2023 NJC H2 Mathematics Preliminary Examination Paper 2

Section A: Pure Mathematics [40 marks]

1 Relative to the origin, the fixed point A has position vector $\mathbf{a}$. The line l has equation $\mathbf{r} = s\mathbf{d}$, where s is a parameter and $\mathbf{d}$ is a unit vector not parallel to $\mathbf{a}$.

(a) Interpret geometrically the vector equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$, $t \in \mathbb{R}$. [2]

(b) Find, in terms of $\mathbf{a}$ and $\mathbf{d}$, the position vector of the foot of perpendicular from A to l. [1]

(c) A variable point P with position vector $\mathbf{p}$ lies on the plane containing A and l. Describe the locus of P if $|\mathbf{p} \times \mathbf{d}| = |\mathbf{a} \times \mathbf{d}|$. [2]

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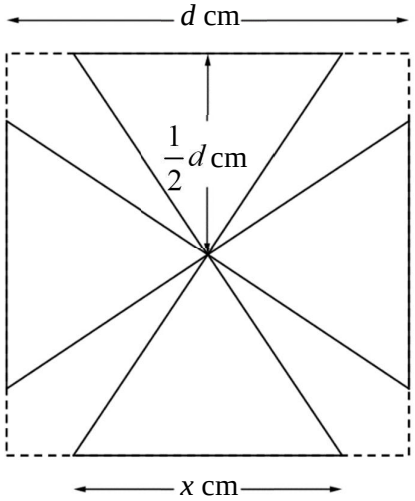


Figure 1

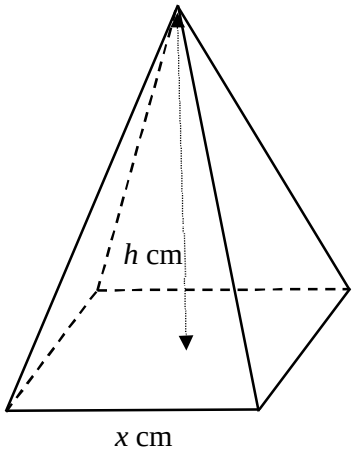


Figure 2

Figure 1 shows the net of an open square-based pyramid cut from a square of cardboard of fixed side length d cm. The net consists of four congruent isosceles triangles, each with base x cm and perpendicular height $\frac{1}{2}d$ cm. The net is folded to form a pyramid which has a square base of side length x cm and vertical height h cm, as shown in Figure 2.

Use differentiation to find the maximum volume of the pyramid in terms of d. [6]

You do not need to show that this value is a maximum volume.

[The volume of a pyramid is given by $\frac{1}{3} \times \text{base area} \times \text{height}$.]

3 **(a)** Using integration by parts on $\int \frac{(\ln x)^2}{x^2}\,\mathrm{d}x$, show that

$$\int \frac{2\ln x}{x^2}\,\mathrm{d}x - \int \frac{(\ln x)^2}{x^2}\,\mathrm{d}x = \frac{(\ln x)^2}{x} + c,$$

where c is an arbitrary constant. [3]

(b) Region R is bounded by the curves $y = \frac{\sqrt{2\ln x}}{x}$ and $y = \frac{\ln x}{x}$. Find the exact volume generated by rotating R about the x-axis through 2π radians. [5]

4 In this question, a and b are real numbers such that $a < 0$ and $b > 0$.

The function f is defined as

$$\mathrm{f} : x \mapsto \frac{a}{x-b}, \ x \neq b.$$

(a) Sketch the graph of $y = \mathrm{f}(x)$ and explain why f^2 does not exist. [3]

(b) The function g is defined as

$$\mathrm{g} : x \mapsto -\frac{b^2}{x-b}, \ x \neq 0 \text{ and } x \neq b.$$

(i) Show that g^2 exists and has the same rule as g^{-1}. [5]

(ii) Hence find $\mathrm{g}^{2024}(2b)$ in terms of b. [2]

5 A complex number w is given by $2 + k\mathrm{i}$, where k is a real constant and $k > 0$.

(a) Find w^3 in the cartesian form $x + \mathrm{i}y$, where x and y are real numbers in terms of k. [2]

(b) Show that w is a root of the equation $z^3 - (k+4)z^2 + (k+2)^2 z - (k^3 + 4k) = 0$. [3]

(c) Hence find the other two roots of the equation in part **(b)** in terms of k. [4]

(d) Find the exact value of k such that the points representing the 3 roots of the equation in part **(b)** are the vertices of an equilateral triangle in the Argand diagram. [2]

Section B: Statistics [60 marks]

6 The random variable X is normally distributed with mean μ and variance σ^2.

(a) Find $\mathrm{P}(X > \mu + 0.05\sigma)$. [2]

(b) The mean of a random sample of n independent observations of X is denoted by $\overline{X}$.

Find the least value of n such that the probability that $\overline{X}$ differs from μ by at least 0.05σ is less than 0.4. [3]

(c) In another random sample of m independent observations of X, the probability that at least 4 of these observations exceeds $\mu + 0.05\sigma$ each is at most 0.95. Find the greatest value of m. [3]

7 The Ang family, the Bao family and the Chew family have three, five and four members respectively. The three families are invited to a wedding dinner.

Due to venue constraints, only 2 members from each family can attend the dinner.

(a) Find the number of ways to select the 6 attendees. [1]

(b) The 6 attendees selected in part **(a)** are randomly arranged to sit at a round table. Find the probability that the 2 members from the Ang family are not next to each other. [3]

The host decides to change the layout of the venue to accommodate more people, so that a total of 7 people can now attend the dinner.

(c) Find the number of ways to select the 7 attendees with at least 2 members from each family. [3]

(d) The 7 attendees selected in part **(c)** are randomly arranged to sit in a row during photo-taking. Find the probability that the people from the same family are seated together. [2]

8 The events A, B and C are such that $\mathrm{P}(A) = 0.1$, $\mathrm{P}(B) = 0.2$ and $\mathrm{P}(C) = c$. A and B are independent events. A and C are mutually exclusive events.

(a) State the values of $\mathrm{P}(A \mid B)$ and $\mathrm{P}(A \mid C)$. [2]

It is given that $\mathrm{P}(A \cup B) = \mathrm{P}(A' \cap C')$.

(b) Find the value of c. [3]

(c) Find the maximum and minimum possible values of $\mathrm{P}(A' \cap B' \cap C')$. [4]

9 It is believed that the probability p of a randomly chosen pregnant woman giving birth to a Down's Syndrome child is related to the women's age x, in years. The table gives the observed values of p for 5 different values of x.

x	25	30	35	40	45
p	0.00067	0.00125	0.00333	0.01000	0.03330

(i) Sketch the scatter diagram representing the data and comment if $p = mx + c$ is a good model for this set of data. [2]

(ii) By finding the product-moment correlation coefficient for each of the following models, determine which of the following would be an appropriate model to represent the above data:

A: $p = a + \dfrac{b}{x}$,

B: $p = a + be^{-x}$,

C: $\ln p = a + bx$,

where a and b are constants. [3]

(iii) For the appropriate model, calculate the values of a and b. [2]

(iv) Obtain an estimate of the probability of a 19-year-old woman giving birth to a Down's Syndrome child. Comment on the reliability of your answer. [2]

10 A factory produces sausage rolls of length 10 cm. The production manager wishes to take a random sample of sausage rolls produced by a particular machine for quality control.

(a) State what it means for a sample to be random in this context. [1]

The lengths, x cm, of a random sample of 60 sausage rolls are summarized as follows.

$$\sum (x-10) = -2.5 \qquad \sum (x-10)^2 = 4.3$$

(b) Suggest with a reason why, in this context, the given data is summarised in terms of $(x-10)$ rather than x. [1]

(c) Calculate unbiased estimates of the population mean and variance of the length of the sausage rolls. [2]

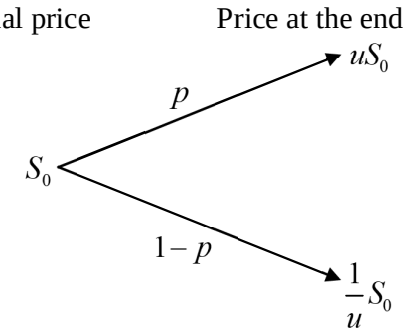
(d) The production manager claims that the sausage rolls produced by the machine are shorter than 10 cm on average. Test, at the 10% level of significance, whether the manager's claim is valid. You should state your hypotheses and define any symbols you use. [5]

After comments from customers, the production manager wishes to test that the mean length of the sausage rolls produced by the factory is in fact 10 cm. Another random sample of 80 sausage rolls is taken.

It is now known that the population variance is 5 cm^2.

Given that the production manager concludes that there is no reason to reject the null hypothesis at the 10% level of significance, find the range of possible values of the sample mean used in calculating the test statistics. [4]

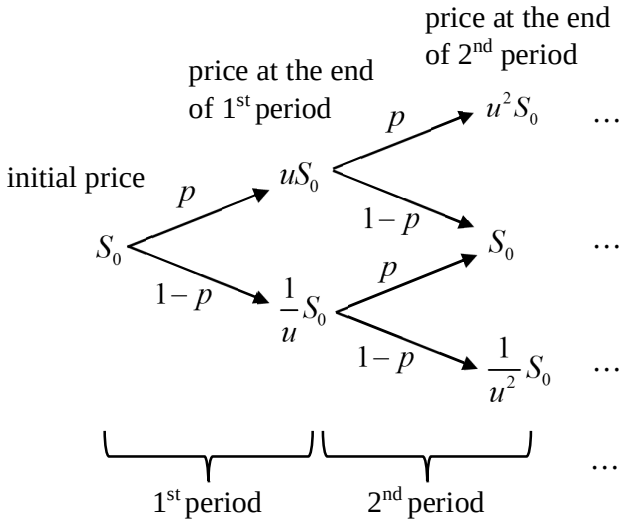
11 **(a)** In a one-period binomial model, the stock price initially at S_0, either rises to uS_0 with a probability p or drops to $\frac{1}{u}S_0$ with a probability $(1-p)$ at the end of one period, where $u>1$.



Suppose that $S_0=10$, $u=1.2$ and $p=0.55$.

Show that the expected value of the stock price S at the end of the period is 10.35 and find the variance of the stock price. [3]

(b) The above model can be extended to a n-period binomial model. For each period, the stock price S either rises to uS with a probability p or goes drops to $\frac{1}{u}S$ with a probability $1-p$ at the end of this period.



(i) In the first quarter, the initial price of the stock at the start of the first period is S_0. Suppose the stock price rises 4 times and drops twice over 6 periods. Find the stock price in terms of S_0 and u at the end of the 6 periods. [1]

(ii) Show that the probability for part **(b)(i)** to happen is $15p^4(1-p)^2$. [1]

(iii) In the second quarter, the initial price of the stock at the start of the first period is S_1. Find the probability that the stock price is higher than S_1 at the end of 6 periods in terms of p. [2]

(c) A *contract* that gives investors the right to buy a stock at a specified *price* at the end of a specified period is termed "call option". The specified price is called "strike price" and the amount needed to buy this contract is called "premium".

Each unit of a stock's call option has a return value V at the end of the period. Its value is given by

$$V = \begin{cases} S - S_{\text{strike}} & \text{if } S > S_{\text{strike}}, \\ 0 & \text{if } S \leq S_{\text{strike}}, \end{cases}$$

where S_{strike} is the strike price and S is the stock price at the end of the period.

Stock K, currently priced at \$10, is modelled by the n-period binomial model in part **(b)** with $n = 6$, $u = 1.2$ and $p = 0.55$. A call option of stock K has a strike price of \$10.

(i) Fill in the missing numbers in the following probability distribution table of V for this call option, giving the values to a suitable degree of accuracy. [2]

Number of times for the stock price to rise	3 or below	4	5	6
v	0		10.736	19.85984
$\mathrm{P}(V = v)$			0.13589	0.02768

The call option currently has a premium of \$2.50 per unit. Peter plans to invest \$10 000 on purchasing 4000 units of the call option and sell them at the end of the 6 periods.

(ii) Find the expected return value of the investment, giving your answer to the nearest dollar. [2]

(iii) Comment on whether Peter should proceed with this investment plan. [1]