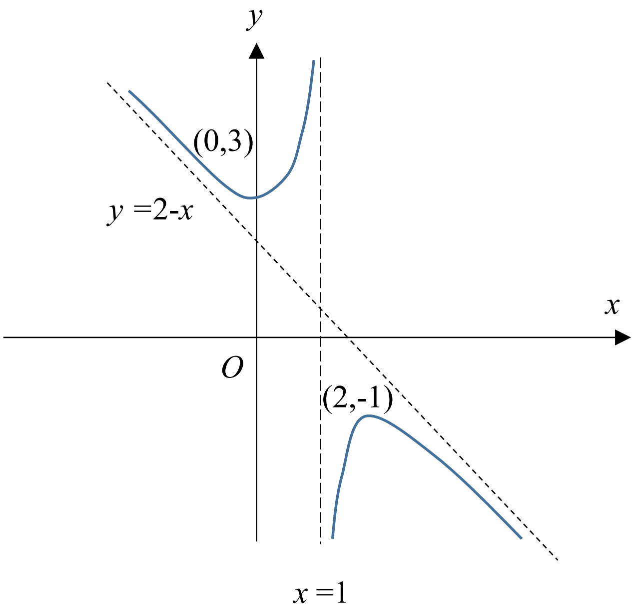
DHS 2023 Year 6 H2 Mathematics Prelim Paper 1 Solutions and Comments

Qn	Suggested Solution	
1	$\int_0^a \left\|2x^2-3x-2\right\| \, dx$ $= \int_0^2 -(2x^2-3x-2)\, dx + \int_2^a (2x^2-3x-2)\, dx$ $= \left[\frac{2}{3}x^3-\frac{3}{2}x^2-2x\right]_2^0 + \left[\frac{2}{3}x^3-\frac{3}{2}x^2-2x\right]_2^a$ $= (\frac{14}{3}) + (\frac{2}{3}a^3-\frac{3}{2}a^2-2a+\frac{14}{3})$ $= \frac{2}{3}a^3-\frac{3}{2}a^2-2a+\frac{28}{3}$ $= \frac{1}{6}(4a^3-9a^2-12a+56)$	

Qn	Suggested Solutions	
2(a)	$y=\frac{x^2-3x+3}{1-x}=2-x+\frac{1}{1-x}$ Asymptotes: $y=2-x$, $x=1$  	

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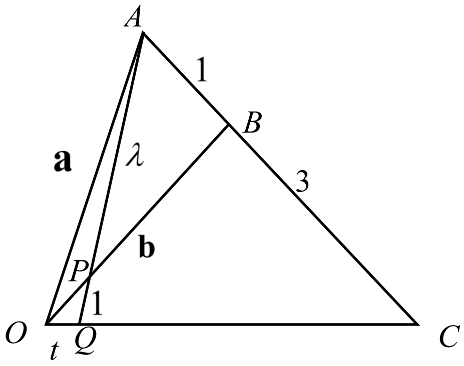
(b)	$\dfrac{x^2-3x+3}{1-x}=kx$ $x^2-3x+3=kx(1-x)$ $(1+k)x^2-(3+k)x+3=0$ For two points of intersection, discriminant > 0. $(3+k)^2-4(1+k)(3)>0$ $9+6k+k^2-12-12k>0$ $k^2-6k-3>0$ $(k-3)^2-12>0$ $k<3-2\sqrt{3}$ or $k>3+2\sqrt{3}$ $y=\dfrac{x^2-3x+3}{1-x}=2-x+\dfrac{1}{1-x}$ Consider the oblique asymptote of the curve C is $y=2-x$, for two points of intersection between the curve and the line, the set of values of k is $\{k\in\mathbb{R}: k<3-2\sqrt{3} \text{ or } k>3+2\sqrt{3}, k\neq -1\}$.	

Qn	Suggested Solution	
3(a)	By Conjugate Root Theorem, another root is $z=1-a\mathrm{i}$.	
(b)	Let $z^3-2z+k=(z-[1+a\mathrm{i}])(z-[1-a\mathrm{i}])(z-c)$ where c is a real constant. z^3-2z+k $=([z-1]-a\mathrm{i})([z-1]+a\mathrm{i})(z-c)$ $=([z-1]^2-[a\mathrm{i}]^2)(z-c)$ $=(z^2-2z+[1+a^2])(z-c)$ Comparing the coefficients of z^2: $-c-2=0 \Rightarrow c=-2$ Comparing the coefficients of z: $a^2+1+2c=-2 \Rightarrow a=1$ since $a>0$ So, $k=-c(1+a^2)=4$	
(c)	Area $=\dfrac{1}{2}(2)(3)=3$ square units	

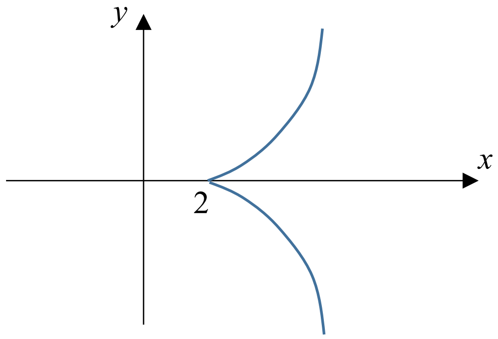
Qn	Suggested Solution	
4(a)	$\frac{2-\mathrm{i}\sin 2\alpha}{1+2\mathrm{i}\sin 2\alpha}\times\frac{1-2\mathrm{i}\sin 2\alpha}{1-2\mathrm{i}\sin 2\alpha}$ $=\frac{(2-\mathrm{i}\sin 2\alpha)(1-2\mathrm{i}\sin 2\alpha)}{1+4\sin^2 2\alpha}$ $=\frac{2-2\sin 2\alpha-5\mathrm{i}\sin 2\alpha}{1+4\sin^2 2\alpha}$ Since the expression is real, $\frac{-5\mathrm{i}\sin 2\alpha}{1+4\sin^2 2\alpha}=0$ $-5\mathrm{i}\sin 2\alpha=0$ $\sin 2\alpha=0$ $2\alpha=k\pi,\ k\in\mathbb{Z}$ $\alpha=\frac{k\pi}{2}$ $\therefore\left\{\alpha\in\mathbb{R}\mid\alpha=\frac{k\pi}{2}\right\}$	
(b)	$\left\|\frac{3(w-z)}{1-z^*w}\right\|$ $=3\left\|\frac{(w-z)}{1-z^*w}\right\|$ $=3\left\|\frac{(w-z)}{w^*w-z^*w}\right\|\quad\left(\because w^*w=\|w\|^2=1\right)$ $=3\left\|\frac{(w-z)}{w(w^*-z^*)}\right\|$ $=\frac{3}{\|w\|}\frac{\|(w-z)\|}{\|(w-z)^*\|}$ $=3\frac{\|w-z\|}{\|w-z\|}\quad\left(\because\left\|(w-z)^*\right\|=\|w-z\|\right)$ $=3$ **Alternative 1** Let $z=re^{\mathrm{i}\theta},\ w=re^{\mathrm{i}\phi}$,	

	$\left\|\dfrac{3(w-z)}{1-z^*w}\right\| = 3\left\|\dfrac{e^{i\phi}-re^{i\theta}}{1-re^{-i\theta}e^{i\phi}}\right\|$ $= 3\left\|\dfrac{e^{i\phi}(1-re^{i(\theta-\phi)})}{1-re^{-i(\theta-\phi)}}\right\|$ $= 3\left\|e^{i\phi}\right\|\left\|\dfrac{1-re^{i(\theta-\phi)}}{(1-re^{i(\theta-\phi)})^*}\right\|$ $= 3$ **Alternative 2** $\left\|\dfrac{3(w-z)}{1-z^*w}\right\|$ $= 3\left\|\dfrac{w(1-zw^*)}{1-z^*w}\right\| \quad \left(\because w^*w = \|w\|^2 = 1, \text{ so } \dfrac{1}{w} = w^*\right)$ $= 3\|w\|\left\|\dfrac{(1-z^*w)^*}{1-z^*w}\right\|$ $= 3$	

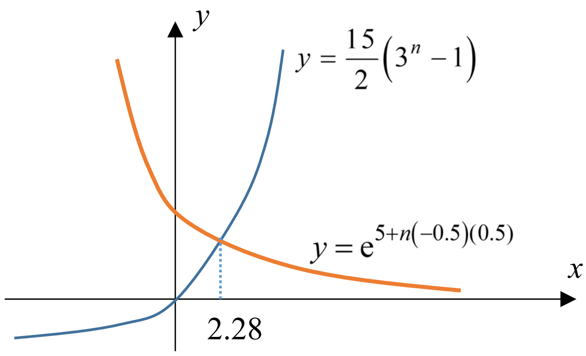
Qn	Suggested Solution	
5(a)	Since A, B and C are collinear $\overrightarrow{BC} \ // \ \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ $\overrightarrow{BC} = 3\mathbf{b} - \mu\mathbf{a} = 3(\mathbf{b} - \mathbf{a})$ $\therefore \mu = 3$ $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{BC}$ $= \mathbf{b} + (\mathbf{3b} - 3\mathbf{a})$ $= 4\mathbf{b} - 3\mathbf{a}$	
(b)	 Given $\overrightarrow{OQ} = t\overrightarrow{OC}$ Using RT: $\overrightarrow{OP} = \dfrac{\lambda\overrightarrow{OQ} + \overrightarrow{OA}}{1+\lambda}$ $= \dfrac{1}{1+\lambda}\left[\lambda t\,\overrightarrow{OC} + \mathbf{a}\right]$ $= \dfrac{1}{1+\lambda}\left[\lambda t(\mathbf{4b} - 3\mathbf{a}) + \mathbf{a}\right]$ $= \dfrac{1}{1+\lambda}\left[4\lambda t\,\mathbf{b} + (1 - 3\lambda t)\mathbf{a}\right]$ Since $\overrightarrow{OP} \ // \ \mathbf{b}$ $1 - 3\lambda t = 0$ $\therefore \lambda t = \dfrac{1}{3}$	
(c)	When $\lambda = 5$ $\overrightarrow{OP} = \dfrac{1}{1+\lambda}[4\lambda t]\mathbf{b} = \dfrac{1}{1+5}\left[\dfrac{4}{3}\right]\mathbf{b} = \dfrac{2}{9}\mathbf{b}$ $\therefore OP : PB = \dfrac{2}{9} : \dfrac{7}{9} = 2 : 7$	

Qn	Suggested Solution	
6(a)	$\sin 3x=(3x)-\frac{(3x)^3}{3!}+...=3x-\frac{9x^3}{2}+...$ $f(x)=e^{\sin 3x}=1+(\sin 3x)+\frac{(\sin 3x)^2}{2!}+\frac{(\sin 3x)^3}{3!}+...$ $=1+(3x-\frac{9x^3}{2}+...)+\frac{1}{2}(3x-\frac{9x^3}{2}+...)^2+\frac{1}{6}(3x-\frac{9x^3}{2}+...)^3+...$ $=1+3x-\frac{9x^3}{2}+\frac{1}{2}[(3x)^2]+\frac{1}{6}[(3x)^3]+...$ $\approx 1+3x+\frac{9}{2}x^2+0x^3$ (independent of x^3) **Alternative (by differentiation)** let $y=e^{\sin 3x}$ $\frac{dy}{dx}=3\cos 3x\cdot e^{\sin 3x}=3\cos 3x\cdot y$ $\frac{d^2y}{dx^2}=3\cos 3x\frac{dy}{dx}-9\sin 3x\cdot y$ $\frac{d^3y}{dx^3}=3\cos 3x\frac{d^2y}{dx^2}-9\sin 3x\frac{dy}{dx}-9\sin 3x\cdot\frac{dy}{dx}-27\cos 3x\cdot y$ When $x=0$, $y=1,\ \frac{dy}{dx}=3,\ \frac{d^2y}{dx^2}=9,\ \frac{d^3y}{dx^3}=0$ $\therefore y=1+3x+\frac{9}{2}x^2+0x^3+\dots$	
(b)	$\int\frac{e^{\sin 3x}}{x^2}\,dx\approx\int(x^{-2}+3x^{-1}+\frac{9}{2})\,dx$ $=-x^{-1}+3\ln\lvert x\rvert+\frac{9}{2}x+C$ where C is an arbitrary constant $\int_{0.1}^{0.2}(\frac{2}{x})^2e^{\sin 3x}\,dx=\int_{0.1}^{0.2}\frac{4e^{\sin 3x}}{x^2}\,dx$ $=4[-x^{-1}+3\ln x+\frac{9}{2}x]_{0.1}^{0.2}$ $=30.1178$ (4 d.p.)	
(c)	Using GC, $\int_{0.1}^{0.2}(\frac{2}{x})^2e^{\sin 3x}\,dx=29.9995$ (4 d.p.)	
(d)	The **approximation is accurate** as the **values of *x*** (between 0.1 and 0.2) **are close to 0** for the magnitude of x^4 and higher powers of x to be neglected. **Alternative:**	

	% error $=\frac{\lvert 30.1178-29.9995\rvert}{29.9995}\times 100=0.3943\%$ Since percentage error is small, approximation is accurate.	

Qn	Suggested Solutions	
7(a)		
(b)	At $(6,8)$, $t=2$. $\frac{dx}{dt}=2t$ and $\frac{dy}{dt}=3t^2$ $\frac{dy}{dx}=3t^2\times\frac{1}{2t}=\frac{3}{2}t$ When $t=2$, $\frac{dy}{dx}=3$ Equation of tangent is $y-8=3(x-6)$ i.e., $y=3x-10$	
(c)	Since tangent to C at the point $(6,8)$ meets the curve C again at point P, $t^3=3(t^2+2)-10$ $t^3-3t^2+4=0$ Using GC, $t=2$ or $t=-1$ (Point $(6,8)$) **Alt** $(t-2)(t^2-t-2)=0$ $(t-2)(t-2)(t+1)=0$ $t=2$ or $t=-1$ At $t=-1$, $x=3$ and $y=-1$. The coordinates of point P are $(3,-1)$.	
(d)	At $t=m$, the normal to the curve is $y-m^3=-\frac{2}{3m}(x-m^2-2)$ i.e., $y=-\frac{2}{3m}x+\frac{2m}{3}+\frac{4}{3m}+m^3$ When $x=0$, $y=\frac{2m}{3}+\frac{4}{3m}+m^3$ (Point R) When $y=0$, $x=\frac{3m^4}{2}+m^2+2$ (Point Q)	

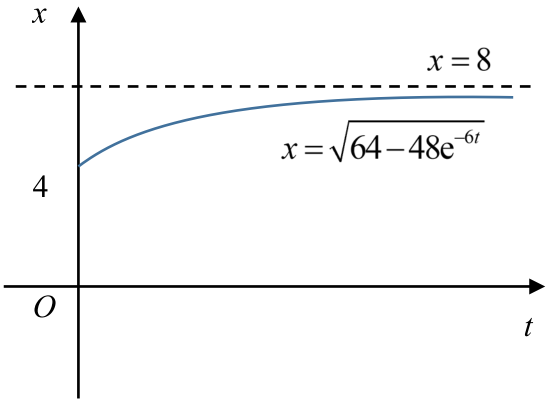
	The mid-point F is $\left(\frac{3m^4}{4}+\frac{m^2}{2}+1, \frac{m}{3}+\frac{2}{3m}+\frac{m^3}{2}\right)$.	
Qn	**Suggested Solutions**	
8(a)	$S_n = 3n(n+2)$ $u_n = S_n - S_{n-1}$ $= 3n(n+2)-3(n-1)(n+1)$ $= 6n+3$ $u_n - u_{n-1} = 6n+3-(6(n-1)+3)$ $= 6n+3-6n+3$ $= 6 \ (\text{constant})$ Since the difference between two consecutive terms is a constant, the series is an arithmetic progression. The common difference is 6.	
(b)	$v_1 = u_2 = 6(2)+3 = 15$ $v_2 = u_7 = 6(7)+3 = 45$ common ratio, $r = \frac{45}{15} = 3$ $v_3 = 15(3)^2 = 135$ The m^{th} term of the series in (i), $135 = 6(m)+3$ $m = \frac{135-3}{6} = 22$ Since $r = 3$ does not lie within $-1 < r < 1$, the sum to infinity of v_n does not exist.	
(c)	common ratio $= \frac{w_n}{w_{n-1}}$ $= \frac{e^{5+nx(x+1)}}{e^{5+(n-1)x(x+1)}}$ $= \frac{e^5 e^{nx(x+1)}}{e^5 e^{(n-1)x(x+1)}}$ $= e^{nx(x+1)-(n-1)x(x+1)}$ $= e^{x(x+1)}$ For the series to converge, $\left\|e^{x(x+1)}\right\| < 1$, $x(x+1) < 0$ The range of values of x is $-1 < x < 0$.	

(d)	Sum of first n terms of v_n, S_{v_n} $=\frac{15\left(3^n-1\right)}{3-1}=\frac{15}{2}\left(3^n-1\right)$ $S_{v_n} > w_n$ using $x=-0.5$, $\frac{15}{2}\left(3^n-1\right) > \mathrm{e}^{5+n(-0.5)(0.5)}$  From the graph, the least value of n is 3. **Alternative (table method)**	

Qn	Suggested Solution	
9(a)	$\int_0^p \sin^{-1}\frac{t}{3}\ \mathrm{d}t = \left[t\sin^{-1}\frac{t}{3}\right]_0^p - \int_0^p t\cdot\frac{\frac{1}{3}}{\sqrt{1-\left(\frac{1}{3}t\right)^2}}\ \mathrm{d}t$ $= \left[t\sin^{-1}\frac{t}{3}\right]_0^p - \int_0^p t\cdot\frac{1}{\sqrt{9-t^2}}\ \mathrm{d}t$ $= p\sin^{-1}\frac{p}{3} + \frac{1}{2}\int_0^p \frac{-2t}{\sqrt{9-t^2}}\ \mathrm{d}t$ $= p\sin^{-1}\frac{p}{3} + \frac{1}{2}\left[\frac{\sqrt{9-t^2}}{\left(\frac{1}{2}\right)}\right]_0^p$ $= p\sin^{-1}\frac{p}{3} + \left[\sqrt{9-p^2}-\sqrt{9}\right]$ $= p\sin^{-1}\frac{p}{3} + \sqrt{9-p^2} - 3$ (Shown)	
(b)	When $x=\sqrt{\frac{\pi}{8}}$, on the curve $y=x^2$, $y=\left(\sqrt{\frac{\pi}{8}}\right)^2 = \frac{\pi}{8}$. on the curve $y=3\sin\left(2x^2\right)$, $y = 3\sin\left[2\left(\sqrt{\frac{\pi}{8}}\right)^2\right] = 3\sin\left(\frac{\pi}{4}\right) = \frac{3\sqrt{2}}{2}$. Also, $y=3\sin\left(2x^2\right) \Rightarrow \sin\left(2x^2\right)=\frac{y}{3} \Rightarrow x^2 = \frac{1}{2}\sin^{-1}\frac{y}{3}$ Volume generated by region *R* rotated about the *y*-axis $= \pi\left(\sqrt{\frac{\pi}{8}}\right)^2\left(\frac{3\sqrt{2}}{2}-\frac{\pi}{8}\right) + \pi\int_0^{\frac{\pi}{8}} y\ \mathrm{d}y - \pi\int_0^{\frac{3\sqrt{2}}{2}} \frac{1}{2}\sin^{-1}\frac{y}{3}\ \mathrm{d}y$ $= \frac{\pi^2}{8}\left(\frac{3\sqrt{2}}{2}-\frac{\pi}{8}\right) + \pi\left[\frac{y^2}{2}\right]_0^{\frac{\pi}{8}} - \frac{\pi}{2}\int_0^{\frac{3\sqrt{2}}{2}} \sin^{-1}\frac{y}{3}\ \mathrm{d}y$ $= \frac{\pi^2}{8}\left(\frac{3\sqrt{2}}{2}-\frac{\pi}{8}\right) + \frac{\pi}{2}\left(\frac{\pi}{8}\right)^2 - \frac{\pi}{2}\left[\frac{3\sqrt{2}}{2}\sin^{-1}\frac{\left(\frac{3\sqrt{2}}{2}\right)}{3} + 3\sqrt{1-\frac{1}{9}\left(\frac{3\sqrt{2}}{2}\right)^2} - 3\right]$ $= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{64} + \frac{\pi^3}{128} - \frac{\pi}{2}\left[\frac{3\sqrt{2}}{2}\sin^{-1}\left(\frac{\sqrt{2}}{2}\right) + 3\sqrt{1-\frac{1}{9}\left(\frac{9}{2}\right)} - 3\right]$ $= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{128} - \frac{\pi}{2}\left[\frac{3\sqrt{2}}{2}\left(\frac{\pi}{4}\right) + 3\sqrt{\frac{1}{2}} - 3\right]$ $= \frac{3\sqrt{2}\pi^2}{16} - \frac{\pi^3}{128} - \frac{3\sqrt{2}\pi^2}{16} - \frac{3\sqrt{2}\pi}{4} + \frac{3\pi}{2} = \frac{3\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right) - \frac{\pi^3}{128}$ cubic metres	

(c)	Point of intersection: (0, 0), (1, 4) Area of region Q $= \int_0^1 (\frac{8\sqrt{x}}{1+x^3} - 4x^{\frac{7}{2}})\,\mathrm{d}x = 3.299901526 = 3.3$ (1 d.p.) square metres	
(d)	Volume of main structure = $\frac{3\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)-\frac{\pi^3}{128}$ m^3 Mass of main structure $=\left[\frac{3\pi}{2}\left(1-\frac{\sqrt{2}}{2}\right)-\frac{\pi^3}{128}\right]\times 1463.46 = 1665.403197$ kg Let h m be the thickness of the prism base. Volume of prism base = $3.299901526h$ m^3 Mass of prism structure = $3.299901526h \times 2550 = 8414.748892h$ kg Total mass = $(8414.748892h + 1665.403197)$ kg Expected weight per square metre of the monument $= \frac{(8414.748892h + 1665.403197)\times \frac{9.81}{1000}}{3.299901526}$ $= 25.0155h + 4.950937242$ $25.0155h + 4.950937242 \le 20$ $h \le 0.60159$ Hence, only if the thickness of the prism base is less than 60.159 cm, then no special foundation will be needed. Should it be between 60.159 cm and 70 cm, a special foundation is needed. Hence, the engineer's claim is not correct. **Alternative method** (find total mass using thickness 70cm)	

Qn	Suggested Solution	
10(a)	$\frac{dy}{dt} = -ay$, a positive $x^2 + y^2 = 8^2$ $2x + 2y\frac{dy}{dx} = 0$ $\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$ $= -\frac{y}{x}(-ay)$ $= \frac{a(64 - x^2)}{x} = \frac{k(64 - x^2)}{x}$ (shown) **Alternative:** $\frac{dy}{dt} = -ky$ $x^2 + y^2 = 8^2$ $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\frac{dx}{dt} = -\frac{y}{x}(-ky) = \frac{k(64 - x^2)}{x}$	
(b)	$\frac{dx}{dt} = \frac{3(64 - x^2)}{x}$ $\frac{x}{64 - x^2}\frac{dx}{dt} = 3$ $\int \frac{x}{64 - x^2}\, dx = \int 3\, dt$ $-\frac{1}{2}\ln(64 - x^2) = 3t + C$ (since $x \le 8$) $x^2 = 64 - Ae^{-6t}$ where $A = e^{-2C}$ $x = \sqrt{64 - Ae^{-6t}}$ When $t = 0,\ x = 4$: $4 = \sqrt{64 - A}$ $A = 48$ $x = \sqrt{64 - 48e^{-6t}}$ When $y = 3,\ x = \sqrt{64 - 3^2} = \sqrt{55}$	

<table>
<tr><td></td><td>$\sqrt{55}=\sqrt{64-48\mathrm{e}^{-6t}}$
$\mathrm{e}^{-6t}=\dfrac{9}{48}$
$t=0.279=0.3\text{ s}\ \ (1\text{ d.p.})$</td><td></td></tr>
<tr><td>(c)</td><td>
</td><td></td></tr>
<tr><td>(d)</td><td>Based on Jim's conjecture, the rod will never be flat on the ground; thus not appropriate.</td><td></td></tr>
<tr><td></td><td></td><td></td></tr>
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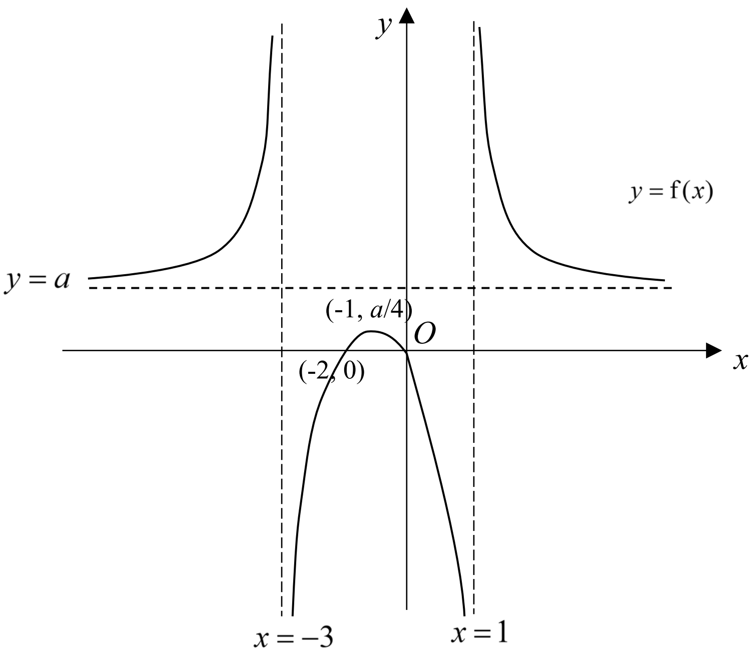
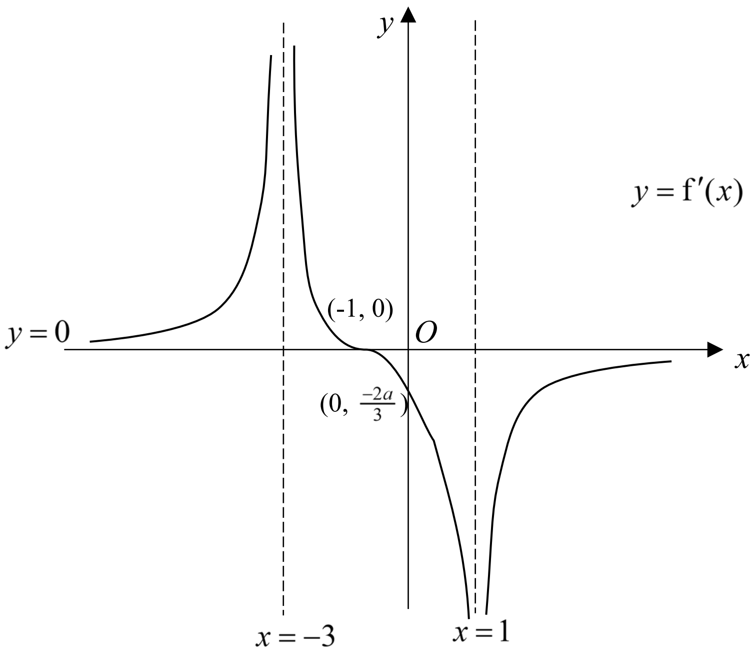
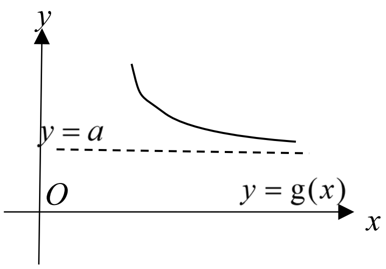
<table>
<tr><th>Qn</th><th>Suggested Solution</th><th></th></tr>
<tr><td>11(a)</td><td>$\mathrm{f}(x)=a+\dfrac{3a}{(x+3)(x-1)}$
$\mathrm{f}'(x)=\dfrac{-3a\left[1(x-1)+1(x+3)\right]}{\left[(x+3)(x-1)\right]^2}=\dfrac{-3a\left[2x+2\right]}{\left[(x+3)(x-1)\right]^2}$
At stationary point,
$\mathrm{f}'(x)=0$
$\dfrac{-6a\left[x+1\right]}{\left[(x+3)(x-1)\right]^2}=0\Rightarrow x=-1$</td><td></td></tr>
</table>

(b)	y y = f(x) y = a (-1, a/4) O x (-2, 0) x = −3 x = 1	
(c)	$-3 < x < -1$	
(d)	y y = f′(x) y = 0 (-1, 0) O x (0, −2a/3) x = −3 x = 1	

(e)	$D_g = [2, \infty)$ $x = 2,\ y = a + \dfrac{3a}{(2+3)(2-1)} = \dfrac{8}{5}a$ $R_g = (a, \frac{8}{5}a]$ For gg to exist, $R_g \subseteq D_g$. Hence, the condition on a is $a \geq 2$.	
(f)	 $[2, \infty) \to (a, \frac{8}{5}a] \to [g(\frac{8}{5}a), g(a))$ For $a = 5$, $g(\frac{8}{5}a) = g(8) = 5 + \dfrac{3(5)}{(8+3)(8-1)} = 5\frac{15}{77} \quad (= \frac{400}{77})$ $g(a) = g(5) = 5 + \dfrac{3(5)}{(5+3)(5-1)} = 5\frac{15}{32} \quad (= \frac{175}{32})$ $\therefore R_{gg} = [5\frac{15}{77}, 5\frac{15}{32})$	