Class | Index Number

Name : ____________________________________

METHODIST GIRLS' SCHOOL

Founded in 1887

PRELIMINARY EXAMINATION 2023
Secondary 4

Monday

21 August 2023

ADDITIONAL MATHEMATICS

PAPER 2 Solution

4049/02

2 hours 15 minutes

Candidates answer on the Question Paper.
No Additional Materials are required.

READ THESE INSTRUCTIONS FIRST
Write your class, index number and name in the spaces at the top of this page.
Write in dark blue or black pen
You may use a HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers correct to 3 significant figure, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

The total number of marks for this paper is 90.

90

1. ALGEBRA

Quadratic Equation

For the quadratic equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial Expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)...(n-r+1)}{r!}$.

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$

$$\sec^2 A = 1 + \tan^2 A$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

Formulae for ΔABC

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1. **(i)** A prism has a volume of $\left(3x^2+8x+1\right)$ cm^3 and a cross-sectional area of $\left(x^2+2x+1\right)$ cm^2. Write down an expression for the height of the prism, in the form $A+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$. [4]

Height, $h=\frac{3x^2+8x+1}{x^2+2x+1}=3+\frac{2x-2}{x^2+2x+1}$ B1

$$\frac{2x-2}{x^2+2x+1}=\frac{B}{(x+1)}+\frac{C}{(x+1)^2}$$

$x=-1$ | $x=0$ M1

$2(-2)=C$ | $-2=B-4$

$C=-4$ B1 | $B=2$

Height, $h=3+\frac{2}{x+1}-\frac{4}{(x+1)^2}$ B1

(ii) Find $\int A+\frac{B}{(x+1)}+\frac{C}{(x+1)^2}\,\mathrm{d}x$. [3]

$$\int 3+\frac{2}{(x+1)}-\frac{4}{(x+1)^2}\,\mathrm{d}x$$

$$=3x+2\ln(x+1)+\frac{4}{(x+1)^2}+C$$

B1 B1 B1

where C is an arbitrary constant

2. A circle C passes through the point $P(4,-3)$ and has the same centre as the circle $x^2+y^2+4x-2y-1=0$.

(i) Find the equation of the circle C. [3]

$$x^2+y^2+4x-2y-1=0$$
$$(x+2)^2-4+(y-1)^2-1=1$$
$$(x+2)^2+(y-1)^2=6$$

Centre of circle = $(-2,1)$ B1

Radius of circle = $\sqrt{(4+2)^2+(-3-1)^2}=\sqrt{52}$ B1

Equation of circle is $(x+2)^2+(y-1)^2=52$ or $x^2+y^2+4x-2y-47=0$ A1

(ii) Find the equation of the tangent to the circle C at the point $P(4,-3)$. [2]

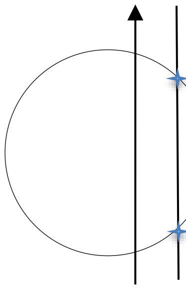
Gradient of P to the centre $=\dfrac{-3-1}{4+2}=\dfrac{-4}{6}=-\dfrac{2}{3}$

Gradient of tangent $=\dfrac{3}{2}$ M1

Equation of tangent at P is

$$y+3=\frac{3}{2}(x-4)$$
$$y=\frac{3}{2}x-6-3$$
$$y=\frac{3}{2}x-9$$ A1

(iii) Another point $Q(4, q)$ which lies on the circle C, is the same distance from the y-axis as the point P. Find the coordinates of the point Q. [2]



Subs $x = 4$ into equation of circle

$$(4+2)^2+(y-1)^2=52$$

M1

$$(y-1)^2=52-36$$

$$y-1=\pm 4$$

$$y=5 \quad \text{or} \quad y=-3$$

Q(4, 5) A1

Alt Mtd : Since perp bisector of chord passes through centre of circle,

$$\frac{q-3}{2}=1$$

$$\therefore q=5$$

Q(4, 5)

3. **(i)** Given that $y=(x-1)\sqrt{4x+1}$, show that $\frac{\text{dy}}{\text{dx}}=\frac{6x-1}{\sqrt{4x+1}}$. [3]

$$\frac{dy}{dx}=(x-1)\frac{1}{2\sqrt{4x+1}}\times 4+\sqrt{4x+1}$$ M1

$$=\frac{2(x-1)}{\sqrt{4x+1}}+\frac{4x+1}{\sqrt{4x+1}}$$ M1

$$=\frac{6x-1}{\sqrt{4x+1}}$$ A1

(ii) Hence, evaluate $\int_1^2 \frac{6x-5}{\sqrt{4x+1}}\,dx$. [4]

$$\int_1^2 \frac{6x-5}{\sqrt{4x+1}}\,dx = \int_1^2 \frac{6x-1-4}{\sqrt{4x+1}}\,dx$$

$$= \int_1^2 \frac{dy}{dx}\,dx - \int_1^2 \frac{4}{\sqrt{4x+1}}\,dx$$ M1

$$= \left[(x-1)\sqrt{4x+1}\right]_1^2 - \left[\frac{4\times 2}{4}\sqrt{4x+1}\right]_1^2$$ M1

$$= \left[\sqrt{9}-0\right] - 2\left[\sqrt{9}-\sqrt{5}\right]$$ M1

$$= 2\sqrt{5}-3$$

$$= 1.47$$

A1

4. **(a)** Express $2\sin\theta - 1.5\cos\theta$ in the form $R\sin(\theta-\alpha)$, where R > 0 and $0 < \alpha < \frac{\pi}{2}$. [3]

$$2\sin\theta - 1.5\cos\theta = R\sin\theta\cos\alpha - R\cos\theta\sin\alpha$$
$$2 = R\cos\alpha$$
$$1.5 = R\sin\alpha$$

$$R^2 = 2^2 + (1.5)^2$$ B1
$$R = \frac{5}{2}$$

$$\tan\alpha = \frac{1.5}{2}$$
$$\alpha = 0.644$$ B1

$$2\sin\theta - 1.5\cos\theta = \frac{5}{2}\sin(\theta - 0.644)$$

A1

(b) John models the height of sea water level, H metres, on a particular day by the equation $H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right)$, $0 \le t < 12$, where t hours is the number of hours after midday. Using this model, calculate

(i) the maximum possible height of sea water level and the value of t, to 2 decimal places, when it occurs. [3]

$$H = 6 + 2\sin\left(\frac{4\pi t}{25}\right) - 1.5\cos\left(\frac{4\pi t}{25}\right)$$

$$\max H = 6 + \frac{5}{2}\sin\left(\frac{4\pi t}{25} - 0.6435\right)$$

$$= 6 + \frac{5}{2}$$

$$= 8.5 \text{ m}$$ B1

$$\sin\left(\frac{4\pi t}{25} - 0.643501\right) = 1$$ M1

It occurs when $\frac{4\pi t}{25} - 0.643501 = \frac{\pi}{2}$

$$t = 4.405204$$

$$t = 4.41$$ A1

4 **(ii)** the first time when the height of the sea water is 7 metres, leaving your answer correct to the nearest minute. [4]

$$7 = 6 + \frac{5}{2}\sin\left(\frac{4\pi t}{25} - 0.643501\right)$$ M1

$$\frac{2}{5} = \sin\left(\frac{4\pi t}{25} - 0.643501\right)$$ M1

$$t = 2.09889$$

$$= 125.933 \text{ mins}$$ A1

$$= 126 \text{ mins}$$

First time = 14 06 or 2.06 pm B1

5. The temperature, θ °C, of an oven t minutes after it was switched on is given by $\theta = 300 - 280e^{-0.05t}$, $t \geq 0$.

(i) State the initial temperature of the oven. [1]

Initial temperature = $300 - 280 = 20°C$ B1

(ii) Find the value of t when the temperature of the oven reaches 160 °C. [2]

$$300 - 280e^{-0.05t} = 160$$
$$280e^{-0.05t} = 140$$
$$e^{-0.05t} = \frac{1}{2}$$ M1
$$-0.05t = \ln\left(\frac{1}{2}\right)$$
$$t = 13.8629$$
$$= 13.9 \text{ mins}$$ A1

(iii) Determine, with justification, the maximum temperature that this oven can approach. [2]

As t increase, $e^{-0.05t}$ approaches zero. B1

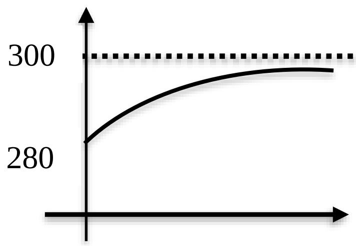
Hence, maximum temperature that the oven can reach = $300°C$

A1

<u>Alt Mtd :</u>

$e^{-0.05t} > 0$
$-280e^{-0.05t} < 0$
$300 - 280e^{-0.05t} < 300$
$\therefore \theta < 300$

or



Since $\theta = 300$ is an asymptote, θ approaches 300 as t increases.

5. The temperature, θ °C, of another oven t minutes after it was switched on is given by $\theta = 250 - 230e^{-0.1t}$, $t \geq 0$.

(iv) Assuming that both ovens are switched on at the same time, find the time when both ovens will have the same temperature since they were switched on. [5]

$$250 - 230e^{-0.1t} = 300 - 280e^{-0.05t}$$

$$280e^{-0.05t} - 230e^{-0.1t} = 50$$

$$-28e^{-0.05t} + 23e^{-0.1t} + 5 = 0$$ M1

Let $e^{-0.05t} = p$

$$23e^{(-0.05t)^2} - 28e^{-0.05t} + 5 = 0$$ M1

$$23p^2 - 28p + 5 = 0$$

$$(23p - 5)(p - 1) = 0$$ M1

$$p = \frac{5}{23}$$

$$e^{-0.05t} = \frac{5}{23}$$

$$t = 30.5211$$ A1

$$= 30.5$$

$$p = 1$$

$$e^{-0.05t} = 1$$ A1

$$-0.05t = 0$$

(NA)

6. **(i)** Prove that $\sin(A+B)\sin(A-B)=(\sin A+\sin B)(\sin A-\sin B)$. [3]

$$\sin(A+B)\sin(A-B)=(\sin A\cos B+\cos A\sin B)(\sin A\cos B-\cos A\sin B)$$
$$=\sin^2 A\cos^2 B-\cos^2 A\sin^2 B$$ M1
$$=\sin^2 A(1-\sin^2 B)-\sin^2 B(1-\sin^2 A)$$ M1
$$=\sin^2 A-\sin^2 A\sin^2 B-\sin^2 B+\sin^2 A\sin^2 B$$
$$=\sin^2 A-\sin^2 B$$ A1
$$=(\sin A+\sin B)(\sin A-\sin B)$$

(ii) Given that $\sin A=\frac{1}{2}$, find all the values of B for $-\pi\le B\le\pi$ that satisfy the equation $\sin(A+B)\sin(A-B)=0$. [4]

$$\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B$$ M1
$$=0$$

From (i),
$$\sin^2 A=\sin^2 B$$
$$\sin^2 B=\frac{1}{4}$$
$$\sin B=\pm\frac{1}{2}$$ M1

Angle B lies in all 4 quadrants

basic angle, $\alpha=\frac{\pi}{6}$

$$\angle B=-\frac{5\pi}{6},-\frac{\pi}{6},\frac{\pi}{6},\frac{5\pi}{6}$$
or $-2.62,-0.524,0.524,2.62$

A1 A1

7. **(a)** Find the range of values of k for which $y=(k-6)x^2-8x+k$ cuts the x – axis at two distinct points and has a minimum point. [4]

$$b^2-4ac>0$$

$$(-8)^2-4(k-6)(k)>0$$ M1

$$64-4k^2+24k>0$$

$$k^2-6k-16<0$$ M1

$$(k-8)(k+2)<0$$

M1 $-2<k<8$

For curve to have minimum point, $k-6>0$

$$k>6$$

$\therefore 6<k<8$ A1

(b) Given that the line $y=5x+c$ is a tangent to the curve $y=2x^2+bx$, show that c cannot be positive. [4]

$$5x+c=2x^2+bx$$

$$2x^2+(b-5)x-c=0$$ M1

Since line is a tangent,

$$b^2-4ac=o$$

$$(b-5)^2-4(2)(-c)=0$$ M1

$$(b-5)^2=-8c$$

since $(b-5)^2\geq 0$ M1

$$-8c\geq 0$$

$$c\leq 0$$ A1

Hence c cannot be positive.

8. The table below shows experimental values of two variables, x and y, which are connected by an equation of the form $y=\frac{a}{x+b}$, where a and b are constants.

x	0.1	0.4	1.0	2.0	3.0
y	8.0	6.0	4.0	2.6	1.9

(i) Draw the graph of x against $\frac{1}{y}$, using a scale of 5 cm to 1 unit on the x-axis and a scale of 2 cm to 0.1 unit on the $\frac{1}{y}$-axis on the graph paper in the next page. [3]

x	0.1	0.4	1.0	2.0	3.0
$\frac{1}{y}$	0.125	0.167	0.25	0.385	0.526

M1

(ii) Use your graph to estimate the value of a and of b. [4]

$$y=\frac{a}{x+b}$$
$$y(x+b)=a$$
$$x+b=\frac{a}{y}$$ M1
$$x=a\left(\frac{1}{y}\right)-b$$

vertical intercept $=-0.8$

$b=0.8$ B1 Range: {0.74 to 0.84 }

gradient $=\frac{2.4+0.8}{0.44-0}$ M1

$a=7.27$ A1

Range: {7.0 to 7.4}

An alternative method for obtaining a straight line graph for the equation $y=\frac{a}{x+b}$ is to plot y against xy.

(iii) Using answers of a and b in part **(ii)** estimate the gradient and vertical intercept of the graph of y plotted against xy. [3]

$$y=\frac{a}{x+b}$$
$$xy+by=a$$
$$by=-xy+a$$
$$y=-\frac{1}{b}xy+\frac{a}{b}$$ M1

gradient $=-\frac{1}{b}$

$=-\frac{1}{0.8}$

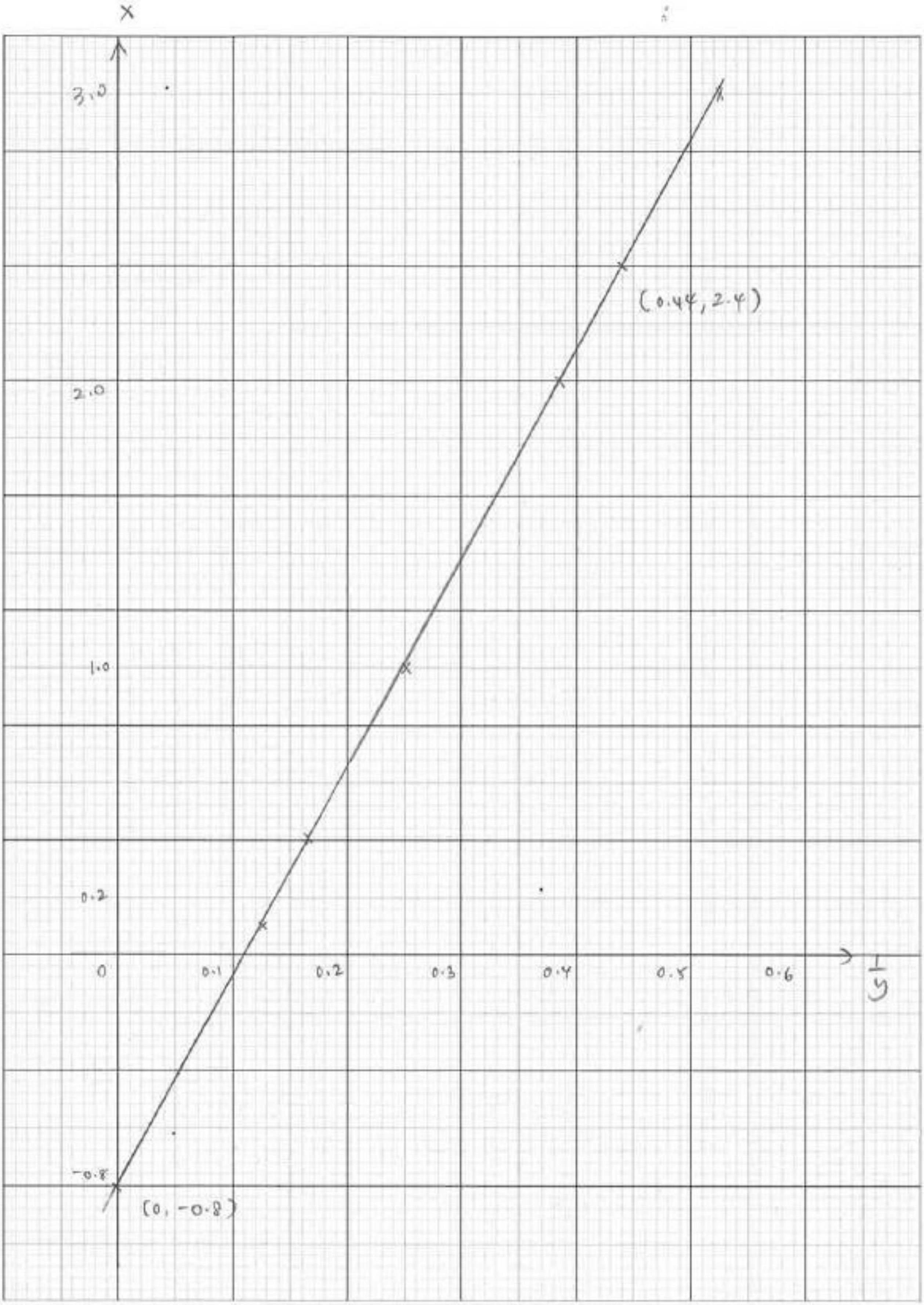
$=-\frac{5}{4}$ B1

vertical intercept $=\frac{a}{b}$

$=\frac{7.22}{0.8}$

$=9.025$ B1

Range: {9.025 to 9.1}


x
3.0
2.0
1.0
0.2
0
-0.8
0.1
0.2
0.3
0.4
0.5
0.6
1/y
(0.44, 2.4)
(0, -0.8)

9. A particle moves in a straight line so that t seconds after passing through a fixed point O, its velocity, v cm/s, is given by.

(i) Find the value of t at the instant when the acceleration is -1 cm/s^2. [2]

$$v = 3t^2 - 7t + 2$$

$$a = \frac{\mathrm{dv}}{\mathrm{dt}}$$

$$= 6t - 7$$

M1

$$-1 = 6t - 7$$

$$6 = 6t$$

$$t = 1$$

A1

(ii) Find the values of t at which the particle is instantaneously at rest. [2]

$$3t^2 - 7t + 2 = 0$$

$$(3t - 1)(t - 2) = 0$$

M1

$$t = \frac{1}{3} \qquad t = 2$$

A1

9. **(iii)** Find the total distance travelled by the particle during the first 2 seconds. [3]

$$s=\int\left(3t^2-7t+2\right)\,\mathrm{d}t$$

$$s=\frac{3t^3}{3}-\frac{7t^2}{2}+2t+c$$

At $t = 0$, $s = 0$ and $c = 0$ M1

$$s=\frac{3t^3}{3}-\frac{7t^2}{2}+2t$$

$$t=\frac{1}{3},\ s=\frac{17}{54}$$ M1

$$t=2,\ s=-2$$

Total distance $=2\left(\frac{17}{54}\right)+2=2\frac{17}{27}$ A1

(iv) Find the time(s) when the particle returns to O. [3]

$$\frac{3t^3}{3}-\frac{7t^2}{2}+2t=0$$

$$t\left(t^2-\frac{7}{2}t+2\right)=0$$ M1

$$\frac{1}{2}t\left(2t^2-7t+4\right)=0$$

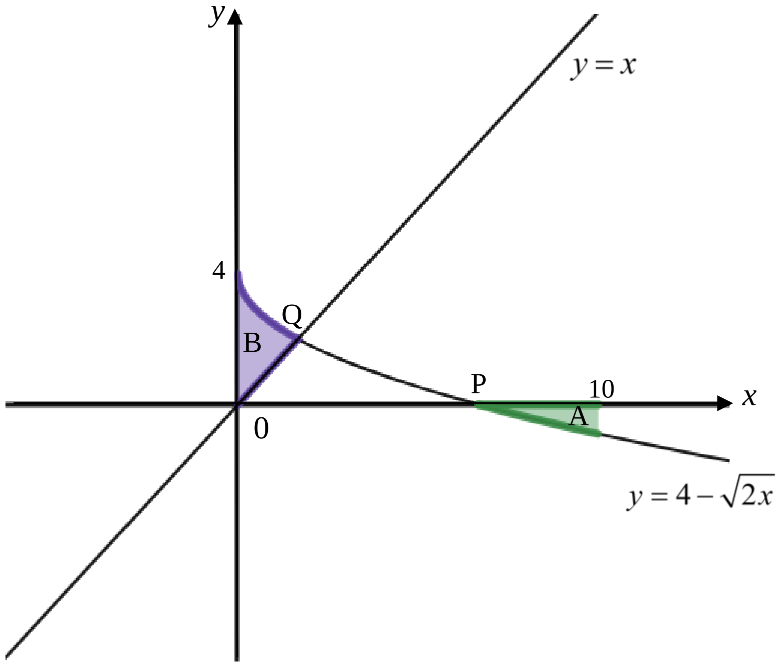
$$2t^2-7t+4=0$$

$t=0$ A1 NA

$$t=\frac{7\pm\sqrt{17}}{4}$$

$=2.78$ or 0.719 A1

10. The diagram shows part of the curve $y = 4 - \sqrt{2x}$ and the line $y = x$. The curve cuts the x – axis at the point P and the line intersects the curve at the point Q.



(i) Show that the *x*- coordinate of the point *P* is 8. [2]

At P, $y = 0$

$4 - \sqrt{2x} = 0$ M1

$16 = 2x$

$x = 8$ A1

(ii) Find the coordinates of the point *Q*. [3]

$x = 4 - \sqrt{2x}$

$\sqrt{2x} = 4 - x$ M1

$2x = 16 - 8x + x^2$

$x^2 - 10x + 16 = 0$ M1

$(x-2)(x-8) = 0$

$x = 2$ or $x = 8$ (point P)

$y = 2 \quad \therefore Q(2,2)$ A1

10. **(iii)** Find the area of the region A, bounded by the curve $y = 4 - \sqrt{2x}$, the x axis and the line $x = 8$ and $x = 10$. [3]

$$A = -\int_8^{10} \left(4 - \sqrt{2x}\right) \text{dx}$$

$$= -\left[4x - \frac{2}{3}\frac{(2x)^{3/2}}{2}\right]_8^{10}$$ M1

$$= -\left[4x - \frac{1}{3}(2x)^{3/2}\right]_8^{10}$$

$$= -\left[10.1857 - \frac{32}{3}\right]$$ M1

$$= 0.480926$$

$$= 0.481$$

A1

(iv) Find the area of the region B, bounded by the straight line $x - y = 0$, the curve $y = 4 - \sqrt{2x}$ and the y – axis. [4]

$$A = \int_0^2 \left(4 - \sqrt{2x}\right) \text{dx} - \frac{1}{2}(2)(2)$$ M1

$$= \left[4x - \frac{1}{3}(2x)^{3/2}\right]_0^2 - 2$$ M1

$$= \frac{16}{3} - 2$$ M1

$$= 3\frac{1}{3}$$ A1

(v) If $\int_k^{10} \left(4 - \sqrt{2x}\right) \text{dx} = 0$, where $2 < k < 8$, explain what this result implies about the curve $y = 4 - \sqrt{2x}$. [2]

B1

The area bounded by the curve $x = k$, $x = 8$ and x-axis lies above the x axis and it is **equal** to the area of region A, which is area bounded by the curve $x = 8$, $x = 10$ and x-axis. This is area below the x axis. B1

End of Paper

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