Dx^2$$

	Method 4: $\int z\ \mathrm{dz} = \int \frac{\mathrm{e}^x - 1}{\mathrm{e}^x + 1}\ \mathrm{d}x$ Let $u = \mathrm{e}^x$ $\int z\ \mathrm{dz} = \int \left(\frac{u-1}{u+1}\frac{1}{u}\right) \mathrm{d}u$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{e}^x$ $\int z\ \mathrm{dz} = \int \left(\frac{u-1}{u(u+1)}\right) \mathrm{d}u$ $\int z\ \mathrm{dz} = \int \left(\frac{2}{u+1} - \frac{1}{u}\right) \mathrm{d}x$ $\frac{z^2}{2} = 2\ln\|u+1\| - \ln\|u\| + C$, where C is an arbitrary constant $\frac{z^2}{2} = 2\ln\left(\mathrm{e}^x + 1\right) - \ln\left(\mathrm{e}^x\right) + C$ $z^2 = 4\ln\left(\mathrm{e}^x + 1\right) - 2\ln\left(\mathrm{e}^x\right) + D,\ D = 2C$ $\left(\frac{y}{x}\right)^2 = 4\ln\left(\mathrm{e}^x + 1\right) - 2\ln\left(\mathrm{e}^x\right) + D$ $y^2 = 4x^2\ln\left(\mathrm{e}^x + 1\right) - 2x^2\ln\left(\mathrm{e}^x\right) + Dx^2$
(b)	$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{A}{9-x} - \frac{x}{20}$ When $x = 4$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 0,$ $0 = \frac{A}{5} - \frac{4}{20}$ $A = 1$ $\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{x}{20} + \frac{1}{9-x}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{-x(9-x)+20}{20(9-x)}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{x^2 - 9x + 20}{20(9-x)}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{(x-4)(x-5)}{20(9-x)}$ (shown)

	$\int \frac{9-x}{(x-4)(x-5)}\,\mathrm{d}x = \int \frac{1}{20}\,\mathrm{d}t$ $\int \frac{-5}{x-4} + \frac{4}{x-5}\,\mathrm{d}x = \frac{t}{20} + C'$ $-5\ln\lvert x-4\rvert + 4\ln\lvert x-5\rvert = \frac{t}{20} + C'$ $t = -100\ln\lvert x-4\rvert + 80\ln\lvert x-5\rvert + C$ where $C = 20C'$ When $t=0,\ x=0,$ $0 = -100\ln 4 + 80\ln 5 + C$ $C = 100\ln 4 - 80\ln 5$ $t = -100\ln\lvert x-4\rvert + 80\ln\lvert x-5\rvert + 100\ln 4 - 80\ln 5$ $t = -100\ln\left\lvert\frac{x-4}{4}\right\rvert + 80\ln\left\lvert\frac{x-5}{5}\right\rvert$ When $x = 2,$ $t = 28.449$ $t = 28.4$ months (3 s.f.)

15	(i) For increasing population, $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$ $\Rightarrow 0.02P(100-P) > 0$ $\therefore 0 < P < 100$ (ii) $\frac{\mathrm{d}P}{\mathrm{d}t} = 0.02P(100-P)$ $\int \frac{1}{P(100-P)}\,\mathrm{d}P = \int 0.02\,\mathrm{d}t$ $\frac{1}{100}\int \frac{1}{P} + \frac{1}{100-P}\,\mathrm{d}P = 0.02t + C$ $\ln P - \ln\lvert 100-P\rvert = 2t + C'$ $\ln\lvert 100-P\rvert - \ln P = -2t + C''$ $\ln\left\lvert\frac{100-P}{P}\right\rvert = -2t + C''$ $\left\lvert\frac{100-P}{P}\right\rvert = \mathrm{e}^{-2t+C''}$

$$\frac{100-P}{P}=\pm e^{C''}e^{-2t}$$

$$\frac{100-P}{P}=Ae^{-2t}$$

$$P=\frac{100}{1+Ae^{-2t}}$$

Given P = 20 when t = 0, $\Rightarrow 20=\dfrac{100}{1+A}$

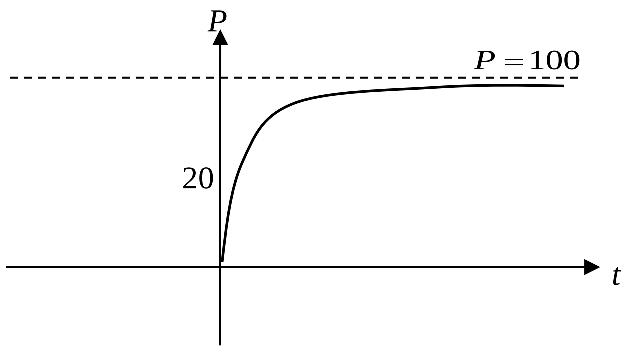
$$\Rightarrow 1+A=5$$

$$\therefore A=4$$

$P=\dfrac{100}{1+4e^{-2t}}$ (shown)

(iii)

P is an increasing function and for large values of t, $P\to 100$.



16

$$\frac{dV}{dt}=4-kV$$

When $V=6$, $\dfrac{dV}{dt}=1$,

$1=4-k(6)$

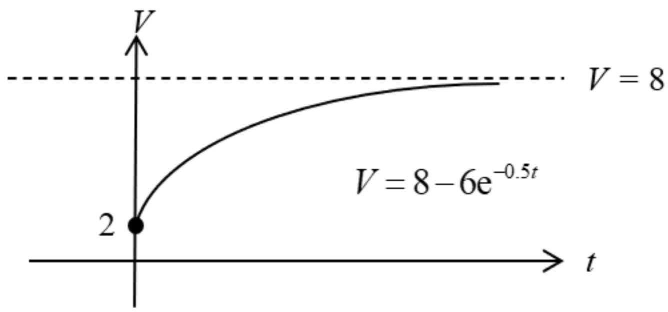
$3=\ k(6)$

$k=0.5$

$$\frac{dV}{dt}=4-0.5V$$

$$\int\frac{1}{4-0.5V}\,dV=\int 1\,dt$$

$$-\frac{1}{0.5}\ln\left|4-0.5V\right|=t+c$$

	$\ln\|4-0.5V\|=-0.5t-0.5c$ $\|4-0.5V\|=\mathrm{e}^{-0.5t-0.5c}$ $4-0.5V=\pm\mathrm{e}^{-0.5t-0.5c}$ $4-0.5V=A\mathrm{e}^{-0.5t}$ where $A=\pm\,\mathrm{e}^{-0.5c}$ When $t=0, V=2$ $4-0.5(2)=A$ $A=3$ $4-0.5V=3\mathrm{e}^{-0.5t}$ $-0.5V=3\mathrm{e}^{-0.5t}-4$ $V=8-6\mathrm{e}^{-0.5t}$
(i)	As $t\to\infty, \mathrm{e}^{-0.5t}\to 0,\ V\to 8$. Therefore, the volume of water in the tank will increase and converge to 8 m^3.
(ii)	

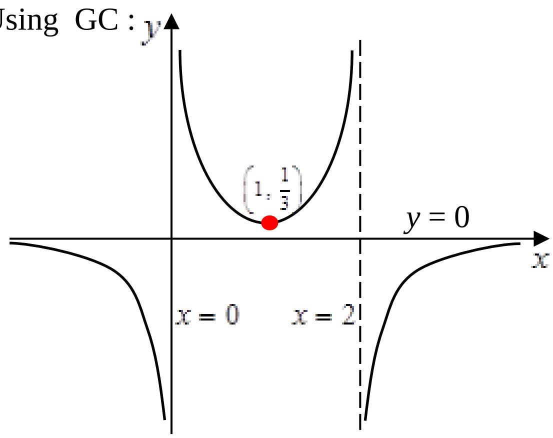
17(i)	Model A: $\frac{\mathrm{d}^2R}{\mathrm{d}\theta^2}=\frac{1}{10000}\mathrm{e}^{0.01\theta}$ $\Rightarrow\frac{\mathrm{d}R}{\mathrm{d}\theta}=\frac{1}{100}\mathrm{e}^{0.01\theta}+A$ i.e. $R=\mathrm{e}^{0.01\theta}+A\theta+B$ Model B: $\frac{\mathrm{d}R}{\mathrm{d}\theta}=R\sec^2 2\theta$ $\Rightarrow\int\frac{1}{R}\,\mathrm{d}R=\frac{1}{2}\int 2\sec^2 2\theta\ \mathrm{d}\theta$ $\Rightarrow\ln R=\frac{1}{2}\tan 2\theta+C'$ i.e. $R=C\,\mathrm{e}^{\frac{1}{2}\tan 2\theta}$

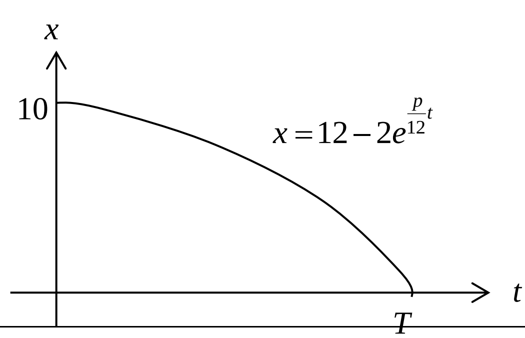
(ii)	The reasons are: (a) R is not increasing with θ (or R is only increasing with θ for disjoint intervals of θ). (b) R is not defined at some values of θ (i.e. at $\theta = \frac{(2k+1)\pi}{4}$, $k \in \mathbb{Z}^+$).
(iii)	Substitute $\frac{\mathrm{d}R}{\mathrm{d}\theta} = 0.1$ & $\theta = 0$ into $\frac{\mathrm{d}R}{\mathrm{d}\theta} = \frac{1}{100}\mathrm{e}^{0.01\theta} + A$, we have $A = 0.09$. Substitute $R = 100$ & $\theta = 0$ into $R = \mathrm{e}^{0.01\theta} + A\theta + B$, we have $B = 99$. $\therefore R = \mathrm{e}^{0.01\theta} + 0.09\theta + 99$ When $R = 200$, $\theta = 415$ (3 s.f.) [using GC]

18	$x^2 \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = x^2 - 1$ $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{x^2 - 1}{x^2}$ $= 1 - \frac{1}{x^2}$ $= 1 - x^{-2}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = x + \frac{1}{x} + A$ $y = \frac{x^2}{2} + \ln\lvert x\rvert + Ax + B$

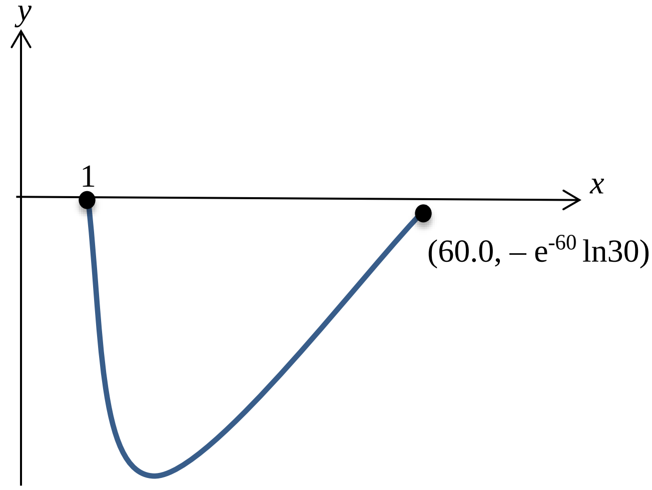
19(i)	$\frac{dv}{dt} = \frac{1}{2}\left(a^2 - v^2\right) \quad \Rightarrow \quad \int \frac{dv}{\left(a^2 - v^2\right)} = \frac{1}{2}\int dt$ $\Rightarrow \quad \frac{1}{2a}\ln\left\lvert\frac{a+v}{a-v}\right\rvert = \frac{1}{2}t + C$ $\Rightarrow \quad \left\lvert\frac{a+v}{a-v}\right\rvert = e^{at+2Ca}$ $\Rightarrow \quad \frac{a+v}{a-v} = Ae^{at}$ where $A = \pm e^{2Ca}$ When $t = 0$, $v = 0$ so $1 = A$ $\Rightarrow \quad \frac{a+v}{a-v} = e^{at}$ $\Rightarrow \quad a + v = e^{at}(a - v)$ $\Rightarrow \quad v = \frac{a\left(e^{at} - 1\right)}{1 + e^{at}}$

(ii)	$v=\dfrac{a\left(e^{at}-1\right)}{1+e^{at}} \Rightarrow v=\dfrac{a\left(1-e^{-at}\right)}{e^{-at}+1}$ When $t\to\infty$, $e^{-at}\to 0$, $v\to\dfrac{a(1-0)}{0+1}=a$ Thus $v_f = a$.

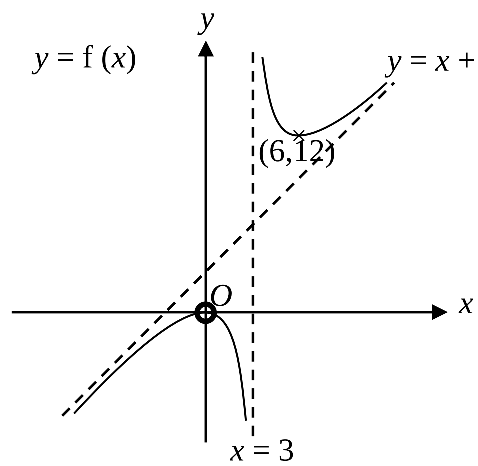
20	$x\dfrac{dy}{dx}+y-3(xy)^2=0$ ....(1) Given $u=xy$: $\dfrac{du}{dx}=x\dfrac{dy}{dx}+y$ Substitute into (1): $\dfrac{du}{dx}-3u^2=0$
	$\dfrac{du}{dx}=3u^2$ $\int \dfrac{1}{u^2}\,du=\int 3\,dx$ $\Rightarrow -\dfrac{1}{u}=3x+C$ (C arbitrary constant) or $u=-\dfrac{1}{3x+C}$ $\therefore y=-\dfrac{1}{x(3x+C)}$ (*)
(i)	Given $\left(1,\dfrac{1}{3}\right)$ $\dfrac{1}{3}=-\dfrac{1}{1(3+C)} \Rightarrow C=-6$ Hence, $y=-\dfrac{1}{3x(x-2)}$ Using GC : 
(ii)	y has no turning point when $C=0$, i.e. particular solution is $y=-\dfrac{1}{3x^2}$

21(a)	$\dfrac{d^2y}{dx^2} = ae^{-2x}$ $\dfrac{dy}{dx} = \dfrac{ae^{-2x}}{-2} + c$ $y = \dfrac{ae^{-2x}}{4} + cx + d$
(b)	$\dfrac{dx}{dt} = kx - p$ Given that $\dfrac{dx}{dt} = 0$ when $x = 12$. $\Rightarrow 12k - p = 0 \Rightarrow k = \dfrac{p}{12}$ $\therefore \quad \dfrac{dx}{dt} = \dfrac{px}{12} - p$ $\Rightarrow \quad \dfrac{dx}{dt} = \dfrac{p}{12}(x-12)$ (Shown)
(i)	$\int \dfrac{1}{x-12}\, dx = \int \dfrac{p}{12}\, dt$ $\ln\lvert x-12\rvert = \dfrac{p}{12}t + c$ $x - 12 = \pm\, e^{\frac{p}{12}t + c}$ $x = 12 + Ae^{\frac{p}{12}t}$, where $A = \pm e^c$ When $t = 0$, $x = 10$ $\Rightarrow$ $A = -2$ $\therefore \quad x = 12 - 2e^{\frac{p}{12}t}$
(ii)	When $x = 0$, $t = T$ $\Rightarrow 12 - 2e^{\frac{p}{12}T} = 0$ $\Rightarrow \dfrac{p}{12}T = \ln 6$ $\Rightarrow T = \dfrac{12}{p}\ln 6$
(iii)	

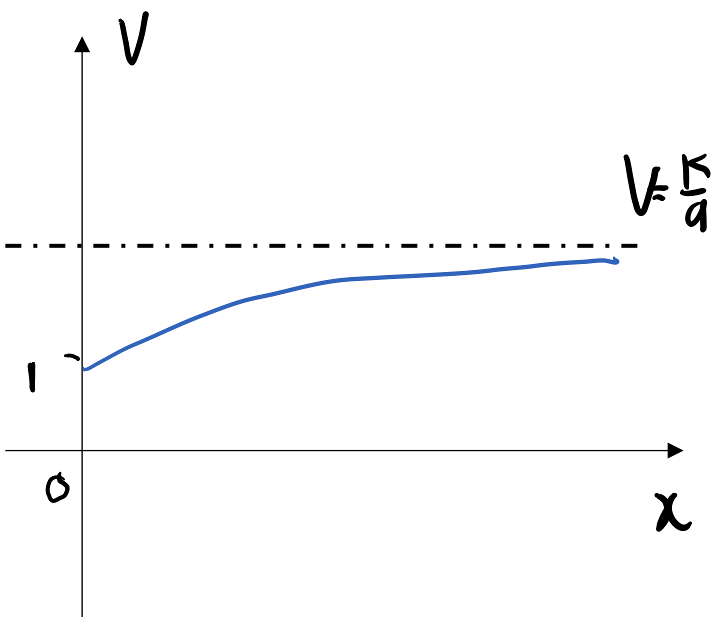
22	i) Let $w = ye^t$, $\frac{\mathrm{d}w}{\mathrm{d}t} = e^t \frac{\mathrm{d}y}{\mathrm{d}t} + ye^t$ $\frac{\mathrm{d}y}{\mathrm{d}t} + y = \frac{e^{-t}}{t-30}$ $e^t \frac{\mathrm{d}y}{\mathrm{d}t} + ye^t = \frac{1}{t-30}$ $\frac{\mathrm{d}w}{\mathrm{d}t} = \frac{1}{t-30}$ $\int \mathrm{d}w = \int \frac{1}{t-30}\mathrm{d}t = -\int \frac{1}{30-t}\mathrm{d}t$ since $t < 20$ $w = \ln(30-t) + \mathrm{C}$ $ye^t = \ln(30-t) + \mathrm{C}$ $t = 0, y = 0 \Rightarrow 0 = \ln 30 + \mathrm{C}$ So, C$= -\ln 30$ $y = e^{-t}\left[\ln(30-t) - \ln 30\right]$ $y = e^{-t}\left[\ln\left(\frac{30-t}{30}\right)\right]$
	ii) $\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \mathrm{e}^{-t}$ $\frac{\mathrm{d}x}{\mathrm{d}t} = -\mathrm{e}^{-t} + \mathrm{A}$ $x = \mathrm{e}^{-t} + \mathrm{A}t + \mathrm{B}$ $t = 0$, $\frac{\mathrm{d}x}{\mathrm{d}t} = 2 \Rightarrow 2 = -1 + \mathrm{A}$ So, A = 3 $t = 0, x = 1 \Rightarrow$ B = 0 Hence, $x = \mathrm{e}^{-t} + 3t$

22	iii) 

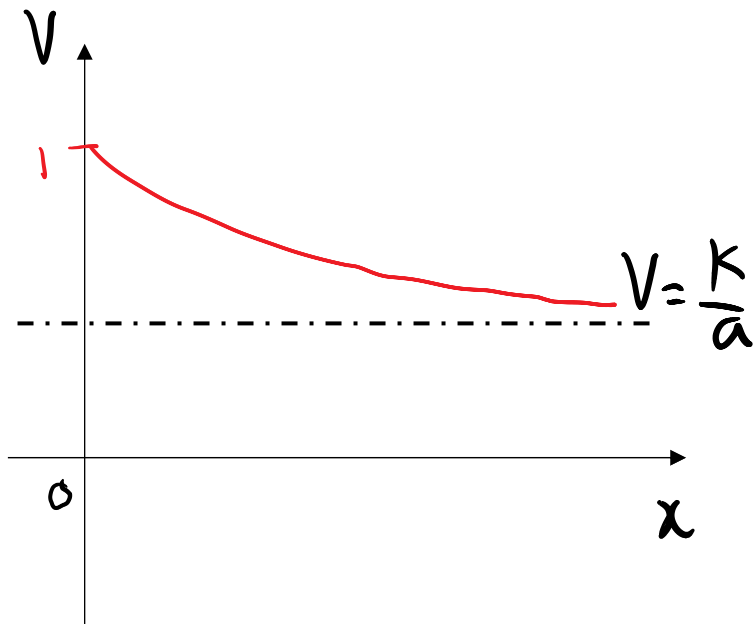
DHS Prelim 9758/2018/01/Q6	
23 **(a)**	$x^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 2xy - y^2$ ...(1) Given $y = zx^2$: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xz + x^2 \frac{\mathrm{d}z}{\mathrm{d}x}$ ...(2) Substitute (2) into (1): $x^2\left(2xz + x^2 \frac{\mathrm{d}z}{\mathrm{d}x}\right) = 2x\left(zx^2\right) - \left(zx^2\right)^2$ $x^4 \frac{\mathrm{d}z}{\mathrm{d}x} = -z^2 x^4$ $\frac{\mathrm{d}z}{\mathrm{d}x} = -z^2$ $\int \frac{1}{z^2}\ \mathrm{d}z = -\int 1 \mathrm{d}x$ $-\frac{1}{z} = -x + C$ $-\frac{x^2}{y} = -x + C$ $y = \frac{x^2}{x - C}$

23**(i)**	Given $(2,-4)$, $-4=\frac{2^2}{2-C}\Rightarrow C=3$ Hence, $y=\frac{x^2}{x-3}=x+3+\frac{9}{x-3}$ 
(ii)	When $C=0$, particular solution is $y=\frac{x^2}{x-0}=x$ which is a straight line and has no turning point.
(b)	Given $y=4x-1+De^{-x}\Rightarrow\frac{dy}{dx}=4-De^{-x}$ $\frac{dy}{dx}+y=\left(4-De^{-x}\right)+\left(4x-1+De^{-x}\right)=4x+3$ $\therefore p=4,\ q=3$

24(i)	Let the volume of water in the tank be V cubic metres at t seconds. $\frac{dV}{dt}=\frac{dV_{IN}}{dt}-\frac{dV_{OUT}}{dt}$ $=k-aV$ $=k\left(1-\frac{a}{k}V\right)$. (Shown) where $k>0$, and $a>0$.

	$\int \frac{1}{1-\frac{a}{k}V}\,\mathrm{d}V = \int k\,\mathrm{d}t$ $-\frac{k}{a}\ln\left|1-\frac{a}{k}V\right| = kt + C$ $\ln\left|1-\frac{a}{k}V\right| = -at - \frac{aC}{k}$ $1-\frac{a}{k}V = \pm e^{-\frac{ac}{k}}e^{-at} = Ae^{-at}$ $V = \frac{k}{a}\left(1-Ae^{-at}\right)$ C is an arbitrary constant and $A = \pm e^{-\frac{aC}{k}}$ At $t = 0$, $V = 1$, $A = 1-\frac{a}{k}$ $V = \frac{k}{a}\left(1-(1-\frac{a}{k})e^{-at}\right) = \frac{1}{a}(k-(k-a)e^{-at})$ **Case (i) if *a* < *k***  The volume of water in the water tank, V, increases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually.

	Case (ii) if *a* > *k*

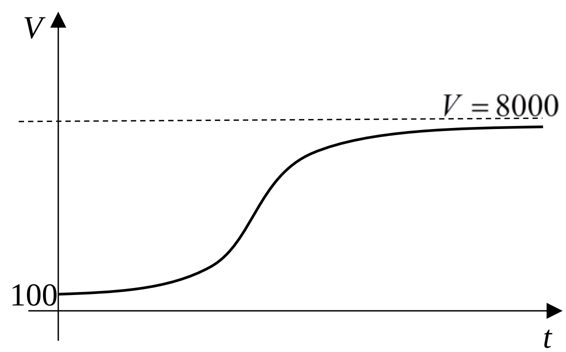


The volume of water in the water tank, V, decreases from one cubic meter and approach $\frac{k}{a}$ cubic meters eventually.

25 (i)	$\frac{d^2x}{dt^2}=9-5t \Rightarrow \frac{dx}{dt}=9t-\frac{5}{2}t^2+b$ $\Rightarrow x=\frac{9}{2}t^2-\frac{5}{6}t^3+bt+c$, where b and c are arbitrary constant.
(ii)	When $t = 0$, $x = 1$, $x=\frac{9}{2}t^2-\frac{5}{6}t^3-\frac{9}{2}t+c \Rightarrow c=1$. When $t = 3$, $\frac{dx}{dt}=0$, $\frac{dx}{dt}=9t-\frac{5}{2}t^2+b \Rightarrow b=-\frac{9}{2}$. Hence, $x=\frac{9}{2}t^2-\frac{5}{6}t^3-\frac{9}{2}t+1$.
(iii)	$\frac{dx}{dt}=\frac{k}{x^2}$

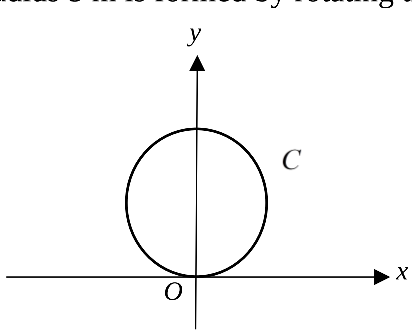
(iv)	$\frac{dx}{dt} = \frac{k}{x^2}$ $\Rightarrow x^2 \frac{dx}{dt} = k$ $\Rightarrow \int x^2 \, dx = \int k \, dt$ $\Rightarrow \frac{x^3}{3} = kt + c$ When $t = 0$, $x = 1$, $c = \frac{1}{3}$ When $t = 1$, $x = 4$, $\frac{4^3}{3} = k + \frac{1}{3} \Rightarrow k = 21$ Hence, $\frac{x^3}{3} = 21t + \frac{1}{3}$ $\Rightarrow x^3 = 63t + 1$. $\Rightarrow t = \frac{1}{63}\left(x^3 - 1\right)$ When $x = 16$, $t = \frac{1}{63}\left(16^3 - 1\right) = 65$. It takes 65 months for the insect population to reach 16000.

26(i)	As $V \to K$, $\ln\left(\frac{K}{V}\right) \to 0$. $\therefore \frac{dV}{dt} \to 0$
26(ii)	$u = \ln\left(\frac{K}{V}\right) \Rightarrow \frac{du}{dt} = \left(\frac{V}{K}\right)\left(-\frac{K}{V^2}\right)\frac{dV}{dt}$ $\Rightarrow \frac{dV}{dt} = -V\frac{du}{dt}$ $\frac{dV}{dt} = aV\ln\left(\frac{K}{V}\right)$ $-V\frac{du}{dt} = aVu$ $\frac{du}{dt} = -au$ $\int \frac{1}{u} \, du = \int -a \, dt$

	$\int \frac{1}{u}\,\mathrm{d}u = \int -a\,\mathrm{d}t$ $\ln\lvert u\rvert = -at + c \Rightarrow \lvert u\rvert = \mathrm{e}^{-at+c}$ $u = A\mathrm{e}^{-at}$, where $A = \pm \mathrm{e}^{c}$ $\ln\left(\frac{K}{V}\right) = A\mathrm{e}^{-at}$ $\frac{K}{V} = \mathrm{e}^{A\mathrm{e}^{-at}} \Rightarrow V = K\mathrm{e}^{-A\mathrm{e}^{-at}}$
26(iii)	As $t \to \infty$, $\mathrm{e}^{-at} \to 0$ and hence $\mathrm{e}^{-A\mathrm{e}^{-at}} \to \mathrm{e}^{0} = 1$ $\therefore V \to K$ The size of the lung tumour approaches $K\ (\mathrm{mm}^3)$. K is the maximum possible size of the lung tumour that can be achieved.
26(iv)	$V = 8000\mathrm{e}^{-A\mathrm{e}^{-0.01t}}$ When $t = 0$, $V = 100$. $100 = 8000\mathrm{e}^{-A\mathrm{e}^{0}} \Rightarrow \frac{1}{80} = \mathrm{e}^{-A} \Rightarrow A = \ln 80$. When $t = 100$, $V = 8000\mathrm{e}^{-\ln 80\,\mathrm{e}^{-1}} = 1595.81 \approx 1596\ \mathrm{mm}^3$
26(v)	 $V = 8000\mathrm{e}^{-\ln 80\,\mathrm{e}^{-0.01t}}$

27. EJC/2022/I/Q10

A spherical container of radius 5 m is formed by rotating the following circle C about the y-axis.



The container has negligible thickness, and the circle C passes through the origin O.

(a) State a cartesian equation of C. [1]

Initially the spherical container is completely filled with water. Two engineers are calculating the time needed for the container to be completely drained from a small circular hole at the bottom. The volume of water in the container at time t seconds is denoted by V m^3.

[The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

(b) The first engineer proposes that the rate of change of V with respect to t is a constant k.

(i) Write down a differential equation relating V, t and k. [1]

(ii) Determine, with justification, the sign of k. [1]

(iii) Find V in terms of t and k, leaving your answer in exact form. [2]

(c) The second engineer argues that the rate at which water flows out from the hole will be at its greatest in the beginning, and decreases as the depth of water in the container decreases. He suggests using Torricelli's law, which says that

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\alpha\sqrt{20h},$$

where α m^2 is the area of the circular hole at the bottom of the container, and h m is the depth of water in the container at time t seconds. The radius of the hole at the bottom of the container is found to be constant at 1 cm.

(i) Show that $\left(10h^{\frac{1}{2}} - h^{\frac{3}{2}}\right)\frac{\mathrm{d}h}{\mathrm{d}t} = -\frac{\sqrt{5}}{5000}$. [4]

(ii) Hence find the numerical value of t when the container is completely drained. [4]

DHS Prelim 9758/2022/01/Q10

Qn	Suggested Solution
(a)	$\frac{\mathrm{d}v}{\mathrm{d}t} = 5 - 0.2v^2$ $\int \frac{1}{5-0.2v^2}\,\mathrm{d}v = \int \mathrm{d}t$ $5\int \frac{1}{25-v^2}\,\mathrm{d}v = \int \mathrm{d}t$ $5\left[\frac{1}{2(5)}\ln\left\lvert\frac{5+v}{5-v}\right\rvert\right] + C = t$ Since $\frac{\mathrm{d}v}{\mathrm{d}t} = 5 - 0.2v^2 > 0$ $\Rightarrow 0 \le v < 5$ as $v \ge 0$ $5\left[\frac{1}{2(5)}\ln\left(\frac{5+v}{5-v}\right)\right] + C = t$, where C is an arbitrary constant

	$\therefore\ t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)+C$ Since at $t=0$, $v=0$; $0=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)+C$ $\Rightarrow C=0$ $\therefore t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)$
(b)	Since $\frac{\mathrm{d}v}{\mathrm{d}t}=5-0.2v^2$, The velocity of the object is increasing at a decreasing rate. **Method 1** $t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)\Rightarrow\ v=\frac{5(\mathrm{e}^{2t}-1)}{\mathrm{e}^{2t}+1}=\frac{5(1-\mathrm{e}^{-2t})}{1+\mathrm{e}^{-2t}}$ As $t\to\infty,\ v\to 5$. In the long run, the velocity of the object will travel at 5 m/s. **Method 2** $t=\frac{1}{2}\ln\left(\frac{5+v}{5-v}\right)\Rightarrow \mathrm{e}^{2t}=\frac{5+v}{5-v}$ As $t\to\infty\Rightarrow 5-v\to 0\Rightarrow v\to 5$
(c)	$\int_0^m v\ \mathrm{d}t$ $=\int_0^m \frac{5(\mathrm{e}^{2t}-1)}{(\mathrm{e}^{2t}+1)}\ \mathrm{d}t$ $=5\left(\int_0^m \frac{(\mathrm{e}^{2t}+1)-2}{\mathrm{e}^{2t}+1}\ \mathrm{d}t\right)$ $=5\left(\int_0^m 1-\frac{2}{\mathrm{e}^{2t}+1}\ \mathrm{d}t\right)$

	$= 5\left(\int_0^m 1 - \frac{2e^{-2t}}{e^{-2t}\left(e^{2t}+1\right)}\ dt\right)$ $= 5\left(\int_0^m 1 - \frac{2e^{-2t}}{1+e^{-2t}}\ dt\right)$ $= 5\left[t + \ln(1+e^{-2t})\right]_0^m$ $= 5\left[m + \ln(1+e^{-2m}) - \ln(2)\right]$ $= 5\left[\ln(e^m) + \ln(1+e^{-2m}) - \ln(2)\right]$ $= 5\left[\ln\left(\frac{e^m(1+e^{-2m})}{2}\right)\right]$ $= 5\ln\left(\frac{e^m + e^{-m}}{2}\right)$ (shown) **Alternative method:** $\int_0^m v\ dt$ $= \int_0^m \frac{5(e^{2t}-1)}{(e^{2t}+1)}\ dt$ $= 5\left(\int_0^m \frac{e^{2t}}{e^{2t}+1}\ dt - \int_0^m \frac{1}{e^{2t}+1}\ dt\right)$ $= 5\left(\int_0^m \frac{e^{2t}}{e^{2t}+1}\ dt - \int_0^m \frac{e^{-2t}}{e^{-2t}\left(e^{2t}+1\right)}\ dt\right)$ $= 5\left(\frac{1}{2}\int_0^m \frac{2e^{2t}}{e^{2t}+1}\ dt - \int_0^m \frac{e^{-2t}}{1+e^{-2t}}\ dt\right)$ $= 5\left[\frac{1}{2}\ln\left(e^{2t}+1\right)\right]_0^m + \frac{5}{2}\int_0^m \frac{-2e^{-2t}}{1+e^{-2t}}\ dt$ $= 5\left[\frac{1}{2}\ln\left(e^{2t}+1\right) + \frac{1}{2}\ln(1+e^{-2t})\right]_0^m$

	$=\frac{5}{2}\left[\ln\left(\mathrm{e}^{2t}+2+\mathrm{e}^{-2t}\right)\right]_0^m$ $=\frac{5}{2}\ln\left(\frac{\mathrm{e}^{2m}+2+\mathrm{e}^{-2m}}{4}\right)$ $=\frac{5}{2}\ln\left(\frac{\left(\mathrm{e}^{m}+\mathrm{e}^{-m}\right)^2}{2^2}\right)$ $=5\ln\left(\frac{e^{m}+e^{-m}}{2}\right)$ (shown) $\because \frac{e^{m}+e^{-m}}{2}>0$