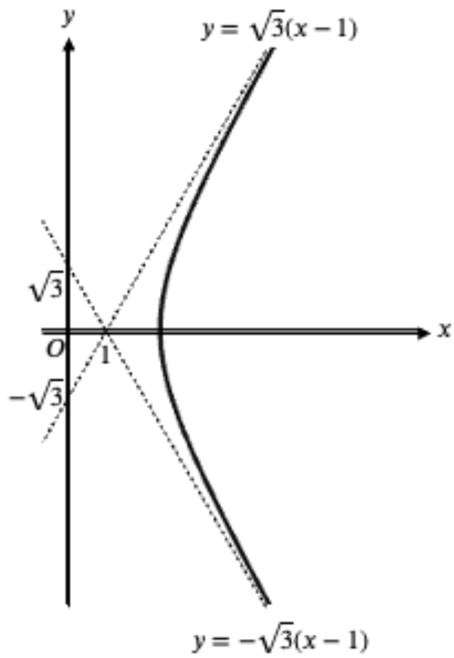
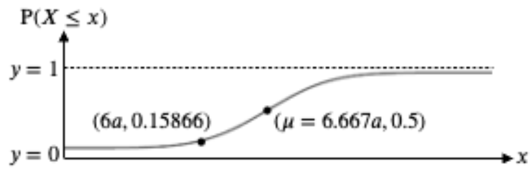
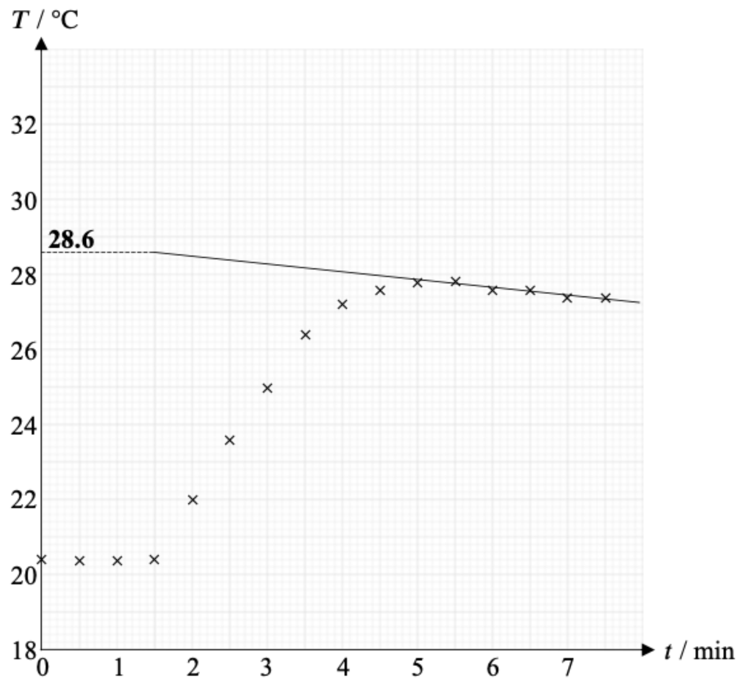
MATHEMATICS

Paper 9758/02
Set I – Paper 2

Topic identification and short answers

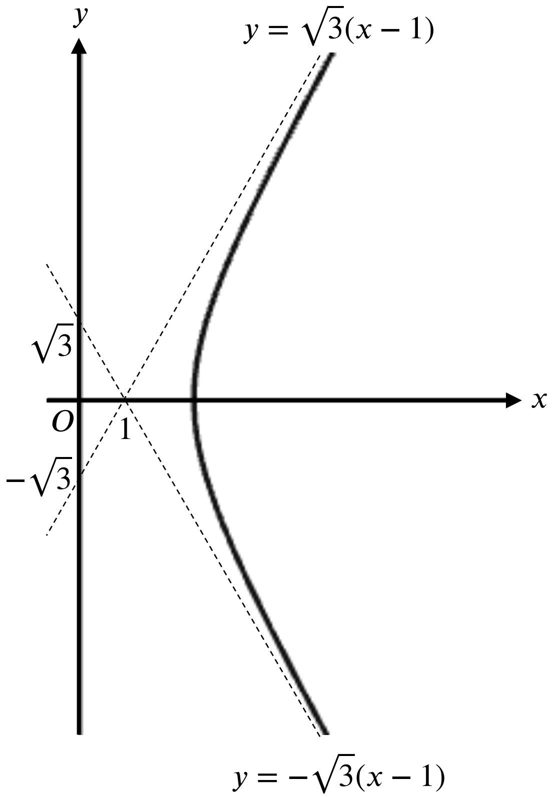
Qn	Topic(s)	Part	Answers
Section A: Pure Mathematics [40 marks]			
1	Complex numbers (cartesian form)		$A = -1$ $B = 1$
2	Complex numbers (modulus, real and imaginary, argument); Graph (sketching, conics section, transformation)	(i)	[shown] $y = \pm\sqrt{3}(x-1)$
		(ii)	
		(iii)	$-\frac{\pi}{3} < \arg(z-1) < \frac{\pi}{3}$
3	Differentiation (maxima and minima)		$h = \sqrt{3}r$ gives minimum volume. Minimum volume $= 4\sqrt{3}r^3$ units^3
4	Complex numbers (cartesian form, conjugate)	(i)	[shown]
		(ii)	$y = 1 + 2x + \left(2 - \frac{a^2}{2}\right)x^2 + \left(\frac{4}{3} - a^2\right)x^3 + \cdots$
		(iii)	$e^{2x}\sin ax = ax + 2ax^2 + \cdots$

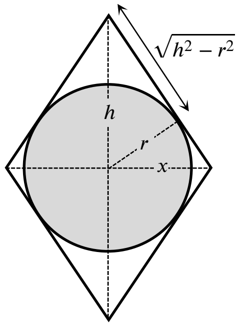
5	Vectors (three dimensions, vector product, magnitude, angle);	**(i)**	[shown] $b = 9.5$
		(ii)	$p = \dfrac{7}{8};\ q = \dfrac{7}{8};\ r = \dfrac{9}{32}$ Area ≈ 132.98 units^2 ≈ 133 units^2
		(iii)	Distance $\approx 14.60407 \approx 14.6$ units
Section B: Probability and Statistics [60 marks]			
6	Normal distribution	**(i)**	$\mu = 6.667a;\ \sigma = 0.667a$
		(ii)	
7	Binomial distribution	**(i)**	[shown]
		(ii)	$p \approx 0.968$ or 0.252
		(iii)	$\dfrac{1}{2} < p < 1$
8	Discrete random variables	**(i)**	[shown]
		(ii)	$\mathrm{E}(X) = \dfrac{4N^2 + 3N - 1}{6N}$
		(iii)	[shown] $m = 71$
9	Probability	**(i)**	$\mathrm{P}(B) = \dfrac{2}{5};\ \mathrm{P}(A \cap B) = \dfrac{7}{30}$
		(ii)	[shown]
		(iii)	$\mathrm{P}(A \cap B \cap C) = \dfrac{1}{15}$
		(iv)	$\dfrac{7}{20} \le \mathrm{P}(A' \cap B' \cap C') \le \dfrac{13}{30}$
10	Sampling; Hypothesis testing	**(i)**	• The sample is taken from the population of 1 000 chips of the newest model. • The sample size (or number of chips) n is greater than 30. • The n chips are obtained randomly (or independently of one another with equal probability).
		(ii)	[shown] $\mathrm{Var}(x) = 98.75$
		(iii)	$H_1\colon \mu < 40;\ \ 6.66 \le \alpha < 13.3$

11	Linear regression	**(i)**	
		(ii)	[shown] The square of the residual horizontal distances between the relevant data points and the best-fit line is minimised.
		(iii)	Required value = 8.2 degree Celsius (allow marginal error) This value is unreliable.
		(iv)	$t = 417.40 - \dfrac{25}{9}T$ $r = -0.9561828875 \approx\ 0.956$
		(v)	$k = 3$ (allow marginal error) The data is selected in the range of t where T is strictly increasing and such that the product moment correlation coefficient is as close to 1 as possible.

Suggested solutions and post-mortem

Qn	Suggested Solutions	Comments
	Section A: Pure Mathematics [40 marks]	
1 **[3]**	By remainder theorem, $P(z)=\left(z^2+1\right)g(z)+Az+B$ Divided by $(z+\mathrm{i})$, remainder is $1+\mathrm{i} \to P(-\mathrm{i})=-A\mathrm{i}+B=1+\mathrm{i}$ Divided by $(z-\mathrm{i})$, remainder is $1-\mathrm{i} \to P(\mathrm{i})=A\mathrm{i}+B=1-\mathrm{i}$ Adding the two results to eliminate $A\mathrm{i}$, $-A\mathrm{i}+B+A\mathrm{i}+B=1+\mathrm{i}+1-\mathrm{i}$ $2B=2 \to B=1$ Substituting $B=1$ into previous result, $-A\mathrm{i}+1=1+\mathrm{i} \to A=-1$	While this question mainly tests on complex numbers expressed in cartesian form, it also demands familiarity with factor-remainder theorem from O–Level Additional Mathematics. After using the remainders to deduce a system of two complex equations relating A and B, their values can be found accordingly with algebraic elimination and substitution.
2 **(i)** **[3]**	Writing z as $x+\mathrm{i}y$, where $x=\mathrm{Re}(z)$ and $y=\mathrm{Im}(z)$, $\lvert x+\mathrm{i}y+3\rvert=\lvert(x+3)+\mathrm{i}y\rvert=2x$ $\therefore \sqrt{(x+3)^2+y^2}=2x$ Squaring both sides, $x^2+6x+9+y^2=4x^2$ $3x^2-6x-9-y^2=0$ $3x^2-6x+3-y^2=12$ $3\left(x^2-2x+1\right)-y^2=12$ $\dfrac{(x-1)^2}{4}-\dfrac{y^2}{12}=1$ Finding asymptotes, $\dfrac{(x-1)^2}{4}=\dfrac{y^2}{12}$ $y=\pm\sqrt{3}(x-1)$ $\therefore$ x and y are related by the equation of the hyperbola with asymptotes $y=\pm\sqrt{3}(x-1)$.	It is perhaps most straightforward to begin by writing z in its cartesian form to uncover the relationship between x and y from the given complex equation. Afterwards, the absence of an imaginary number and the presence of a squared term in the cartesian equation of a hyperbola should sufficiently hint the use of modulus operation. The remaining part on finding the asymptotes is routine in A–Levels and can be deduced from this cartesian equation.

2 (ii) [2]		As hinted by the key phrase "part of the hyperbola", not all points on in **(i)** will satisfy the given relationship in z. The left-hand side of the equation $\lvert z+3\rvert = 2\text{Re}(z)$ contains a modulus, which can only be positive, and therefore implies for the right-hand side that $\text{Re}(z) \geq 0$. Since $x = \text{Re}(z)$, it should be apparent that the part of the hyperbola that is relevant in the sketch is the part where $x \geq 0$. A handy technique when sketching a graph with asymptotes is to **start with the asymptotes, followed by the graph itself, and finally the two axes**. When done in this order, the asymptotical behaviour of the graph is prioritised in the sketch and not compromised by any existing axes.
2 (iii) [2]	From the sketch in **(ii)**, it can be seen that $\arg(z-1)$ has an asymptotic behaviour. $\therefore -\tan^{-1}\left(\sqrt{3}\right) < \arg(z-1) < \tan^{-1}\left(\sqrt{3}\right)$ $\therefore -\frac{\pi}{3} < \arg(z-1) < \frac{\pi}{3}$	It may be useful to consider graph transformation in this question. Note that the value of $\arg z$ is measured with respect to the origin O, which implies that $\arg(z-1)$ takes reference from the point $(1, 0)$. The range of values can then be deduced through the appropriate translation. The relevant angles can be obtained from the gradient of the asymptotes. Candidates may find it noteworthy that angle calculations using gradient values are becoming more common in recent A–Levels.

3 **[8]**	Consider the cross-section of the octahedron where it is tangent to the inscribed sphere.  By similar triangles, $\frac{x}{r} = \frac{h}{\sqrt{h^2 - r^2}} \to x = \frac{rh}{\sqrt{h^2 - r^2}}$ $\therefore$ The volume of the inscribing tetrahedron, V $= 2\left(\frac{1}{3}\right)(2x)^2(h)$ $= \frac{2}{3}\left(\frac{4r^2h^2}{h^2 - r^2}\right)(h)$ $= \frac{8r^2}{3}\left(\frac{h^3}{h^2 - r^2}\right)$ Differentiating V with respect to h, $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{8r^2}{3}\left(\frac{3h^2\left(h^2 - r^2\right) - h^3(2h)}{(h^2 - r^2)^2}\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{8r^2}{3}\left(\frac{h^2\left(3h^2 - 3r^2 - 2h^2\right)}{(h^2 - r^2)^2}\right)$ $\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{8}{3}r^2\left(\frac{h^2\left(h^2 - 3r^2\right)}{(h^2 - r^2)^2}\right)$	Questions on maxima and minima and justification of the nature of stationary values are the bread and butter of A–Levels and school examinations. As is the trend for most optimisation problems, this question concerns packing geometrical shapes, and hence would call for some level of fluency in geometry and trigonometry that is considered routine in H2 Mathematics. When finding an expression for the volume of the octahedron in the beginning, **caution is advised when relying on only the diagram given in the question**, as there is risk of misinterpretation – in this case, potentially mistaking r to be equal to half of the side of the pyramid's square base. It is advisable to produce a preliminary cross-sectional diagram to cross-check the contact points of the inscribed sphere with the octahedron. All relevant lengths can be deduced using geometry, and eventually the volume of the inscribing octahedron can be obtained in terms of h and r, which is to be differentiated accordingly. When differentiating expressions involving many letters, **candidates must be attentive in distinguishing the variable of differentiation from the remaining letters which represent constants** – in this case, h is the variable and r is a constant.

[Continued]

For stationary value of $V, \dfrac{\mathrm{d}V}{\mathrm{d}h} = 0$

$\rightarrow h^2\left(h^2 - 3r^2\right) = 0$

$\rightarrow h^2 = 0$ or $h^2 = 3r^2$

$\rightarrow h = 0$ or $h = \pm\sqrt{3}r$

Since $h > 0$, $h = \sqrt{3}r$

Considering sign test, notice that

$$\frac{\mathrm{d}V}{\mathrm{d}h} = \frac{8}{3}r^2\left(\frac{h^2\left(h^2 - 3r^2\right)}{(h^2 - r^2)^2}\right) = \frac{8}{3}\left(\frac{rh}{h^2 - r^2}\right)^2\left(h^2 - 3r^2\right)$$

h	$\left(\sqrt{3}r\right)^-$	$\sqrt{3}r$	$\left(\sqrt{3}r\right)^+$
$h^2 - 3r^2$	$-$	0	$+$
$\dfrac{\mathrm{d}V}{\mathrm{d}\theta} = \dfrac{8}{3}\left(\dfrac{rh}{h^2 - r^2}\right)^2\left(h^2 - 3r^2\right)$	$-$	0	$+$
Slope	\	—	/

$\therefore h = \sqrt{3}r$ gives the minimum volume for the inscribing tetrahedron.

Minimum volume

$$= \frac{8r^2}{3}\left(\frac{\left(\sqrt{3}r\right)^3}{\left(\sqrt{3}r\right)^2 - r^2}\right) = \frac{8r^2}{3}\left(\frac{3\sqrt{3}r^3}{3r^2 - r^2}\right) = \frac{8r^2}{3}\left(\frac{3\sqrt{3}r}{2}\right) = 4\sqrt{3}r^3 \text{ units}^3$$

As per routine, stationary values are found through the roots of the first derivative. In that process, **it is considered good practice to justify any root rejections or acceptance in context**, which would otherwise be a common cause of penalisation.

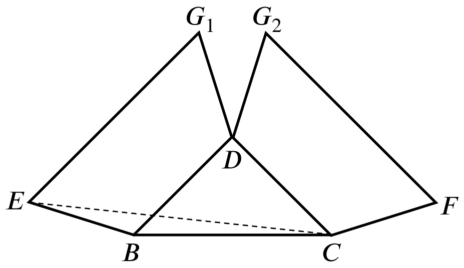
Proving that the volume found is indeed minimum can be done in a couple of ways: second differentiation or sign test. In this case, the expression for the first derivative lends itself to a factored form that works best with sign test. **Be advised that a proper sign test must show how all significant factors contribute to the sign change.** If the constants are **not** unknown, justifications using derivative values around the stationary point as quoted from a calculator would also be acceptable.

4 (i) [3]	$y = e^{2x}\cos ax$ Differentiating with respect to x, $\frac{dy}{dx} = 2e^{2x}\cos ax - ae^{2x}\sin ax$ $\frac{dy}{dx} = 2y - ae^{2x}\sin ax$ Differentiating with respect to x for the second time, $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2ae^{2x}\sin ax - a^2e^{2x}\cos ax$ $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 2\left(2y - \frac{dy}{dx}\right) - a^2y$ $\frac{d^2y}{dx^2} = 2\frac{dy}{dx} - 4y + 2\frac{dy}{dx} - a^2y$ $\frac{d^2y}{dx^2} = 4\frac{dy}{dx} - \left(a^2+4\right)y$	Like most questions on Maclaurin expansions in A–Levels, the bulk of the mark is attributed to the rigour of performing repeated differentiation. While the process is admittedly tedious and error-prone, having it posed as a "show" question such as this one provides opportunity to retrospectively identify any mistakes. It would be wise to secure the many marks that are usually allocated to Maclaurin expansion questions, especially those with parts asking to "show" a given result. Of note in this question is the use of intermediate results as substitutions, which is a common exploit in Maclaurin expansion questions in A–Levels – in this case, substituting $ae^{2x}\sin ax = 2y - \frac{dy}{dx}$, which may not be initially apparent.
4 (ii) [4]	Differentiating with respect to x for the third time, $\frac{d^3y}{dx^3} = 4\frac{d^2y}{dx^2} - \left(a^2+4\right)\frac{dy}{dx}$ When $x = 0$, $\rightarrow y = 1$, $\rightarrow \frac{dy}{dx} = 2(1) - a(1)(0) = 2$, $\rightarrow \frac{d^2y}{dx^2} = 4(2) - \left(a^2+4\right)(1) = 4 - a^2$ $\rightarrow \frac{d^3y}{dx^3} = 4\left(4-a^2\right) - \left(a^2+4\right)(2) = 16 - 4a^2 - 2a^2 - 8 = 8 - 6a^2$ $\therefore$ Maclaurin expansion of y is given by $y = 1 + 2x + \frac{1}{2}\left(4-a^2\right)x^2 + \frac{1}{6}\left(8-6a^2\right)x^3 + \cdots$ $y = 1 + 2x + \left(2 - \frac{a^2}{2}\right)x^2 + \left(\frac{4}{3} - a^2\right)x^3 + \cdots$	This second part requires as much rigour and attentiveness as the first part, especially since this Maclaurin expansion involves coefficients in terms of an unknown constant.

4 (iii) [2]	From first derivative, we have: $\frac{\mathrm{d}y}{\mathrm{d}x} = 2y - a\mathrm{e}^{2x}\sin ax$ $\therefore \mathrm{e}^{2x}\sin ax$ $= \frac{2}{a}y - \frac{1}{a}\frac{\mathrm{d}y}{\mathrm{d}x}$ $= \frac{2}{a}\left[1 + 2x + \left(2 - \frac{a^2}{2}\right)x^2 + \cdots\right] - \frac{1}{a}\left[2 + 2\left(2 - \frac{a^2}{2}\right)x + 3\left(\frac{4}{3} - a^2\right)x^2 + \cdots\right]$ $= \frac{2}{a} + \frac{4}{a}x + \left(\frac{4}{a} - a\right)x^2 + \cdots - \frac{2}{a} - \left(\frac{4}{a} - a\right)x - \left(\frac{4}{a} - 3a\right)x^2 - \cdots$ $= ax + 2ax^2 + \cdots$	As this final part asks to "hence" deduce the expansion for another function, it would be only fitting to start by referring to results found in earlier parts, particularly where $\mathrm{e}^{2x}\sin ax$ may have appeared in some intermediary results. The subsequent steps call for differentiation involving unknown constants, which should lead to the required result if done carefully and correctly.
5 (i) [2]	Since *BCD* and *EFG* are parallel, *EG* is parallel to *BD*. $\overrightarrow{EG} = k\overrightarrow{BD}, k \in \mathbb{R}$. $\begin{pmatrix} 4 \\ -1 \\ 10 \end{pmatrix} = k\begin{pmatrix} 3 \\ a-7 \\ b-2 \end{pmatrix}$ From the **i**–component, $4 = 3k \rightarrow k = \frac{4}{3}$ From the **j**–component, $-1 = \frac{4}{3}(a-7) \rightarrow a = 7 - \frac{3}{4} = \frac{25}{4} = 6.25$ From the **k**–component, $10 = \frac{4}{3}(b-2) \rightarrow b = 2 + \frac{15}{2} = \frac{19}{2} = 9.5$	Questions on three-dimensional vectors may initially seem perplexing, with many points, lines and planes thrown into the cartesian space without any clear explanation pertaining their relationship to each other. Fortunately, starting off with a "show" question may provide a strong initial footing to help verify and unravel these relationships, and therefore should not be overlooked to the detriment of marks. This initial part aims to reveal the relationship between relevant parallel edges. After further deducing that *EG* is a scalar multiple of *BD*, the remaining results will eventually follow.

5 (ii) [6]	From **(i)**, it can be deduced that $BD = \frac{3}{4}EG$ and $CD = \frac{3}{4}FG$ Area of face $BDGE$ $= \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert}{\lvert\overrightarrow{EG}\rvert}\right)\left(\lvert\overrightarrow{BD}\rvert+\lvert\overrightarrow{EG}\rvert\right) = \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert}{\lvert\overrightarrow{EG}\rvert}\right)\left(\frac{7}{4}\lvert\overrightarrow{EG}\rvert\right) = \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert$ Area of face $CDFG$ $= \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert}{\lvert\overrightarrow{FG}\rvert}\right)\left(\lvert\overrightarrow{CD}\rvert+\lvert\overrightarrow{FG}\rvert\right) = \frac{1}{2}\left(\frac{\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert}{\lvert\overrightarrow{FG}\rvert}\right)\left(\frac{7}{4}\lvert\overrightarrow{FG}\rvert\right) = \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert$ Area of face BCD $= \frac{1}{2}\lvert\overrightarrow{BD}\times\overrightarrow{CD}\rvert = \frac{1}{2}\left\lvert\frac{3}{4}\overrightarrow{EG}\times\frac{3}{4}\overrightarrow{FG}\right\rvert = \frac{9}{32}\lvert\overrightarrow{EG}\times\overrightarrow{FG}\rvert$ $\therefore$ Area of carboard used $= \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{EG}\rvert + \frac{7}{8}\lvert\overrightarrow{DG}\times\overrightarrow{FG}\rvert + \frac{9}{32}\lvert\overrightarrow{EG}\times\overrightarrow{FG}\rvert$ $= \frac{7}{8}\left\lvert\begin{pmatrix}-1\\-6.25\\-1.5\end{pmatrix}\times\begin{pmatrix}-5\\-2\\7\end{pmatrix}\right\rvert + \frac{7}{8}\left\lvert\begin{pmatrix}-1\\-6.25\\-1.5\end{pmatrix}\times\begin{pmatrix}4\\-1\\10\end{pmatrix}\right\rvert + \frac{9}{32}\left\lvert\begin{pmatrix}4\\-1\\10\end{pmatrix}\times\begin{pmatrix}-5\\-2\\7\end{pmatrix}\right\rvert$ $= \frac{7}{8}\left\lvert\begin{pmatrix}-46.75\\14.5\\-29.25\end{pmatrix}\right\rvert + \frac{7}{8}\left\lvert\begin{pmatrix}-64\\4\\26\end{pmatrix}\right\rvert + \frac{9}{32}\left\lvert\begin{pmatrix}13\\-78\\-13\end{pmatrix}\right\rvert$ $\approx 132.98 \text{ units}^2 \approx 133 \text{ units}^2$	This part on the whole tests on the ability to perform cross product to calculate areas defined by vertices on a three-dimensional space. The cardboard area consists of three faces: two trapezoids $BDGE$ and $CDFG$, and one triangle BCD. When finding the trapezoid areas, note that, after establishing in **(i)** the scale factor relating the parallel faces in, the sum of parallel edges can be further simplified. The altitude can be found using cross product, and the area expression for the trapezoids soon follow. Finding an area expression for triangle BCD should not pose as much of an issue. The second part on numerically evaluating the cross products may prove tedious, but nonetheless can be done correctly with careful calculation.

5 (iii) [5]



Considering the shape of the flat surface, the required distance is $|\overrightarrow{CE}|$. Using cosine rule,

$$|\overrightarrow{CE}| = \sqrt{|\overrightarrow{BE}|^2 + |\overrightarrow{BC}|^2 - 2|\overrightarrow{BE}||\overrightarrow{BC}|\cos(\angle EBD + \angle DBC)}$$

Finding $\angle EBD$,

$$\angle EBD = \cos^{-1}\left(\frac{\overrightarrow{BE}\cdot\overrightarrow{BD}}{|\overrightarrow{BE}||\overrightarrow{BD}|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}-2\\-6\\-4\end{pmatrix}\cdot\begin{pmatrix}3\\-0.75\\7.5\end{pmatrix}}{\left|\begin{pmatrix}-2\\-6\\-4\end{pmatrix}\right|\left|\begin{pmatrix}3\\-0.75\\7.5\end{pmatrix}\right|}\right) = \cos^{-1}\left(\frac{-31.5}{|\sqrt{56}||\sqrt{65.8125}|}\right) = 121.26^\circ$$

Finding $\angle DBC$, and considering $\Delta BCD \cong \Delta EFG$ on the original folded shape,

$$\angle DBC = \angle GEF = \cos^{-1}\left(\frac{\overrightarrow{EG}\cdot\overrightarrow{EF}}{|\overrightarrow{EG}||\overrightarrow{EF}|}\right) = \cos^{-1}\left(\frac{\begin{pmatrix}4\\-1\\10\end{pmatrix}\cdot\begin{pmatrix}9\\1\\3\end{pmatrix}}{\left|\begin{pmatrix}4\\-1\\10\end{pmatrix}\right|\left|\begin{pmatrix}9\\1\\3\end{pmatrix}\right|}\right) = \cos^{-1}\left(\frac{65}{|\sqrt{117}||\sqrt{91}|}\right) = 50.95^\circ$$

$\therefore$ Required distance

$$= \sqrt{|\overrightarrow{BE}|^2 + \left(\frac{3}{4}|\overrightarrow{EF}|\right)^2 - 2|\overrightarrow{BE}|\left(\frac{3}{4}|\overrightarrow{EF}|\right)\cos(121.26^\circ + 50.95^\circ)}$$

$$= \sqrt{56 + \frac{9}{16}(91) - \frac{3}{2}\left(\sqrt{56}\right)\left(\sqrt{91}\right)\cos(172.21^\circ)}$$

≈ 14.60407 units ≈ 14.6 units

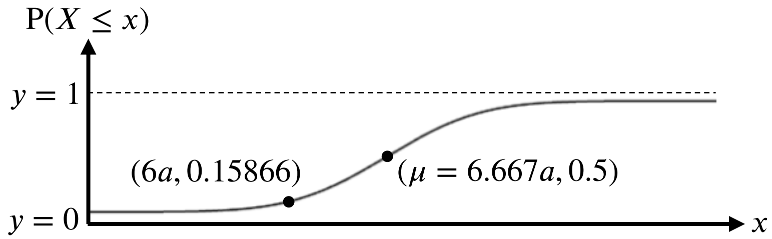
When approaching this part, a preliminary diagram of the flattened surface may prove helpful. The diagram should sufficiently hint at applying the cosine rule to the flat triangle BCE to find the flattened length of CE.

The cosine rule calls for the values of angles and edge lengths on faces $BDGE$ and BCD, which can be deduced by proxy of the original folded shape, since these angles and edges are preserved in both states of the shape.

To find angles and edges involving the vertex C, which has not been asked to be determined, a suggested method is considering similar triangles in the folded shape. This method is reasonably efficient as it relieves the need of discovering a vertex whose coordinates may turn out to be complicated. Still, proceeding by finding C beforehand is acceptable, and in fact there may be other alternatives that can be considered apart from these two.

Throughout the working, some vectors may need to be calculated beforehand, such as $\overrightarrow{BE}$, $\overrightarrow{BD}$ and $\overrightarrow{EF}$, but the rest can be recalled from previous parts. Equally important to note in this answer is the scalar relationship $\overrightarrow{BC} = \frac{3}{4}|\overrightarrow{EF}|$, which is relevant in the final calculation.

Section B: Probability and Statistics [60 marks]		
6 (i) [5]	$P(X \le 6a) = 0.15866$ $\rightarrow P\left(Z \le \frac{6a-\mu}{\sigma}\right) = 0.15866$ $\rightarrow \frac{6a-\mu}{\sigma} = -0.99998$ $\rightarrow 6a - \mu + 0.99998\sigma = 0$ — (1) $\frac{X_1+X_2+X_3+X_4}{4} = \bar{X} \sim N\left(\mu, \frac{\sigma^2}{4}\right)$ $P\left(6a \le \bar{X} \le 7a\right) = 0.81859$ $\rightarrow P\left(\bar{X} \le 7a\right) - P\left(\bar{X} \le 6a\right) = 0.81859$ $\rightarrow P\left(Z \le \frac{7a-\mu}{\frac{\sigma}{2}}\right) - P\left(Z \le \frac{6a-\mu}{\frac{\sigma}{2}}\right) = 0.81859$ $\rightarrow P\left(Z \le \frac{7a-\mu}{\frac{\sigma}{2}}\right) - P\big(Z \le 2(-0.99998)\big) = 0.81859$ $\rightarrow P\left(Z \le \frac{7a-\mu}{\frac{\sigma}{2}}\right) = P\big(Z \le 2(-0.99998)\big) + 0.81859 = 0.02275 + 0.81859 = 0.84134$ $\rightarrow \frac{7a-\mu}{\frac{\sigma}{2}} = 0.99998$ $\rightarrow 7a - \mu - 0.49999\sigma = 0$ — (2) Using GC to solve (1) and (2) simultaneously, $\mu = 6.667a$ and $\sigma = 0.667a$	This question mainly tests on forming relationships between μ, σ and a using the inverse of standard normal distribution. The two probability values yield a system of two equations in terms of the three variables, which can be solved using calculator to obtain the desired final expressions. A key observation here is recognising that the intermediate value obtained for $\frac{6a-\mu}{\sigma}$ is relevant in evaluating the similar looking $P\left(Z \le \frac{6a-\mu}{\frac{\sigma}{2}}\right)$.

6 **(ii)** **[2]**		This part is a different take on the more familiar question which asks for the sketch of a normal distribution (also known as *probability density function*), which seems to be the trend in recent A–Level papers. For the case of the graph of $\mathrm{P}(X \le x)$ against x (also known as *cumulative distribution function*), there are some important features to include: • asymptotic behaviours to indicate that cumulative probabilities can never equal 0 or 1, and • the steepest slope at the mean value.
7 **(i)** **[1]**	Let X denote the number of times that the computer decides to add 1 to the sum. $\therefore X \sim \mathrm{B}(n, p)$. For the case where $n = 9$, P(sum = 3) = P(6 additions and 3 subtractions in any order) $= \binom{9}{6} p^6 q^3$ $= 84 p^6 q^3$	This question tests on binomial distribution modelling given an event in context. This part also serves as a primer to subsequent parts which require similar models.
7 **(ii)** **[3]**	For the case where $n = 5$, P(sum = 3) = P(4 additions and 1 subtraction in any order) $= \binom{5}{4} p^4 q$ $= 5 p^4 q$ Since it is equally likely that the sum is 3 after 9 actions as it is after 5 actions, $84 p^6 q^3 = 5 p^4 q$ $p^2 (1 - p^2) = \frac{5}{84}$ $p^4 - p^2 + \frac{5}{84} = 0$ Using GC to find the positive roots of p, $p \approx 0.968$ or 0.252	This part follows immediately after finding the expression in **(i)**. With further adjustment to the previously used binomial model, a second expression can be found, and equating the two expressions will yield the result.

7 (iii) [5]	The sum can only be positive if computer does addition for more than half of the times: • For $2k-1$ actions, at least k additions must occur. • For $2k+1$ actions, at least $k+1$ additions must occur. Let $X \sim \mathrm{B}(2k-1, p)$. The probability that the sum is positive after $2k-1$ actions, $\mathrm{P}(\mathrm{sum}_{2k-1} > 0)$ $= \mathrm{P}(X \geq k)$ $= \mathrm{P}(X = k) + \mathrm{P}(X \geq k+1)$ The probability that the sum is positive after $2k+1$ actions $\mathrm{P}\left(\mathrm{sum}_{2\mathrm{k}+1} > 0\right)$ $= \mathrm{P}(X = k-1) \times \mathrm{P}(\text{all 2 remaining adds}) + \mathrm{P}(X = k) \times \mathrm{P}(\text{1 of 2 remaining adds}) + \mathrm{P}(X \geq k+1)$ $= \mathrm{P}(X = k-1)p^2 + \mathrm{P}(X = k)\left(1-(1-p)^2\right) + \mathrm{P}(X \geq k+1)$ The following inequality is obtained: $\mathrm{P}\left(\mathrm{sum}_{2\mathrm{k}+1} > 0\right) > \mathrm{P}(\mathrm{sum}_{2k-1} > 0)$ $\mathrm{P}(X = k-1)p^2 + \mathrm{P}(X = k)\left(1-(1-p)^2\right) + \mathrm{P}(X \geq k+1) > \mathrm{P}(X = k) + \mathrm{P}(X \geq k+1)$ $\mathrm{P}(X = k-1)p^2 - \mathrm{P}(X = k)(1-p)^2 > 0$ $\left[\binom{2k-1}{k-1}p^{k-1}(1-p)^k\right]p^2 - \left[\binom{2k-1}{k}p^k(1-p)^{k-1}\right](1-p)^2 > 0$ $\left[\frac{(2k-1)!}{(k-1)!\,k!}p^k(1-p)^k\right]p - \left[\frac{(2k-1)!}{k!\,(k-1)!}p^k(1-p)^k\right](1-p) > 0$ $\left[\frac{(2k-1)!}{(k-1)!\,k!}p^k(1-p)^k\right](2p-1) > 0$ Given that $0 < p < 1$, $p^k(1-p)^k > 0$ for all possible k $\rightarrow 2p - 1 > 0$ $\rightarrow p > \frac{1}{2}$ $\therefore \frac{1}{2} < p < 1$ (since $0 < p < 1$)	This part may prove challenging. To begin this part, the number of additions required for the sum to be positive for each $(2k-1)$ and $(2k+1)$ actions must first be considered. The subsequent part entails forming an inequality and requires making use of a common random variable to model both the case of $(2k-1)$ actions and $(2k+1)$ actions. In this case, both cases are modelled after $X \sim \mathrm{B}(2k-1, p)$. This model directly models the case of $(2k-1)$ actions. For $(2k+1)$ actions, X models the first $(2k-1)$ actions, and the remaining two actions are considered case-by-case. The remaining intermediate steps simplify the inequality by utilising the expression for $P(X = x)$ for binomial distributions found in the List of Formulae (MF26). With proper factorisation and justification, the required range of p is subsequently obtained. Note that the final range of values of p to be obtained must also account for the fact that $0 < p < 1$, as given by the question – while the decision to penalise may vary based on strictness of the marking scheme, it is still considered erroneous to only conclude that $p > \frac{1}{2}$.

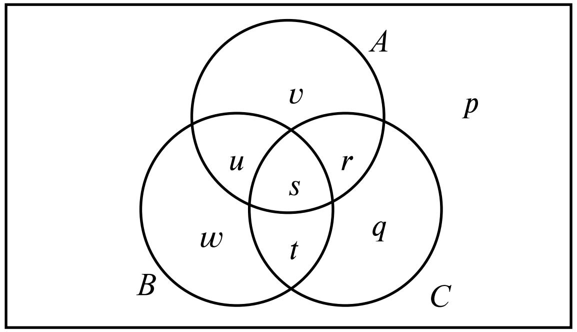
8 (i) [2]	$\mathrm{P}(X=r)$ $= \mathrm{P}(\text{pick ball } r \text{ from a box and pick ball} \le r \text{ from another box}) - \mathrm{P}(\text{pick equal ball from both})$ $= 2\left(\frac{1}{N}\right)\left(\frac{r}{N}\right) - \frac{2}{2!}\left(\frac{1}{N}\right)\left(\frac{1}{N}\right)$ $= \frac{2r}{N^2} - \frac{1}{N^2} = \frac{2r-1}{N^2}$	This question tests on formulating probability of events given a context. Note that picking the two balls can be done in any order (i.e., Box ***A*** then Box ***B***, or Box ***B*** first then Box ***A***) and thus the probability is doubled to give $2\left(\frac{1}{N}\right)\left(\frac{r}{N}\right)$. Equally of note is the fact that picking two balls with equal value is considered the same event, and thus the probability should be deducted accordingly.
8 (ii) [2]	$\mathrm{E}(X)$ $= \sum_{r=1}^{N} r\,\mathrm{P}(X=r)$ $= \sum_{r=1}^{N} r\left(\frac{2r-1}{N^2}\right)$ $= \frac{1}{N^2}\sum_{r=1}^{N} 2r^2 - r$ $= \frac{1}{N^2}\left(2\sum_{r=1}^{N} r^2 - \sum_{r=1}^{N} r\right)$ $= \frac{1}{N^2}\left(\frac{1}{3}N(N+1)(2N+1) - \frac{1}{2}N(N+1)\right)$ $= \frac{2N(N+1)(2N+1) - 3N(N+1)}{6N^2}$ $= \frac{(N+1)(4N-1)}{6N}$ $= \frac{4N^2+3N-1}{6N}$	Finding the expectation of an event is considered routine though A–Levels tend to make the probability distribution of the event apparent, such as in a tabulated form that is either provided or to be found. This question provides a different way of representing the probability distribution without tables, which may blur the relationship $\mathrm{E}(X) = \sum_{r=1}^{N} r\,\mathrm{P}(X=r)$ and the significance of the result in **(i)**.

8 (iii) [5]	$\mathrm{P}(X \le m) = \sum_{r=1}^{m} \frac{2r-1}{N^2} = \frac{1}{N^2}\left(\frac{2m(m+1)}{2} - m\right) = \frac{m^2}{N^2}$ $\mathrm{P}(X \ge m) = 1 - \mathrm{P}(X \le m-1) = 1 - \frac{(m-1)^2}{N^2}$ $\mathrm{P}(X \ge m) = 1 - \frac{(m-1)^2}{N^2} \ge \frac{1}{2}$ $\to \frac{(m-1)^2}{N^2} \le \frac{1}{2}$ $\to m - 1 \le \frac{1}{\sqrt{2}}N = \frac{\sqrt{2}}{2}N$ $\to m \le \frac{\sqrt{2}}{2}N + 1$ $\mathrm{P}(X \le m) = \frac{m^2}{N^2} \ge \frac{1}{2}$ $\to m \ge \frac{1}{\sqrt{2}}N = \frac{\sqrt{2}}{2}N$ $\therefore \frac{\sqrt{2}}{2}N \le m \le \frac{\sqrt{2}}{2}N + 1$ Using the given $\mathrm{E}(X)$ value to find N and m, $\frac{4N^2+3N-1}{6N} = \frac{40299}{600}$ $\to N = 100$ (using GC / by observation) $\to 50\sqrt{2} \le m \le 50\sqrt{2} + 1$ $\to 70.7 \le m \le 71.7$ $\to m = 71$	Upon a quick glance, there are three requirements in this part that can be noticed: • finding the expressions for $\mathrm{P}(X \le m)$ and $\mathrm{P}(X \ge m)$, • using these expressions to deduce the required inequality in m, and • finding m for a specific case of N. Finding an expression for $\mathrm{P}(X \le m)$ entails using the result in **(i)** as a summation of probabilities and the use of the arithmetic series formula. Finding an expression for $\mathrm{P}(X \ge m)$ entails similar working, though also of note here is the use of complementary events. Eventually, the two expressions work out to yield the maximum and minimum values of m, which can be combined to give the required inequality. The value of N can be easily deduced from $\mathrm{E}(X)$, and the remaining steps to deduce the integer value m follow.

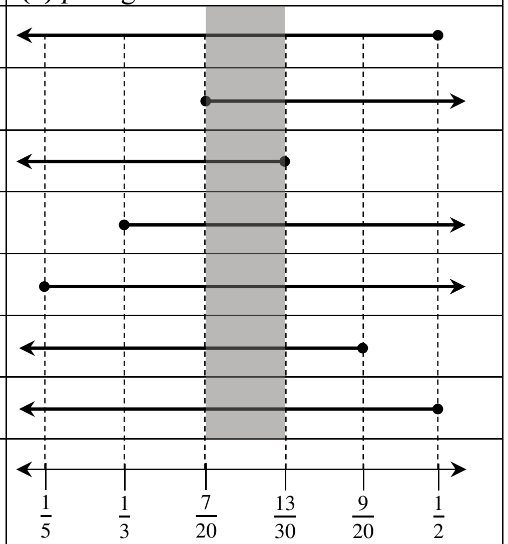
9 (i) [3]	$\mathrm{P}(A\|B') = \dfrac{\mathrm{P}(A \cap B')}{\mathrm{P}(B')}$ $\dfrac{1}{6} = \dfrac{\mathrm{P}(A \cup B) - \mathrm{P}(B)}{1 - \mathrm{P}(B)} = \dfrac{\frac{1}{2} - \mathrm{P}(B)}{1 - \mathrm{P}(B)}$ $\dfrac{1}{6} - \dfrac{1}{6}\mathrm{P}(B) = \dfrac{1}{2} - \mathrm{P}(B)$ $\dfrac{5}{6}\mathrm{P}(B) = \dfrac{1}{2} - \dfrac{1}{6} = \dfrac{1}{3}$ $\therefore \mathrm{P}(B) = \dfrac{2}{5}$ $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$ $\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{2}{5} - P(A \cap B)$ $\therefore \mathrm{P}(A \cap B) = \dfrac{1}{3} + \dfrac{2}{5} - \dfrac{1}{2} = \dfrac{7}{30}$	This question particularly assesses on the ability to manipulate probability values using various techniques, especially those pertaining complementary events, union events, and expressions for conditional probabilities.
9 (ii) [1]	$\mathrm{P}(A \cup B \cup C) = \mathrm{P}\big(A \cup (B \cup C)\big)$ $= \mathrm{P}(A) + \mathrm{P}(B \cup C) - \mathrm{P}\big(A \cap (B \cup C)\big)$ $= \mathrm{P}(A) + \mathrm{P}(B \cup C) - \mathrm{P}\big((A \cap B) \cup (A \cap C)\big)$ $= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(B \cap C) - [\mathrm{P}(A \cap B) + \mathrm{P}(A \cap C) - \mathrm{P}(A \cap B \cap A \cap C)]$ $= \mathrm{P}(A) + \mathrm{P}(B) + \mathrm{P}(C) - \mathrm{P}(A \cap B) - \mathrm{P}(A \cap C) - \mathrm{P}(B \cap C) + \mathrm{P}(A \cap B \cap C)$	This part makes use of the probability for the union of two events and applies it to the extended case of the union of three events. As a side, $\mathrm{P}\big(A \cap (B \cup C)\big) = \mathrm{P}\big((A \cap B) \cup (A \cap C)\big)$ is known as *De Morgan's law*, a particularly useful and fundamental result in the realm of propositional logic and Boolean algebra.
9 (iii) [2]	Using the result in **(ii)**, $1 - \dfrac{11}{30} = \dfrac{1}{3} + \dfrac{2}{5} + \dfrac{1}{4} - \dfrac{7}{30} - \left(\dfrac{1}{3}\right)\left(\dfrac{1}{4}\right) - \left(\dfrac{2}{5}\right)\left(\dfrac{1}{4}\right) + \mathrm{P}(A \cap B \cap C)$ $\therefore \mathrm{P}(A \cap B \cap C) = \dfrac{1}{15}$	This part entails using the result in **(ii)**, but not before recognising that the value of $\mathrm{P}(A' \cap B' \cap C')$ is simply the complement of $\mathrm{P}(A \cup B \cup C)$. Equally important to notice in this question is the condition of independent events.

9 (iv) [4]

Let $P(A' \cap B' \cap C') = p$, where $p \geq 0$. Labelling the Venn diagram as follows:



Region	**(1)** Probability of region ≥ 0	**(2)** p range	**(3)** p range in a number line
q	$1 - P(A \cup B) - p = \frac{1}{2} - p \geq 0$	$p \leq \frac{1}{2}$	
r	$P(C) - q - P(B \cap C) = p - \frac{7}{20} \geq 0$	$p \geq \frac{7}{20}$	
s	$P(A \cap C) - r = \frac{13}{30} - p \geq 0$	$p \leq \frac{13}{30}$	
t	$P(B \cap C) - s = p - \frac{1}{3} \geq 0$	$p \geq \frac{1}{3}$	
u	$P(A \cap B) - s = p - \frac{1}{5} \geq 0$	$p \geq \frac{1}{5}$	
v	$P(A) - r - s - u = \frac{9}{20} - p \geq 0$	$p \leq \frac{9}{20}$	
w	$P(B) - s - t - u = \frac{1}{2} - p \geq 0$	$p \leq \frac{1}{2}$	



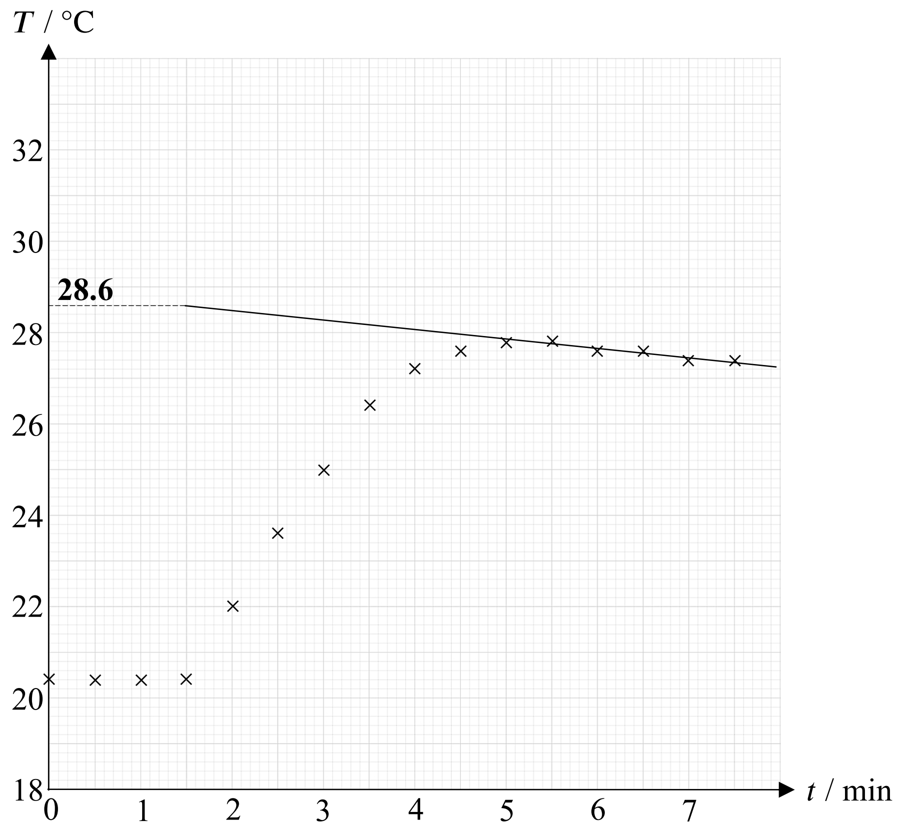
$$\therefore \frac{7}{20} \leq P(A' \cap B' \cap C') \leq \frac{13}{30}$$

Questions on maximum and minimum probabilities involving Venn Diagrams have currently witnessed a rising trend in both JC papers and A–Levels. As such, it would be wise to get familiar with methods to tackle such questions.

This suggested solution offers a possible strategy. The key idea in this approach is maintaining positive (or non-negative) values on regions in the Venn diagram that are not explicitly known to be zero.

10 (i) [3]	• The sample is taken from the population of 1 000 chips of the newest model. • The sample size (or number of chips) n is greater than 30. • The n chips are obtained randomly (or independently of one another with equal probability).	This question tests on sampling methods. Of note in this part are the significant factors of consideration when obtaining a sample that is appropriate for z-test: • **Sample vs. Population**: this is subtle, but it must be acknowledged that the sample should **only** be obtained from the new 1 000 chips. • **Central Limit Theorem**: given that the sample is taken from a population with **an unknown distribution**, and yet it is already appropriate for use in z-test, it must be that Central Limit Theorem applies here and thus its size must be greater than 30. • **Randomness**: random selection must be done to remove bias (e.g., each selection should **not** depend on the performance of the previously selected chip).
10 (ii) [2]	$\mathrm{E}(t)$ $= \dfrac{15(4)+25(14)+35(37)+45(33)+55(12)}{100}$ $= 38.5$ $\mathrm{E}\left(t^2\right)$ $= \dfrac{15^2(4)+25^2(14)+35^2(37)+45^2(33)+55^2(12)}{100}$ $= 1581$ $\mathrm{Var}(t)$ $= \mathrm{E}\left(t^2\right) - \mathrm{E}(t)^2$ $= 1581 - 38.5^2$ $= 98.75$	When given a tiered data summary, the calculations for expectation values makes use of the midpoint of each tier. The remainder of this question is standard practice in A–Levels, dealing with expectations and variance of a sample.

10 (iii) [7]	Unbiased estimate of the population mean $= \mathrm{E}(t) = 38.5$ Unbiased estimate of the population variance $= \dfrac{n}{n-1}\mathrm{Var}(t) = \dfrac{100}{99}(98.75)$ Let T be the time taken, in minutes, for the device to carry out the tasks with a random new chip. $H_0: \mu = 40$ H_1: (to be found) Test at $\alpha\%$ significance level. Under H_0, since $n = 100$ is large, by Central Limit Theorem, $\bar{T} \sim \mathrm{N}\left(40, \dfrac{98.75}{99}\right)$ approximately. Test statistics is given by $Z = \dfrac{\bar{T} - 40}{\sqrt{\dfrac{98.75}{99}}} \sim \mathrm{N}(0,1)$ With $\bar{t} = 38.5$, z–value $= -1.501897534$ For rejection, z–value must lie in the critical region. The unique alternative hypothesis condition requires that, given a value of α, z–value must lie in **only one** out of three critical regions possible. Consider right-tailed test or two-tailed test, $H_1: \mu > 40$ or $H_1: \mu \neq 40$. → If there is α such that the right-tailed critical region or two-tailed critical region includes the z–value respectively, that α also yields a left-tailed critical region big enough to include the z–value. → ∴ The alternative hypothesis is not $H_1: \mu > 40$ or $H_1: \mu \neq 40$. Consider left-tailed test, $H_1: \mu < 40$. → The left-tailed critical value should be more than the z–value. → $\therefore \alpha\% \geq \mathrm{P}(Z \leq -1.501897534) = 0.0666$ → The α must also be such that the two-tailed critical region excludes the z–value. → $\therefore \alpha\% < 2(0.0666) = 0.133$ $\therefore H_1: \mu < 40, \quad 6.66 \leq \alpha < 13.3$	As hinted by the question, a feasible approach to this question is first finding the test statistic: the z–value. Care must be taken during the following: • finding unbiased estimate of the population variance • stating Central Limit Theorem • writing down the parameters for $\bar{T}$, especially its variance This test statistic is subsequently used to find H_1 by considering possible critical regions and proceeding with elimination. A step-by-step thought process follows: 1. **List down the possible cases for H_1**: since *A*–Levels only tests on two-tailed tests and one-tailed tests, there are effectively only three cases of H_1. 2. **Consider the respective critical region for each H_1 that "just nice" rejects H_0:** how does the critical region look like for each case of H_1 that "just nice" contains the z–value? What should the α–value be in this case? 3. **Deduce whether there is only one H_1 that cause rejection under this "just nice" α–value**: if the "just nice" critical region is large, the required α–value will also be large, making it more likely for other critical regions with the same area to contain the z–value and thus also cause rejection. After deducing the correct alternative hypothesis, the range of α can be deduced from considering the "just nice" critical regions.

11 (i) [1]	[The following diagram also bears suggested sketches for **11(ii)** and **11(iii)**.]  	A strategic grid sketch considers scale of axes beforehand. Upon preliminary reading of the subsequent question parts, it can be noticed that in **(iii)** the sketch will need to be used to find an extrapolated value of T. An appropriate scale would consider the case that the extrapolated value lies beyond the data range for T.
11 (ii) [2]	To ensure that the line is best-fit, the square of the residual **horizontal** distances between the relevant data points and the best-fit line is minimised. For this data set, this line passes through midpoints of pairs of data that has the same value of T (i.e., passes through midpoint of (5, 27.8) and (5.5, 27.8), etc.).	This question mainly aims to test on awareness of the type of residual distances used, horizontal or vertical, for a given regression line.
11 (iii) [3]	Required value = 28.6 − 20.4 = 8.2 degree Celsius This value is unreliable. The temperature obtained at $t = 1.5$ from L is an extrapolation as the t value lies outside the data range $5 \le t \le 7.5$ from which L was drawn.	Note that the temperature difference using the maximum T data is less than using the extrapolated T value. This shows that while extrapolation is mathematically unreliable, it is still serviceable in certain applications.

<table>
<tr>
<td>11
(iv)
[4]</td>
<td>
The following data is used:

t	5	5.5	6	6.5	7	7.5
T	27.8	27.8	27.6	27.6	27.4	27.4

For the given data, regression line of t on T:
$t = 144.25 - 5T$

If T were given in degrees Rankine, convert to Celsius first:
$t = 144.25 - 5\left(\frac{5(T - 491.67)}{9}\right)$
$t = 417.40 - \frac{25}{9}T$

$r = -0.9561828875 \approx\ 0.956$
This applies to both cases where T is given in degrees Celsius or degrees Rankine.
</td>
<td>
This question tests on three concepts:

- calculation of regression line using calculator,
- conversion of regression line in the context of transformed data, and
- understanding that product moment correlation coefficient of a regression line remains unaffected after translation and scaling

Great care must be taken during the Celsius–Rankine conversion. It would be remiss to just substitute T in the regression line equation with $\frac{9}{5}T + 491.67$.
</td>
</tr>
<tr>
<td>11
(v)
[3]</td>
<td>
The following data is used:

t	1.5	2	2.5	3	3.5
T	20.4	22.0	23.6	25.0	26.4

For the given data, regression line of T on t:
$T = 15.98 + 3t$
$\rightarrow k = 3$

The data is selected in the range of t where T is strictly increasing and such that **the product moment correlation coefficient is as close to 1 as possible**. For the regression line used above, $r = 0.99938 \approx 0.999$, which indicates a very strong positive linear relationship between t and T, and thus its gradient accurately represents the rate at which T increases with respect to t.
</td>
<td>
This question, atop testing on calculating the regression line using calculator, also tests on the ability to justify the reliability of the behaviour of the data based on the r–value of its regression line.

To obtain an accurate value of k in this question, the regression line must first be suitably obtained from an appropriate range which indicates a **constant** rate of increase of T (since k is constant). This constant rate of increase can be observed from the scatter diagram in **(i)** in the range $1.5 \leq t \leq 3.5$. Any data outside this range not only reduces the r–value (and hence affecting the reliability of the answer) but also suggests that T does not increase at a constant rate.
</td>
</tr>
</table>