Chapter 4: Graphing Techniques

Table of Content

Objectives:
At the end of the chapter, you should be able to

- Use Graphing Calculator to graph a given function
- Understand important characteristics of graphs such as symmetry, intersections with the axes, turning points and asymptotes of the following:
  $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$
  $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$; $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$
  $y=\frac{ax+b}{cx+d}$
  $y=\frac{ax^2+bx+c}{dx+e}$
- Determine the equations of asymptotes, axes of symmetry and restrictions on the possible values of x and/or y
- Understand and draw simple parametric equations and their graphs.

Prerequisite Knowledge:
You should be able to

- Perform long division,
- Able to complete the square for quadratic expressions,
- Apply differentiation techniques to find stationary points,
- Applying second derivative test to find nature of stationary points,
- Sketch the graphs of basic graphs such as linear function $(y=mx+c)$, quadratic function $(y=ax^2+bx+c)$ and their properties such as estimation of gradient, maximum and minimum points and symmetry,
- Sketch the graphs of exponential function $\left(y=ka^x\right)$ and logarithmic function $\left(y=\log_a x\right)$ where a is a positive integer.

1 Basic Features of Graphs

Learning points

Purpose of a Graph
A graph illustrates diagrammatically the relationship between two variables, usually denoted as x and y. This helps us to 'visualise' the relationship between x and y which can aid us in understanding the relationship between x and y.

In secondary mathematics, we learnt to plot the graph of a function. It is done by plotting points on a graph paper.

Unlike plotting a graph, in curve sketching we are not required to find all points of the function. Instead, we are required to show important features of the graph, such as

- Asymptotes (A)
- Intersection with axes (I)
- Stationary points (S)
- Symmetry

In particular, the first three are the more commonly seen and important basic features of a graph and you can remember them using the acronym "**A.I.S**".

1.1 Asymptotes (A)

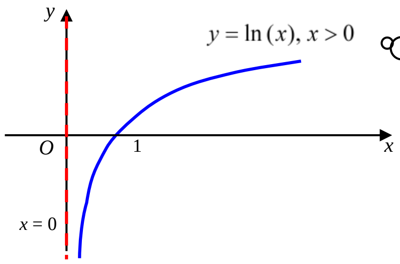
In our syllabus, we will learn three types of asymptotes, namely, vertical asymptote, horizontal asymptote and oblique asymptote. Asymptotes are usually drawn as **dotted lines**.

(a) Vertical Asymptote

In the graph of $y = \text{f}(x)$, if there exist a constant a such that $x \to a,\ y \to +\infty$ or as $x \to a,\ y \to -\infty$, then the line $x = a$ is a vertical asymptote of the graph.

Notation:
If x approaches a value a from the right-hand side but never reaching a, we write $x \to a^+$.
Similarly, if x approaches a value a from the left-hand side but never reaching a, we write $x \to a^-$.

For example, in the graph of $y = \ln x,\ x > 0$, the value of y tends to $-\infty$ when x gets closer and closer to 0. (In fact, x approaches to 0 from the right side, i.e. $x \to 0^+$). Then the line ___________ is a vertical asymptote of the graph $y = \ln x$.



Have you ever wondered why $y = \ln x$ does not touch the y – axis?

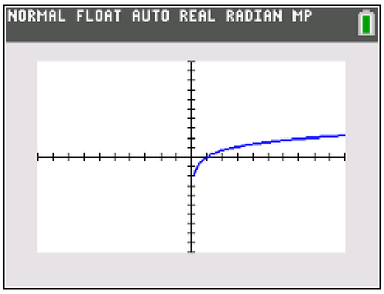
Note:

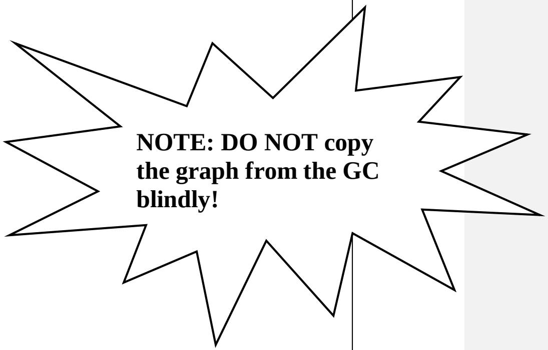
1. A graph does not touch its vertical asymptote(s) <u>at all times</u>.
2. GC **does not** indicate the presence of asymptotes.

   For example, the graph $y = \ln x$, obtained from GC is shown, appears as if the graph discontinues approximately at the point (0.15, – 2)!

   This is due to the limitation of GC.

Learning points ✍





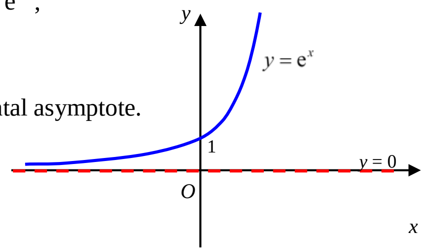
Example 1
Find the <u>equation of vertical asymptotes</u> (if any) of the following graphs with equation **(a)** $y = -\dfrac{2}{1+x}$ **(b)** $y = \dfrac{x^3}{x^2+5}$ **(c)** $y = \ln(1-2x)$

Solution:

(b) Horizontal Asymptote

In the graph of $y = \text{f}(x)$, if there exist a constant k such that $x \to +\infty, y \to k$ **or** as $x \to -\infty, y \to k$, then the line $y = k$ is a horizontal asymptote of the graph.

For example, for the graph of $y = \text{e}^x$,
As $x \to \infty, y = \text{e}^x \to \infty$,
but $x \to -\infty, y = \text{e}^x \to 0$.
The line ___________ is a horizontal asymptote.



Learning points

Remark:
A graph **may cut** across a horizontal asymptote. However, positive or negative infinity, the graph does not touch the horizontal asymptote.

Commented [YWJ1]: Changed from and to or

(c) Oblique Asymptote

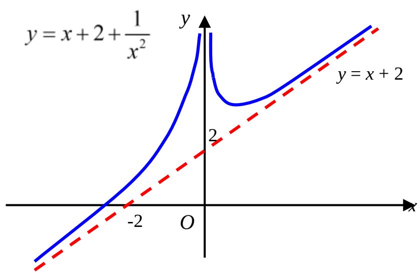
If the graph of $y = \text{f}(x)$ approaches the line $y = mx + c$ (where $m \neq 0$) as $x \to -\infty$ and $x \to \infty$, then the line $y = mx + c$ is an oblique asymptote for the curve $y = \text{f}(x)$.

Consider the graph of $y = x + 2 + \frac{1}{x^2}$, as the x-value gets larger and larger, the y-value seems to get closer and closer to a certain non-horizontal, non-vertical straight line.

As $x \to -\infty$,

As $x \to \infty$,

Therefore we say that _________________is an oblique asymptote.



Learning points ✍

1.2 Axial Intercepts (x and y intercepts) (I)

These are the point where the curve intersects x- and y-axes. Obtain the x- and y-intercepts by setting $y = 0$ and $x = 0$ respectively.

Example 2

Find the axial intercepts of $y = x^2 e^x - 1$.

Solution:

y-intercept: When $x = 0$, $y = -1$.

x-intercept:

Hence, the axial intercepts are

Self-learning activity:

See **ANNEX 1- AXIAL INTERCEPTS** (pg 35) to find out how to use GC to find x- and y-intercepts on a graph.

1.3 Stationary Points (S)

If $\frac{dy}{dx} = 0$ when $x = a$, then the stationary points occurs at $x = a$.

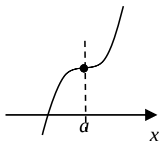
To determine the nature of the stationary points we can use the second derivative test.

Maximum point	Minimum point
$\left.\frac{d^2y}{dx^2}\right\|_{x=a} < 0$	$\left.\frac{d^2y}{dx^2}\right\|_{x=a} > 0$

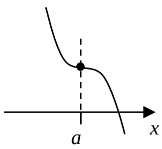
However, for stationary points of inflexion, $\left.\frac{d^2y}{dx^2}\right|_{x=a} = 0$

Therefore we can use the first derivative test.

Stationary point of inflexion



x	a^-	a	a^+
$\frac{dy}{dx}$	+	0	+
Sketch of tangent	/	—	/



x	a^-	a	a^+
$\frac{dy}{dx}$	–	0	–
Sketch of tangent	\	—	\

Example 3 (Worked)

Find the exact stationary point(s) of $y = x^2e^x - 1$, if any, and state the nature of the stationary point(s).

Solution

$$\frac{dy}{dx} = 2xe^x + x^2e^x = xe^x(2+x)$$

At stationary points, $\frac{dy}{dx} = 0 \quad \Rightarrow \quad xe^x(2+x) = 0$

$\Rightarrow\ x = 0 \quad$ or $\quad x = -2$

$\Rightarrow\ y = -1 \quad$ or $\quad y = 4e^{-2} - 1$

$$\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2e^x$$

$$= e^x(2 + 4x + x^2)$$

Method 1: First Derivative Test

x	-1	0	1
$\frac{dy}{dx}$		0	
Sketch of tangent			

Therefore $(0,\ -1)$ is a minimum point.

x	-3	-2	-1
$\frac{dy}{dx}$		0	
Sketch of tangent			

Therefore $\left(-2,\ 4e^{-2} - 1\right)$ is a maximum point.

Learning points ✍

Recall Product Rule

$$\frac{dy}{dx} = x^2\frac{d}{dx}\left(e^x\right) + e^x\frac{d}{dx}\left(x^2\right)$$
$$= x^2e^x + e^x\left(2x\right)$$

Question to self:
How do we test for the nature of stationary points?

To test for the nature of stationary points:

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=0} = \mathrm{e}^0(2+0+0)$$
$$= 2 > 0$$

Therefore $(0, \ -1)$ is a minimum point.

$$\left.\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right|_{x=-2} = \mathrm{e}^{-2}(2-8+4)$$
$$= -2\mathrm{e}^{-2} < 0$$

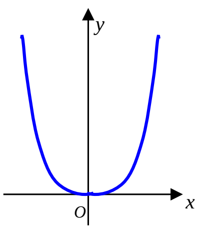
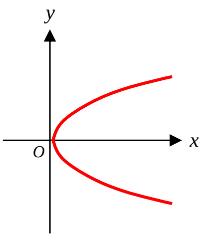
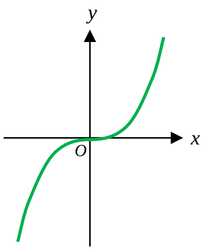
Therefore $\left(-2,\, 4\mathrm{e}^{-2}-1\right)$ is a maximum point.

So, which test should we use and why?

First derivative test is recommended when:

Self-learning activity:
See **ANNEX 1- STATIONARY POINTS** (pg 33) to find out how to use GC to find stationary points on a graph.

1.4 Symmetries

Axis/Point of Symmetry	***y*-axis**	***x*-axis**	**origin**
Algebraically	If (x, y) is a point on the graph, then $(-x, y)$ is also a point on the same graph.	If (x, y) is a point on the graph, then $(x, -y)$ is also a point on the same graph.	If (x, y) is a point on the graph, then $(-x, -y)$ is also a point on the same graph.
Graph	eg. $y = x^4$ 	eg. $y^2 = x$ 	eg. $y = x^3$ 

Learning points ✍

2 Graph of Rational Functions

A rational function is of the form $\mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where P(x) and Q(x) are **polynomials**[1] with Q(x) ≠ 0.

Expressions like $\dfrac{x-1}{x+2}$, $\dfrac{x-1}{x^2+4}$ and $\dfrac{x^2+1}{2x^3-2x+3}$ are rational functions.

In general, if there is a vertical asymptote for the graph of a rational function, there will be a solution for x for the equation $Q(x)=0$.

2.1 If degree of P(x) < degree of Q(x)

Graph of the form $y=\dfrac{a}{bx+c}$, $a\neq 0, b\neq 0, c\neq 0$.

Example 4

Sketch $y=-\dfrac{2}{1+x}$, giving the equations of asymptotes, exact coordinates of any points of intersection with the axes.

Solution:

Asymptotes:

Vertical asymptote [From **Eg 1**]: As $x+1\to 0$, $y\to\infty$ or $y\to-\infty$.

Let $x+1=0\Rightarrow x=-1$

Hence, $x=-1$ is the **vertical** asymptote.

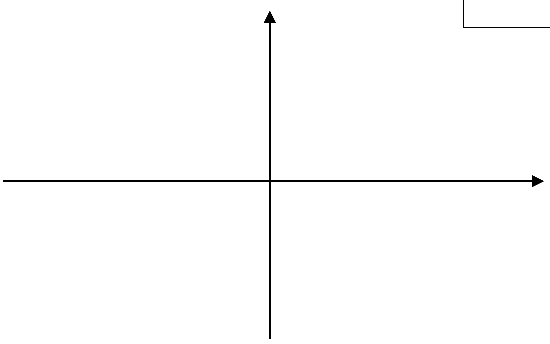
Horizontal asymptote :

How to prove there are NO stationary (turning) points?

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2}{(1+x)^2}\neq 0,\ \forall x\in\mathbb{R}$

Hence, there are no turning points.

Intercepts:



[1] A **polynomial function** is a **function** of the form: $f(x) = a_0 + a_1x^1 + a_2x^2 + ... + a_nx^n$
Example: linear, quadratic, cubic, quartic functions are polynomial functions.

Learning points

2.2 If degree of P(x) = degree of Q(x)

Graph of the form $y=\dfrac{ax+b}{cx+d}$, $a\neq 0, c\neq 0, ad-bc\neq 0$

Example 5

Sketch $y=\dfrac{x}{1+x}$, giving the equations of asymptotes, exact coordinates of points of intersection with the axes and turning points (if any).

Solution:

$y=\dfrac{x}{1+x}=1-\dfrac{1}{1+x}$ (by long division. Why?)

Asymptotes:

Vertical asymptote : As $x+1\to 0$, $y\to\infty$ or $y\to-\infty$.

Let

Horizontal asymptote:

Intercepts with axes:

Stationary (turning) points:

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{1}{(1+x)^2}$

Using GC to obtain the sketch:

2.3 If degree of P(x) > degree of Q(x)

Learning points ✍

Graph of the form $y=\dfrac{ax^2+bx+c}{dx+e}$, $a\neq 0,\ d\neq 0$

Example 6

Sketch $y=\dfrac{x^2+4x+3}{x+2}$, giving the equations of asymptotes and exact coordinates of any points of intersection with the axes.
Determine if there are any turning points for the graph.

Solution:

By long division, $y=\dfrac{x^2+4x+3}{x+2}=x+2-\dfrac{1}{x+2}$ (why long division?)

Asymptotes:

Vertical asymptote :

Oblique asymptote :

Intercepts:

Question: What does the derivative tell you about the graph?

Stationary points:

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=1+\dfrac{1}{(x+2)^2}$

Learning points ✍

In summary:

	Type of Rational Function	Asymptote
Case 1	If degree of P(x) < degree of Q(x) $\Rightarrow \mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is a proper fraction	Horizontal asymptote: $y = 0$
Case 2	If degree of P(x) $\geq$ degree of Q(x)	
	$\Rightarrow \mathrm{f}(x)=\dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is an improper fraction. Divide P(x) by Q(x) to obtain $\mathrm{f}(x)= ax+b+\dfrac{\mathrm{R}(x)}{\mathrm{Q}(x)}$	If $a=0$, there is an horizontal asymptote $y = b$. If $a \neq 0$, there is an oblique asymptote $y = ax+b$.

Note: In all cases, check if there is a solution for x in $Q(x) = 0$. If there is, it means that there exists a vertical asymptote for the graph of $y = \mathrm{f}(x)$.

<u>Drill Practice 1</u>

Use long division to simplify the following rational functions:

(i) $\dfrac{27x^3+9x^2-3x-9}{3x-2}$ (ii) $\dfrac{3x^3+4x+11}{x^2-3x+2}$

Example 7

Consider the graph $y=\dfrac{2x}{x^2+1}$.

(i) Find the stationary points of the graph analytically.

(ii) Sketch the graph. giving the equations of asymptotes and coordinates of any points of intersection with the axes.

(iii) Comment on the symmetry of the curve.

Note to lecturers:
Please highlight to students that analytically means to make use of calculus as the approach.

Solution:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2(x^2+1)-2x(2x)}{(x^2+1)^2}$.

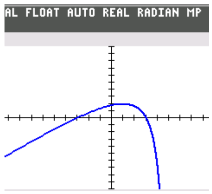
Example 8

The graph C with equation $y=\dfrac{x^2-a}{x-b}$ has an asymptote $x = 5$ and stationary point (1,2). Find the values of a and b. Give the equations of asymptote(s) and coordinates of turning point(s). Sketch the graph C.
Hence, determine the possible values that y can take.

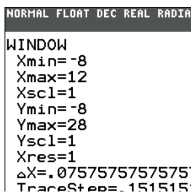
Solution:

Learning points ✍

When using 6:Zstandard to obtain the graph, GC only shows one part of the graph.



You need to adjust window setting to capture the entire graph!



Learning points ✍

Sometimes, the question will require you to find the range of the graph, which essentially are all the possible values of y which the graph admits as a possible value. The idea of range will be discussed in greater depth in **Chapter 5: Functions.**

Example 9 (Algrabraic Approach to Example 8)

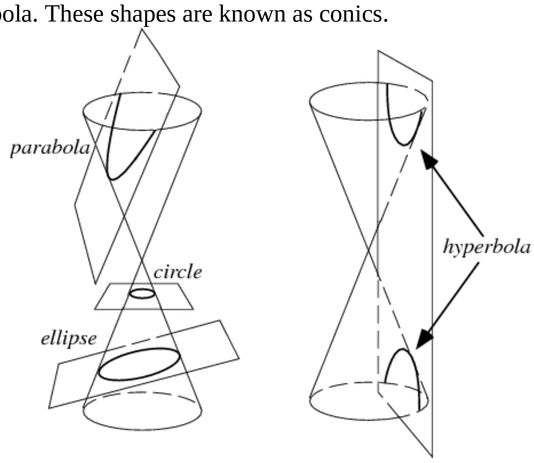
Find, using an algebraic method, the range of the graph $y = \dfrac{x^2 - 9}{x - 5}$.

Solution:

Suppose there is a horizontal line $y = k$.

3 Conic Sections

A conic section is the intersection of a plane and a cone as shown in the diagram below. This creates the shapes parabola, circle, ellipse and hyperbola. These shapes are known as conics.



Applet on conics: https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG

3.1 Parabola

Learning points ✍

The equation of a parabola with vertex (0, 0) is as follows.

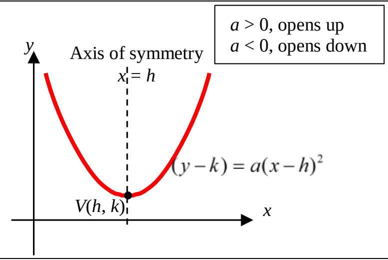
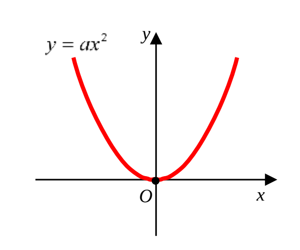
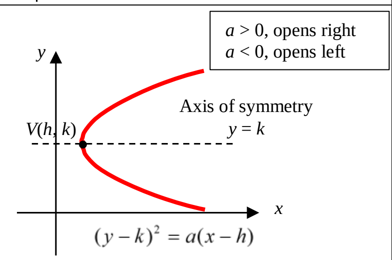
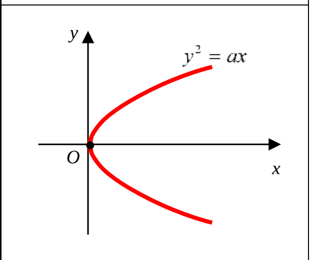
$y = ax^2$ axis of symmetry: y – axis

$y^2 = ax$ axis of symmetry: x – axis

The standard form of the equation of a parabola with vertex (h, k) is as follows.

$(y-k) = a(x-h)^2,\quad a \neq 0,$ axis of symmetry: $x = h$

$(y-k)^2 = a(x-h),\quad a \neq 0,$ axis of symmetry: $y = k$

	Vertex (*h*, *k*)	**Vertex (0, 0)**
Vertical Axis		
Horizontal Axis		

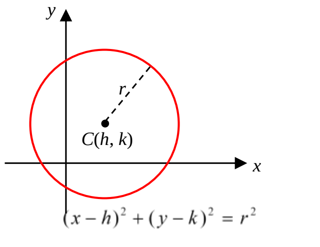
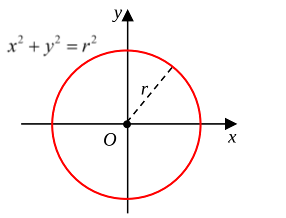
Example 10 Sketch the graph of $(y-2)^2 = 4(x+1)$ and state the axis of symmetry

Solution:

Learning points ✍

3.2 Circle

Recall from O levels Mathematics, the equation of the circle is $\left(x-h\right)^2+\left(y-k\right)^2=r^2$, centered at ($h$, k) with radius r. In particular, the equation of the circle is $x^2+y^2=r^2$ centered at (0, 0) with radius r.

Centre (h, k)	**Centre (0, 0)**
	

Example 11

Sketch the graphs of $x^2+y^2-2x+4y=4$.

Solution

$$x^2+y^2-2x+4y=4$$
$$(x^2-2x)+(y^2+4y)=4$$

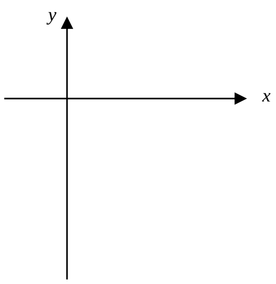
(In this example, we need to complete the square for x^2 *and* y^2 *to express in the form* $\left(x-h\right)^2+\left(y-k\right)^2=r^2$ *)*

Completing the squares, we have

$$(x-1)^2-1+(y+2)^2-4=4$$

Converting to standard form:

$$(x-1)^2+(y+2)^2=3^2$$



Learning points ✍

3.3 Ellipse

The standard form of the equation of an ellipse with centre (h, k) and major and minor axes of lengths 2a and 2b respectively, is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad , \quad a > b, \qquad \text{Major axis is horizontal.}$$

The major and minor axes of lengths 2b and 2a respectively is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1 \quad , \quad b > a, \qquad \text{Major axis is vertical.}$$

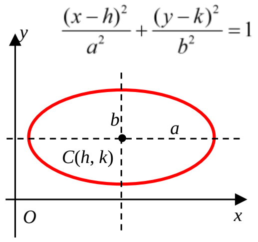
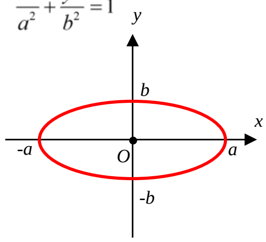
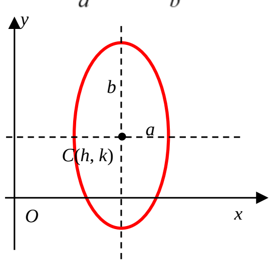
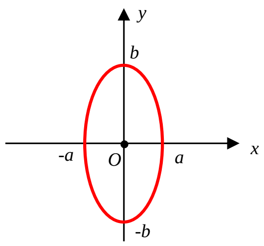
These ellipse are symmetrical about the lines $x=h$ and $y=k$.

When the centre of the ellipse is at the origin (0, 0), the equation of the ellipse takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a > b, \qquad \text{Major axis is horizontal}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad b > a, \qquad \text{Major axis is vertical}$$

These ellipses are symmetrical about the x and y axes.

	Centre (h, k)	**Centre (0, 0)**
$a > b$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 
$a < b$	$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$ 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ 

Learning points

Example 12

Sketch the graph of $4x^2 + y^2 - 6y - 16 = 0$.

Solution:

$4x^2 + y^2 - 6y - 16 = 0$

$4x^2 + (y^2 - 6y) - 16 = 0$

(Similarly, we need to complete the square for x^2 *and* y^2 *to express in the form* $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ *)*

Note: You can also use the GC to sketch this graphs in Example 11 and 12. Refer to **Examples 19** and **20** in **Section 6 Self-Reading Examples.**

Learning points

3.4 Hyperbola

The standard form of the equation of a hyperbola with centre (h, k) is of the form:

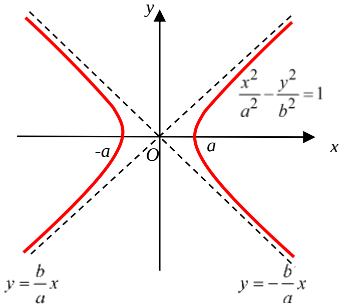
$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad ,$$

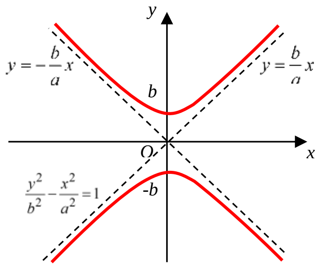
$$\frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1 \quad ,$$

These hyperbolas are symmetrical about the lines $x=h$ and $y=k$.
When the centre of the hyperbola is at the origin (0, 0), the equation of takes the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad \frac{y^2}{b^2}-\frac{x^2}{a^2}=1,$$

These hyperbola are symmetrical about the x and y axes.





From the diagram, we can see that each hyperbola has two asymptotes that intersect at the centre of the hyperbola.

To obtain the equation of these asymptotes, we arrange the equation such that

$$\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$$

$$\frac{y^2}{b^2}=\frac{x^2}{a^2}+1$$

As $x\to\pm\infty$, $\frac{x^2}{a^2}+1\to\frac{x^2}{a^2}$ (since 1 becomes negligible.)

Therefore,

$$\frac{y^2}{b^2}\to\frac{x^2}{a^2}$$

$$y^2\to\frac{b^2}{a^2}x^2$$

Thus we have $y\to\pm\frac{b}{a}x$ as $x\to\pm\infty$ and we get the lines $y=\pm\frac{b}{a}x$ as the asymptotes of the graphs.

Learning points ✍

Example 13

Sketch the graph of $\frac{x^2}{9}-\frac{y^2}{4}=1$ and state the equations of the asymptotes, if any.

Solution:

See **Example 15 (page 21**) for steps to using GC to sketch the hyperbola, but we need to find the oblique asymptotes for the hyperbola.

$\frac{x^2}{9}-\frac{y^2}{4}=1 \quad \Leftrightarrow \quad \frac{x^2}{3^2}-\frac{y^2}{2^2}=1$

Example 14 (non standard)

Express $2x^2+4x-y^2+6y=9$ in the form $\frac{(x-c)^2}{a^2}-\frac{(y-d)^2}{b^2}=1$, where *a*, *b*, *c*, *d* are integers to be determined. Hence, determine the asymptotes of this graph.

Solution:

$(2x^2+4x)-(y^2-6y)=9$

(*Similarly, we need to complete the square for* x^2 *and* y^2 *to express in the form* $\frac{(x-c)^2}{a^2}-\frac{(y-d)^2}{b^2}=1$)

$(2x^2+4x)-(y^2-6y)=9$

$(x+1)^2-\frac{(y-3)^2}{2}=1$

$(y-3)^2=2(x+1)^2-2$

Drill Questions 2

Identify and sketch the curves represented by the following equations.

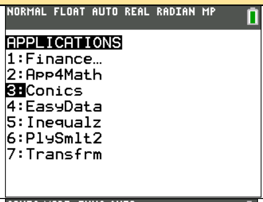
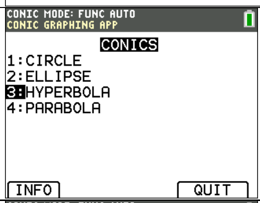
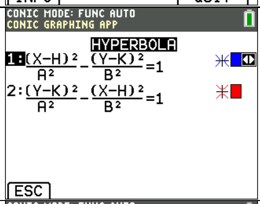
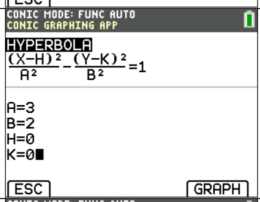
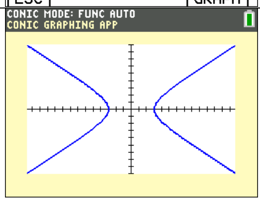
(a) $x^2 - 7x + y^2 + 6 = 0$ (b) $y = \sqrt{4 - x^2}$

(c) $(12x)^2 + (5y)^2 = 13^2$ (d) $y = \dfrac{x+1}{x+2}$

3.5 Conics App in GC

Limitation of the CONICS Application: It can give us the shape of the conic but it cannot be used to find the intersection points with another graph. [Refer to **Example 23** on how to find the intersection of a conic with another graph]

Example 15 Sketch the graph $\dfrac{x^2}{3^2} - \dfrac{y^2}{2^2} = 1$

Steps	Screenshot	Remarks
Press [APPS] Select the Conics mode press [ENTER]		
Press [4] to select HYPERBOLA Press [ENTER]		
Press [1]		
Key in the appropriate values for *A*, *B*, *H* and *K*. In this case $A = 3$, $B = 2$, $H = 0$, $K = 0$ Press [GRAPH]		
		This is the correct graph of the hyperbola Centre of hyperbola = $(0,0)$ Asymptotes: $y = \pm\dfrac{2}{3}x$ *x*-intecepts are (3, 0) , (-3, 0) [Note that the asymptotes are not shown on the GC.]

Learning points ✍

4 Parametric Equations
4.1 Graph of Parametric Equations

There are two forms of equations for curves:
Cartesian equations and parametric equations.

Cartesian equations are equations expressed in term of x and y.
For examples, $y = 2x^2$, $y = x^3 + e^{x-1}$, $y = \ln(x-1)$ are Cartesian equations.

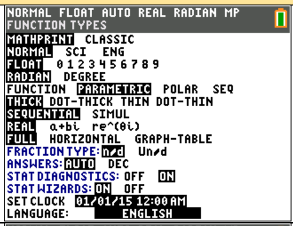
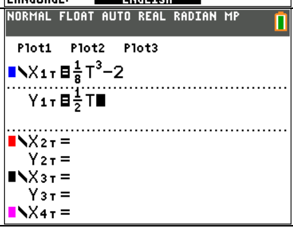
In parametric equations, x and y are expressed separately in terms of a third variable, for example, t, which is called a parameter.
For example, $x = t,\ y = t^2,\ t > -1$ is a pair of parametric equation.

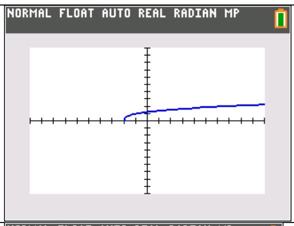
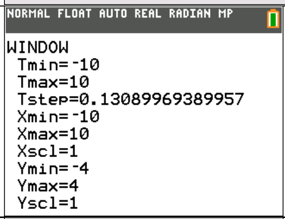
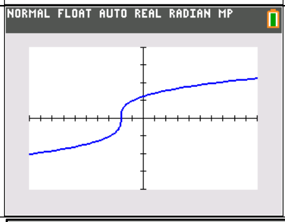
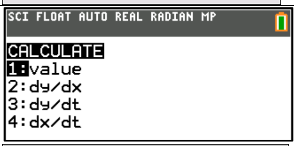
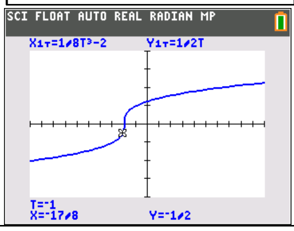
The coordinates of a point on the curve defined by the parametric equations $x = t,\ y = t^2,\ t > -1$ is of the form $\left(t, t^2\right)$.

t	$x = t$	$y = t^2$	$(x, y) = (t,\ t^2)$
1	1	1	(1, 1)
2	2	4	(2, 4)
3	3	9	(3, 9)
4	4	16	(4, 16)
5	5	25	(5, 25)
⋮			

Example 16

Use the graphic calculator to sketch a parametric curve $x = \frac{1}{8}t^3 - 2$, $y = \frac{1}{2}t$, where $t \in \mathbb{R}$. Find the exact axial intercepts and label its coordinates on the curve.

Steps	Screenshot	Remarks
1. Press [MODE] Select the parametric mode by toggling cursor to PARAMETRIC then press [ENTER]		
2. Press [Y=] to access the parametric equation input window. The key in the parametric equations.		

3. Press [ZOOM][6] or [GRAPH]		The graph on the left is for $t > 0$ and not $t \in \mathbb{R}$. This is because in the default setting the smallest value of the parameter t is '0' . i.e. $T_{min} = 0$.
4. Press [WINDOW] Change window setting		As $t \in \mathbb{R}$, we ought to change T_{min} to some number which is a small negative number, e.g. $T_{min} = -10$.
5. Press [ZOOM][2]		This is the correct graph of $x = \frac{1}{8}t^3 - 2$ and $y = \frac{1}{2}t$ where $t \in \mathbb{R}$.
6. Press [2nd][CALC], press [1] Enter a value of t , e,g $t = -1$. Press [ENTER].	 	Instead of using the arrow key to move the cursor until it reaches the particular value of t, .e.g. $t = -1$, we can enter the value of our choice.

GC limitation: Although the GC can give us the general shape of the graph, it is unable to give us the exact axes intercepts of the graph and we need to calculate it manually!

x-intercepts $\Rightarrow y = 0$:

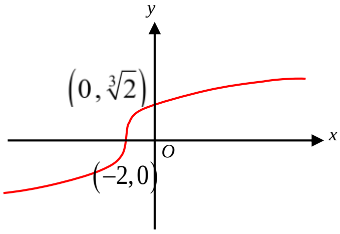
Set $y = \frac{1}{2}t \Rightarrow t = 0$.

Then substitute $t = 0$ into $x = \frac{1}{8}(0)^3 - 2 = -2$.

y-intercepts $\Rightarrow x = 0$:

Set $x = \frac{1}{8}t^3 - 2 \Rightarrow \frac{1}{8}t^3 = 2 \Rightarrow t^3 = 16 \Rightarrow t = 2\sqrt[3]{2}$

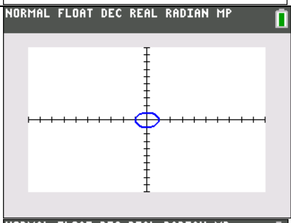
Then substitute $t = 2\sqrt[3]{2}$ into $y = \frac{1}{2}\left(2\sqrt[3]{2}\right) = \sqrt[3]{2}$.

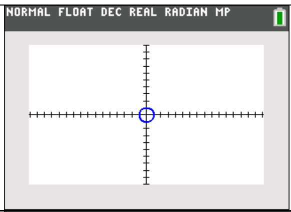
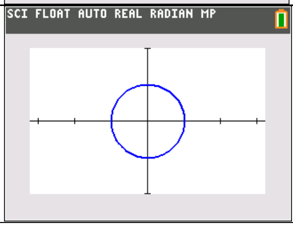


Example 17

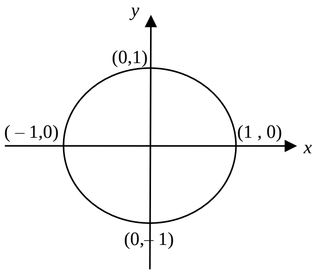
Sketch the curve defined by the parametric equations $x = \cos\theta, y = \sin\theta$ for $-\pi < \theta \le \pi$, labelling the coordinates of all axial intercepts and asymptotes if any.

Finding y intercepts:	Finding the x intercepts:

Steps	Screenshot	Remarks
Press [Y=] to access the parametric equation input window. The key in the parametric equations.	NORMAL FLOAT DEC REAL RADIAN MP Plot1 Plot2 Plot3 \X1T=cos(T) Y1T=sin(T) \X2T= Y2T= \X3T= Y3T= \X4T= Y4T= \X5T=	
Press [WINDOW] to change the values of t: $T_{min} = -\pi$ and $T_{max} = \pi$.	NORMAL FLOAT DEC REAL RADIAN MP WINDOW Tmin=-3.141592654 Tmax=3.141592654 Tstep=.13089969389957 Xmin=-10 Xmax=10 Xscl=1 Ymin=-10 Ymax=10 Yscl=1	
Press [GRAPH]	 NORMAL FLOAT DEC REAL RADIAN MP ZOOM MEMORY 1:ZBox 2:Zoom In 3:Zoom Out 4:ZDecimal 5:ZSquare 6:ZStandard 7:ZTrig 8:ZInteger 9↓ZoomStat	Question to ponder: Does this necessarily mean that $x = \cos\theta, y = \sin\theta$ has the shape of an ellipse? Answer: No, why? By default, the graphic calculator scale for x and y axes are not the same. In order to check whether the shape is an ellipse or circle, we need to change the axes scale to be the same. This can be done by pressing [ZOOM][5] to select ZSquare.

		As we can see from the previous graph, the *y*-values are very small, we can change the value of X_{min}, X_{max}, Y_{min} and Y_{max} to enlarge the graph. Alternatively, press [ZOOM][2][ENTER]
Press [ZOOM][2]		This is the correct graph of $x = \cos\theta$ and $y = \sin\theta$ where $-\pi < \theta \le \pi$.

This is how the final sketch of the graph looks like:



Learning points

Note:
When using a graphic calculator to sketch a parametric curve,

(i) if the domain of *t* is not specified, we take it to be $t \in \mathbb{R}$.

(ii) ensure that the scale is the same for the *x* and *y* axes for a circle and an ellipse. [If not, circles may appear to look like ellipses or vice versa].

Drill Questions 3

Sketch the following parametric equations, indicating the axes intercepts on the graphs.

(a) $x = 2\cos\theta, y = \sin\theta$ for $0 \le \theta \le \pi$.

(b) $x = t^2, y = 2t$ for $t \in \Re$

(c) $x = \cos^2 t, y = \sin^3 t$ for $0 \le t \le \frac{\pi}{2}$

Learning points ✍

4.2 Converting Parametric Equations to Cartesian Equations

To convert parametric equations to a Cartesian equation (equation involving only *x* and *y*), elimination of parameter *t* is necessary. The common method used are:

(a) Making the parameter *t* the subject from one of the equations and substituting into *t* into the other equation
(b) Using trigonometry identities, e.g. $\sin^2\theta+\cos^2\theta=1$.
(c) Using some algebraic approach.

Note: It is not always possible to convert parametric equations to Cartesian equations.

Example 18
Convert the following pairs of parametric equations to Cartesian Equation:

(a) $x=3t-1,\quad y=\frac{3}{t}$;

(b) $x=3\cos\theta-4, y=\sin\theta+2$.

(c) $x=t+\frac{1}{t}, y=t-\frac{1}{t}$

Solution:

(a) Given:

$$\left.\begin{aligned} x=3t-1 \quad &\Rightarrow \quad t=\frac{x+1}{3}\ ...(1) \\ y=\frac{3}{t} \quad &\Rightarrow \quad t=\frac{3}{y}\ ....(2) \end{aligned}\right\}$$ Making the parameter, *t* the subject

Equating (1) & (2),

$$\left.\frac{3}{y}=\frac{(x+1)}{3}\right\}$$ Elimination of *t*

The Cartesian equation is $y=\frac{9}{x+1}$.

(b) Given:

$$x=3\cos\theta-4 \quad \Rightarrow \quad \cos\theta=\frac{x+4}{3}\cdots(1)$$

$$y=\sin\theta+2 \quad \Rightarrow \quad \sin\theta=y-2\cdots(2)$$

(c)

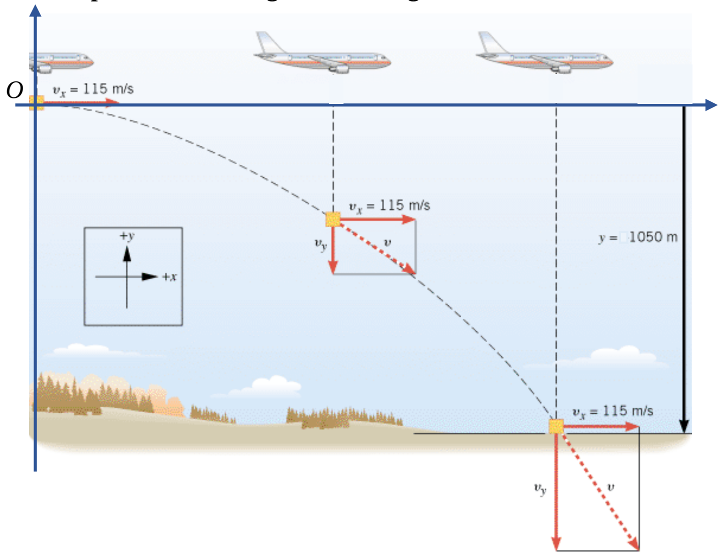
Learning points ✍

Learning points ✍

5 Modelling the Projectile Motion using Parametric Equations

Aim: To show how parametric equations arise naturally from real-world situations, model motion of objects encountered in the real-world and interpret the parametric equations in context.

Example 19 : A Falling Care Package



The figure above shows an airplane moving horizontally releases a 'care package' to the ground along a curved trajectory. The constant velocity of the airplane is 115 ms^{-1} at an altitude of 1050 m. Ignoring air resistance and assuming the acceleration due to gravity is 9.8 ms^{-2},

(i) What coordinates system is used in this scenarios?

(ii) Express the position of the care package t seconds after being dropped from the plane in the form of parametric equations.

(iii) How long will the 'care package' take to reach the ground?

Solution:

(i) We use the Cartesian coordinates system (i.e. x-y plane) by taking the 'care packing' dropping point from the plane as the origin, the positive x-direction as rightwards and positive y-direction as upwards.

(ii) Let (x, y) be the position of the care package at time t. Along the horizontal direction, the object moves with constant velocity 115 ms^{-1}

$\therefore$ The horizontal distance at time t is

Learning points ✍

In the vertical motion, due to gravity, the object undergoes an acceleration of – 9.8 ms^{-2},

Using the formula $s = ut + \frac{1}{2}at^2$, where u = initial vertical velocity and a = acceleration = – 9.8 ms^{-2}

The vertical distance at time t is

The negative sign indicates that the object is below the x-axis.

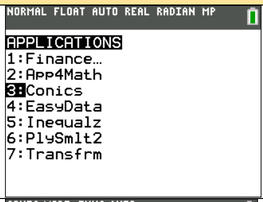
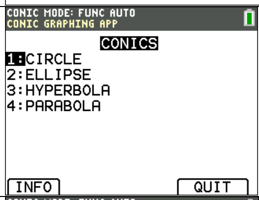
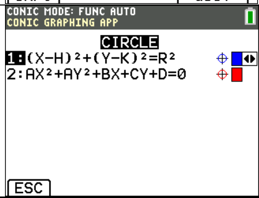
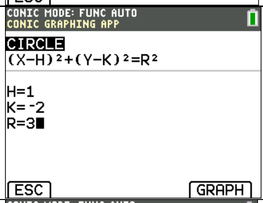
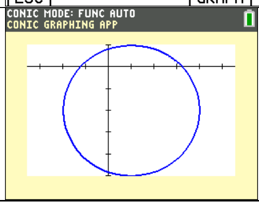
Hence the position of the care package at time t is (x, y) where

(iii) As the plane is at the height 1050 m above the ground, so the object at time t is at a height $h = 1050 - 4.9t^2$ above the ground. When the care package reaches the ground, $h = 0$

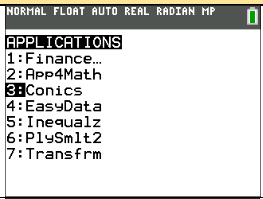
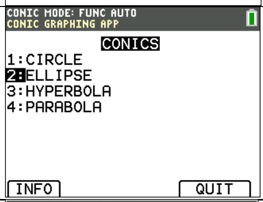
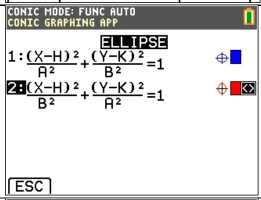
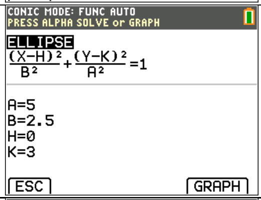
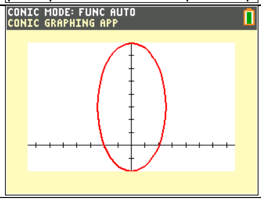
6 Self-Reading Examples

Example 20 Sketch the graph of $(x-1)^2+(y+2)^2=9$.

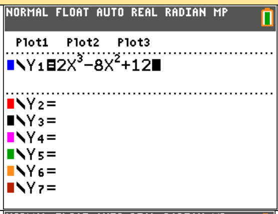
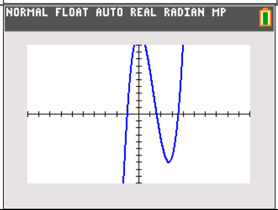
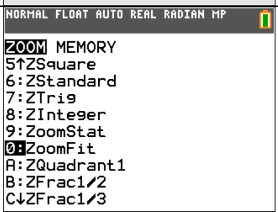
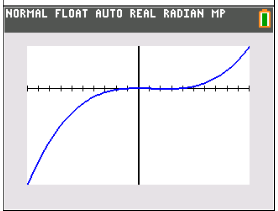
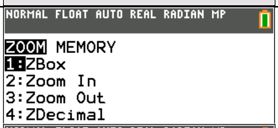
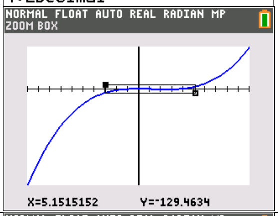
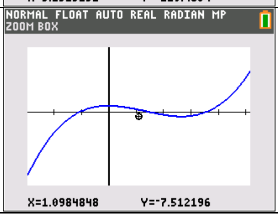
Write the equation in standard form : $(x-1)^2+(y-(-2))^2=3^2$

Steps	Screenshot	Remarks
Press [APPS] Select the Conics mode press [ENTER]		
Press [1] to select CIRCLE		
Press [ENTER]		
Key in the appropriate values. In this case, H = 1, K = -2, R = 3		
Press [GRAPH]		This is the correct graph of the circle with Centre = $(1,-2)$ Radius = 3

Example 201 Sketch the graph of $\dfrac{x^2}{\left(\frac{5}{2}\right)^2}+\dfrac{(y-3)^2}{5^2}=1$

Steps	Screenshot	Remarks
Press [APPS] Select the Conics mode press [ENTER]		
Press [2] to select ELLIPSE Press [ENTER]		
Press [2]		If A > B, select [1].
Key in the appropriate values for *A*, *B*, *H* and *K*. In this case $A=5$, $B=\frac{5}{2}$, $H=0$, $K=3$ Press [GRAPH]		
		This is the correct graph of the ellipse. Centre of ellipse = $(0,3)$ Horizontal Radius = 5/2 = 2.5 Vertical Radius = 5

Example 22 Sketch the graph of $y = 2x^3 - 8x^2 + 12$.

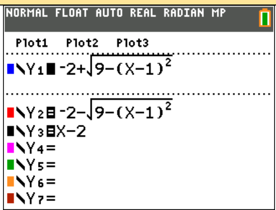
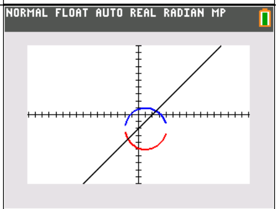
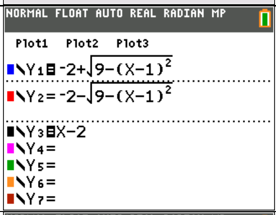
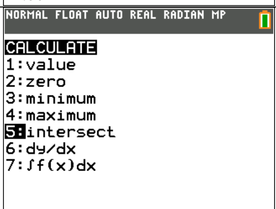
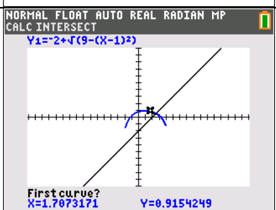
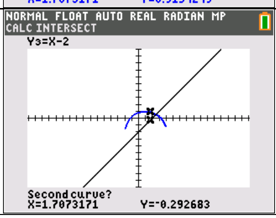
Steps	Screenshot	Notes
Press [Y=] and enter the expression into Y_1 and press [GRAPH]		
Press [ZOOM][6]		Notice that the cubic graph seems incomplete. Therefore, students need to learn how to adjust the window themselves in order to see the complete graph.
Press [ZOOM][0] Using ZoomFit GC to automatically		Using ZoomFit, GC automatically sets the window setting for you. Notice that some characteristics/shape of the graph is lost when you use this feature (ZoomFit) and so this feature may not be so useful in some scenarios such as in this case!
Press [ZOOM][1] To set a "box" on the screen, press [ENTER] to start a point. You will see a '+'. Use the arrow keys by first moving the cursor from left to right and then down to up to create the box.		Using **ZBox**, we can view a section of the graph that we want to focus on.
Press [ENTER] to view the 'cut' section.		Now the turning points of the curve can be seen clearly. You can use GC to find the coordinates of the turning points and axial intercepts.

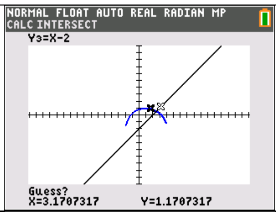
Example 23

Find the intersection between the circle $(x-1)^2+(y+2)^2=9$ and the line $y=x-2$.

We cannot use the Conics App here to sketch the circle, this is because in the Conics App, we cannot find the point of intersection.

First, we make y the subject of the equation: $y=-2\pm\sqrt{9-(x-1)^2}$.

Steps	Screenshot	Notes
Press [Y=] and enter the expression $-2+\sqrt{9-(x-1)^2}$ into Y_1, $-2-\sqrt{9-(x-1)^2}$ into Y_2, and $x-2$ into Y_3.		
Press [ZOOM][5]		Notice that there is a gap between two parts of the circle, this is due to limitation of GC.
To find the point of intersection between the upper circle and the line, De-select Y_2 by simply move the cursor on Y_2 then press [ENTER].		
Press [2nd][CALC] select [5]		
		
		

<table>
<tr><td></td><td>
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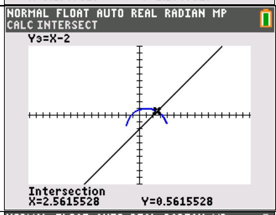
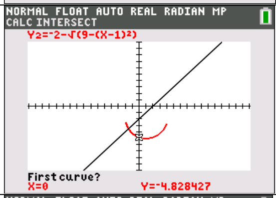
<tr><td></td><td>
</td><td>Repeat the steps to find the intersection between Y_2 and Y_3</td></tr>
<tr><td></td><td>
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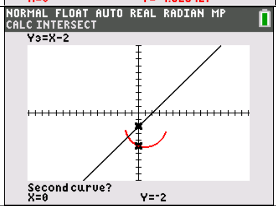
<tr><td></td><td>
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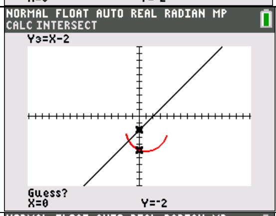
<tr><td></td><td>
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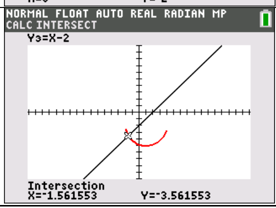
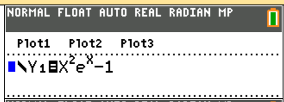
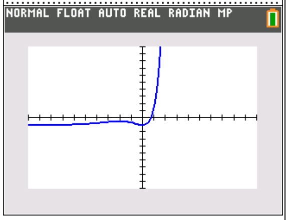
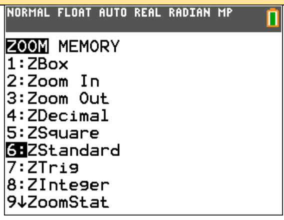
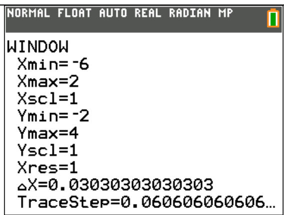
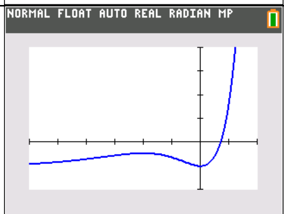
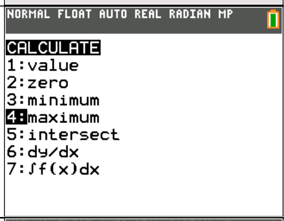
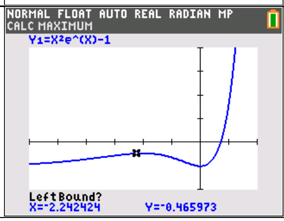
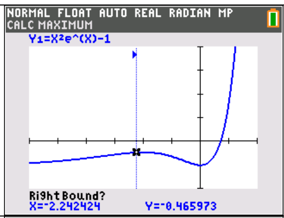
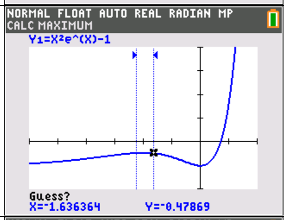
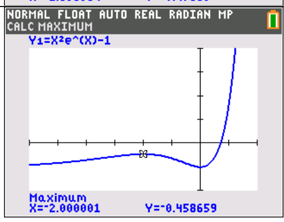
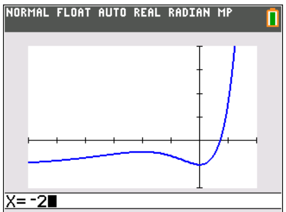
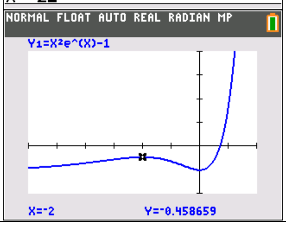
<tr><td></td><td>
</td><td>The points of intersection are $(2.56, 0.562)$ and $(-1.56, -3.56)$.</td></tr>
</table>

ANNEX 1 – STATIONARY POINTS:

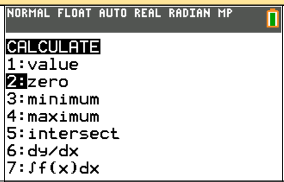
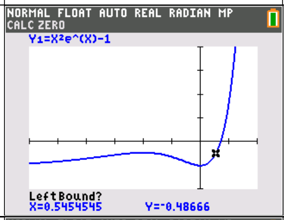
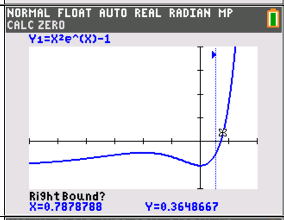
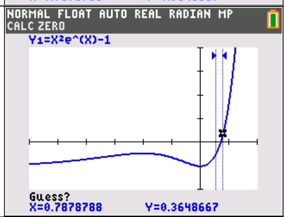
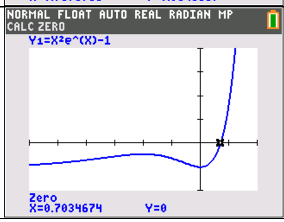
Using GC to find the maximum point (Example 3)

Steps	Screenshot	Notes
1. Press [Y=] and enter the expression $x^2e^x - 1$ and press [GRAPH] or [ZOOM][6]		 The standard window settings shows the graph for $-10 \le x \le 10$ and $-10 \le y \le 10$.
2. Press [WINDOW] to change window setting Press [GRAPH]		
3. Press [GRAPH]		
4. Press [2nd][CALC] Select 4: maximum or simply press [4] Press [ENTER]		
5. Respond to "Left bound?", move the cursor to the left of the maximum point and press [ENTER]		

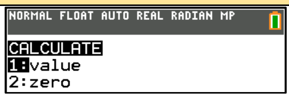
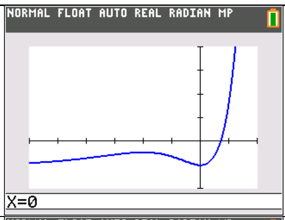
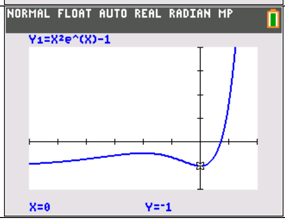
6. Respond to "Right bound?", move the cursor to the right of the maximum point and press [ENTER]		
7. When prompted "Guess?", press [ENTER]		
8. The x-value and the y-value of the maximum point is shown on the screen.		The maximum point that we found from the analytical method is $\left(-2,\ 4e^{-2}-1\right)$. Notice that sometimes GC does not give the exact value. In this case, repeat the steps 3 – 6 till the screen "Guess?", instead of pressing [ENTER] , input $x = -2$ followed by [ENTER], then you will see the screen below which confirms the fact that the maximum point indeed occurs when $x = -2$.  

ANNEX 1 – AXIAL INTERCEPTS:

To find x - intercepts:

Steps	Screenshot	Notes
1. Press [2nd][CALC] Select 2: zero or simply press [2]		
2. Respond to "Left bound?", move the cursor to the left of the x-intercept and press [ENTER]		
3. Respond to "Right bound?", move the cursor to the right of the x-intercept and press [ENTER]		
4. When prompted "Guess?", press [ENTER]		
5. The x-intercept is $x = 0.7034674$.		

To find *y*-intercept:

Steps	Screenshot	Notes
1. Press [2nd][CALC] Select 1: value Press [ENTER]		
2. Enter the value $x = 0$. Press [ENTER]		
3. The *y*-intercept is (0, – 1)		

Graphing Techniques – A Checklist of Concepts

- ☐ I understand the characteristics of graphs such as Stationary points (**S**), intersections with the axes (**I**) and asymptotes (**A**). [Refer to **Section 1**]
- ☐ I am able to determine equations of asymptote (horizontal, vertical and oblique), axes of symmetry and restrictions of possible values of *x*/or *y* of a graph; [Refer to **Examples 4, 5 and 6**]
- ☐ I know how to use a graphic calculator to draw a given function, locating the turning points, axial intercepts, and adjusting the 'window setting' that would display the essential features of the functions [Refer to **Example 8** and **Annex 1**]
- ☐ I am able to sketch the graphs of rational functions such as

  $y=\dfrac{ax+b}{cx+d}$ and $y=\dfrac{ax^2+bx+c}{dx+e}$ [Refer to **Example 5** and **6**]
- ☐ I can recognize different types of conics such as circle, ellipse and hyperbola and sketch their graphs with or without the use of a graphing calculator; [Refer to **Section 3.1 to 3.4**]
- ☐ I understand the concept of parametric equations and sketch their graphs [Refer to **Examples 16** and **17**]
- ☐ I can convert parametric equations into Cartesian equations. [Refer to **Example 18**]

Relevant websites

1) Applet on conics: https://www.geogebra.org/m/GmTngth7#material/T8TV2JqG
2) Applet on rational function:

   https://www.geogebra.org/m/Fsnt4mRk

   https://www.geogebra.org/m/naKtmybb

   https://www.geogebra.org/m/dFP5VqR7
3) Applet on graphs of parametric equations: https://www.geogebra.org/m/z3sh6xSE
4) Applet on graphs of various functions: http://www.analyzemath.com/