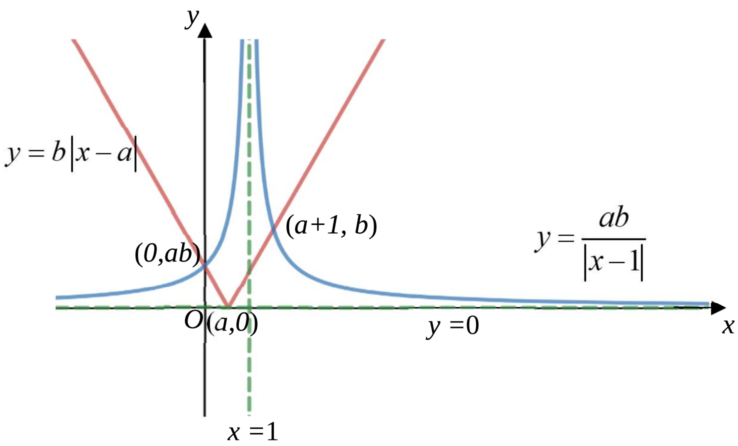
Suggested solution
$x^2 + y^2 = y(x-3)$ **----------(1)** **Differentiating throughout with respect to *x*, we have** $2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = y + \frac{\mathrm{d}y}{\mathrm{d}x}(x-3)$. $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}(2y - x + 3) = y - 2x$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y-2x}{2y-x+3}$ **For tangent to be parallel to *y*-axis, gradient must be undefined.** **So** $2y - x + 3 = 0$. $\Rightarrow x = 2y + 3$ **Sub into (1):** $(2y+3)^2 + y^2 = y(2y)$ $\Rightarrow 3y^2 + 12y + 9 = 0$ $\Rightarrow (y+3)(y+1) = 0$ $\Rightarrow y = -3$ **or** -1 **The points at which the tangents are parallel to the *y*-axis are** $(-3,\ -3)$ **and** $(1,\ -1)$. **Alternatively:** $2y - x + 3 = 0$ $\Rightarrow y = \frac{x-3}{2}$ **Sub into (1):** $x^2 + \left(\frac{x-3}{2}\right)^2 = \left(\frac{x-3}{2}\right)(x-3)$ $\Rightarrow x^2 = \left(\frac{x^2-6x+9}{4}\right) \Rightarrow 3x^2 + 6x - 9 = 0$ $\Rightarrow x^2 + 2x - 3 = 0$ $\Rightarrow x = 1 \text{ or } -3$ $\Rightarrow y = -1 \text{ or } -3$ **The points at which the tangents are parallel to the *y*-axis are** $(-3,\ -3)$ **and** $(1,\ -1)$

Suggested solution
$\frac{\mathrm{d}\left(\tan x^2\right)}{\mathrm{d}x} = 2x\sec^2 x^2$ $\int x^3 \sec^2 x^2 \mathrm{d}x = \frac{1}{2}\int x^2\left(2x\sec^2 x^2\right)\mathrm{d}\theta$ $u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ $\frac{\mathrm{d}v}{\mathrm{d}x} = 2x\sec^2 x^2 \Rightarrow v = \tan x^2$ Therefore, $\int x^3 \sec^2 x^2 \mathrm{d}x = \frac{1}{2}\left[x^2\tan x^2 - \int 2x\tan x^2 \mathrm{d}\theta\right] = \frac{1}{2}\left(x^2 \tan x^2 - \ln\left\|\sec x^2\right\|\right) + c$ where c is an arbitrary constant

2

Suggested solution
(i) $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\mathrm{d}r}{\mathrm{d}V}\times\frac{\mathrm{d}V}{\mathrm{d}t}$ Given $\frac{\mathrm{d}V}{\mathrm{d}t} = 12$ (constant) $\frac{\mathrm{dr}}{\mathrm{d}t} = \frac{12}{4\pi r^2} = \frac{3}{\pi r^2}$ At r= 5, $\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{3}{\pi(5)^2} = \frac{3}{25\pi} = 0.0382$ cm/min
(ii) $A = 4\pi r^2$ $\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$ $\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r}\times\frac{dr}{\mathrm{d}t}$ $= \left(8\pi r\right)\frac{3}{\pi r^2} = \frac{24}{r}$ In 10 min, the volume of the balloon $V = 12\times 10 = 120$ cm^3. $V = \frac{4}{3}\pi r^3 = 120$ $r = \sqrt[3]{\frac{90}{\pi}} \approx 3.0598$ At t=10 min, $\frac{\mathrm{d}A}{\mathrm{d}t} = 24\sqrt[3]{\frac{\pi}{90}} \approx \frac{24}{3.0598} \approx 7.84$ cm/min

3

Suggested solution
(i) 
(ii)Multiplying b to $\lvert x-a\rvert\le\dfrac{a}{\lvert x-1\rvert}$ gives $b\lvert x-a\rvert\le\dfrac{ab}{\lvert x-1\rvert}$ $(\because b>0)$ **Hence from the graph,** $\lvert x-a\rvert\le\dfrac{b}{\lvert x-1\rvert}\Rightarrow 0\le x\le a+1, x\ne 1$

4

Suggested solution
(i) $y=\mathrm{e}^x\Rightarrow\dfrac{\mathrm{d}y}{\mathrm{d}x}=\mathrm{e}^x$ $F=\displaystyle\int\frac{y}{\left(y+\frac{1}{y}\right)}\left(\frac{1}{y}\right)\mathrm{d}y=\frac{1}{2}\int\frac{2y}{y^2+1}\,\mathrm{d}y$ $=\dfrac{1}{2}\ln\left(y^2+1\right)+d\ \left(\because y^2+1>0\right)$ $=\dfrac{1}{2}\ln\left(\mathrm{e}^{2x}+1\right)+d$ where d is an arbitrary constant.
(ii) $\mathrm{e}^x=\dfrac{1}{2}(\mathrm{e}^x+\mathrm{e}^{-x})+\dfrac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x})$ $F=\dfrac{1}{2}\displaystyle\int\frac{(\mathrm{e}^x+\mathrm{e}^{-x})+(\mathrm{e}^x-\mathrm{e}^{-x})}{\mathrm{e}^x+\mathrm{e}^{-x}}\,\mathrm{d}x=\frac{1}{2}\int\left(1+\frac{\mathrm{e}^x-\mathrm{e}^{-x}}{\mathrm{e}^x+\mathrm{e}^{-x}}\right)\mathrm{d}x$ $=\dfrac{1}{2}\left[x+\ln(\mathrm{e}^x+\mathrm{e}^{-x})\right]+c\qquad\left(\because \mathrm{e}^x+\mathrm{e}^{-x}>0\right)$ where c is an arbitrary constant.

(iii) From (ii),

$$F=\frac{1}{2}\left[x+\ln(e^{x}+e^{-x})\right]+c=\frac{1}{2}x+\frac{1}{2}\ln\left(\frac{e^{2x}+1}{e^{x}}\right)+c$$

$$=\frac{1}{2}x+\frac{1}{2}\ln(e^{2x}+1)-\frac{1}{2}\ln(e^{x})+c=\frac{1}{2}x+\frac{1}{2}\ln(e^{2x}+1)-\frac{1}{2}x+c$$

$$=\frac{1}{2}\ln(e^{2x}+1)+c$$

The difference is $|c-d|$, where c and d are the arbitrary constants for the answers in (ii) and (i) respectively.

5

Suggested solution

(i)

Let $y=a+\dfrac{2}{3(x-a)}$

$$y-a=\frac{2}{3(x-a)}$$

$$x-a=\frac{2}{3(y-a)}$$

$$x=a+\frac{2}{3(y-a)}$$

$D_{f^{-1}}=R_f=\mathbb{R}\setminus\{a\}$

$f^{-1}:x\mapsto a+\dfrac{2}{3(x-a)},\ x\in\mathbb{R},\ x\neq a,$

Method 1

Since $f^{-1}=f$, $f^{2}(x)=ff^{-1}(x)=x$

Method 2

$$f^{2}(x)=f\left(a+\frac{2}{3(x-a)}\right)$$

$$=a+\frac{2}{3\left(a+\dfrac{2}{3(x-a)}-a\right)}=a+\frac{2}{3\left(\dfrac{2}{3(x-a)}\right)}=a+(x-a)=x$$

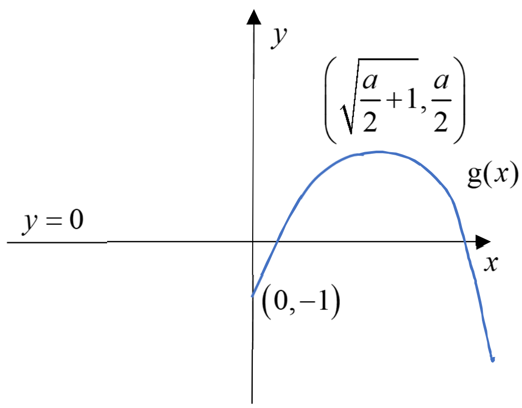
(ii)

$$f^{2k+1}(x)=ff^{2k}(x)=f(x)$$

$$f^{2k+1}(2a)=f(2a)=a+\frac{2}{3a}$$

(iii)

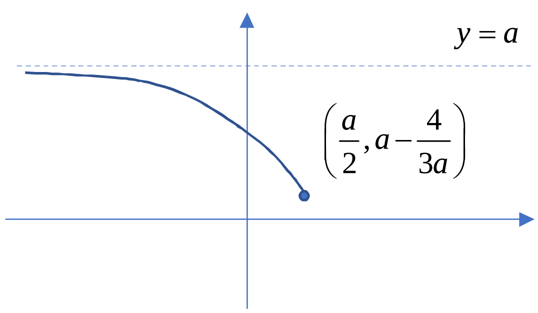
For maximum value of $g(x)=\dfrac{a}{2}-\left(x-\sqrt{\dfrac{a}{2}+1}\right)^{2}$, It occurs at $x=\sqrt{\frac{a}{2}+1}$ [when $\left(x-\sqrt{\frac{a}{2}+1}\right)^2=0$]



From the sketch, $\mathrm{R_g}=\left(-\infty,\frac{a}{2}\right]\subset\mathbb{R}\setminus\{a\}=\mathrm{D_f}$

Therefore, fg exists

When we restrict $\mathrm{D_f}=\left(-\infty,\frac{a}{2}\right]$



Range of fg = $\left[a-\frac{4}{3a},a\right)$

6

Suggested solution
$S_1 = 5 = 3a + b + c$ $S_2 = 14 = 9a + 2b + c$ $S_3 = 47 = 27a + 3b + c$ OR $u_1 = S_1 \Rightarrow 5 = 3a + b + c$ $u_2 = S_2 - S_1 \Rightarrow 9 = 6a + b$ $u_3 = S_3 - S_2 \Rightarrow 33 = 18a + b$ Using GC, $a = 2, b = -3$ and $c = 2$ Hence $S_n = 2\left(3^n\right) - 3n + 2$
$u_{n+1} = S_{n+1} - S_n$ $= 2\left(3^{n+1}\right) - 3(n+1) + 2 - \left[2\left(3^n\right) - 3n + 2\right]$ $= 2\left(3^{n+1}\right) - 3n - 3 + 2 - 2\left(3^n\right) + 3n - 2$ $= 2(3-1)\left(3^n\right) - 3$ $= 4\left(3^n\right) - 3$
$\sum_{r=2}^{n} u_{r+1} = \sum_{r=2}^{n}\left[4\left(3^r\right) - 3\right]$ $= 4\sum_{r=2}^{n}\left(3^r\right) - 3\sum_{r=2}^{n} 1$ $= 4\left[\frac{3^2\left(3^{n-1} - 1\right)}{3-1}\right] - 3(n - 2 + 1)$ $= 18\left(3^{n-1} - 1\right) - 3(n-1)$ $= 6\left(3^n\right) - 3n - 15$

Suggested solution

(a)(i)

Since $z_1 = -1+\mathrm{i}$ is a root,

$$\left(-1+\mathrm{i}\right)^2 + a\left(-1+\mathrm{i}\right) + \left(1-\sqrt{3}\right) + b\mathrm{i} = 0$$

$$-2\mathrm{i} + a\left(-1+\mathrm{i}\right) + \left(1-\sqrt{3}\right) + b\mathrm{i} = 0$$

$$-a + \left(1-\sqrt{3}\right) + \left(a+b-2\right)\mathrm{i} = 0$$

Comparing Re and Im parts

$$-a + \left(1-\sqrt{3}\right) = 0 \Rightarrow a = 1-\sqrt{3}$$

$$a+b-2 = 0 \Rightarrow b = 1+\sqrt{3}$$

(ii)

$$z^2 + \left(1-\sqrt{3}\right)z + \left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i} = 0$$

$$z^2 + \left(1-\sqrt{3}\right)z + \left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i} = \left[z - \left(-1+\mathrm{i}\right)\right]\left(z - z_2\right)$$

Method 1: Comparing z

$$1-\sqrt{3} = -z_2 - \left(-1+\mathrm{i}\right) \Rightarrow z_2 = \sqrt{3} - \mathrm{i}$$

Method 2: Comparing "constant"

$$\left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i} = z_2\left(-1+\mathrm{i}\right)$$

$$\Rightarrow z_2 = \frac{\left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i}}{\left(-1+\mathrm{i}\right)} = \frac{\left[\left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i}\right]\left[-1-\mathrm{i}\right]}{2}$$

$$= \frac{-\left[\left(1-\sqrt{3}\right) + \left(1+\sqrt{3}\right)\mathrm{i}\right]\left[1+\mathrm{i}\right]}{2} = \sqrt{3} - \mathrm{i}$$

Method 3: Sum of roots

Sum of roots = $-\left(1-\sqrt{3}\right)$

$$-1+\mathrm{i}+z_2 = -\left(1-\sqrt{3}\right)$$

$$z_2 = \sqrt{3} - \mathrm{i}$$

Method 4: General formula

$$z_2=\frac{-\left(1-\sqrt{3}\right)\pm\sqrt{\left(1-\sqrt{3}\right)^2-4(1)\left[\left(1-\sqrt{3}\right)+\left(1+\sqrt{3}\right)\mathrm{i}\right]}}{2}$$

$$=\frac{-\left(1-\sqrt{3}\right)\pm\sqrt{1-2\sqrt{3}+3-4+4\sqrt{3}-4\mathrm{i}-4\sqrt{3}\mathrm{i}}}{2}$$

$$=\frac{-\left(1-\sqrt{3}\right)\pm\sqrt{2\sqrt{3}-4\sqrt{3}\mathrm{i}-4\mathrm{i}}}{2}$$

$$=\frac{-\left(1-\sqrt{3}\right)\pm\sqrt{\left(1+\sqrt{3}-2\mathrm{i}\right)^2}}{2}$$

$$=\frac{-\left(1-\sqrt{3}\right)\pm\left(1+\sqrt{3}-2\mathrm{i}\right)}{2}$$

$$=-1+\mathrm{i}\ \text{(rej)}\quad\text{or}\quad\sqrt{3}-\mathrm{i}$$

(b)**(i)**

Method 1:

$$w_1=2-2\mathrm{i}=2\sqrt{2}e^{-\frac{\pi}{4}\mathrm{i}}\text{ or }2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+\mathrm{i}\sin\left(-\frac{\pi}{4}\right)\right)$$

$$w_2=-\sqrt{3}+\mathrm{i}=2e^{\frac{5\pi}{6}\mathrm{i}}\text{ or }2\left(\cos\frac{5\pi}{6}+\mathrm{i}\sin\frac{5\pi}{6}\right)$$

$$w_1w_2=4\sqrt{2}e^{\left(-\frac{\pi}{4}+\frac{5\pi}{6}\right)\mathrm{i}}=4\sqrt{2}e^{\frac{7\pi}{12}\mathrm{i}}$$

$$\left|w_1w_2\right|=4\sqrt{2}\ \text{ and }\ \arg\left(w_1w_2\right)=\frac{7\pi}{12}$$

Method 2:

$$w_1w_2=2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$$

$$\left|w_1w_2\right|=\sqrt{4\left(1-\sqrt{3}\right)^2+4\left(1+\sqrt{3}\right)^2}=\sqrt{32}=4\sqrt{2}$$

$$\arg\left(w_1w_2\right)=\pi-\tan^{-1}\frac{\left(1+\sqrt{3}\right)}{\left(\sqrt{3}-1\right)}=\frac{7}{12}\pi$$

(ii)

Method 1:

From (ii),

$$w_1w_2=4\sqrt{2}e^{\frac{7\pi}{12}\mathrm{i}}\text{ or }4\sqrt{2}\left(\cos\left(\frac{7\pi}{12}\right)+\mathrm{i}\sin\left(\frac{7\pi}{12}\right)\right)$$

$$w_1w_2=2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$$

Hence $4\sqrt{2}\cos\frac{7}{12}\pi = 2\left(1-\sqrt{3}\right) \Rightarrow \cos\frac{7}{12}\pi = \frac{1-\sqrt{3}}{2\sqrt{2}}$ Otherwise **Method 2:** Student using geometry approach on $w_1w_2 = 2\left(1-\sqrt{3}\right)+2\left(1+\sqrt{3}\right)\mathrm{i}$ **Method 3:** Student using special angles and addition formula

8 **(a)**

Suggested solution
(a)(i) To prove $x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=k---(*)$ Consider $y=\frac{k\left(\ln x+\alpha\right)}{x} \Rightarrow xy=k\left(\ln x+\alpha\right)--(1)$ Diff (1) wrt x, $x\frac{\mathrm{d}y}{\mathrm{d}x}+y=k\left(\frac{1}{x}\right) \Rightarrow x^2\frac{\mathrm{d}y}{\mathrm{d}x}+xy=k$ [shown]
(a)(ii) $y=\frac{k\left(\ln x+\alpha\right)}{x}$ At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow xy=k$ [from (*)] So $\frac{k}{x}=\frac{k\left(\ln x+\alpha\right)}{x} \Rightarrow \ln x=1-\alpha \Rightarrow x=\mathrm{e}^{1-\alpha}$ When $x=\mathrm{e}^{1-\alpha}$, $y=\frac{k}{x}=\frac{k}{\mathrm{e}^{1-\alpha}}=k\mathrm{e}^{\alpha-1}$. Therefore, $\left(\mathrm{e}^{1-\alpha},\ k\mathrm{e}^{\alpha-1}\right)$ is a stationary point of the curve $y=\frac{k\left(\ln x+\alpha\right)}{x}$.
(b)(i) Given $y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\sqrt{x^2+y^2}$ ---(**) $v=x^2+y^2 \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x}=2x+2y\frac{\mathrm{d}y}{\mathrm{d}x}$ Sub into (**): $y\frac{\mathrm{d}y}{\mathrm{d}x}+x=\sqrt{x^2+y^2} \Rightarrow \frac{1}{2}\frac{\mathrm{d}v}{\mathrm{d}x}=\sqrt{v} \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x}=2\sqrt{v}$ [shown]
(b)(ii) $\frac{\mathrm{d}v}{\mathrm{d}x}=2\sqrt{v} \Rightarrow \frac{1}{2\sqrt{v}}\frac{\mathrm{d}v}{\mathrm{d}x}=1 \Rightarrow \int\frac{1}{2\sqrt{v}}\ \mathrm{d}v=\int 1\ \mathrm{d}x$ $\Rightarrow \sqrt{v}=x+C$ Since $y=0$ when $x=-2$, we have $v=4$.

$\sqrt{4} = -2 + C \Rightarrow C = 4$

$\sqrt{v} = x + 4 \Rightarrow v = (x+4)^2$

$\Rightarrow y^2 = (x+4)^2 - x^2 = 8x + 16$

Hence $\mathrm{f}(x) = 8x + 16$.

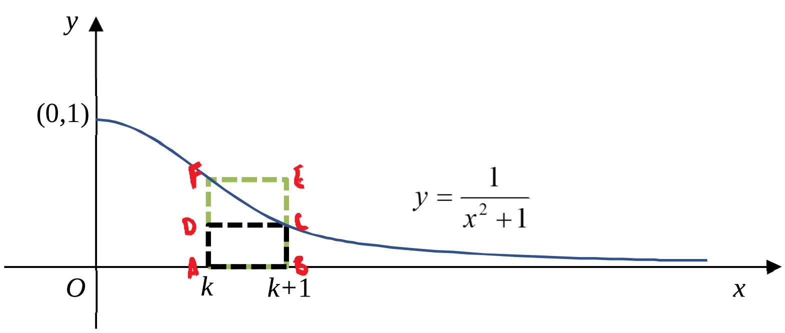
9

Suggested solution

(i)

$$\int_k^{k+1} \frac{1}{x^2+1}\,\mathrm{d}x = \left[\tan^{-1}\frac{x}{1}\right]_k^{k+1} = \tan^{-1}(k+1) - \tan^{-1} k$$

(ii)



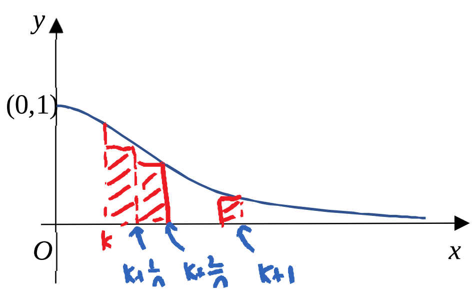
From the diagram, we can see that

Area of rectangle *ABCD* < Area under curve from $x=k$ to $x=k+1$ < Area of rectangle *ABEF*

Hence $\frac{1}{(k+1)^2+1}(1) < \int_k^{k+1} \frac{1}{x^2+1}\,\mathrm{d}x < \frac{1}{k^2+1}(1)$

$\Rightarrow \frac{1}{(k+1)^2+1} < \tan^{-1}(k+1) - \tan^{-1} k < \frac{1}{k^2+1}$ **[from (i)] [shown]**

Alternative (For one side)



$$\frac{1}{n}\left[\mathrm{f}\left(k+\frac{1}{n}\right) + \mathrm{f}\left(k+\frac{2}{n}\right) + \ldots \mathrm{f}(k+1)\right] < \int_k^{k+1} \mathrm{f}(x)\,\mathrm{d}x \text{ --- } \mathbf{(1)}$$

Note that

$$\frac{1}{n}[\mathrm{f}\left(k+\frac{1}{n}\right)+\mathrm{f}\left(k+\frac{2}{n}\right)+\ldots.\mathrm{f}\left(k+1\right)]$$

$$>\frac{1}{n}[n\mathrm{f}\left(k+1\right)]=\mathrm{f}\left(k+1\right)=\frac{1}{\left(k+1\right)^2+1}---(2)$$

Since $\mathrm{f}\left(k+\frac{1}{n}\right),\mathrm{f}\left(k+\frac{2}{n}\right),\ldots\mathrm{f}\left(k+\frac{n-1}{n}\right)>\mathrm{f}\left(k+1\right)$

Thus from (1) and (2), we have $\frac{1}{\left(k+1\right)^2+1}<\int_k^{k+1}\mathrm{f}\left(x\right)\mathrm{d}x$.

(iii)

Let $A=\tan^{-1}x; B=\tan^{-1}y$

$$\tan\left(A-B\right)=\tan\left(\tan^{-1}x-\tan^{-1}y\right)$$

$$=\frac{\tan\left(\tan^{-1}x\right)-\tan\left(\tan^{-1}y\right)}{1+\tan\left(\tan^{-1}x\right)\tan\left(\tan^{-1}y\right)}=\frac{x-y}{1+xy}$$

Hence $\tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)$ [shown]

(iv)

With $\frac{1}{\left(k+1\right)^2+1}<\tan^{-1}\left(k+1\right)-\tan^{-1}k<\frac{1}{k^2+1}$

Sum the inequalities for $k=1$ to n. (This way we are actually considering the area under the curve from $x=1$ to $x=n+1$, which can be divided to n sections, each with unit base.)

$$\sum_{k=1}^{n}\frac{1}{\left(k+1\right)^2+1}<\sum_{k=1}^{n}\left(\tan^{-1}\left(k+1\right)-\tan^{-1}k\right)<\sum_{k=1}^{n}\frac{1}{k^2+1}$$

$$\sum_{k=1}^{n}\left(\tan^{-1}\left(k+1\right)-\tan^{-1}k\right)$$

$$=\left(\begin{array}{c}\cancel{\tan^{-1}(2)}-\tan^{-1}(1)\\ +\cancel{\tan^{-1}(3)}-\cancel{\tan^{-1}(2)}\\ +\cancel{\tan^{-1}(4)}-\cancel{\tan^{-1}(3)}\\ \vdots\\ +\cancel{\tan^{-1}(n)}-\cancel{\tan^{-1}(n-1)}\\ +\tan^{-1}(n+1)-\cancel{\tan^{-1}(n)}\end{array}\right)$$

$$=\tan^{-1}\left(n+1\right)-\tan^{-1}\left(1\right)$$

$$=\tan^{-1}\left(\frac{\left(n+1\right)-1}{1+\left(n+1\right)\left(1\right)}\right)$$

$=\tan^{-1}\left(\frac{n}{n+2}\right)$ From (iii)

Hence

$\sum_{k=1}^{n}\frac{1}{\left(k+1\right)^2+1}<\tan^{-1}\left(\frac{n}{n+2}\right)<\sum_{k=1}^{n}\frac{1}{k^2+1}$ **[proven]**

Suggested solution
(a)(i) $\tan\left(\frac{\phi}{2}\right)=\frac{d}{2D}$ Since ϕ is small, $\frac{\phi}{2}\approx\frac{d}{2D}\Rightarrow\phi D=d$
(ii) $d=(0.00873)\left(9.46\times10^{12}\right)=8.26\times10^{10}$ km
(b)(i) $R^2+y^2=\left(x+R\right)^2$ $R^2+y^2=x^2+2xR+R^2$ $y^2=x^2+2xR$ $y^2=x^2\left(1+\frac{2R}{x}\right)\Rightarrow y=x\left(1+\frac{2R}{x}\right)^{\frac{1}{2}}$
(ii) $\tan\theta=\frac{R}{y}=\frac{R}{x}\left(1+\frac{2R}{x}\right)^{-\frac{1}{2}}=\alpha\left(1+2\alpha\right)^{-\frac{1}{2}}$ Since R is small relative to x, then $\alpha=\frac{R}{x}$ is small $\tan\theta=\alpha\left(1+2\alpha\right)^{-\frac{1}{2}}$ $=\alpha\left(1+\left(-\frac{1}{2}\right)\left(2\alpha\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(2\alpha\right)^2+\ldots\right)$ $\approx\alpha\left(1-\alpha+1.5\alpha^2\right)$ $=\alpha-\alpha^2+1.5\alpha^3$
(iii) $\theta=0.0345$ $\tan\left(0.0345\right)=\alpha-\alpha^2+1.5\alpha^3$ From GC, we have $\alpha=0.0357$ $\frac{R}{x}=0.0357$ $R=0.0357\left(180000\right)=6426$ km

	Suggested solution
(i)	$\overrightarrow{AB}=\overrightarrow{OC}=\begin{pmatrix}-1\\9\\1\end{pmatrix}$, $\overrightarrow{AV}=\begin{pmatrix}-3\\3\\15\end{pmatrix}$ $\overrightarrow{AB}\times\overrightarrow{AV}=\begin{pmatrix}-1\\9\\1\end{pmatrix}\times\begin{pmatrix}-3\\3\\15\end{pmatrix}=12\begin{pmatrix}11\\1\\2\end{pmatrix}$ A normal to the face *ABV* is $\begin{pmatrix}11\\1\\2\end{pmatrix}$. $\begin{pmatrix}8\\1\\-1\end{pmatrix}\bullet\begin{pmatrix}11\\1\\2\end{pmatrix}=87$ A Cartesian equation of the face *ABV* is $11x+y+2z=87$.
(ii)	$\overrightarrow{MS}=\begin{pmatrix}14.5\\7.5\\a+0.5\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\9\\1\end{pmatrix}=\begin{pmatrix}15\\3\\a\end{pmatrix}$ Acute angle between line *MS* and horizontal plane (or *xy*-plane) $=\sin^{-1}\frac{\left\lvert\begin{pmatrix}15\\3\\a\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}{\left\lvert\begin{pmatrix}15\\3\\a\end{pmatrix}\right\rvert\left\lvert\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rvert}$ $=\sin^{-1}\frac{a}{\sqrt{234+a^2}}$ Since $0°\le\sin^{-1}\frac{a}{\sqrt{234+a^2}}\le 30°$, $0\le\frac{a}{\sqrt{234+a^2}}\le\frac{1}{2}$. $\Rightarrow 0\le 4a^2\le 234+a^2$ $\Rightarrow 0\le a\le\sqrt{78}$
(iii)	$\overrightarrow{MS}=\begin{pmatrix}14.5\\7.5\\3.5\end{pmatrix}-\frac{1}{2}\begin{pmatrix}-1\\9\\1\end{pmatrix}=3\begin{pmatrix}5\\1\\1\end{pmatrix}$ A vector equation of line *MS* is $\mathbf{r}=\begin{pmatrix}-0.5\\4.5\\0.5\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$. ----------(1)

	Plane ABV: $\mathbf{r}\bullet\begin{pmatrix}11\\1\\2\end{pmatrix}=87$ ----------(2) Sub (1) into (2): $\left[\begin{pmatrix}-0.5\\4.5\\0.5\end{pmatrix}+\lambda\begin{pmatrix}5\\1\\1\end{pmatrix}\right]\bullet\begin{pmatrix}11\\1\\2\end{pmatrix}=87$ $\Rightarrow 58\lambda=87$ $\Rightarrow \lambda=\dfrac{3}{2}$ $\therefore \quad X$ is $(7,\ 6,\ 2)$.
(iv)	A normal to the plane OCV is $\mathbf{n}=\overrightarrow{OC}\times\overrightarrow{OV}=\begin{pmatrix}-1\\9\\1\end{pmatrix}\times\begin{pmatrix}5\\4\\14\end{pmatrix}=\begin{pmatrix}122\\19\\-49\end{pmatrix}$. Shortest distance from X to plane OCV $=\left\lvert\overrightarrow{OX}\bullet\hat{\mathbf{n}}\right\rvert=\dfrac{\left\lvert\begin{pmatrix}7\\6\\2\end{pmatrix}\bullet\begin{pmatrix}122\\19\\-49\end{pmatrix}\right\rvert}{\sqrt{122^2+19^2+49^2}}=\dfrac{870}{\sqrt{17646}}$ $=6.55$ (3 s.f.). Desired length of rope is 6.55 metres.