RIVER VALLEY HIGH SCHOOL

2022 JC 1 H2 Mathematics (9758)

End-of-Year Revision Package
(Solutions)

1 Inequalities & Equations

Basic Skills

1. Find the discriminant of the following quadratic expressions.

(a) x^2+x+1 (b) x^2+4x-5 (c) $-x^2+x-1$

(d) $-2x^2-4x+3$ (e) $\dfrac{x^2}{2}+x-\dfrac{5}{9}$ (f) $-x^2+\dfrac{x}{3}+6$

(a) x^2+x+1 $D=(1)^2-4(1)(1)$ $=-3$	(b) x^2+4x-5 $D=(4)^2-4(1)(-5)$ $=36$	(c) $-x^2+x-1$ $D=(1)^2-4(-1)(-1)$ $=-3$
(d) $-2x^2-4x+3$ $D=(-4)^2-4(-2)(3)$ $=40$	(e) $\dfrac{x^2}{2}+x-\dfrac{5}{9}$ $D=(1)^2-4\left(\dfrac{1}{2}\right)\left(-\dfrac{5}{9}\right)$ $=\dfrac{19}{9}$	(f) $-x^2+\dfrac{x}{3}+6$ $D=\left(\dfrac{1}{3}\right)^2-4(-1)(6)$ $=24\dfrac{1}{9}$

2. Find the exact roots of the following quadratic equations.

(a) $x^2+3x+2=0$ (b) $-x^2-2x+3=0$ (c) $-\dfrac{x^2}{2}+2x-1=0$

(d) $\dfrac{2}{3}x^2+x-\dfrac{1}{4}=0$ (e) $-\pi x^2+x+\pi=1$ (f) $\dfrac{x^2}{2}+\sqrt{6}x+3=0$

(a) $x^2+3x+2=0$ $(x+2)(x+1)=0$ $x=-1\text{ or }-2$	(b) $-x^2-2x+3=0$ $(1-x)(x+3)=0$ $x=1\text{ or }-3$	(c) $-\dfrac{x^2}{2}+2x-1=0$ $x=\dfrac{-2\pm\sqrt{4-4\left(-\frac{1}{2}\right)(-1)}}{2\left(-\frac{1}{2}\right)}$ $=\dfrac{-2\pm\sqrt{2}}{-1}=2\pm\sqrt{2}$
(d) $\dfrac{2}{3}x^2+x-\dfrac{1}{4}=0$ $x=\dfrac{-1\pm\sqrt{1-4\left(\frac{2}{3}\right)\left(-\frac{1}{4}\right)}}{2\left(\frac{2}{3}\right)}$ $=\dfrac{-1\pm\sqrt{\frac{5}{3}}}{\frac{4}{3}}=\dfrac{-3\pm3\sqrt{\frac{5}{3}}}{4}$ $=\dfrac{-3\pm\sqrt{15}}{4}$	(e) $-\pi x^2+x+\pi=1$ $(-\pi)x^2+x+(\pi-1)=0$ $x=\dfrac{-1\pm\sqrt{1-4(-\pi)(\pi-1)}}{2(-\pi)}$ $=\dfrac{1\pm\sqrt{1+4\pi^2-4\pi}}{2\pi}$ $=\dfrac{1\pm\sqrt{(2\pi-1)^2}}{2\pi}$ $=\dfrac{1+2\pi-1}{2\pi}\text{ or }\dfrac{1-2\pi+1}{2\pi}$ $=1\text{ or }\dfrac{1-\pi}{\pi}$	(f) $\dfrac{x^2}{2}+\sqrt{6}x+3=0$ $x=\dfrac{-\sqrt{6}\pm\sqrt{6-4\left(\frac{1}{2}\right)(3)}}{2\left(\frac{1}{2}\right)}$ $=-\sqrt{6}$

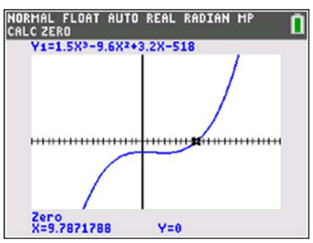
Tutorial Review
Tutorial 1 Questions 4 and 7.

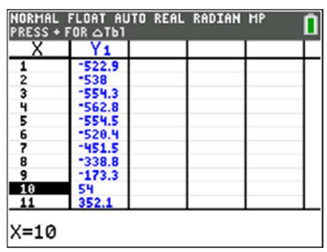
Revision Questions

1. NYJC Promo 9758/2020/Q2

(i) The first four terms of a sequence u_n are given by $u_1 = 32.1$, $u_2 = 17$, $u_3 = 0.7$ and $u_4 = -7.8$. Given that u_n is a cubic polynomial in n, find u_n in terms of n. [3]

(ii) Find the least value of n for which u_n is greater than 555. [2]

<table>
<tr><td>(i)</td><td>Let $u_n = an^3 + bn^2 + cn + d$
$u_1 = a + b + c + d = 32.1$
$u_2 = 8a + 4b + 2c + d = 17$
$u_3 = 27a + 9b + 3c + d = 0.7$
$u_4 = 64a + 16b + 4c + d = -7.8$
Using GC, $a = 1.5$, $b = -9.6$, $c = 3.2$, $d = 37$
$\therefore u_n = 1.5n^3 - 9.6n^2 + 3.2n + 37$</td></tr>
<tr><td>(ii)</td><td>$u_n > 555 \Rightarrow 1.5n^3 - 9.6n^2 + 3.2n - 518 > 0$
<table>
<tr><td>Method 1


From GC graphing, $n > 9.7871$
$\therefore$ least value of $n = 10$</td>
<td>Method 2
From GC Table,


$\therefore$ least value of $n = 10$</td></tr>
</table></td></tr>
</table>

2. YIJC Promo 9758/2020/Q1

Four families, namely Chan, Lee, Tan and Wong, purchase masks, hand sanitisers and thermometers at a pharmacy. The Tan family made their purchase using a voucher, which entitled them to a 20% discount off the total amount paid. The quantity of each item purchased and the total amounts paid are shown in the following table.

	Masks (in boxes)	**Hand sanitisers**	**Thermometers**	**Total Amount paid**
Chan	4	6	5	\$89.30
Lee	2	4	8	\$83.30
Tan	3	5	3	\$50.28
Wong	5	2	4	

Calculate the total amount the Wong family paid. [4]

Solution

Let x, y and z be the selling price of a box of mask, hand sanitiser and thermometer respectively before discount.

Amount that Tan family would have paid before discount

$= \frac{(50.28)}{80} \times 100$

$= \$62.85$

$4x + 6y + 5z = 89.3 --- (1)$

$2x + 4y + 8z = 83.3 --- (2)$

$3x + 5y + 3z = 62.85 --- (3)$

From GC,

$x = 11.25,\ y = 1.80,\ z = 6.70$

Amount paid by Wong family

$= 5 \times \$11.25 + 2 \times \$1.80 + 4 \times \$6.70$

$= \$86.65$

3. 2017/Prelim/NJC/P2/Q1

There are 3 bike-sharing companies in the current market. For each ride, α-bike charges a certain amount per 5 min block or part thereof, β-bike charges a certain amount per 10 min block or part thereof and μ-bike charges a certain amount per 15 min block or part thereof. Rebecca rode each of the bike-sharing companies' bikes once in each month. The table below shows the amount of time Rebecca clocked for each ride and her total spending for each month. In celebration of the company's first anniversary, the pricings in February and March 2017 of μ-bikes are a 5% discount off the immediate previous month's pricing.

	January 2017	**February 2017**	**March 2017**
α-bike	25 min	17 min	36 min
β-bike	30 min	10 min	39 min
μ-bike	15 min	44 min	33 min
Total spending	\$5.70	\$5.72	\$9.71

Determine which bike-sharing company offers the cheapest rate (without any discount) for a 40-min ride. Justify your answer clearly. [4]

Let α, β and μ be the original amount charged per 5 min, 10 min and 15 min block for each ride by α-bike, β-bike and μ-bike respectively.

$5\alpha + 3\beta + \mu = 5.7$ ---------- (1)

$4\alpha + \beta + 3(0.95\mu) = 5.72$ ---------- (2)

$8\alpha + 4\beta + 3(0.95^2\mu) = 9.71$ ---------- (3)

Solving the above 3 equations simultaneously by GC,

$\alpha = \$0.4079329609, \beta = \$0.8402234637, \mu = \$1.139664804$

$\alpha = \$0.41, \beta = \$0.84, \mu = \$1.14$.

Original pricing per 40-min block:

Using calculator values

α-bike: $\$0.4079329609 \times 8 = \$3.26$

β-bike: $\$0.84 \times 4 = \$3.36$

μ-bike: $\$1.14 \times 3 = \$3.42$

Thus, α-bike offers the cheapest rate for a 40-min ride.

4. 2016/Promo/RI/3

Do not use a calculator in answering this question.

Solve the inequality

$$\frac{x+4}{-x^2+2x+3}<1. \qquad [4]$$

$$\frac{x+4}{-x^2+2x+3}<1$$

$$\frac{x+4+x^2-2x-3}{-x^2+2x+3}<0$$

$$\frac{x^2-x+1}{-(x+1)(x-3)}<0,$$

$$-\left(x^2-x+1\right)(x+1)(x-3)<0, \qquad x\neq -1, x\neq 3$$

$$\left(x^2-x+1\right)(x+1)(x-3)>0$$

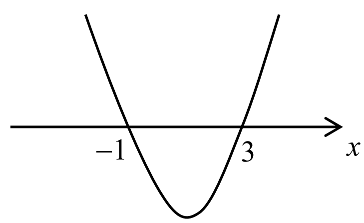
Since

$$x^2-x+1=\left(x-\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2$$

$$=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0 \ \ \forall\, x\in\mathbb{R}$$

Or considering $x^2-x+1=0$, discriminant = $1-4=-3<0$ and coefficient of x^2 is positive, so $x^2-x+1>0\ \forall x\in\mathbb{R}$.

The inequality is reduced to $(x+1)(x-3)>0$, $x\neq -1, x\neq 3$



$\therefore x<-1$ or $x>3$

Deduce the solution of

(i) $\dfrac{x^2+4}{-x^4+2x^2+3}<1,$ [2]

Method 1:	Method 2:
Given $\dfrac{x^2+4}{-x^4+2x^2+3}<1,$ $\dfrac{\left(x^2\right)+4}{-\left(x^2\right)^2+2\left(x^2\right)+3}<1$	Given $\dfrac{x+4}{-x^2+2x+3}<1,$ we replace x with x^2 to obtain $\dfrac{x^2+4}{-x^4+2x^2+3}<1.$ $\therefore x^2<-1$ or $x^2>3$

By replacing x with x^2 in $\dfrac{x+4}{-x^2+2x+3}<1$, $x^2<-1$ or $x^2>3$ Since $x^2\geq 0\ \forall x\in\mathbb{R}$, $x^2<-1$ has no solution. $\therefore x^2>3 \Rightarrow x<-\sqrt{3}$ or $x>\sqrt{3}$	Since $x^2\geq 0\ \forall x\in\mathbb{R}$, $x^2<-1$ has no solution. $\therefore x^2>3 \Rightarrow x<-\sqrt{3}$ or $x>\sqrt{3}$

(ii) $\dfrac{x-4}{x^2+2x-3}<1.$ [2]

<u>Method 1:</u> Given $\dfrac{x-4}{x^2+2x-3}<1$, $\dfrac{-(-x+4)}{-\left(-x^2-2x+3\right)}<1$, $\dfrac{(-x)+4}{-(-x)^2+2(-x)+3}<1$, By replacing x with $-x$ in $\dfrac{x+4}{-x^2+2x+3}<1$, $\therefore -x<-1$ or $-x>3$ $x>1$ or $x<-3$	<u>Method 2:</u> Given $\dfrac{x+4}{-x^2+2x+3}<1$, we replace x with $-x$ to obtain $\dfrac{-x+4}{-(-x)^2+2(-x)+3}<1$ $\dfrac{-(x-4)}{-\left[x^2+2x-3\right]}<1$ $\dfrac{x-4}{x^2+2x-3}<1$ $\therefore -x<-1$ or $-x>3$ $x>1$ or $x<-3$

5. 2017/Prelim/HCI/P1/Q3

(i) By first expressing $3x-x^2-4$ in completed square form, show that $3x-x^2-4$ is always negative for all real values of x. [2]

$$3x-x^2-4=-\left(x^2-3x+4\right)$$
$$=-\left(\left(x-\frac{3}{2}\right)^2+\frac{7}{4}\right)$$
$$=-\left(x-\frac{3}{2}\right)^2-\frac{7}{4}$$

Since $\left(x-\dfrac{3}{2}\right)^2\geq 0$ for all $x\in\mathbb{R}$, $-\left(x-\dfrac{3}{2}\right)^2\leq 0$

Hence $3x-x^2-4=-\left(x-\dfrac{3}{2}\right)^2-\dfrac{7}{4}\leq-\dfrac{7}{4}<0$

$\therefore$ $3x-x^2-4$ is always negative for all values of x.

(ii) Hence, or otherwise, without the use of a calculator, solve the inequality

$$\frac{\left(3x-x^2-4\right)\left(x-1\right)^2}{x^2-2x-5}\le 0 \ ,$$

leaving your answer in exact form. [4]

$$\frac{\left(3x-x^2-4\right)\left(x-1\right)^2}{x^2-2x-5}\le 0$$

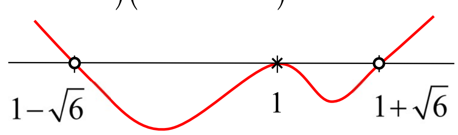
Since $3x-x^2-4$ is always negative, $\dfrac{\left(x-1\right)^2}{x^2-2x-5}\ge 0$

<u>Method 1</u> (Quadratic formula)

Let $x^2-2x-5=0$

$$\therefore\ x=\frac{2\pm\sqrt{(-2)^2-4(1)(-5)}}{2(1)}=\frac{2\pm\sqrt{24}}{2}=1\pm\sqrt{6}$$

Hence $\dfrac{\left(x-1\right)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\ge 0$

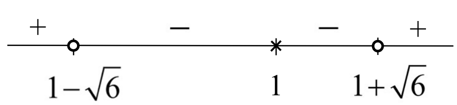


$\therefore\ x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

<u>Method 2</u> (Complete the square)

$$\frac{\left(x-1\right)^2}{\left(x-1\right)^2-6}\ge 0$$

$$\frac{\left(x-1\right)^2}{\left(x-(1-\sqrt{6})\right)\left(x-(1+\sqrt{6})\right)}\ge 0$$



$\therefore\ x<1-\sqrt{6}$ or $x>1+\sqrt{6}$ or $x=1$

6.

ACJC JC1 Promo 9758/2019/Q2

Without using a calculator, solve

(i) $1+\dfrac{6}{x}\geq x$, [3]

(ii) $1+\dfrac{6}{|x|}\geq |x|$. [2]

	(i) $1+\dfrac{6}{x}\geq x\ ,\quad x\neq 0$ $\dfrac{x+6}{x}\geq x$ $\dfrac{x+6}{x}-x\geq 0$ $\dfrac{-x^2+x+6}{x}\geq 0$ $\dfrac{(-x+3)(x+2)}{x}\geq 0$ $(-x+3)(x+2)x\geq 0$ $x\leq -2\quad \text{or}\quad 0<x\leq 3$ **(ii)** Replace *x* with $\lvert x\rvert$, $\lvert x\rvert\leq -2\qquad \text{or}\qquad 0<\lvert x\rvert\leq 3$ (Reject as $\lvert x\rvert>0$) $\qquad \lvert x\rvert\leq 3$ $-3\leq x\leq 3,\ x\neq 0.$

7.

RI JC1 Promo 9758/2019/Q8

(a) Without using a calculator, solve the inequality $\dfrac{2x^2-x}{x^2+3x-4}>1$. [4]

(b) **(i)** On the same axes, sketch the graphs of $y=2+\dfrac{a}{x}$ and $y=2-|x|$, where a is a constant such that $1<a<2$. [3]

(ii) Hence, or otherwise, solve the inequality $2+\dfrac{a}{x}<2-|x|$. [2]

(a) [4]

$$\frac{2x^2-x}{x^2+3x-4}>1$$

$$\frac{2x^2-x-(x^2+3x-4)}{x^2+3x-4}>0$$

$$\frac{x^2-4x+4}{x^2+3x-4}>0$$

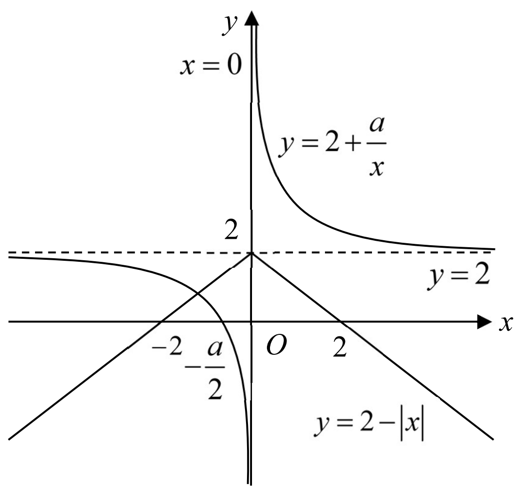
$$\frac{(x-2)^2}{(x-1)(x+4)}>0$$

Since $(x-2)^2>0\ \ \forall\, x\in\mathbb{R}\setminus\{2\}$,

$(x-1)(x+4)>0,\quad x\neq 2$

$\therefore\ x<-4 \quad \text{or} \quad 1<x<2 \quad \text{or} \quad x>2$

(b)
(i) [3]



(b)
(ii)
[2]

At point of intersection,

$$2+\frac{a}{x}=2+x$$

$$x^2=a$$

$$x=-\sqrt{a}\ \ (\because x<0)$$

The solution to the inequality is $-\sqrt{a}<x<0$.

8. MI Promo 9758/2020/PU1/Q2 (*modified (ii)*)

(i) Without the use of a calculator, solve the inequality

$$\frac{3}{x-1} \geq 2x+3 . \qquad [4]$$

(ii) Hence solve exactly the inequality $\dfrac{3}{\left(e^x-1\right)\left(2e^x+3\right)} \geq 1$. [2]

Qn	Solution
(i)	$\dfrac{3}{x-1} \geq 2x+3$ $\dfrac{3}{x-1}-(2x+3) \geq 0$ $\dfrac{3-(2x+3)(x-1)}{x-1} \geq 0$ $\dfrac{3-\left(2x^2+x-3\right)}{x-1} \geq 0$ $\dfrac{-2x^2-x+6}{x-1} \geq 0$ $\dfrac{2x^2+x-6}{x-1} \leq 0$ **Method 1: Factorisation** $\dfrac{(2x-3)(x+2)}{x-1} \leq 0$ ----------------------- **Method 2: Quadratic formula** Let $2x^2+x-6=0$ $x=\dfrac{-1 \pm \sqrt{1^2-4(2)(-6)}}{2(2)}$ $=\dfrac{-1 \pm 7}{4}$ $=\dfrac{3}{2}$ or -2 --------------------------- **Method 3: Completing the square** $2x^2+x-6=2\left(x^2+\dfrac{x}{2}\right)-6$ $=2\left[\left(x+\dfrac{1}{4}\right)^2-\dfrac{1}{16}\right]-6$ $=2\left(x+\dfrac{1}{4}\right)^2-\dfrac{49}{8}$ Let $2x^2+x-6=0$

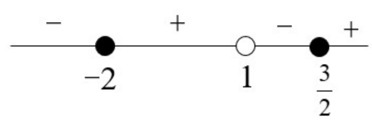
Qn	Solution

i.e. $2\left(x+\frac{1}{4}\right)^2-\frac{49}{8}=0$

$\Rightarrow \left(x+\frac{1}{4}\right)^2=\frac{49}{16}$

$\Rightarrow x+\frac{1}{4}=\pm\frac{7}{4}$

$\Rightarrow x=\frac{3}{2} \text{ or } -2$



Therefore, $x \le -2$ or $1 < x \le \frac{3}{2}$.

(ii)

$\frac{3}{\left(e^x-1\right)\left(2e^x+3\right)} \ge 1$

$\frac{3}{\left(e^x-1\right)} \ge 2e^x+3 \quad (\text{since } 2e^x+3>0)$

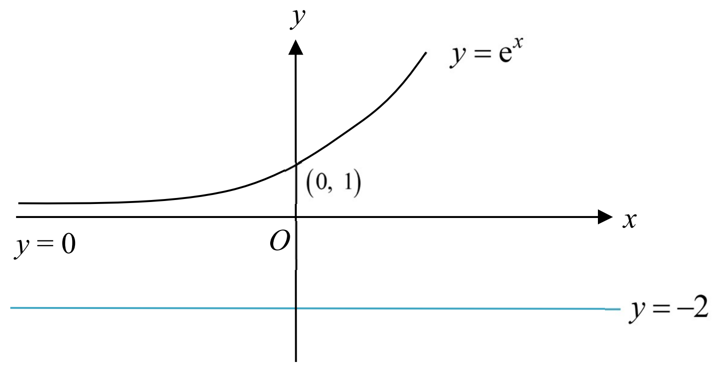
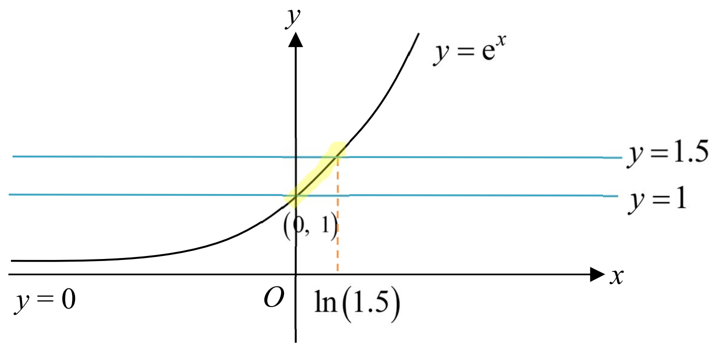
Replace x with e^x (from (i))

Hence $e^x \le -2$ **or** $1 < e^x \le \frac{3}{2}$

$e^x \le -2$ **(rejected)**	$1 < e^x \le \frac{3}{2}$
No solution since $e^x > 0$ for all real values of x)	$1 < e^x$ <u>**AND**</u> $e^x \le \frac{3}{2}$ $x > 0$ <u>**AND**</u> $x \le \ln\left(\frac{3}{2}\right)$ (or 0.40547) **Solution:** $0 < x \le \ln\left(\frac{3}{2}\right)$ (or 0.405)

Refer below for graphical explanations to the above solutions: (LEARN THIS TOO!!!)

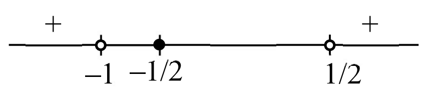
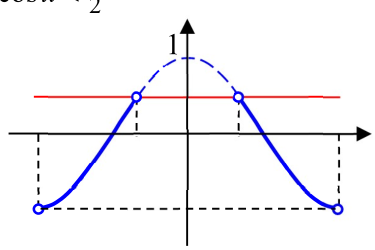
For $e^x \le -2$:

Qn	Solution
	 For $1 < e^x \le \frac{3}{2}$: 

9. 2021/Prelim/HCI/P1/Q

(i) Without using a calculator, solve the inequality $\dfrac{2x^2+3x}{2x^2+x-1} \le \dfrac{1}{2x+2}$. [4]

(ii) Using your answer to part **(i)**, deduce the values of x for the inequality $\dfrac{2\cos^2 x+3\cos x}{2\cos^2 x+\cos x-1} \le \dfrac{1}{2\cos x+2}$, where $-\pi \le x \le \pi$, leaving your answer in exact form. [3]

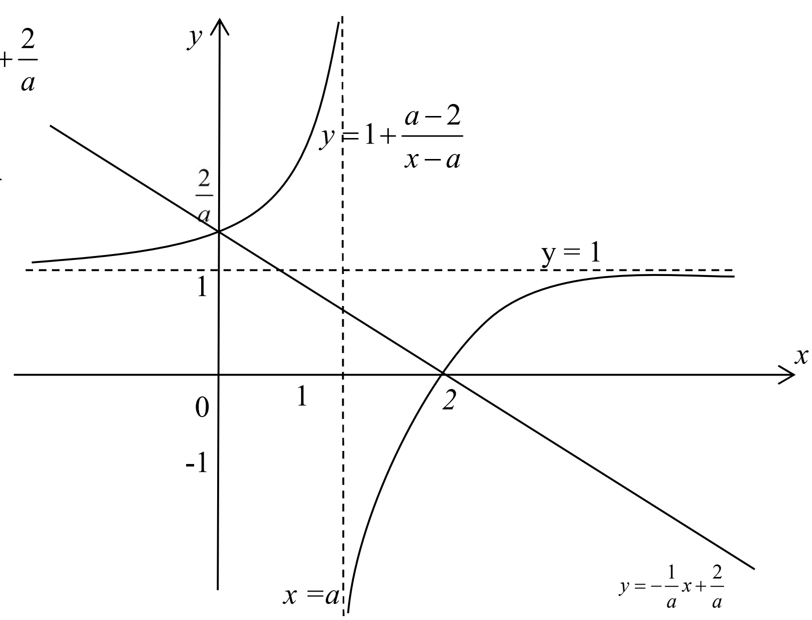
3(i)	$\dfrac{2x^2+3x}{2x^2+x-1} \le \dfrac{1}{2x+2}$ $\dfrac{2x^2+3x}{(2x-1)(x+1)} - \dfrac{1}{2(x+1)} \le 0$ $\dfrac{4x^2+6x-(2x-1)}{2(2x-1)(x+1)} \le 0$ $\dfrac{4x^2+4x+1}{2(2x-1)(x+1)} \le 0$ $\dfrac{(2x+1)^2}{2(2x-1)(x+1)} \le 0$  $\therefore\ -1 < x < \frac{1}{2}$	
3(ii)	Replace x with $\cos x$, $\therefore\ -1 < \cos x < \frac{1}{2}$  When $\cos x = \frac{1}{2}$, $\therefore\ x = \pm\frac{\pi}{3}$ Hence required values of x are $-\pi < x < -\frac{\pi}{3}$ or $\frac{\pi}{3} < x < \pi$	

10. 2021/Prelim/SAJC/P1/Q4

Sketch the graphs of $y=1+\dfrac{a-2}{x-a}$, and $y=-\dfrac{1}{a}x+\dfrac{2}{a}$ on a single diagram, where a is a positive constant and $1<a<2$, showing all asymptotes and axial intercepts clearly. **[4]**

(i) Using the graphs, solve, in terms of a, $1+\dfrac{a-2}{x-a}>-\dfrac{1}{a}x+\dfrac{2}{a}$. **[1]**

(ii) Hence, solve $1+\dfrac{ax-2x}{1-ax}>-\dfrac{1}{ax}+\dfrac{2}{a}$. **[3]**

4 (i)	$y=1+\dfrac{a-2}{x-a}, \quad x\neq a$ Asymptotes: $x=a$, $y=1$ When $x=0$, $y=\dfrac{2}{a}$. When $y=0$, $x=2$ $y=-\dfrac{1}{a}x+\dfrac{2}{a}$ When $x=0$, $y=\dfrac{2}{a}$ When $y=0$, $x=2$
(i)	From the graph, $1+\dfrac{a-2}{x-a}>-\dfrac{1}{a}x+\dfrac{2}{a}$. Ans: $0<x<a$ or $x>2$
(ii)	$1+\dfrac{ax-2x}{1-ax}>-\dfrac{1}{ax}+\dfrac{2}{a}$. $1+\dfrac{x(a-2)}{x(\frac{1}{x}-a)}>-\dfrac{1}{a}\left(\dfrac{1}{x}\right)+\dfrac{2}{a}$ $1+\dfrac{a-2}{\frac{1}{x}-a}>-\dfrac{1}{a}\left(\dfrac{1}{x}\right)+\dfrac{2}{a}$ Let $y=\dfrac{1}{x}$, $1+\dfrac{a-2}{y-a}>-\dfrac{1}{a}y+\dfrac{2}{a}$. From (i), $0<y<a$ or $y>2$ $0<\dfrac{1}{x}<a$ or $\dfrac{1}{x}>2$ $\therefore x>\dfrac{1}{a}$ or $0<x<\dfrac{1}{2}$ 

2 Sequences & Series

Basic Skills

Simplify the following expressions:

(a) $\left(\dfrac{8^x \cdot 4^x}{2^{1-x}}\right)^3$ (b) $\sqrt{\dfrac{e^{4x-1}}{e^{2x-1}}}$ (c) $\left(\dfrac{1}{\sqrt{e^{-2x}}}\right)^x$

(d) $\ln(x-1)+\ln(x-2)$ (e) $\ln(x^3+1)-\ln(x+1)$ (f) $2\ln(x)-\ln(2x)-\ln(x+1)$

(a) $(2^{3x}.2^{2x}.2^{x-1})^3$ $= 2^{18x-3}$	(b) $\left(e^{(4x-1)-(2x-1)}\right)^{\frac{1}{2}}$ $= e^x$	(c) $\left(e^{-(-2x)\left(\frac{1}{2}\right)(x)}\right)$ $= e^{x^2}$
(d) $\ln(x^2-3x+2)$ OR $\ln[(x-1)(x-2)]$	(e) $\ln(x^2-x+1)$	(f) $\ln\frac{x^2}{2x(x+1)} = \ln\frac{x}{2(x+1)}$

Find the partial fraction decomposition for the following:

(a) $\dfrac{2x+1}{x^2+x}$ (b) $\dfrac{4x^2-2x-1}{2x^3+x^2}$ (c) $\dfrac{x^3-x-2}{x^4-1}$

(a) $\frac{1}{x}+\frac{1}{x+1}$	(b) $\dfrac{4}{2x+1}-\dfrac{1}{x^2}$	(c) $\frac{x+1}{x^2+1}-\frac{1}{2(x-1)}+\frac{1}{2(x+1)}$

Tutorial Review

Tutorial 2A Question 3.

Tutorial 2B Questions 5, 7 and 10.

Revision Questions

1. CJC JC1 Promo 9758/2019/Q6

It is given that $f(x)=\dfrac{1}{\sqrt[3]{8-3x}}$.

(i) Find the binomial expansion for $f(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form.
State the range of values of x for which the expansion is valid. [4]

(ii) By putting $x=\dfrac{1}{16}$ into the expansion found in part **(i)**, find an approximate value of $\sqrt[3]{16}$. Leave your answer in the form of $\dfrac{a}{b}$ in its lowest term, where a and b are positive integers to be determined. [3]

(i)	$(8-3x)^{-\frac{1}{3}}$ $=8^{-\frac{1}{3}}\left(1-\frac{3}{8}x\right)^{-\frac{1}{3}}$ $=\frac{1}{2}\left(1-\frac{3}{8}x\right)^{-\frac{1}{3}}$ $=\frac{1}{2}\left(1+\left(-\frac{1}{3}\right)\left(-\frac{3}{8}x\right)+\frac{\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{2}\left(-\frac{3}{8}x\right)^{2}+\ldots\right)$ $=\frac{1}{2}\left(1+\frac{1}{8}x+\frac{2}{64}x^{2}+\ldots\right)$ $=\frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^{2}+\ldots$ $\left\lvert -\frac{3}{8}x\right\rvert<1$ $\Leftrightarrow -\frac{8}{3}<x<\frac{8}{3}$
	<u>Method 2:</u> $\mathrm{f}(x)=(8-3x)^{-\frac{1}{3}}$ $\mathrm{f}'(x)=(8-3x)^{-\frac{4}{3}}$ $\mathrm{f}''(x)=4(8-3x)^{-\frac{7}{3}}$ $\mathrm{f}(0)=\frac{1}{2}$ $\mathrm{f}'(0)=\frac{1}{16}$ $\mathrm{f}''(0)=\frac{1}{32}$ $(8-3x)^{-\frac{1}{3}}=\frac{1}{2}+\frac{1}{16}x+\frac{1}{64}x^{2}+\ldots$ $\left\lvert -\frac{3}{8}x\right\rvert<1\Leftrightarrow -\frac{8}{3}<x<\frac{8}{3}$

(ii)

$$\frac{1}{\sqrt[3]{8-3\left(\frac{1}{16}\right)}}=\frac{1}{2}+\frac{1}{16}\left(\frac{1}{16}\right)+\frac{1}{64}\left(\frac{1}{16}\right)^2+\ldots$$

$$\frac{1}{\sqrt[3]{\frac{125}{16}}}=\frac{1}{2}+\frac{1}{256}+\frac{1}{16384}+\ldots$$

$$\frac{\sqrt[3]{16}}{5}\approx\frac{8192+64+1}{16384}$$

$$\sqrt[3]{16}\approx\frac{41285}{16384}$$

2. 2017/Promo/ACJC/Q2

Expand $(4-x)^{\frac{1}{2}}$ in ascending powers of x, up to and including the term in x^2, and state the set of values of x for which the expansion is valid. [3]

Use the substitution $x=\frac{4}{5}$ to find an approximate value for $\sqrt{5}$ in fraction form. [2]

$$(4-x)^{1/2}=2\left(1-\frac{x}{4}\right)^{1/2}$$

$$=2\left(1-\frac{x}{8}+\frac{\frac{1}{2}.-\frac{1}{2}}{2}\left(-\frac{x}{4}\right)^2+\ldots\right)$$

$$\approx 2\left(1-\frac{x}{8}-\frac{x^2}{128}\right)\ or\ 2-\frac{x}{4}-\frac{x^2}{64}$$

Expansion valid for $\left|\frac{x}{4}\right|<1 \ \Rightarrow$ set is $\{x\in\mathbb{R}:-4<x<4\}$

When $x=\frac{4}{5}$,

$$\left(4-\frac{4}{5}\right)^{1/2}\approx 2-\frac{1}{4}\left(\frac{4}{5}\right)-\frac{1}{64}\left(\frac{4}{5}\right)^2$$

$$\left(\frac{16}{5}\right)^{1/2}\approx 2-\frac{1}{5}-\frac{1}{100}$$

$$\frac{4}{\sqrt{5}}\approx\frac{179}{100}$$

$$\sqrt{5}\approx\frac{400}{179}$$

3. DHS JC1 Promo 9758/2019/Q1

(i) Obtain the expansion of $(1+x)^{\frac{1}{2}}$ up to and including the term in x^2. [1]

(ii) In the triangle ABC, $AC = 1$, $BC = \sqrt{3}$ and angle $ACB = \theta + \frac{\pi}{6}$ radians. Given that θ is a sufficiently small angle, show that

$$AB \approx 1 + p\theta + q\theta^2,$$

where p and q are constants to be determined in exact form. [5]

(i)

$$(1+x)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \ldots$$

ii)

Using cosine rule,

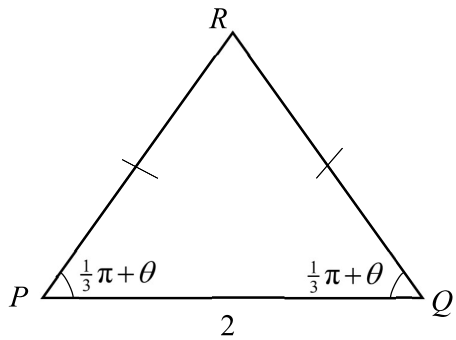
$$AB^2 = 1^2 + \left(\sqrt{3}\right)^2 - 2(1)(\sqrt{3})\cos\left(\theta + \frac{\pi}{6}\right)$$

$$= 4 - 2\sqrt{3}\left(\cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}\right)$$
$$= 4 - 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$$
$$= 4 - 3\cos\theta + \sqrt{3}\sin\theta$$
$$\approx 4 - 3\left(1 - \frac{1}{2}\theta^2\right) + \sqrt{3}\theta$$
$$= 1 + \sqrt{3}\theta + \frac{3}{2}\theta^2$$

$$AB = \left(1 + \sqrt{3}\theta + \frac{3}{2}\theta^2\right)^{\frac{1}{2}}$$
$$= 1 + \frac{1}{2}\left(\sqrt{3}\theta + \frac{3}{2}\theta^2\right) - \frac{1}{8}\left(\sqrt{3}\theta + \frac{3}{2}\theta^2\right)^2 + \ldots$$
$$= 1 + \frac{\sqrt{3}}{2}\theta + \frac{3}{4}\theta^2 - \frac{3}{8}\theta^2 + \ldots$$
$$\approx 1 + \frac{\sqrt{3}}{2}\theta + \frac{3}{8}\theta^2$$

where $p = \frac{\sqrt{3}}{2}$, $q = \frac{3}{8}$

4. 2017/Prelim/DHS/P1/Q4

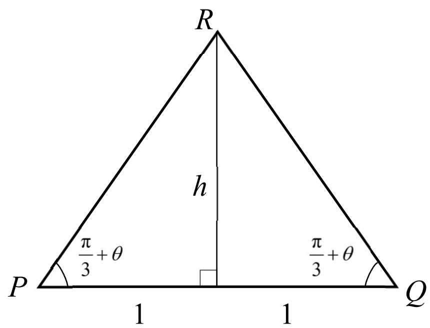


In the isosceles triangle PQR, $PQ = 2$ and the angle QPR = angle $PQR = \left(\frac{1}{3}\pi+\theta\right)$ radians. The area of triangle PQR is denoted by A.

Given that θ is a sufficiently small angle, show that

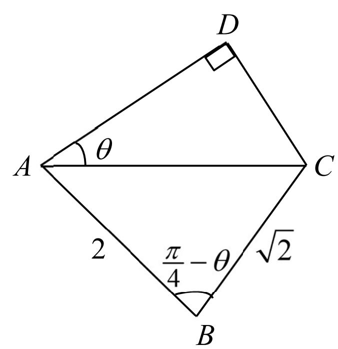
$$A = \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}(\tan\theta)} \approx a+b\theta+c\theta^2,$$

for constants a, b and c to be determined in exact form. [5]



	$h=\tan\left(\frac{\pi}{3}+\theta\right)$ $A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$ $=\dfrac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\dfrac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}$ (shown) $\approx\dfrac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$ $=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$ $\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$ $=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$

5. 2017/Prelim/AJC/P1/Q3



The diagram above shows a quadrilateral $ABCD$, where AB = 2, $BC=\sqrt{2}$, angle $ABC=\frac{\pi}{4}-\theta$ radians and angle $CAD=\theta$ radians.
Show that

$$AC=\sqrt{6-4\cos\theta-4\sin\theta}.$$ [2]

Given that θ is small enough for θ^3 and higher powers of θ to be neglected, show that

$$AD\approx a+b\theta+c\theta^2,$$

where a, b and c are constants to be determined. [5]

Consider triangle ABC,

$$AC^2 = 4+2-2(2)\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$$

$$= 6-4\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right) = 6-4\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$

$AC = \sqrt{6-4\cos\theta-4\sin\theta}$ (shown)

Consider triangle ACD,

$$\cos\theta = \frac{AD}{AC}$$

$$AD = \cos\theta\sqrt{6-4\cos\theta-4\sin\theta}$$

Since θ is small, $\sin\theta \approx \theta$, $\cos\theta \approx 1-\dfrac{\theta^2}{2}$,

$$AD \approx \left(1-\frac{\theta^2}{2}\right)\sqrt{6-4\left(1-\frac{\theta^2}{2}\right)-4\theta}$$

$$= \left(1-\frac{\theta^2}{2}\right)\left(2+2\theta^2-4\theta\right)^{\frac{1}{2}}$$

$$= \left(1-\frac{\theta^2}{2}\right)\sqrt{2}\left(1+\theta^2-2\theta\right)^{\frac{1}{2}}$$

$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}(\theta^2-2\theta)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\theta^2-2\theta)^2+\ldots\right)$$

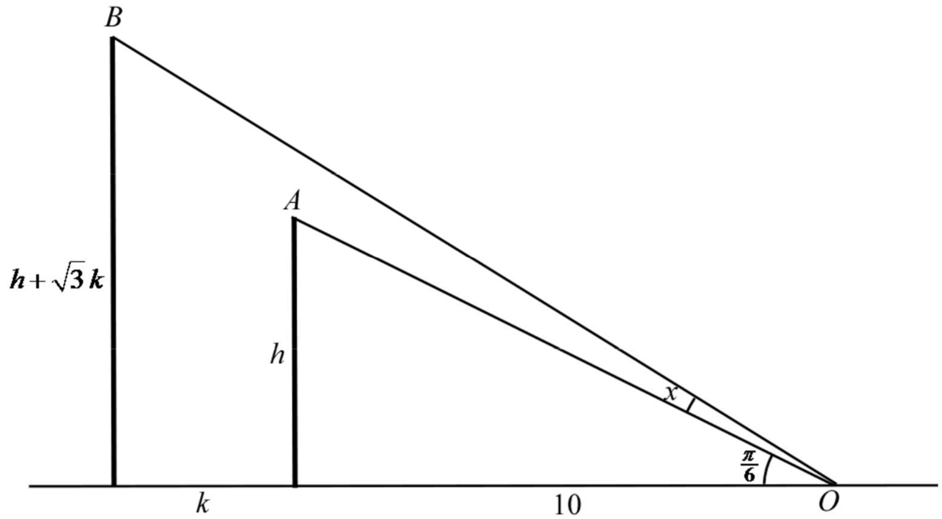
$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}\theta^2-\theta-\frac{1}{2}\theta^2+\ldots\right)$$

$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1-\theta+\ldots\right)$$

$$= \sqrt{2}\left(1-\theta-\frac{\theta^2}{2}+\ldots\right)$$

$$\approx \sqrt{2}-\sqrt{2}\theta-\frac{\sqrt{2}}{2}\theta^2$$

6. 2017/Prelim/JJC/P1/Q10



A laser from a fixed point O on a flat ground projects light beams to the top of two vertical structures A and B as shown above. To project the beam to the top of A, the laser makes an angle of elevation of $\frac{\pi}{6}$ radians. To project the beam to the top of B, the laser makes an angle of elevation of $\left(\frac{\pi}{6}+x\right)$ radians. The two structures A and B are of heights h m and $\left(h+\sqrt{3}k\right)$ m respectively and are 10 m and $(10+k)$ m away from O respectively.

(i) Show that the length of the straight beam from O to A is $\frac{20}{\sqrt{3}}$ m. [1]

(ii) Show that the length of AB is $2k$ m and that the angle of elevation of B from A is $\frac{\pi}{3}$ radians. [3]

(iii) Hence, using the sine rule, show that $k=\dfrac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$. [2]

(iv) If x is sufficiently small, show that $k\approx\frac{20}{\sqrt{3}}\left(x+ax^2\right)$, where a is a constant to be determined. [6]

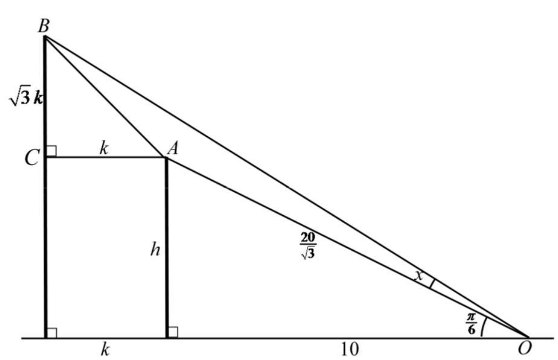
(i)

$$\cos\frac{\pi}{6}=\frac{10}{OA}$$

$$\frac{\sqrt{3}}{2}=\frac{10}{OA}$$

$$OA=\frac{20}{\sqrt{3}}\text{ m}$$

(Shown)



(ii)

$$AB = \sqrt{k^2+3k^2} = \sqrt{4k^2} = 2k \qquad \text{(Shown)}$$

$$\angle BAC = \tan^{-1}\frac{\sqrt{3}\,k}{k} = \tan^{-1}\sqrt{3}$$

$$= \frac{\pi}{3} \qquad \text{(Shown)}$$

(iii)

$$\angle CBO = \tfrac{\pi}{2} - \left(\tfrac{\pi}{6}+x\right) = \tfrac{\pi}{3} - x$$

$$\angle CBA = \tfrac{\pi}{2} - \tfrac{\pi}{3} = \tfrac{\pi}{6}$$

$$\angle ABO = \tfrac{\pi}{3} - x - \tfrac{\pi}{6} = \tfrac{\pi}{6} - x$$

Or:

$$\angle BAO = 2\pi - \tfrac{\pi}{2} - \tfrac{\pi}{3} - \tfrac{\pi}{3} \quad (\angle \text{ at a pt})$$

$$= \tfrac{5\pi}{6}$$

$$\angle ABO = \pi - x - \tfrac{5\pi}{6} = \tfrac{\pi}{6} - x$$

In ΔABO,

$$\frac{2k}{\sin x} = \frac{\frac{20}{\sqrt{3}}}{\sin\left(\frac{\pi}{6}-x\right)}$$

$$k = \frac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$$

(iv)

$$k = \frac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$$

$$= \frac{10\sin x}{\sqrt{3}\left(\sin\frac{\pi}{6}\cos x - \cos\frac{\pi}{6}\sin x\right)}$$

$$\approx \frac{10x}{\sqrt{3}\left[\frac{1}{2}\left(1-\frac{x^2}{2}\right) - \frac{\sqrt{3}}{2}x\right]}$$

$$= \frac{10x}{\frac{\sqrt{3}}{2}\left[\left(1-\frac{x^2}{2}\right) - \sqrt{3}\,x\right]}$$

$$= \frac{20x}{\sqrt{3}}\left[1-\left(\sqrt{3}x+\frac{x^2}{2}\right)\right]^{-1}$$

$$\approx \frac{20x}{\sqrt{3}}\left(1+\sqrt{3}\,x\right)$$

$$= \underline{\underline{\frac{20}{\sqrt{3}}\left(x+\sqrt{3}\,x^2\right)}}$$

7. 2017/Prelim/TPJC/P1/Q7

(i) Express $\dfrac{1}{r^2-1}$ in partial fractions, and deduce that

$$\frac{1}{r(r^2-1)}=\frac{1}{2}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right].$$ [2]

$$\frac{1}{r^2-1}=\frac{1}{2(r-1)}-\frac{1}{2(r+1)}$$

$$\begin{aligned}\frac{1}{r(r^2-1)}&=\frac{1}{r}\left[\frac{1}{2(r-1)}-\frac{1}{2(r+1)}\right]\\&=\frac{1}{2}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right]\end{aligned}$$

(ii) Hence, find the sum, S_n, of the first n terms of the series

$$\frac{1}{2\times3}+\frac{1}{3\times8}+\frac{1}{4\times15}+\ldots\,.$$

[4]

$$\begin{aligned}S_n&=\frac{1}{2\times3}+\frac{1}{3\times8}+\frac{1}{4\times15}+\ldots+(n\text{th term})\\
&=\sum_{r=2}^{n+1}\frac{1}{r(r^2-1)}\\
&=\frac{1}{2}\sum_{r=2}^{n+1}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right]\\
&=\frac{1}{2}\Bigg[\frac{1}{2\times1}-\frac{1}{2\times3}\\
&\quad+\frac{1}{3\times2}-\frac{1}{3\times4}\\
&\quad+\frac{1}{4\times3}-\frac{1}{4\times5}\\
&\quad\bullet\\
&\quad\bullet\\
&\quad\bullet\\
&\quad+\frac{1}{(n-1)\times(n-2)}-\frac{1}{(n-1)\times n}\\
&\quad+\frac{1}{(n)\times(n-1)}-\frac{1}{n\times(n+1)}\\
&\quad+\frac{1}{(n+1)\times n}-\frac{1}{(n+1)\times(n+2)}\Bigg]\\
&=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]\\
&=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}\end{aligned}$$

(iii) Explain why the series converges, and write down the value of the sum to infinity. [2]

As $n \to \infty$, $\dfrac{1}{2(n+1)(n+2)} \to 0$.

$$\frac{1}{4} - \frac{1}{2(n+1)(n+2)} \to \frac{1}{4}$$

Sum to infinity $= \dfrac{1}{4}$

(iv) Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]

$$(0<)\frac{1}{4} - S_n < 0.0025$$

$$\Rightarrow (0<)\frac{1}{4} - \left[\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right] < 0.0025$$

$$\Rightarrow (0<)\frac{1}{2(n+1)(n+2)} < 0.0025$$

$$\Rightarrow (n+1)(n+2) > 200$$

Using G.C.

$n < -15.651$ or $n > 12.651$

Since $n \in \mathbb{Z}^+$,

Smallest value of $n = 13$

8. Given that $u_n = \ln\left(\dfrac{1+x^{n+1}}{1+x^n}\right)$, where $x > -1$.

(i) Show that $\displaystyle\sum_{n=1}^{N} u_n = \ln\left(\dfrac{1+x^{N+1}}{1+x}\right)$.

(ii) Hence find $\displaystyle\sum_{n=1}^{\infty} u_n$ in terms of x when $-1 < x < 1$.

. (i)

Method 1:

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln\left(\frac{1+x^{n+1}}{1+x^n}\right)$$
$$= \sum_{n=1}^{N}\left[\ln\left(1+x^{n+1}\right) - \ln\left(1+x^n\right)\right]$$
$$= \ln\left(1+x^2\right) - \ln\left(1+x\right)$$
$$+\ln\left(1+x^3\right) - \ln\left(1+x^2\right)$$
$$+ \quad \ldots\ldots\ldots\ldots \qquad \textbf{M1}$$
$$+\ln\left(1+x^N\right) - \ln\left(1+x^{N-1}\right)$$
$$+\ln\left(1+x^{N+1}\right) - \ln\left(1+x^N\right)$$
$$= \ln\left(1+x^{N+1}\right) - \ln\left(1+x\right) \qquad \textbf{M0.5}$$
$$= \ln\left(\frac{1+x^{N+1}}{1+x}\right) \quad \text{(Shown)} \qquad \textbf{M0.5}$$

Method 2:

$$\sum_{n=1}^{N} u_n = \sum_{n=1}^{N} \ln\left(\frac{1+x^{n+1}}{1+x^n}\right)$$
$$= \ln\left(\frac{1+x^2}{1+x}\right) + \ln\left(\frac{1+x^3}{1+x^2}\right) + \ln\left(\frac{1+x^4}{1+x^3}\right)$$
$$+ \ldots + \ln\left(\frac{1+x^N}{1+x^{N-1}}\right) + \ln\left(\frac{1+x^{N+1}}{1+x^N}\right)$$
$$= \ln\left[\left(\frac{1+x^2}{1+x}\right)\times\left(\frac{1+x^3}{1+x^2}\right)\times\left(\frac{1+x^4}{1+x^5}\right)\times\cdots\right.$$
$$\left.\times\left(\frac{1+x^N}{1+x^{N-1}}\right)\left(\frac{1+x^{N+1}}{1+x^N}\right)\right] \qquad \textbf{M1}$$
$$= \ln\left(\frac{1+x^{N+1}}{1+x}\right) \quad \text{(Shown)} \qquad \textbf{A0.5}$$

(ii) $\displaystyle\sum_{n=1}^{\infty} u_n = \lim_{N\to\infty}\sum_{n=1}^{N} u_n = \lim_{N\to\infty}\left[\ln\left(\frac{1+x^{N+1}}{1+x}\right)\right] = \ln\left(\frac{1}{1+x}\right)$ or $-\ln(1+x)$

9. 2018/Prelim/SAJC/P1/Q3

(i) Show that $\mathrm{e}^{-r} - 2\mathrm{e}^{-r+1} + \mathrm{e}^{-r+2} = \dfrac{(\mathrm{e}-1)^2}{\mathrm{e}^r}$. [1]

(ii) Hence find $\displaystyle\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of N. [4]

(iii) Using your result in part **(ii)**, find $\displaystyle\sum_{r=9}^{N+1}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}$ in terms of e. [2]

(i)	$\mathrm{e}^{-r} - 2\mathrm{e}^{-r+1} + \mathrm{e}^{-r+2}$ $= \mathrm{e}^{-r}\left(1 - 2\mathrm{e} + \mathrm{e}^2\right)$ $= \dfrac{(\mathrm{e}-1)^2}{\mathrm{e}^r}$

(ii)	$\begin{aligned} &\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}} \\ &=\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r}\mathrm{e}^{1}} \\ &=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r}} \\ &=\frac{1}{\mathrm{e}}\sum_{r=1}^{N}\left(\mathrm{e}^{-r}-2\mathrm{e}^{-r+1}+\mathrm{e}^{-r+2}\right) \\ &=\frac{1}{\mathrm{e}}[\mathrm{e}^{-1}-2\mathrm{e}^{0}\ \ +\mathrm{e}^{1} \\ &\quad+\mathrm{e}^{-2}-2\mathrm{e}^{-1}\ \ +\mathrm{e}^{0} \\ &\quad+\mathrm{e}^{-3}-2\mathrm{e}^{-2}\ \ +\mathrm{e}^{-1} \\ &\quad+\mathrm{e}^{-4}-2\mathrm{e}^{-3}\ \ +\mathrm{e}^{-2} \\ &\quad+\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad+\mathrm{e}^{-N+2}-2\mathrm{e}^{-N+3}\ \ +\mathrm{e}^{-N+4} \\ &\quad+\mathrm{e}^{-N+1}-2\mathrm{e}^{-N+2}\ \ +\mathrm{e}^{-N+3} \\ &\quad+\mathrm{e}^{-N}-2\mathrm{e}^{-N+1}\ \ +\mathrm{e}^{-N+2}] \\ &=\frac{1}{\mathrm{e}}\left(\mathrm{e}-1+\mathrm{e}^{-N}-\mathrm{e}^{-N+1}\right) \\ &=1-\frac{1}{\mathrm{e}}+\mathrm{e}^{-N-1}-\mathrm{e}^{-N} \end{aligned}$
(iii)	$\begin{aligned} \sum_{r=9}^{N+1}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}} &=\sum_{r=1}^{N+1}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}}-\sum_{r=1}^{8}\frac{(\mathrm{e}-1)^2}{\mathrm{e}^{r+1}} \\ &=1-\frac{1}{\mathrm{e}}+\mathrm{e}^{-N-2}-\mathrm{e}^{-N-1}-\left(1-\frac{1}{\mathrm{e}}+\mathrm{e}^{-9}-\mathrm{e}^{-8}\right) \\ &=\mathrm{e}^{-N-2}-\mathrm{e}^{-N-1}-\frac{1}{\mathrm{e}^{9}}+\frac{1}{\mathrm{e}^{8}} \end{aligned}$

10. HCI/2016/Prelim/I/Q4

Prove that $\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}=\sqrt{n^2+2n}-\sqrt{n^2-1}$. [2]

Hence find $\displaystyle\sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$. [3]

(a) Deduce the value of $\displaystyle\sum_{n=2}^{N}\frac{2n-1}{\sqrt{n^2-2n}+\sqrt{n^2-1}}$. [3]

(b) Show that $\displaystyle\sum_{n=1}^{N}\frac{2n+1}{2n-1}>\sqrt{N^2+2N}$. [1]

	<u>Method 1</u> $\dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} \times \dfrac{\sqrt{n^2+2n}-\sqrt{n^2-1}}{\sqrt{n^2+2n}-\sqrt{n^2-1}}$ $= \dfrac{(2n+1)\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)}{\left(n^2+2n\right)-\left(n^2-1\right)}$ $= \sqrt{n^2+2n}-\sqrt{n^2-1}$ <u>Method 2</u> $\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)\left(\sqrt{n^2+2n}+\sqrt{n^2-1}\right)$ $= \left(n^2+2n-\left(n^2-1\right)\right)$ $= 2n+1$ $\therefore \dfrac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \sqrt{n^2+2n}-\sqrt{n^2-1}$
	$\displaystyle\sum_{n=1}^{N} \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $\displaystyle= \sum_{n=1}^{N}\left(\sqrt{n^2+2n}-\sqrt{n^2-1}\right)$ $= \begin{bmatrix} \cancel{\sqrt{3}}-\sqrt{0} \\ +\cancel{\sqrt{8}}-\cancel{\sqrt{3}} \\ +\cancel{\sqrt{15}}-\cancel{\sqrt{8}} \\ \ldots \\ +\sqrt{N^2+2N}-\cancel{\sqrt{N^2-1}} \end{bmatrix}$ $= \sqrt{N^2+2N}$
(a)	Replace n by $n+1$, $\displaystyle\sum_{n=2}^{N} \frac{2n-1}{\sqrt{n^2-1}+\sqrt{n(n-2)}}$ $\displaystyle= \sum_{n=1}^{N-1} \frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}}$ $= \sqrt{(N-1)^2+2(N-1)}$ $= \sqrt{N^2-1}$
(b)	Notice that $\sqrt{n^2+2n} > n$ and $\left(\sqrt{n^2-1}\right)^2-(n-1)^2 = 2n-2 \geq 0$. $\Rightarrow \sqrt{n^2-1} \geq n-1$ $\Rightarrow \sqrt{n^2+2n}+\sqrt{n^2-1} > 2n-1$ $\Rightarrow \dfrac{1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} < \dfrac{1}{2n-1}$ $\displaystyle\therefore \sum_{n=1}^{N}\frac{2n+1}{2n-1} > \sum_{n=1}^{N}\frac{2n+1}{\sqrt{n^2+2n}+\sqrt{n^2-1}} = \sqrt{N^2+2N}$

11. 2018/Prelim/PJC/P2/Q4

The rth term of a sequence is given by $u_r = \dfrac{1}{r!}$.

(i) Show that $u_r - u_{r+1} = \dfrac{1}{r! + (r-1)!}$. [2]

(ii) Hence find $\displaystyle\sum_{r=1}^{N} \frac{1}{r! + (r-1)!}$. [2]

(iii) Give a reason why the series in **(ii)** is convergent and state the sum to infinity. [2]

(iv) Use your answer to **(ii)** to find $\displaystyle\sum_{r=1}^{N+1} \frac{1}{r! + (r+1)!}$. [3]

(v) Deduce that $\displaystyle\sum_{r=1}^{N} \frac{1}{r!} < 2$. [3]

(i)
$$\begin{aligned}
u_r - u_{r+1} &= \frac{1}{r!} - \frac{1}{(r+1)!} \\
&= \frac{1}{r!} - \frac{1}{(r+1)r!} \\
&= \frac{(r+1)-1}{(r+1)r!} \\
&= \frac{r}{(r+1)r!} \\
&= \frac{1}{(r+1)(r-1)!} \\
&= \frac{1}{r(r-1)! + (r-1)!} \\
&= \frac{1}{r! + (r-1)!} \quad \text{(Shown)}
\end{aligned}$$

(ii)
$$\sum_{r=1}^{N} \frac{1}{r! + (r-1)!} = \sum_{r=1}^{N} u_r - u_{r+1}$$

$$\begin{aligned}
&= u_1 - \cancel{u_2} \\
&+ \cancel{u_2} - \cancel{u_3} \\
&+ \cancel{u_3} - u_4 \\
&\quad \ldots.. \\
&+ \cancel{u_{N-2}} - \cancel{u_{N-1}} \\
&+ \cancel{u_{N-1}} - \cancel{u_N} \\
&+ \cancel{u_N} - u_{N+1} \\
&= u_1 - u_{N+1} \\
&= 1 - \frac{1}{(N+1)!}
\end{aligned}$$

(iii) As $N \to \infty$, $\dfrac{1}{(N+1)!} \to 0$ $\displaystyle\sum_{r=1}^{N} \frac{1}{r!+(r-1)!} \to 1$ which is finite, hence

$\displaystyle\sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$ converges and the sum to infinity is 1

(iv)

$$\sum_{r=1}^{N+1} \frac{1}{r!+(r+1)!} = \sum_{r=2}^{N+2} \frac{1}{r!+(r-1)!}$$

$$= \sum_{r=1}^{N+2} \frac{1}{r!+(r-1)!} - \frac{1}{2}$$

$$= 1 - \frac{1}{(N+3)!} - \frac{1}{2}$$

$$= \frac{1}{2} - \frac{1}{(N+3)!}$$

(v) Since $r! > (r-1)!$

$$\frac{1}{r!+r!} < \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{r!+r!} < \sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{2r!} < \sum_{r=1}^{N} \frac{1}{r!+(r-1)!}$$

$$\sum_{r=1}^{N} \frac{1}{r!} < 2\sum_{r=1}^{N} \frac{1}{r!+(r-1)!} = 2\left(1 - \frac{1}{(N+1)!}\right) < 2 \text{ since } \frac{1}{(N+1)!} > 0.$$

12. 2017/Prelim/NYJC/P2/Q2

(a) Find the set of values of θ lying in the interval $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ such that the sum to infinity of the geometric series $1 + \tan\theta + \tan^2\theta + \ldots$ is greater than 2. [5]

For sum to infinity to exist,

$$|\tan\theta| < 1$$

$$-1 < \tan\theta < 1$$

$$-\frac{\pi}{4} < \theta < \frac{\pi}{4}$$

$$\frac{1}{1-\tan\theta} > 2$$

$$0 < 1 - \tan\theta < \frac{1}{2}$$

$$\tan\theta > \frac{1}{2} \Rightarrow \theta > 0.464$$

Since $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, therefore $\{\theta \in \mathbb{R} \mid 0.464 < \theta < 0.786\}$ or $\theta : (0.464, 0.786)$

(b) The sum of the first n terms of a positive arithmetic sequence $\{u_n\}$ is given by the formula $S_n = 4n^2 - 2n$. Three terms of this sequence, u_2, u_m and u_{32}, are consecutive terms in a geometric sequence. Find m. [4]

	$u_1 = S_1 = 2 \Rightarrow a = 8$ $u_2 = S_2 - S_1 = 10 \Rightarrow d = 8$ $u_{32} = a + (32-1)d = 2 + (32-1)8 = 250$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2} = \text{constant}$ $\Rightarrow (u_m)^2 = (10)(250) = 2500$ $u_m = 50$ (since it is a positive sequence) $50 = 2 + (m-1)8 \Rightarrow m = 7$	Alternatively, $u_n = S_n - S_{n-1}$ $= 4n^2 - 2n - \left[4(n-1)^2 - 2(n-1)\right]$ $= 8n - 6$ $\frac{u_{32}}{u_m} = \frac{u_m}{u_2}$ $\frac{8(32)-6}{8m-6} = \frac{8m-6}{8(2)-6}$ $(8m-6)^2 = (250)(10) = 2500$ $m = 7$ or $m = -5.5$ (rejected as m is a positive inte

13. 2021/Prelim/NJC/P1/Q7

A sequence of positive numbers $u_1, u_2, u_3, \ldots$ is a strictly increasing arithmetic progression. It is given that the first term is a and the ninth term is b.

(i) Find u_3 in terms of a and b and show that $u_3 + u_5 + u_7 = \frac{3}{2}(b+a)$. [3]

(ii) Given also that a, u_3 and b are consecutive terms of a geometric progression, express b in terms of a. [3]

(iii) Hence, determine if a sequence that consists of consecutive terms $\ln(u_3), \ln(u_5)$ and $\ln(u_7)$ is an arithmetic progression. [2]

Q7	Suggested Solutions
(i)	$a + (9-1)d = b$ $d = \frac{b-a}{8}$ $u_3 = a + \frac{2(b-a)}{8} = \frac{3a+b}{4}$

	$u_3+u_5+u_7$ $=\left(a+\frac{2(b-a)}{8}\right)+\left(a+\frac{4(b-a)}{8}\right)+\left(a+\frac{6(b-a)}{8}\right)$ $=3a+\frac{12(b-a)}{8}$ $=\frac{3}{2}(b+a)$
(ii)	$\frac{u_3}{a}=\frac{b}{u_3}$ $ab=(u_3)^2$ $=(a+2d)^2$ $=\left(a+\frac{b-a}{4}\right)^2$ $=\left(\frac{3a+b}{4}\right)^2$ $=\frac{9a^2+6ab+b^2}{16}$ $\frac{9a^2+6ab+b^2}{16}-ab=0$ $\frac{9a^2+6ab+b^2-16ab}{16}=0$ $9a^2-10ab+b^2=0$ $(9a-b)(a-b)=0$ Since the arithmetic progression is strictly increasing, $b\neq a$. Hence $b=9a$.
(iii)	$\ln(u_5)-\ln(u_3)=\ln\left(\frac{a+\frac{4(9a-a)}{8}}{a+\frac{2(9a-a)}{8}}\right)$ $=\ln\left(\frac{a+4a}{a+2a}\right)$ $=\ln\left(\frac{5}{3}\right)$

$$\ln(u_7)-\ln(u_5)=\ln\left(\frac{a+\frac{6(9a-a)}{8}}{a+\frac{4(9a-a)}{8}}\right)$$

$$=\ln\left(\frac{a+6a}{a+4a}\right)$$

$$=\ln\left(\frac{7}{5}\right)$$

Since $\ln(u_7)-\ln(u_5)\neq\ln(u_5)-\ln(u_3)$, the terms are not consecutive terms of an arithmetic progression.

14. MI PU2 Promo 9758/2019/01/Q11

Benny bought an apartment that cost \$500,000. He paid a deposit of 5% of the cost to the property developer. For the remaining 95% of the cost, he took a loan from the bank on 1 January 2011 to pay for it. The bank offered him two repayment packages.

For Package A, Benny paid \$2000 to the bank on 1 January 2011. On the first day of each subsequent month, he paid \$100 more than the previous month. Thus on 1 February, he paid \$2100 and on 1 March, he paid \$2200, and so on.

(i) On what date did Benny complete his loan repayment under Package A? [5]

For Package B, Benny paid \$*x* to the bank on the first day of each month, starting from 1 January 2011. The bank charges interest at a rate of 0.5% per month of the outstanding loan amount on the last day of each month, starting on 31 January 2011.

(ii) Show that the outstanding amount Benny owed in dollars, at the end of the *n*th month under Package B can be expressed as $(475000)1.005^n-201x(1.005^n-1)$. [3]

(iii) Hence, find the least amount of monthly payment, $\$x$, if Benny wished to repay his loan completely by the end of 3 years under Package B. Leave your answer to the nearest dollar. [2]

(iv) Using your answer in part **(iii)**, find the amount of interest Benny paid at the end of 3 years under Package B, giving your answer to the nearest dollar. [2]

(i) Loan amount: $\$500,000\times0.95=\$475,000$

To repay his loan completely:

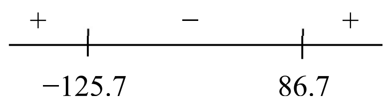
Amount paid $\geq 475,000$

$$\frac{n}{2}\left[4000+(n-1)(100)\right]\geq 475,000$$

$$n(3900+100n)\geq 950,000$$

$$n^2+39n-9500\geq 0$$

Method 1:



$n \le -118.9$ or $n \ge 79.9$

Since n is a positive integer, $\therefore n = 80$ months

Method 2: (Table Method)

n	$n^2+39n-9500$
79	-178
80	20
81	220

$\therefore n = 80$ months

Benny would finish paying up her loan on 1st Aug 2017.

(ii) Package B:

n	At the beginning of month	At the end of month
1	$a-x$, where $a = 475000$	$1.005(a-x)$
2	$1.005(a-x)-x$	$1.005^2(a-x)-1.005x$
3	$1.005^2(a-x)-1.005x-x$	$1.005^3(a-x)-1.005^2x$ $-1.005x$
....		
n	$1.005^{n-1}(a-x)-1.005^{n-2}x$ $-1.005^{n-3}x-\ldots-x$	$1.005^{n}(a-x)-1.005^{n-1}x$ $-1.005^{n-2}x-\ldots-1.005x$

Therefore, the amount of money owed at the end of the nth month:

$= 1.005^n a - 1.005^n x - 1.005^{n-1}x - \ldots - 1.005x$ --- (*)

$= 1.005^n a - x\left(1.005 + 1.005^2 + \ldots + 1.005^{n-1} + 1.005^n\right)$

$= 1.005^n a - 1.005x\left(\dfrac{1.005^n - 1}{1.005 - 1}\right)$

$= 1.005^n a - 201x\left(1.005^n - 1\right)$

$= (475000)1.005^n - 201x\left(1.005^n - 1\right)$ (shown)

(iii) Let $n = 36$,

$(475000)1.005^{36} - 201x\left(1.005^{36} - 1\right) \le 0$

$201x\left(1.005^{36} - 1\right) \ge 568423.2493$

$x \ge 14378.52765$

Minimum amount is \$14,379 (nearest dollar).

(iv) Total repayment amount in Package B = $(\$14,378.53 \times 3 \times 12) = \$517,627.08$

Interest paid in Package B = $\$517,627.08 - \$475,000$

= \$42,627.08

= \$42,627 (nearest dollar)

15. 2017/Prelim/NYJC/P1/Q1

A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where n and i are positive integers.

1			
1	2		
1	2	3	
$\vdots$			$\ddots$

Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

Sum of numbers in kth row = $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

$$\begin{aligned}
\text{Required sum} &= \sum_{k=1}^{n} \frac{k(k+1)}{2} \\
&= \frac{1}{2}\sum_{k=1}^{n}\left(k^2+k\right) \\
&= \frac{1}{12}n(n+1)(2n+1)+\frac{1}{4}n(n+1) \\
&= \frac{1}{12}n(n+1)(2n+1+3) \\
&= \frac{1}{6}n(n+1)(n+2)
\end{aligned}$$

16. 2017/Prelim/NJC/P1/Q4

A researcher is investigating the elasticity of a new material. In the experiment, he stretched an extensible string of length 30 cm using a machine.

Each stretch is followed by a contraction. The initial stretch leads to an elongation of 10 cm and is followed by a contraction of 0.1 cm. The elongation resulting from each subsequent stretch is $\frac{10}{11}$ of the elongation caused by the previous stretch. Each subsequent contraction is 0.001 cm less than the previous contraction.

(i) Show that the length of the string after two stretches is 48.892 cm correct to 3 decimal places. [2]

Stretch count, n	Length of string before stretch	Elongation after stretch, u_n	Contraction after stretch, t_n	Final length of string
1	30	10	0.1	$30+10-0.1=39.9$
2	39.9	$10\left(\frac{10}{11}\right)$	$0.1-0.001$ $=0.099$	$39.9+10\left(\frac{10}{11}\right)-0.099$ $=48.8919$

$$30+10-0.1+10\left(\frac{10}{11}\right)-0.099$$
$$=48.8919$$
$$=48.892\ (3\text{ dp})$$

(ii) Find the length of the string after it has been stretched n times, in terms of n. [3]

Total length of string

$$=30+u_1+u_2+...+u_n-(t_1+t_2+...+t_n)$$
$$=30+10+10\left(\frac{10}{11}\right)+...+10\left(\frac{10}{11}\right)^{n-1}$$
$$-\big(0.1+(0.1-0.001(1))+...+(0.1-0.001(n))\big)$$

$$\sum_{i=1}^{n}u_i=\frac{10\left[1-\left(\frac{10}{11}\right)^n\right]}{1-\frac{10}{11}}=110\left(1-\left(\frac{10}{11}\right)^n\right)$$

$$\sum_{i=1}^{n}t_i=\frac{n}{2}\left[2(0.1)+(n-1)(-0.001)\right]=\frac{n}{2000}(201-n)$$

Length of string after n stretches $=30+110\left(1-\left(\frac{10}{11}\right)^n\right)-\frac{n}{2000}(201-n)$

(iii) The string loses its elasticity completely when contraction exceeds elongation in a stretch. Find the minimum number of stretches for the string to lose its elasticity. [2]

$t_n > u_n$

$0.1+(n-1)(-0.001)>(10)\left(\frac{10}{11}\right)^{n-1}$

$0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}>0$

Using GC,

when $n=58$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=-7.1364\times10^{-4}$

when $n=59$, $0.1+(n-1)(-0.001)-(10)\left(\frac{10}{11}\right)^{n-1}=0.00226$

Therefore, the minimum number of stretches is 59.

(iv) The researcher coats a new string of the same initial length with another material. Now the string does not contract after every stretch while its elongation properties remain unchanged. Justify why it is impossible for the string to be elongated beyond 140 cm. [1]

$S_\infty = 30+\frac{10}{1-\frac{10}{11}}=140$ (since $0<r<1$)

Since the sum to infinity, S_∞ is 140, it is impossible for the string to be stretched beyond 140cm.

OR

The theoretical maximum is 140 cm so it is impossible for the strong to be stretched beyond 140 cm.

17. 2018/Prelim/EJC/P2/Q3

(a) A retirement savings account pays a compound interest of 0.2% per month on the amount of money in the account at the end of each month. A one-time principal amount of \$$P$ is deposited to open the account and a monthly pay-out of \$$x$ is withdrawn from the account at the beginning of each month, starting from the month that the account is opened.

(i) Show that the amount in the account at the end of n months after the interest has been added is given by

$$P(1.002^n)-501x(1.002^n-1).$$ [4]

(ii) Suppose a fixed monthly pay-out of \$2,000 is to be sustained for at least 25 years, find the minimum principal amount required correct to the nearest dollar. [2]

(iii) If a principal amount of \$600,000 is placed in the account, find the number of years for which a monthly pay-out of \$2,000 per month can be sustained, leaving your answer correct to the nearest whole number. [2]

(b) A different retirement savings account provides an increasing amount of monthly pay-out over a period of 25 years. The pay-out in the first month is \$$a$. The pay-out for each subsequent month is an increment of \$$c$ from the pay-out of the previous month.

The pay-out in the final month is \$4,000, and the total pay-out at the end of 25 complete years is \$751,500. Find the month in which the pay-out is \$2,000. [5]

(a)(i)

Month	Beginning of month	Balance at the end of the month
1	$P-x$	$(P-x)(1.002)=P(1.002)-x(1.002)$
2	$P(1.002)-x(1.002)-x$	$P(1.002)^2-x(1.002)^2-x(1.002)$
3	$P(1.002)^2-x(1.002)^2-x(1.002)-x$	$P(1.002)^3-x(1.002)^3-x(1.002)^2-x(1.002)$
n	$P(1.002)^{n-1}-x(1.002)^{n-1}$ $-x(1.002)^{n-2}-\cdots-x(1.002)-x$	$P(1.002)^n-x(1.002)^n$ $-x(1.002)^{n-1}-\cdots-x(1.002)$

Balance after n months

$=P(1.002)^n-x(1.002)^n-x(1.002)^{n-1}-\cdots-x(1.002)$

$=P(1.002)^n-\dfrac{x(1.002)(1.002^n-1)}{1.002-1}$

$=P(1.002)^n-501x(1.002^n-1)$ (shown)

(a)(ii) $P(1.002)^{300}-501(2000)(1.002^{300}-1)\geq 0$

$P\geq\dfrac{501(2000)(1.002^{300}-1)}{1.002^{300}}=451761.1356$

Minimum principal amount required is \$451762.

(a)(iii)

Method 1

$(600000)(1.002)^n-501(2000)(1.002^n-1)\geq 0$

n	LHS
457	$214.38>0$
458	$-1789.2<0$

$\therefore$ No. of months $=457$

Method 2

$(600000)(1.002)^n-501(2000)(1.002^n-1)<2000$

n	LHS
456	$2214>2000$
457	$214.38<2000$

$\therefore$ No. of months $=457$

No. of years for which the pay-out can be sustained is $\dfrac{457}{12}=38.083\approx 38$ years.

(b)

Month	Pay-out
1	a
2	a+c
3	a+2c
$\vdots$	$\vdots$
300	4000

Let $\$a$ be the pay-out in the first month.

$a+299c=4000$ --- (1)

$\frac{300}{2}\left(2a+299c\right)=751500$

$2a+299c=5010$ --- (2)

Solving the simultaneous equations,

$a=1010, \quad c=10$

To find the month n with a pay-out of \$2,000:

$(1010)+(n-1)(10)=2000$

$n=100$

$\therefore$ The pay-out is \$2,000 in the 100th month.

Alternative solution:

Let \$$a$ be the pay-out in the first month.

$a+299c=4000$ --- (1)

$\frac{300}{2}\left(a+4000\right)=751500 \Rightarrow a=1010$.

Hence from (1), $c=10$

Consider $U_n=1010+\left(n-1\right)\left(10\right)=2000 \Rightarrow n=100$

$\therefore$ The pay-out is \$2,000 in the 100th month.

18. 2018/Prelim/MJC/P1/Q10

(a) The sum of the first n terms of a sequence $\{u_n\}$ is given by $S_n=kn^2-3n$, where k is a non-zero real constant.

(i) Prove that the sequence $\{u_n\}$ is an arithmetic sequence. [3]

(ii) Given that u_2, u_3 and u_6 are consecutive terms in a geometric sequence, find the value of k. [3]

(b) A zoology student observes jaguars preying on white-tailed deer in the wild. He observes that when a jaguar spots its prey from a distance of d m away, it starts its chase. At the same time, the white-tailed deer senses danger and starts escaping.

He models the predator-prey movements as follows:

The jaguar starts its chase with a leap distance of 6 m. Subsequently, each leap covers a distance of 0.1 m less than its preceding leap.

The white-tailed deer starts its escape with a leap distance of 9 m. Subsequently, each leap covers a distance of 5% less than its preceding leap.

(i) Find the total distance travelled by a white-tailed deer after n leaps. Deduce the maximum distance travelled by a white-tailed deer. [3]

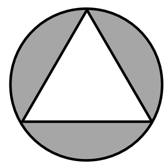
(ii) Assume that both predator and prey complete the same number of leaps in the same duration of time. Given that d = 11 m, find the least value of n for a jaguar to catch a white-tailed deer within n leaps. [3]

(a)(i)	$u_n=S_n-S_{n-1}$ $=kn^2-3n-\left[k\left(n-1\right)^2-3\left(n-1\right)\right]$ $=2kn-k-3$

	$u_n - u_{n-1} = 2kn - k - 3 - \left[2k(n-1) - k - 3\right]$ $= 2k$, a constant $\Rightarrow \{u_n\}$ is in AP.
(ii)	$\dfrac{u_3}{u_2} = \dfrac{u_6}{u_3}$ $(u_3)^2 = (u_2)(u_6)$ $\left[2k(3) - k - 3\right]^2 = \left[2k(2) - k - 3\right]\left[2k(6) - k - 3\right]$ $(5k-3)^2 = (3k-3)(11k-3)$ $8k^2 - 12k = 0$ $4k(2k-3) = 0$ $k = 0$ (rej since $k \neq 0$) or $k = \dfrac{3}{2}$
(b)(i)	Distance travelled by white-tailed deer, $S_n = \dfrac{9(1-0.95^n)}{1-0.95}$ $= 180(1-0.95^n)$ As $n \to \infty$, $S_n \to 180$. $\therefore$ max distance is 180 m.
(ii)	Distance travelled by jaguar after n leaps, $S_n = \dfrac{n}{2}\left[2(6) + (n-1)(-0.1)\right]$ For jaguar to catch white-tailed deer within n leaps, Let $D = \dfrac{n}{2}\left[12 + (n-1)(-0.1)\right] - 180(1-0.95^n) - 11 \geq 0$ When $n = 49, D = -0.21 < 0$ When $n = 50, D = 0.3501 > 0$ $\therefore$ min $n = 50$

19. RI Promo 9758/2020/Q11

A jewellery maker intends to make charms for pendants and accessories. She designed each charm such that its cross section includes an equilateral triangle inscribed in a circle with radius r cm and all three vertices of the triangle touch the circumference of the circle, as shown in the diagram below.



In her design, the portion of the charm with cross section as shaded in the above diagram is filled with glitter resin. It is given that the thickness of each charm she makes is 1 cm.

(i) Show that the volume of glitter resin needed for the charm is $\left(\pi-\frac{3\sqrt{3}}{4}\right)r^2$ cm^3. [3]

After making the first charm with $r=3$, she continues to make charms where each successive charm she makes has a value of r which is 5% less than that of the preceding charm. She then places the first two charms made in the first basket, and places the subsequent five charms made in the second basket, and continues in this manner such that the number of charms placed in each basket after the first is three more than the number of charms placed in the previous basket.

(ii) Show that the radius of the k th charm made is $\frac{60}{19}(0.95)^k$ cm. [1]

(iii) Find, in terms of n, the total number of charms in the first n baskets. [2]

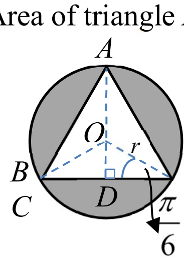
(iv) Find an expression, in terms of n, for the volume of glitter resin needed to make the biggest charm in the n th basket. (You need not simplify your answer.)
Hence, find the volume when $n=6$. [4]

(v) One day, the jewellery maker would like to make a piece of fashion accessory using a charm with an approximate diameter of 0.306 cm. This charm can be found in the m th basket. Find m. [3]

11(i) [3]

<u>Method 1: By considering area of triangle *OBC*:</u>

Area of triangle $ABC = 3(\text{Area of triangle } OBC)$



$$=3\left(\frac{1}{2}r^2\sin\frac{2\pi}{3}\right)$$

$$=3\left(\frac{1}{2}r^2\frac{\sqrt{3}}{2}\right)=\frac{3\sqrt{3}}{4}r^2 \text{ cm}^2$$

<u>Method 2: By considering triangle *OCD*:</u>

$$CD=r\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}r \Rightarrow BC=2CD=\sqrt{3}r$$

	Area of triangle $ABC = \frac{1}{2}\left(\sqrt{3}r\right)^2 \sin\frac{\pi}{3} = \frac{3\sqrt{3}}{4}r^2 \text{ cm}^2$ $\therefore$ Volume of glitter resin needed $= 1\times\left[\pi r^2 - \frac{3\sqrt{3}}{4}r^2\right]$ $= \left(\pi - \frac{3\sqrt{3}}{4}\right)r^2 \text{ cm}^3$ (shown).	
(ii) **[1]**	Radius of k th charm $= 3\left(0.95\right)^{k-1} = 3\left(0.95\right)^{-1}\left(0.95\right)^k = 3\left(\frac{100}{95}\right)\left(0.95\right)^k = \frac{60}{19}\left(0.95\right)^k$ (shown).	
(iii) **[2]**	Total number of charms in the first n baskets $= \frac{n}{2}\left[2(2)+(n-1)(3)\right]$ $= \frac{n}{2}(3n+1)$	
(iv) **[4]**	Using **(iii)**, total number of charms in the first $n-1$ baskets is $\frac{n-1}{2}\left(3(n-1)+1\right) = \frac{1}{2}(n-1)(3n-2)$ So the biggest charm in the n^{th} basket is the $\left(\frac{1}{2}(n-1)(3n-2)+1\right)^{\text{th}}$ charm made. Using **(ii)**, radius of the $\left(\frac{1}{2}(n-1)(3n-2)+1\right)^{\text{th}}$ charm is $\frac{60}{19}\left(0.95\right)^{\left(\frac{1}{2}(n-1)(3n-2)+1\right)}$ Using **(i)**, volume needed $= \left(\pi - \frac{3\sqrt{3}}{4}\right)\left(\frac{60}{19}\left(0.95\right)^{\left(\frac{1}{2}(n-1)(3n-2)+1\right)}\right)^2$ When $n=6$, volume needed $= 0.274 \text{ cm}^3$. (3 s.f.)	
(v) **[3]**	Method 1: $\frac{60}{19}\left(0.95\right)^{\frac{m}{2}(3m+1)} \le 0.153$ $\frac{m}{2}(3m+1) \ge \frac{\ln 0.04845}{\ln 0.95}$ (Note that $\ln 0.95 < 0$) $3m^2 + m - 118.04 \ge 0$ Since $m > 0$, $m \ge 6.1082$ As m is the least positive integer satisfying the inequality, $m = 7$. Method 2: From **(ii)**, the radius of the k th charm made is $\frac{60}{19}\left(0.95\right)^k$.	

Corresponding radius of charm when diameter is 0.306 cm is 0.153 cm.

To solve $\frac{60}{19}(0.95)^k = 0.153$:

From GC,

k	$\frac{60}{19}(0.95)^k$
58	0.16120
59	0.15314
60	0.14548

Hence $k = 59$.

<u>Alternatively,</u>

$$\frac{60}{19}(0.95)^k = 0.153$$

$$k = \frac{\ln\left(0.153 \times \frac{19}{60}\right)}{\ln(0.95)} = 59.0 \text{ (3s.f.)}$$

To solve for m:

From **(iii)**, the total no of charms in the first n baskets is $\frac{n}{2}(3n+1)$.

From GC, when $n = 6,$ total no of charms $= 57.$

when $n = 7,$ total no of charms $= 77.$

Since 57<59<77, the 7th basket contains the 59th charm, i.e. $m = 7$.

20 EJC Promo 9758/2020/Q8

(i) Using the identity $4\cos^3\theta = 3\cos\theta + \cos 3\theta$, show that

$$\frac{4\cos^3\left(3^r x\right)}{(-3)^r} = 3\left[\frac{\cos\left(3^r x\right)}{(-3)^r} - \frac{\cos\left(3^{r+1} x\right)}{(-3)^{r+1}}\right].$$ [2]

(ii) Hence, show that $\sum_{r=1}^{n}\frac{\cos^3\left(3^r x\right)}{(-3)^r} = -\frac{1}{4}\cos 3x - \frac{3}{4}\left[\frac{\cos\left(3^{n+1} x\right)}{(-3)^{n+1}}\right]$. [3]

(iii) The Squeeze Theorem states that if $\frac{p}{\mathrm{f}(n)} \le \frac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}} \le \frac{q}{\mathrm{f}(n)}$ for all positive integers n, and $\lim_{n\to\infty}\frac{p}{\mathrm{f}(n)} = \lim_{n\to\infty}\frac{q}{\mathrm{f}(n)} = 0$, then $\lim_{n\to\infty}\frac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}} = 0$ too. By considering the minimum and maximum values of the cosine function, use the Squeeze Theorem to explain why $\sum_{r=1}^{\infty}\frac{\cos^3\left(3^r x\right)}{(-3)^r} = -\frac{1}{4}\cos 3x$. [3]

(iv) Evaluate $\sum_{r=1}^{\infty}\frac{\cos^3\left(3^{r-1}\pi\right)}{(-3)^r}$. [1]

(i) $$\frac{4\cos^3(3^r x)}{(-3)^r}=\frac{3\cos(3^r x)}{(-3)^r}+\frac{\cos(3^{r+1}x)}{(-3)^r}$$

$$=\frac{3\cos(3^r x)}{(-3)^r}+\frac{(-3)\cos(3^{r+1}x)}{(-3)(-3)^r}=\frac{3\cos(3^r x)}{(-3)^r}-\frac{3\cos(3^{r+1}x)}{(-3)^{r+1}}$$

$$=3\left[\frac{\cos(3^r x)}{(-3)^r}-\frac{\cos(3^{r+1}x)}{(-3)^{r+1}}\right] \text{ (shown)}$$

(ii) $$\sum_{r=1}^{n}\frac{\cos^3(3^r x)}{(-3)^r}=\frac{3}{4}\sum_{r=1}^{n}\left[\frac{\cos(3^r x)}{(-3)^r}-\frac{\cos(3^{r+1}x)}{(-3)^{r+1}}\right]$$

$$\begin{aligned}=\frac{3}{4}\Bigg[&\frac{\cos(3^1 x)}{(-3)^1}-\cancel{\frac{\cos(3^2 x)}{(-3)^2}}\\
&+\cancel{\frac{\cos(3^2 x)}{(-3)^2}}-\cancel{\frac{\cos(3^3 x)}{(-3)^3}}\\
&\quad\vdots\\
&+\cancel{\frac{\cos(3^{n-1} x)}{(-3)^{n-1}}}-\cancel{\frac{\cos(3^n x)}{(-3)^n}}\\
&+\cancel{\frac{\cos(3^n x)}{(-3)^n}}-\frac{\cos(3^{n+1} x)}{(-3)^{n+1}}\Bigg]\end{aligned}$$

$$=\frac{3}{4}\left[\frac{\cos 3x}{-3}-\frac{\cos(3^{n+1}x)}{(-3)^{n+1}}\right]=-\frac{1}{4}\cos 3x-\frac{3}{4}\left[\frac{\cos(3^{n+1}x)}{(-3)^{n+1}}\right] \quad \text{(SHOWN)}$$

(iii) $-1\le\cos\left(3^{n+1}x\right)\le 1$ **for all *n***

Dividing throughout by 3^{n+1}**,**

$$\frac{-1}{3^{n+1}}\le\frac{\cos(3^{n+1}x)}{3^{n+1}}\le\frac{1}{3^{n+1}} \text{......(1)}$$

If n odd, $(-3)^{n+1}=3^{n+1}$,

so $\dfrac{-1}{3^{n+1}}\le\dfrac{\cos(3^{n+1}x)}{(-3)^{n+1}}\le\dfrac{1}{3^{n+1}}$.

If n even, $(-3)^{n+1}=(-1)3^{n+1}$,

so multiplying (1) by -1 throughout,

$\dfrac{-1}{3^{n+1}}\le\dfrac{\cos(3^{n+1}x)}{(-3)^{n+1}}\le\dfrac{1}{3^{n+1}}$ too.

So for all positive integers *n*,

$$\frac{-1}{3^{n+1}}\le\frac{\cos(3^{n+1}x)}{(-3)^{n+1}}\le\frac{1}{3^{n+1}}.$$

Since $\lim_{n\to\infty}\dfrac{-1}{3^{n+1}}=\lim_{n\to\infty}\dfrac{1}{3^{n+1}}=0$**,**

$\lim\limits_{n\to\infty}\dfrac{\cos\left(3^{n+1}x\right)}{(-3)^{n+1}}=0$ **by Squeeze Theorem.** **So** $\displaystyle\sum_{r=1}^{\infty}\frac{\cos^3(3^r x)}{(-3)^r}=\lim_{n\to\infty}\left[-\frac{1}{4}\cos 3x-\frac{3}{4}\left(\frac{\cos(3^{n+1}x)}{(-3)^{n+1}}\right)\right]$ $=-\dfrac{1}{4}\cos 3x-\dfrac{3}{4}(0)$ $=-\dfrac{1}{4}\cos 3x$
(iv) Substituting $x=\dfrac{\pi}{3}$: $\displaystyle\sum_{r=1}^{\infty}\frac{\cos^3(3^{r-1}\pi)}{(-3)^r}=-\frac{\cos\pi}{4}$ $=\dfrac{1}{4}$

21. 2021/Prelim/NJC/P2/Q2

An Art teacher teaches her students to create patterns using squares of different sizes. One possible pattern is to begin with the first square with sides of length 2 mm. The first square is inscribed in the second square, where the corners of the first square coincide with the midpoints of the second square. She continues inscribing squares in this manner where the n^{th} square is inscribed in the $(n+1)^{\text{th}}$ square. **Figure 1** shows a piece of artwork after 4 squares are drawn.

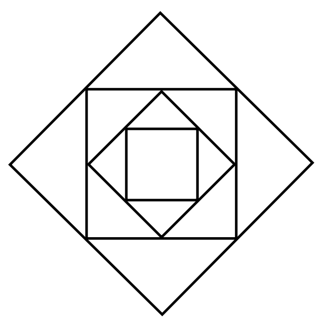


Figure 1

By using this pattern, Student A begins his artwork.

(i) Find, in terms of *n*, the length of the sides of the n^{th} square. [2]

(ii) A standard A4 paper measures 210 mm by 297 mm. Find the maximum number of complete squares that he can draw on the paper. [2]

Student B uses a giant drawing board and decides to make his artwork more eye-catching. He uses the same pattern and measurements as Student A, but he shades the 1^{st} square and also shades on any protruding areas covered by the 4^{th}, 7^{th} ,…, $(3N+1)^{\text{th}}$ squares, where N is a non-negative integer. A protruding area is defined by the region bounded by the newly drawn square and the square immediately preceding it. **Figure 2** shows a piece of artwork if he draws 4 squares.

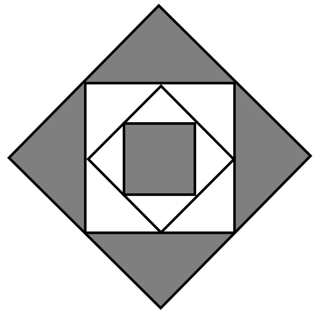


Figure 2

(iii) Find, in mm^2, the total shaded area as shown in **Figure 2**. [2]

(iv) Hence or otherwise, find the total shaded area if he draws 30 squares. Give your answer in m^2. [3]

<table>
<tr><th>Q2</th><th>Suggested Solutions</th></tr>
<tr><td>(i)</td><td>
<table>
<tr><td>n</td><td>Length of n^{th} square</td></tr>
<tr><td>1</td><td>2</td></tr>
<tr><td>2</td><td>$\left(2^{\frac{1}{2}}\right)2 = 2^{\frac{3}{2}}$</td></tr>
<tr><td>3</td><td>$\left(2^{\frac{2}{2}}\right)2 = 2^{\frac{4}{2}}$</td></tr>
<tr><td>:
.</td><td>:
.</td></tr>
<tr><td>n</td><td>$2^{\frac{n+1}{2}}$</td></tr>
</table>
The length of the n^{th} square is $2^{\frac{n+1}{2}}$ mm.</td></tr>
<tr><td>(ii)</td><td>$2^{\frac{n+1}{2}} < 210$
$\frac{n+1}{2} < \frac{\ln(210)}{\ln 2}$
$n < 14.428$
Hence maximum number of square is 14.</td></tr>
<tr><td>(iii)</td><td>From part (i), length of the n^{th} square is $2^{\frac{n+1}{2}}$.
Therefore, area of the n^{th} square $=\left(2^{\frac{n+1}{2}}\right)^2 = 2^{n+1}$.

Area of the 1^{st} square $= 2^2$

Area of the 4^{th} square – Area of the 3^{rd} square</td></tr>
</table>

<table>
<tr><th>Q2</th><th>Suggested Solutions</th></tr>
<tr><td></td><td>$= 2^5 - 2^4$
$= 2^4(2-1)$
$= 2^4$

Total shaded area in Figure $2 = 2^2 + 2^4 = 20$</td></tr>
<tr><td>(iv)</td><td>

n	Area of n^{th} square	Protruding area of n^{th} square
1	2^2	2^2
2	2^3	$2^3 - 2^2 = 2^2(2-1) = 2^2$
3	2^4	$2^4 - 2^3 = 2^3(2-1) = 2^3$
4	2^5	2^4
⋮	⋮	⋮
7	2^8	2^7
⋮	⋮	⋮
28	2^{29}	2^{28}

He will only shade up to the 28th square if he draws 30 squares.

Total shaded area $= 2^2 + 2^4 + 2^7 + \ldots + 2^{28}$

$$= 4 + \frac{2^4(2^{3(9)} - 1)}{(2^3 - 1)}$$

$$= 306{,}783{,}380 \text{ mm}^2$$

$$= 307 \text{ m}^2 \text{ (3 s.f.).}$$

</td></tr>
</table>

3 Graphing Techniques

Basic Skills

Use long division to simplify the following expressions

(a) $\dfrac{x+1}{x+2}$ (b) $\dfrac{x^2+x-2}{x+1}$ (c) $\dfrac{x^2}{x^2+3x+2}$

(d) $\dfrac{2x^2}{3x-2}$ (e) $\dfrac{-\pi x^2+x+\pi}{x+\pi}$ (f) $\dfrac{x^2}{2-x}$

(a) $\dfrac{x+1}{x+2}=1-\dfrac{1}{x+2}$	(b) $\dfrac{x^2+x-2}{x+1}=x-\dfrac{2}{x+1}$	(c) $\dfrac{x^2}{x^2+3x+2}=1-\dfrac{3x+2}{x^2+3x+2}$ Or $1+\dfrac{1}{x+1}-\dfrac{4}{x+2}$
(d) $\dfrac{2x^2}{3x-2}=\dfrac{2}{3}x+\dfrac{4}{9}+\dfrac{8}{9(3x-2)}$	(e) $\pi^2+1-\pi x-\dfrac{\pi^3}{x+\pi}$	(f) $\dfrac{x^2}{2-x}=-x-2+\dfrac{4}{2-x}$ Or $-\dfrac{4}{x-2}-x-2$

Tutorial Review

Tutorial 3A Questions 1 and 8.

Tutorial 3B Questions 1 and 6.

Revision Questions

1. 2017/Prelim/CJC/P1/Q8

The curve C has equation

$$y=\frac{2x^2-3x+5}{x-5}.$$

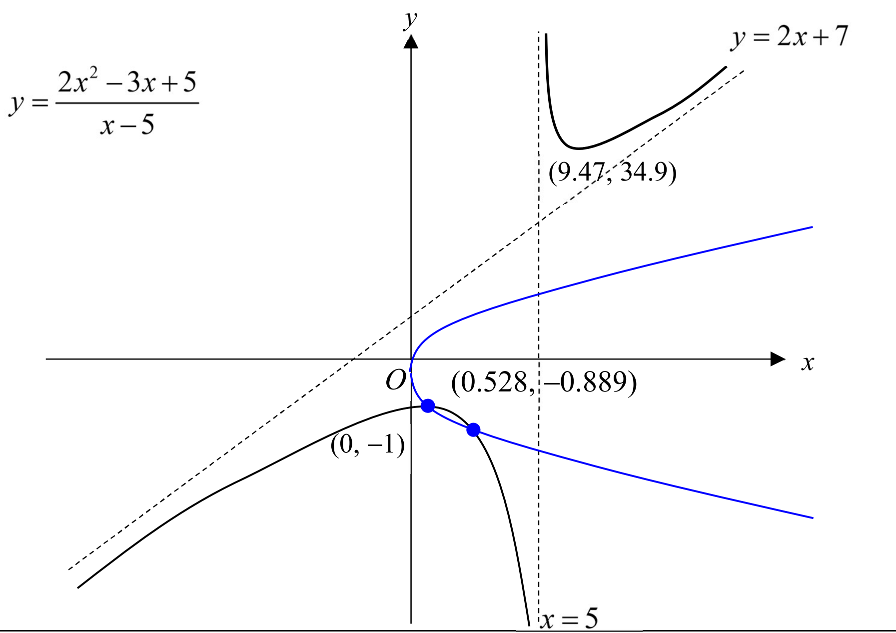
(i) Express y in the form $px+q+\dfrac{r}{x-5}$ where p, q and r are constants to be found. [3]

$y=\dfrac{2x^2-3x+5}{x-5}=2x+7+\dfrac{40}{x-5}$ $\therefore p=2$ $q=7$ $r=40$

$$\begin{array}{r} 2x+7 \\ x-5\overline{)\,2x^2-3x+5} \\ -\underline{(2x^2-10x)} \\ 7x+5 \\ -\underline{(7x-35)} \\ 40 \end{array}$$

(ii) Sketch C, stating the equations of any asymptotes, the coordinates of any stationary points and any points where the curve crosses the x - and y -axes. [4]

Asymptotes: $y = 2x + 7$ and $x = 5$



(iii) By sketching another suitable curve on the same diagram in part **(ii)**, state the number of roots of the equation

$$\left(2x^2 - 3x + 5\right)^2 = 5x\left(x - 5\right)^2 .$$ [3]

$$\left(2x^2 - 3x + 5\right)^2 = 5x\left(x - 5\right)^2$$

$$\left(\frac{2x^2 - 3x + 5}{x - 5}\right)^2 = 5x$$

$$y^2 = 5x$$

$$y = \pm\sqrt{5x}$$

Sketch $y = \pm\sqrt{5x}$ in part **(ii).**

From the diagram, there are 2 points of intersections. Hence, there are 2 roots.

2. SAJC Promo 9758/2020/Q7

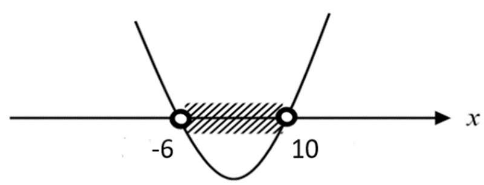
The equation of a curve C is $y=\dfrac{x^2-2x+16}{x-2}$.

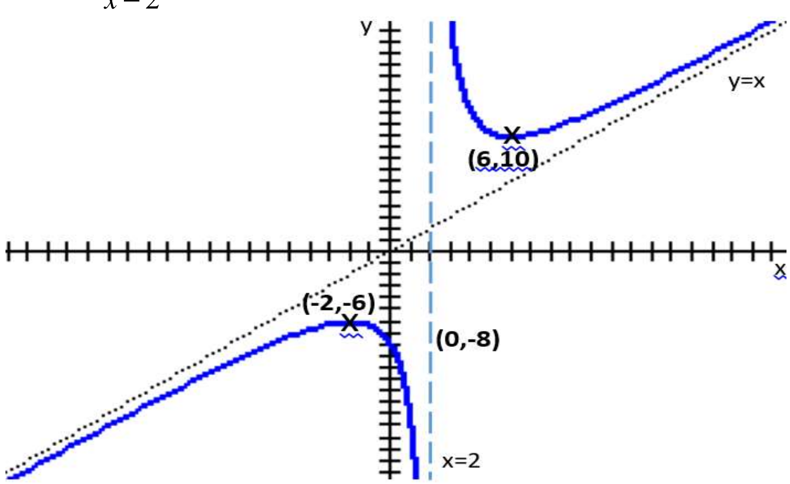
(i) Find algebraically the range of values that y cannot take. **[3]**

(ii) Sketch the graph of C, stating the equations of any asymptotes and the coordinates of any stationary points and points where the curve crosses the axes. **[3]**

(iii) The curve C' has equation $b^2x^2-9y^2=9b^2$, where $b>0$. By finding the asymptotes of C', state the range of values of b such that C' does not intersect C. **[2]**

<table>
<tr><td>(i)</td><td>

$y=\dfrac{x^2-2x+16}{x-2}$
$yx-2y=x^2-2x+16$
$x^2-(2+y)x+16+2y=0 \quad \cdots(1)$
For values that y cannot take, (1) has no solutions so Discriminant <0.
$(2+y)^2-4(1)(2y+16)<0$
$4+4y+y^2-8y-64<0$
$y^2-4y-60<0$
$(y+6)(y-10)<0$

Values that y cannot take are:
$-6<y<10$
</td></tr>
<tr><td>(ii)</td><td>

$y=\dfrac{x^2-2x+16}{x-2}$
$y=x+\dfrac{16}{x-2}$

</td></tr>
</table>

(iii)	$b^2x^2-9y^2=9b^2$ $\frac{x^2}{3^2}-\frac{y^2}{b^2}=1$ This is a hyperbola with x-intercepts (-3, 0) and (3, 0) and asymptotes $y=\pm\frac{b}{3}x$. *Learning point:* **For the hyperbola curve not to intersect curve C, the absolute value of the gradient of the asymptote of the hyperbola must not be greater than that of the oblique asymptote of curve C, so* $\frac{b}{3}\le 1$ and $b>0$ (given by question) $\therefore\ 0<b\le 3$ **Explanation only for learning purposes.*

3. 2017/CT/SRJC/P1/Q3

The curve C is defined by the parametric equations

$$x=2\tan\theta,\quad y=\sqrt{2}\sec\theta+1$$

(i) Find the Cartesian equation of the curve C. [2]

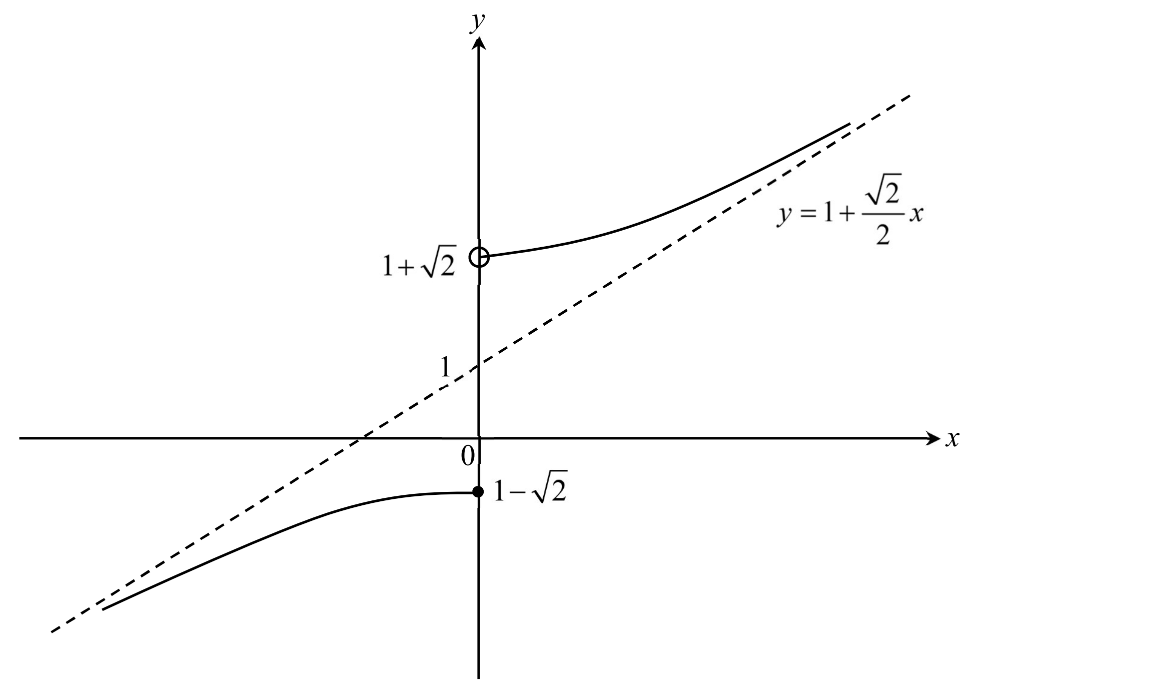
$$\tan\theta=\frac{x}{2},\quad \sec\theta=\frac{y-1}{\sqrt{2}}$$
$$\tan^2\theta+1=\sec^2\theta$$
$$\left(\frac{x}{2}\right)^2+1=\left(\frac{y-1}{\sqrt{2}}\right)^2$$
$$\frac{(y-1)^2}{2}-\frac{x^2}{4}=1$$

(ii) Find the equations of the asymptotes, expressing y in terms of x. [1]

Equations for asymptotes of $\frac{(y-1)^2}{\left(\sqrt{2}\right)^2}-\frac{x^2}{2^2}=1$ are $\frac{y-1}{\sqrt{2}}=\pm\frac{x}{2}\Rightarrow y=1\pm\frac{\sqrt{2}x}{2}$	**<u>Alternative method</u>** Equation of asymptotes: $y=\pm\frac{\sqrt{2}x}{2}+c$ Since the asymptotes pass through (0,1), $c=1$ $\therefore y=\pm\frac{\sqrt{2}x}{2}+1$

For the subsequent parts, take $0 < \theta \leq \pi$.

(iii) Sketch the curve C, showing clearly the exact axial intercepts and the equation of the asymptote. [4]



(iv) Hence, find the range of values of k for which $\frac{k^2x^2}{2} - \frac{x^2}{4} = 1$ has real roots. [2]

$\frac{k^2x^2}{2} - \frac{x^2}{4} = 1 \Rightarrow \frac{(kx)^2}{(\sqrt{2})^2} - \frac{x^2}{2^2} = 1$

$\therefore y - 1 = kx \Rightarrow y = kx + 1$

From graph, $k > \frac{\sqrt{2}}{2}$

4. 2018/Prelim/NJC/P1/Q5

Given that a is a positive constant. A curve C_1 has parametric equations

$$x = \frac{a}{t}, \quad y = 1 + t.$$

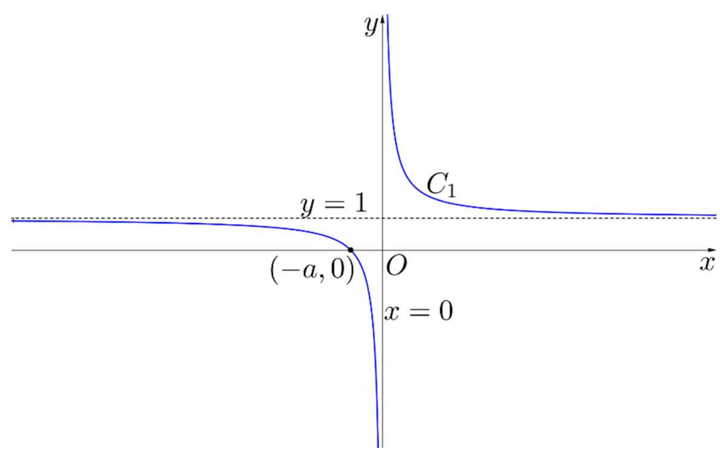
Sketch C_1, labelling the coordinates of the point(s) where the curve crosses the x- and y-axes, and the equations of the asymptote(s) in terms of a, if any. [2]

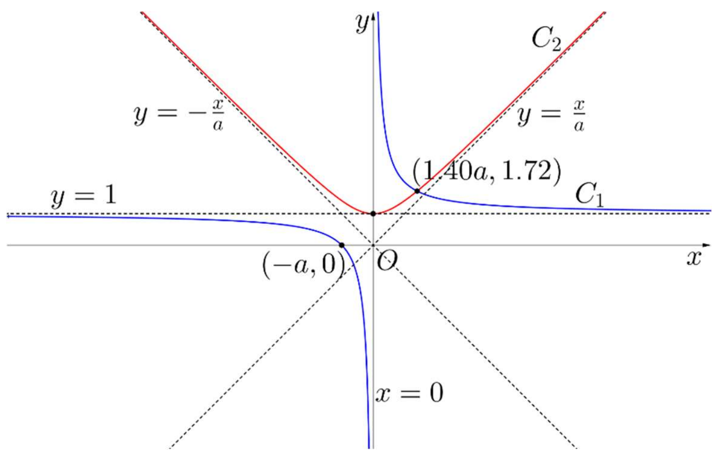
Another curve C_2 has equation $y = \sqrt{1 + \frac{x^2}{a^2}}$.

(i) Show algebraically that the y-coordinates of the point(s) of intersection of C_1 and C_2 satisfies the equation $(y-1)^2(y^2-1) - 1 = 0$. [2]

(ii) Sketch C_2 on the same diagram as C_1 labelling the coordinates of the point(s) where the curve crosses the x- and y-axes, and the equations of the asymptotes in terms of a, if any.

Find the coordinates of point(s) of intersections of C_1 and C_2 and label the coordinates in this diagram, leaving the answers correct to 3 significant figures, in terms of a. [4]

(1st part)	
(i)	$x = \frac{a}{t}, y = 1 + t \Rightarrow x = \frac{a}{y-1}$ Substitute $x = \frac{a}{y-1}$ into $y = \sqrt{1 + \frac{x^2}{a^2}}$: $y = \sqrt{1 + \frac{\left(\frac{a}{y-1}\right)^2}{a^2}}$ $y^2 = 1 + \frac{1}{(y-1)^2}$ $y^2(y-1)^2 = (y-1)^2 + 1$ $y^2(y-1)^2 - (y-1)^2 - 1 = 0$ $(y-1)^2(y^2-1) - 1 = 0$

	Alternatively, $y=\sqrt{1+\frac{\left(\frac{a}{t}\right)^2}{a^2}}=\sqrt{1+\frac{1}{t^2}}$ Squaring both sides, $y^2=1+\frac{1}{t^2}$ $y^2=1+\frac{1}{(y-1)^2}$ $y^2-1=\frac{1}{(y-1)^2}$ $\left(y^2-1\right)(y-1)^2=1$ $\left(y^2-1\right)(y-1)^2-1=0$ (shown)
(ii)	 Using GC to solve, $(y-1)^2\left(y^2-1\right)-1=0$, $y=-1.106919$ (rejected) or 1.716673 When $y\approx 1.716673$, $x\approx 1.39534a$ $\therefore$ The point of intersection is $\left(1.40a, 1.72\right)$.

5. 2017/AJC/Prelim/P1/Q8

The curve C has equation $y = \dfrac{4x^2 - kx + 2}{x - 2}$, where k is a constant.

(i) Show that curve C has stationary points when $k < 9$. [3]

$$y = \frac{4x^2 - kx + 2}{x - 2}$$

By long division, $y = 4x + 8 - k + \dfrac{18 - 2k}{x - 2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(x-2)(8x-k) - (4x^2 - kx + 2)(1)}{(x-2)^2}$$

$$= \frac{4x^2 - 16x + 2k - 2}{(x-2)^2}$$

Let $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 4x^2 - 16x + 2k - 2 = 0$

$\Rightarrow 2x^2 - 8x + k - 1 = 0$

$\Rightarrow x = \dfrac{8 \pm \sqrt{64 - 4(2)(k-1)}}{4} = 2 \pm \sqrt{\dfrac{9-k}{2}}$

C has stationary point when $k \le 9$

However, when $k = 9$, the value x=2 is undefined on the curve.

In fact, the curve C is a straight line, $y = 4x - 1$.

Hence C has stationary point when $k < 9$.

<u>Alternative Presentation 1:</u>

Let $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 4x^2 - 16x + 2k - 2 = 0$

$\Rightarrow 2x^2 - 8x + k - 1 = 0$

For $\dfrac{dy}{dx}$=0 to have real roots, "$b^2 - 4ac \ge 0$"

$\Rightarrow 8^2 - 4(2)(k-1) \ge 0$

$\Rightarrow 64 - 8k + 8 \ge 0$

$\Rightarrow 8k \le 72$

$\Rightarrow k \le 9$

However, when $k = 9$, the value x=2 is undefined on the curve.

In fact, the curve C is a straight line, $y = 4x - 1$.

Hence C has stationary point when $k < 9$.

<u>Alternative Presentation 2:</u>

$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow 4x^2 - 16x + 2k - 2 = 0$

$\Rightarrow 2x^2 - 8x + k - 1 = 0$

$\Rightarrow 2(x-2)^2 + k - 9 = 0$

$\Rightarrow 2(x-2)^2 = 9 - k$

For $\dfrac{dy}{dx} = 0$ to have roots x,

$9 - k \ge 0 \Rightarrow k \le 9$

(ii) Sketch the graph of C for the case where $6 < k < 9$, clearly indicating any asymptotes and points of intersection with the axes. [4]

(ii)

$$y = \frac{4x^2 - kx + 2}{x-2} = 4x + (8-k) + \frac{18-2k}{x-2}$$

Asymptotes of C are $y = 4x + 8 - k$ and $x = 2$

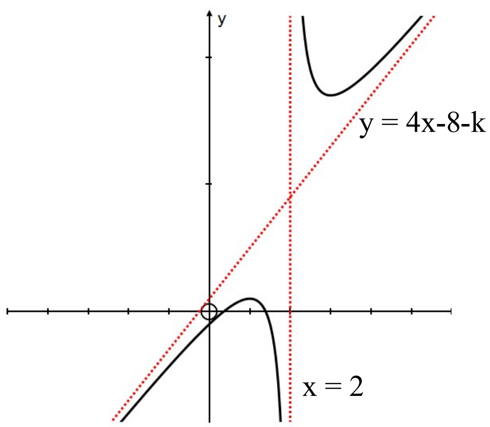
When $x = 0$, $y == -1$.

When $y = 0$, $4x^2 - kx + 2 = 0$

$$\Rightarrow x = \frac{k \pm \sqrt{k^2 - 32}}{8}$$

The axial intercepts are $(0,-1)$, $\left(\frac{k-\sqrt{k^2-32}}{8}, 0\right)$ and $\left(\frac{k+\sqrt{k^2-32}}{8}, 0\right)$.

(ii)



(iii) Describe a sequence of transformations which transforms the graph of $y = 2x + \frac{1}{x}$ to the graph of $y = \frac{4x^2 - 8x + 2}{x-2}$. [3]

(iii)

When k = 8, $y = 4x + (8-8) + \frac{18-2(8)}{x-2} = 4x + \frac{2}{x-2}$

$$y = 2x + \frac{1}{x} \xrightarrow{A} y = 2\left(2x + \frac{1}{x}\right) = 4x + \frac{2}{x}$$

$$y = 4x + \frac{2}{x} \xrightarrow{B} y = 4(x-2) + \frac{2}{(x-2)} = y = 4x - 8 + \frac{2}{(x-2)}$$

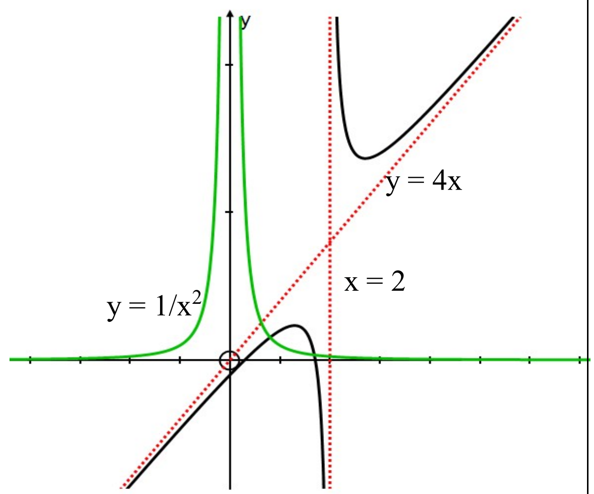
$$y = 4x - 8 + \frac{2}{(x-2)} \xrightarrow{C} y = \left(4x - 8 + \frac{2}{(x-2)}\right) + 8 = 4x + \frac{2}{(x-2)}$$

A – Translate the graph by 2 units in the direction of x-axis
B - Scaling, parallel to the y-axis by a scale factor of 2.
C - Translate the graph by 8 units in the direction of y-axis

<u>Alternate Sequence of Transformations:</u>
A – Translate the graph by 2 units in the direction of x-axis
B - Translate the graph by 4 units in the direction of y-axis
C - Scaling, parallel to the y-axis by a scale factor of 2.

(iv) By drawing a suitable graph on the same diagram as the graph of C, solve the inequality

$$\frac{4x^2-8x+2}{x-2} > \frac{1}{x^2}.$$ [3]

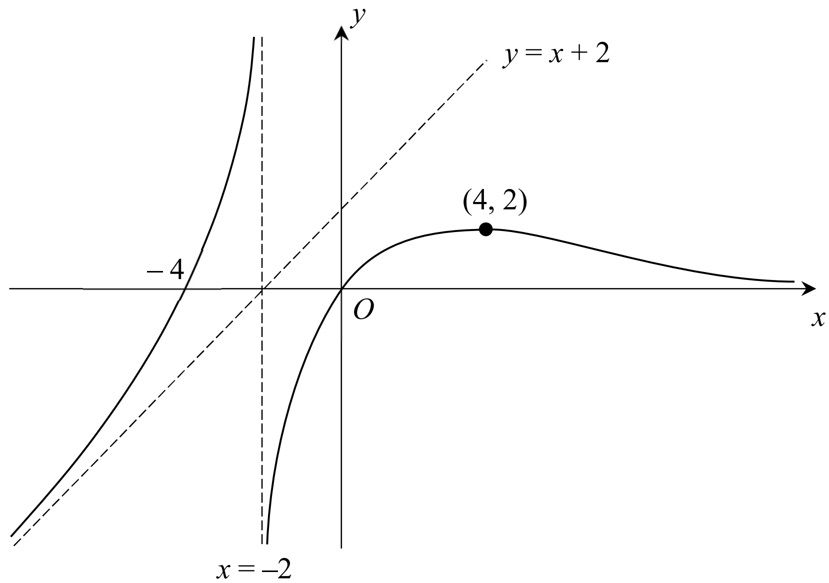
(iv)	When $k=8$, $\dfrac{4x^2-kx+2}{x-2} > \dfrac{1}{x^2}$ $\Rightarrow \dfrac{4x^2-8x+2}{x-2} > \dfrac{1}{x^2}$ From G.C, $0.805 < x < 1.69$ or $x > 2$. 

6. 2017/Prelim/DHS/P1/Q6

(a) State a sequence of transformations that transform the graph of $x^2+\frac{1}{3}(y-2)^2=1$ to the graph of $(x-2)^2+y^2=1$. [3]

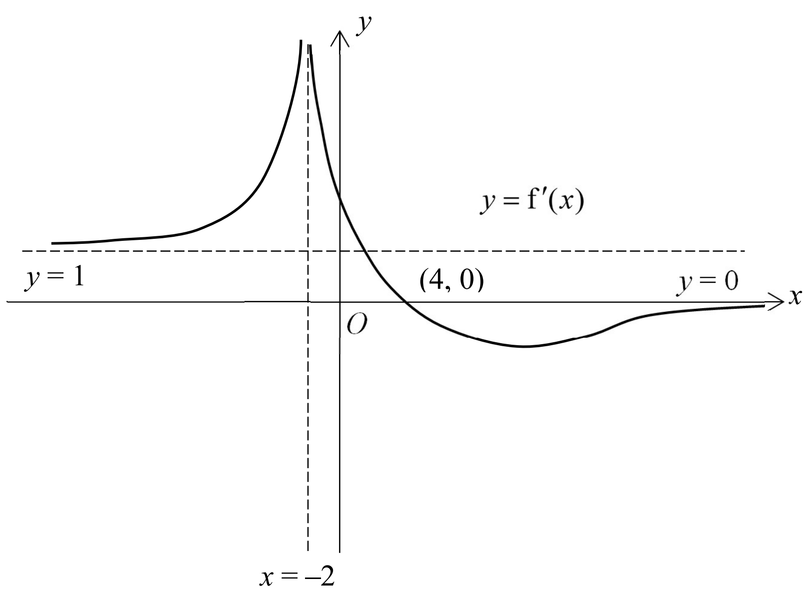
$x^2+\frac{1}{3}(y-2)^2=1$ $\downarrow$ Replace x by $x-2$ $(x-2)^2+\frac{1}{3}(y-2)^2=1$ $\downarrow$ Replace y by $y+2$ $(x-2)^2+\frac{1}{3}(y)^2=1$ $\downarrow$ Replace y by $\sqrt{3}y$ $(x-2)^2+y^2=1$	
1. Translate 2 units in the positive x-direction 2. Translate 2 units in the negative y-direction 3. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction	<u>Alternative:</u> 2. Scale by a factor of $\dfrac{1}{\sqrt{3}}$ parallel to the y-direction 3. Translate $\dfrac{2}{\sqrt{3}}$ units in the negative y-direction

(b) The diagram below shows the curve $y = \mathrm{f}(x)$. It has a maximum point at $(4, 2)$ and intersects the x-axis at $(-4, 0)$ and the origin. The curve has asymptotes $x = -2$, $y = 0$ and $y = x + 2$.

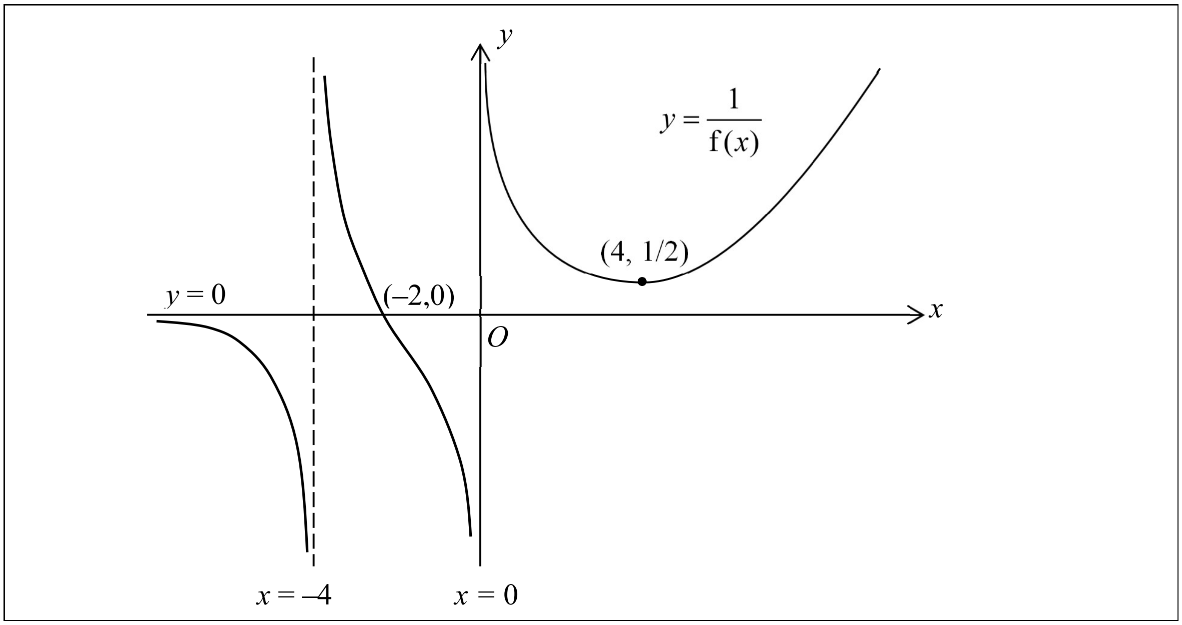


Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}'(x)$, [3]



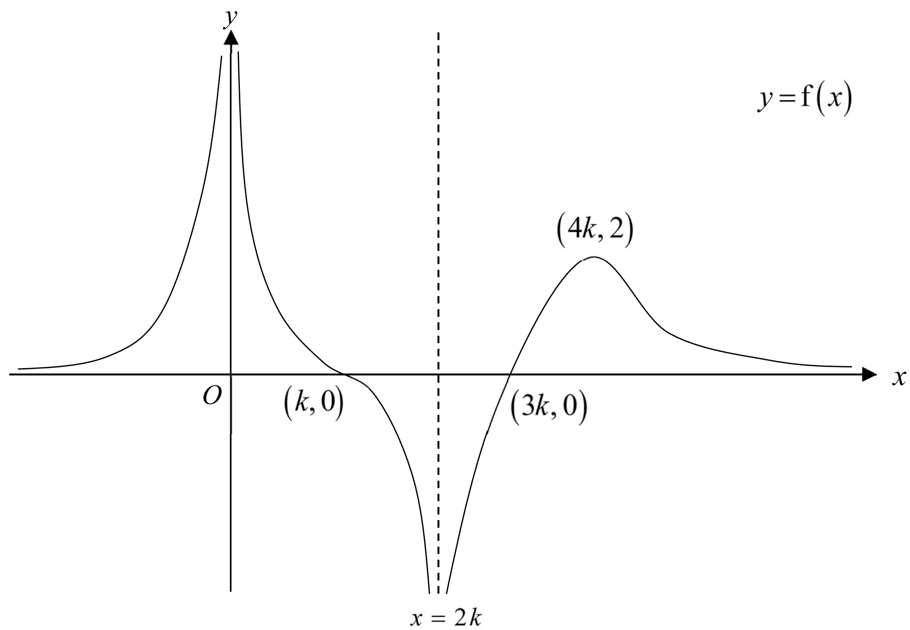
(ii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]



including the coordinates of the points where the graphs cross the axes, the turning points and the equations of any asymptotes, where appropriate.

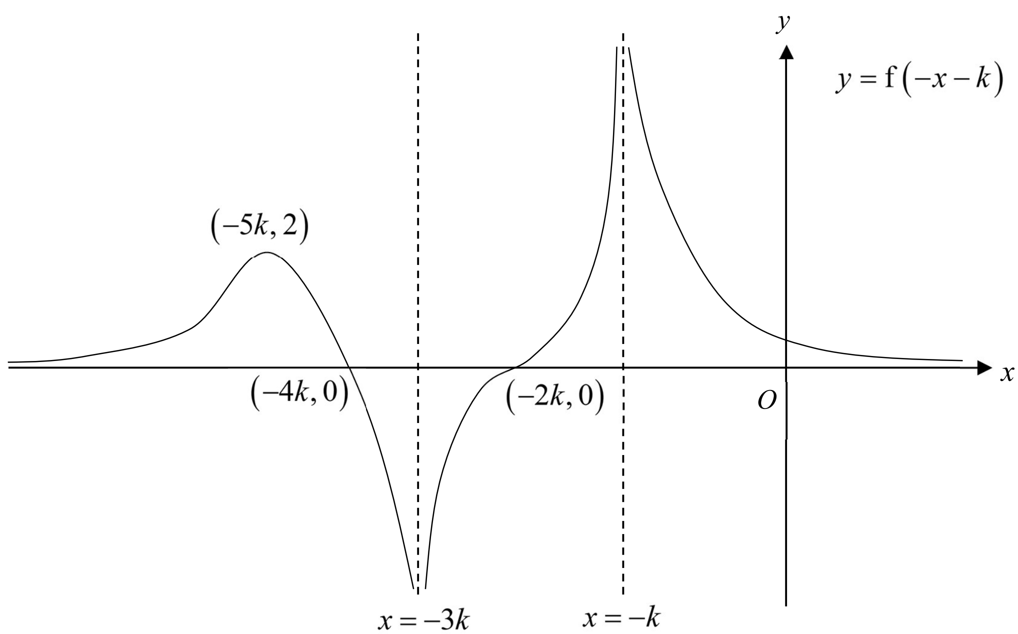
7. 2017/Prelim/CJC/P1/Q5

The diagram below shows the sketch of the graph of $y = \mathrm{f}(x)$ for $k > 0$. The curve passes through the points with coordinates $(k, 0)$ and $(3k, 0)$, and has a maximum point with coordinates $(4k, 2)$. The asymptotes are $x = 0$, $x = 2k$ and $y = 0$.

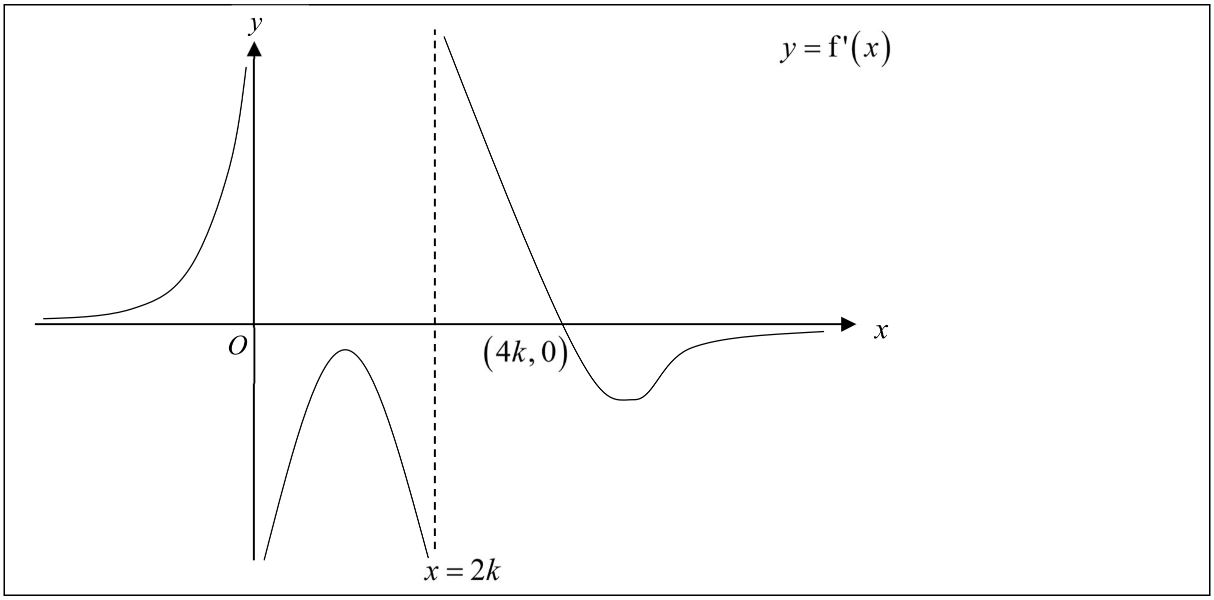


Sketch on separate diagrams, the graphs of

(i) $y = \mathrm{f}(-x-k)$, [2]

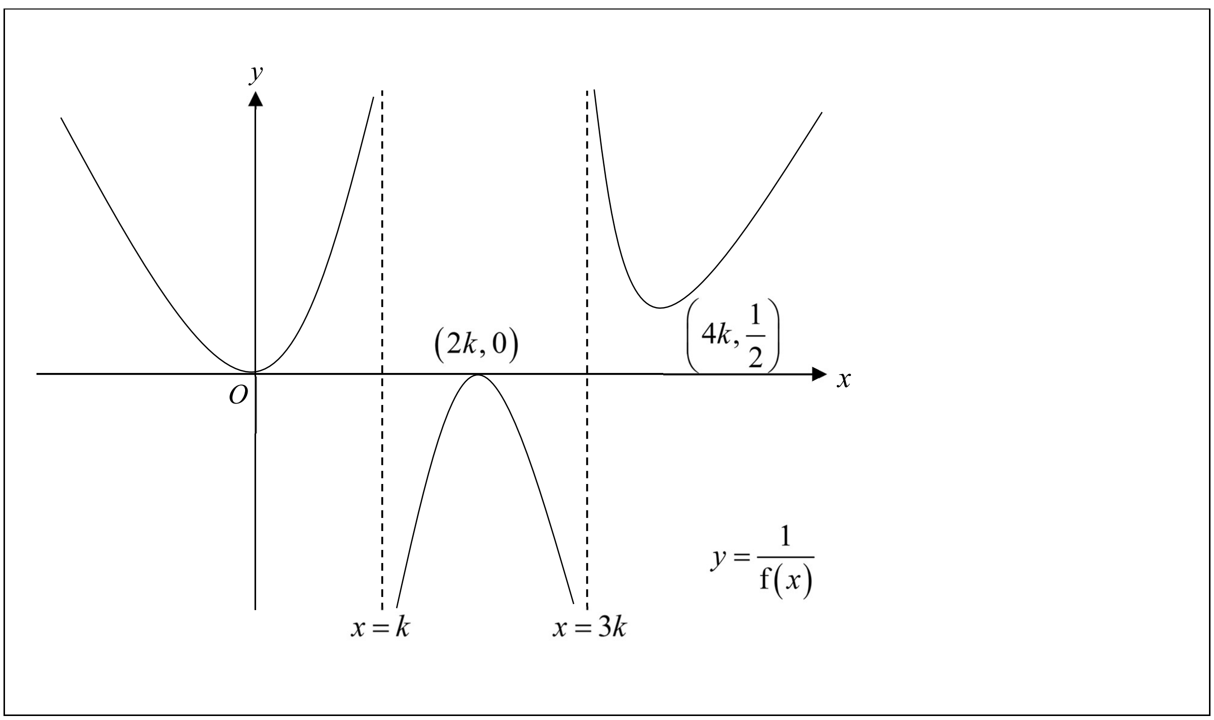


(ii) $y = \mathrm{f}'(x)$, [2]



(iii) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

showing clearly, in terms of k, the equations of any asymptote(s), the coordinates of any turning point(s) and any points where the curve crosses the x - and y -axes.



8. HCI JC1 Promo 9758/2019/Q10

(i) Describe a sequence of transformations which transforms the graph of $y = \dfrac{1}{x}$ to the graph of $y = \dfrac{3x+5}{x+2}$. [3]

(ii) Sketch the graphs of $y = \dfrac{3x+5}{x+2}$ and $16(x+3)^2 + 9(y-4)^2 = 144$ on a single diagram, indicating clearly any axial intercepts, points of intersection of the two graphs and the equations of asymptotes. [5]

(iii) Hence find the set of values of x that satisfies the inequality

$$\frac{3x+5}{x+2} > 4 - \sqrt{\frac{144-16(x+3)^2}{9}}.$$ [2]

(i)

$$y = \frac{3x+5}{x+2} = 3 - \frac{1}{x+2}$$

<u>Method 1:</u>

$$\frac{1}{x} \xrightarrow{1} \frac{1}{x+2} \xrightarrow{2} -\frac{1}{x+2} \xrightarrow{3} 3-\frac{1}{x+2}$$

1. Translate 2 units in the negative x-direction
2. Reflect in x-axis
3. Translate 3 units in the positive y-direction

<u>Method 2:</u>

$$\frac{1}{x} \rightarrow -\frac{1}{x} \rightarrow -\frac{1}{x+2} \rightarrow 3-\frac{1}{x+2}$$

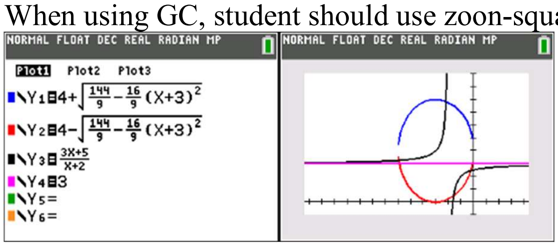
1. Reflect in y-axis
2. Translate 2 units in the negative x-direction
3. Translate 3 units in the positive y-direction

Students should recognize the equation of the ellipse.

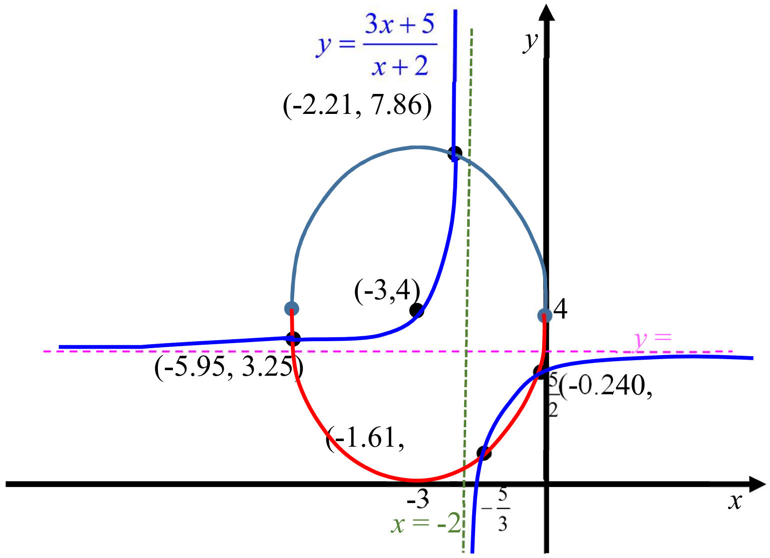
$$16(x+3)^2 + 9(y-4)^2 = 144$$

$$\frac{(x+3)^2}{3^2} + \frac{(y-4)^2}{4^2} = 1$$

When using GC, student should use zoon-square to ensure that the axis are both equal in scale.



(ii)



(iii) From the graph, compare the curve with the lower half of the ellipse,
$\{x \in \mathbb{R}:\ -5.95 < x < -2 \ \text{ or } \ -1.61 < x < -0.240\}$

9. 2021/Prelim/CJC/P1/Q1

A curve has equation $y = \mathrm{f}(x)$, where

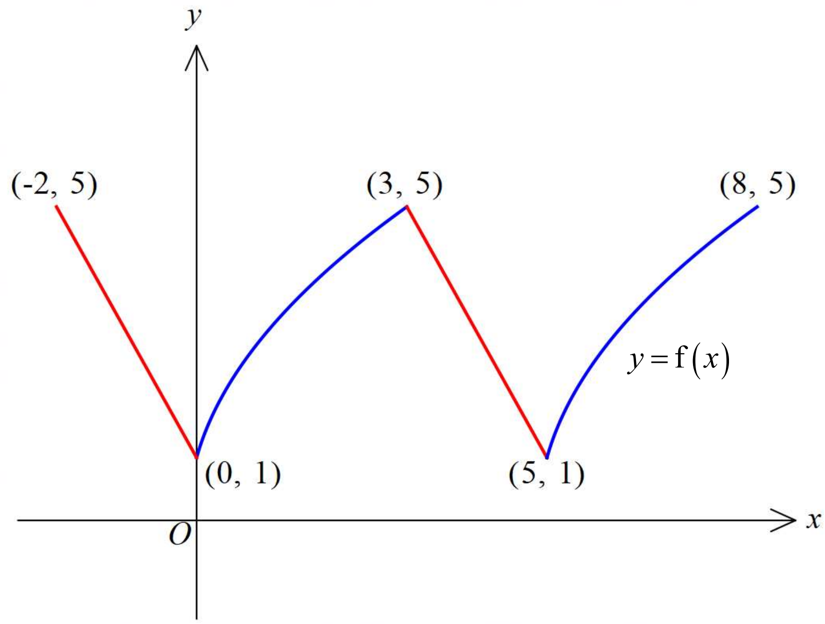
$$\mathrm{f}(x) = \begin{cases} \sqrt{8x+1} & \text{for } 0 \le x \le 3, \\ 11 - 2x & \text{for } 3 < x \le 5, \end{cases}$$

and $\mathrm{f}(x) = \mathrm{f}(x-5)$ for all real values of x.

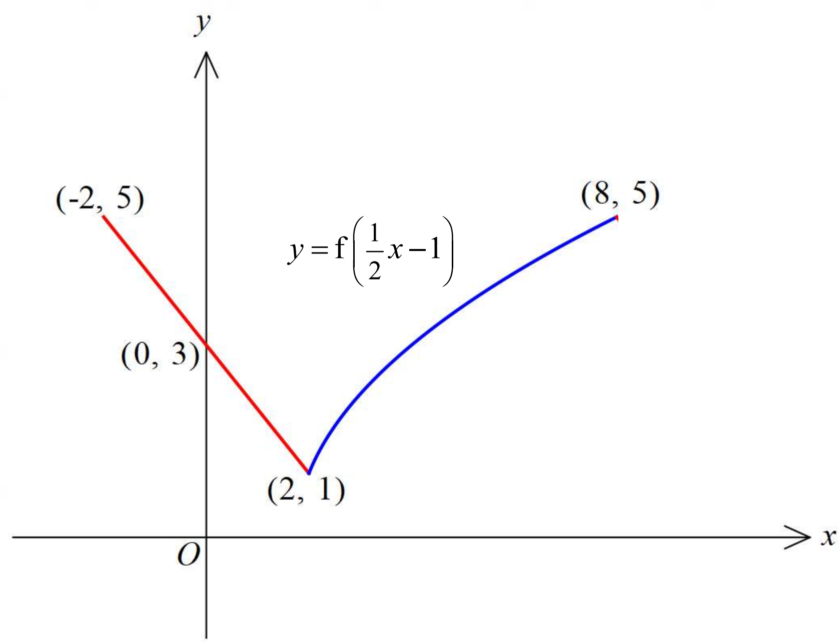
(i) Sketch the curve for $-2 \le x \le 8$. [2]

(ii) On a separate diagram, sketch the curve with equation $y = \mathrm{f}\left(\frac{1}{2}x - 1\right)$, for $-2 \le x \le 8$. [2]

(i)



(ii)



$y=\mathrm{f}(x)$

$\downarrow$ translate 1 unit in the positive x-direction

$y=\mathrm{f}(x-1)$

$\downarrow$ scale parallel to x-axis by a scale factor of 2

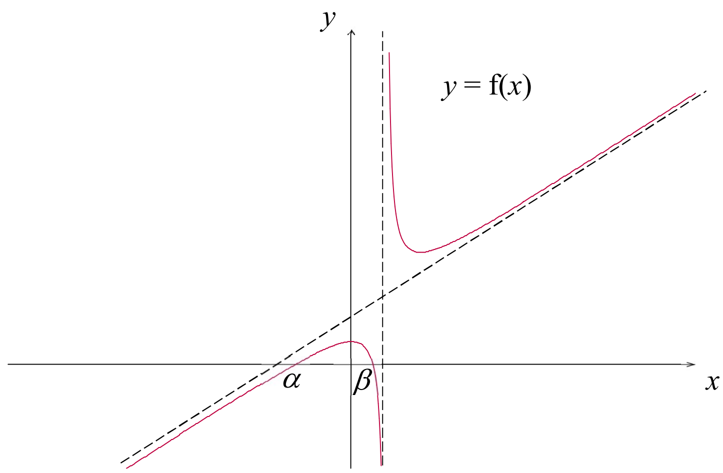
$y=\mathrm{f}\left(\frac{1}{2}x-1\right)$

10. 2021/Prelim/ASRJC/P1/Q7

(i) The curve C_1 with equation $y = \dfrac{(x+2)^2}{x+1}$ is transformed onto the curve C_2 with equation $y = \mathrm{f}(x)$. The curve C_1 has a minimum turning point (0, 4) which corresponds to the point with coordinates (a, b) on the curve C_2, where a, b > 0. Given that f(x) has the form $\dfrac{p^2x^2}{px-1} + q$, where p, q are positive constants, express p and q in terms of a and b.

[4]

(ii) The curve of y = f(x) has a maximum point and a minimum point at $(0, q)$ and (a, b) respectively, and intersects the x-axis at α and β, as shown in the diagram below. The equation of the vertical asymptote is $x = \dfrac{1}{p}$.



Sketch the curve $y = \dfrac{1}{\mathrm{f}(x)}$. Your diagram should indicate clearly, in terms of a, b, α and β, the equations of any asymptote(s) as well as the coordinates of turning points and axial intercepts.

[3]

Solutions:

Question 7(i)

$$y=\frac{(x+2)^2}{x+1}\xrightarrow{\text{Replace } x \text{ with } x-2} y=\frac{\left[(x-2)+2\right]^2}{(x-2)+1}=\frac{x^2}{x-1}$$ (Translation of 2 units in the positive x-direction)

$$\xrightarrow{\text{Replace } x \text{ with } px} y=\frac{(px)^2}{(px)-1}=\frac{p^2x^2}{px-1}$$ (Scaling of scale factor $\frac{1}{p}$ along the x-axis)

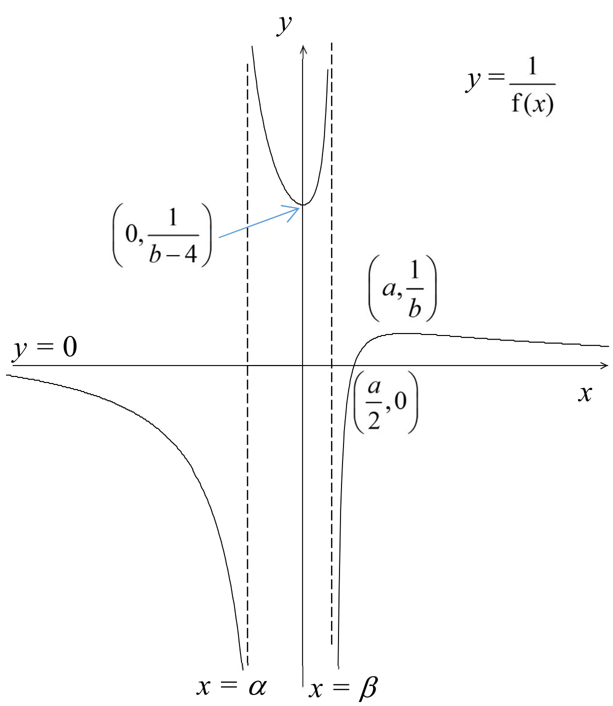
$$\xrightarrow{\text{Replace } y \text{ with } y-q} y=\frac{p^2x^2}{px-1}+q$$ (Translation of q units in the positive y-direction)

The minimum turning point on C_1, $(0, 4)$ corresponds to (a, b) on C_2.

Hence $a=\frac{1}{p}(0+2) \Rightarrow p=\frac{2}{a}$

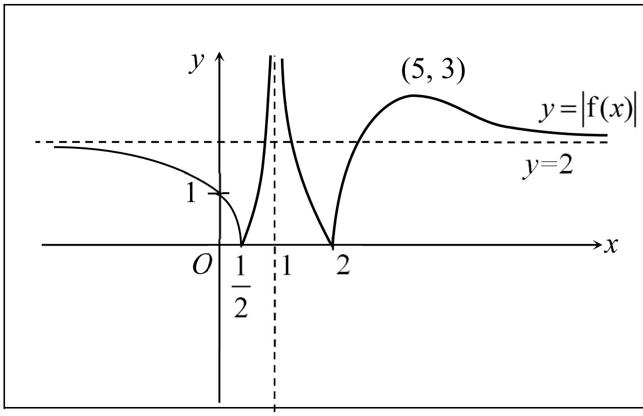
$b=4+q \Rightarrow q=b-4$

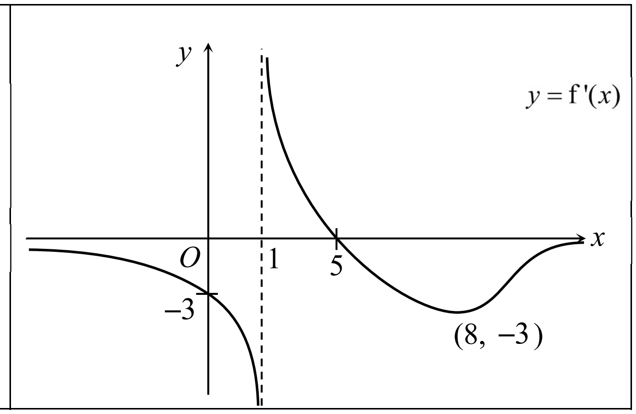
Question 7(ii)



11. 2018/Prelim/EJC/P2/Q2

The diagrams below show the graphs of $y = |f(x)|$ and $y = f'(x)$.





On separate diagrams, sketch the graphs of:

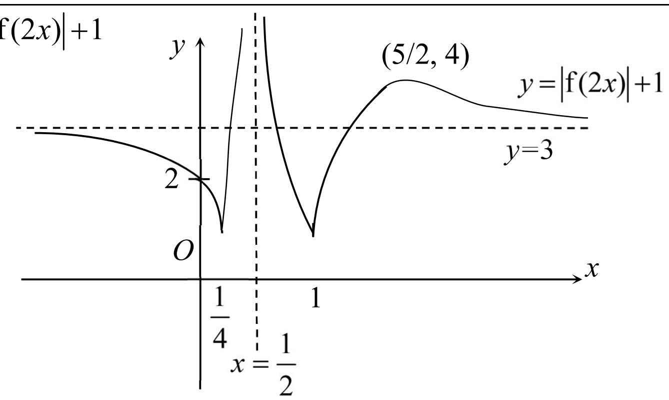
(i) $y = |f(2x)| + 1$, [2]

(ii) $y = \dfrac{1}{f'(x)}$, [3]

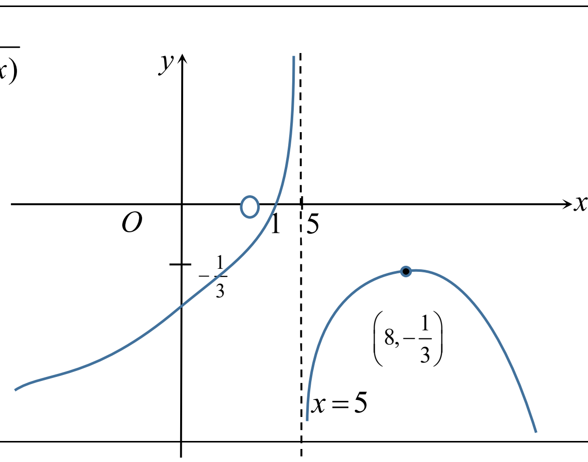
(iii) $y = f(x)$, [3]

showing clearly, in each case, the intersection(s) with the axes, the coordinates of the turning point(s) and the equation(s) of the asymptotes.

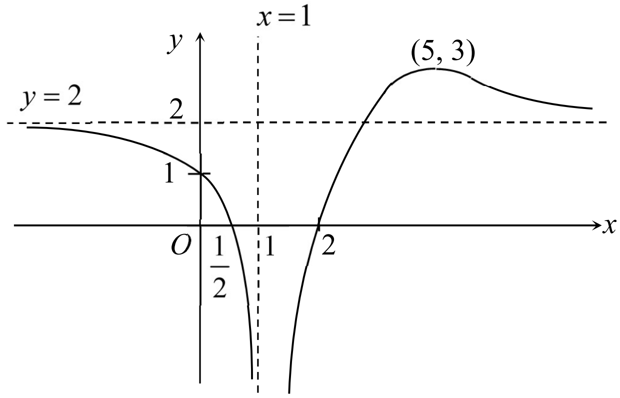
(i) $y = |f(2x)| + 1$



(ii) $y = \dfrac{1}{f'(x)}$



(iii) $y = \mathrm{f}(x)$



12. ASRJC Promo 9758/2020/Q6

The curve C_1 whose equation is $x^2 + y^2 = 16$ undergoes, in succession, the following transformations:

A: Translation of 4 units in the negative x-direction.
B: Translation of 2 units in the positive y-direction.
C: Scaling by factor 2 parallel to the y-axis.

(i) Find the equation of the resulting curve C_2. [2]

(ii) Sketch C_1 and C_2 on the same diagram, indicating clearly the relevant features of the two curves. You need not find the coordinates of axial intercepts. [2]

(iii) State the coordinates of the points of intersection between C_1 and C_2. [1]

(iv) Find the overlapping area between the two curves C_1 and C_2. [2]

(i) $x^2 + y^2 = 16$

Transformation A, replace x with $x+4$: $(x+4)^2 + y^2 = 16$

Transformation B, replace y with $y-2$: $(x+4)^2 + (y-2)^2 = 16$

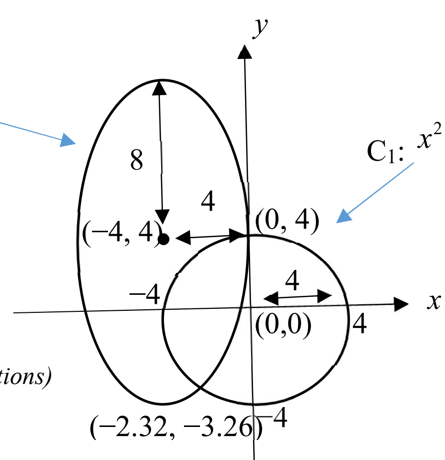
Transformation C, replace y with $\frac{y}{2}$: $(x+4)^2 + \left(\frac{y}{2} - 2\right)^2 = 16$

equation of the resulting curve C_2: $\frac{(x+4)^2}{4^2} + \frac{(y-4)^2}{8^2} = 1$

(ii)

C_2:

$\frac{(x+4)^2}{4^2} + \frac{(y-4)^2}{8^2} = 1$



C_1: $x^2 + y^2 = 16$

(iii) Intersection points are (–2.32, –3.26) and (0, 4)

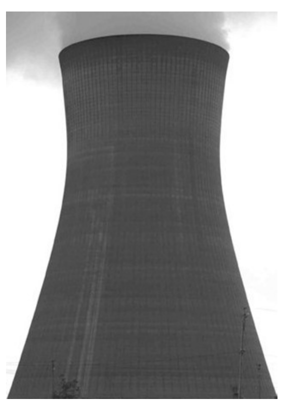
(iv) $\int_{-3.259216}^{4}\left(\left(-4+4\sqrt{\left(1-\frac{(y-4)^2}{8^2}\right)}\right)-\left(-\sqrt{16-y^2}\right)\right)\mathrm{d}y = 19.2\text{ sq units}$

12. NJC Promo 9758/2020/Q12

A curve D has equation $3ax^2 - ay^2 = 1$, where $a > 0$ and $-60 \le y \le 30$.

(i) Sketch D, labelling clearly the axial intercepts and the coordinates of the end-points. [3]

Curve D traces the curved outline of the side view of a cooling tower as shown in the figure below. All units are in metres. The horizontal cross-sections of the cooling tower are circular planes whose centres lie on the y-axis. This hyperbolic form of the cooling tower allows it to withstand extreme winds while requiring less material than any other forms of their size and strength.

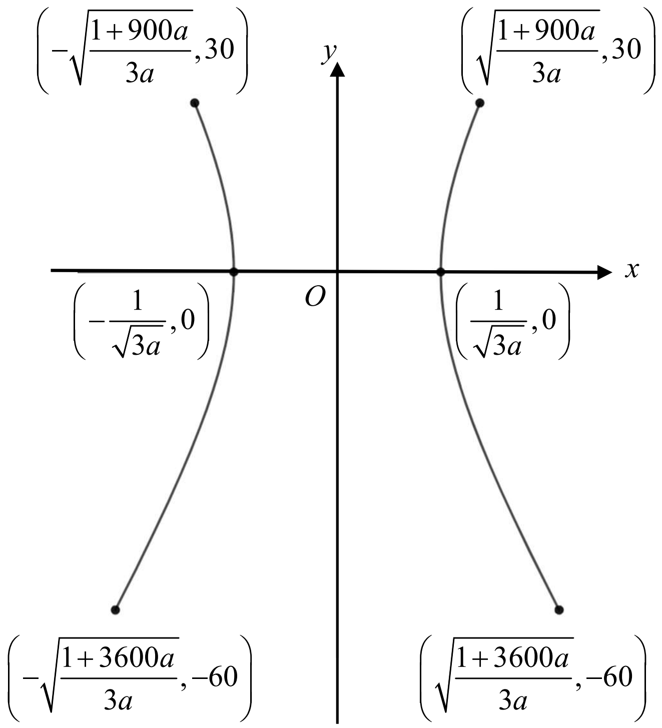


(Source: https://www.pleacher.com/mp/mlessons/calculus/mobaphyp.html)

An engineer is asked to design cooling towers for two different sites.

(ii) For the cooling tower at the first site, $a = 0.01$ is chosen. Find the exact volume contained by the tower. [You do not need to consider the thickness of the tower's walls.] [3]

(iii) For the cooling tower at the second site, another value of a is chosen such that the ratio of its smallest circumference to the circumference of its base is $1:2$. Determine the exact value of a to achieve this design. [2]

Qn.	Solution
12(i)	 When $y=-60$, $3ax^2-a(-60)^2=1$ $\Rightarrow x=\sqrt{\frac{1+3600a}{3a}}$ or $-\sqrt{\frac{1+3600a}{3a}}$ When $y=30$, $3ax^2-a(30)^2=1$ $\Rightarrow x=\sqrt{\frac{1+900a}{3a}}$ or $-\sqrt{\frac{1+900a}{3a}}$
(ii)	Given $a=0.01$. $3(0.01)x^2-(0.01)y^2=1,$ Volume of the tower $=\int_{-60}^{30}\pi x^2\,\mathrm{d}y$ $=\pi\int_{-60}^{30}\frac{1}{3(0.01)}+\frac{y^2}{3}\,\mathrm{d}y$ $=\frac{\pi}{3}\left[100y+\frac{y^3}{3}\right]_{-60}^{30}$ $=30000\pi\ \mathrm{m}^3.$

(iii)	Radius of base $=\sqrt{\frac{1+a(60)^2}{3a}}$ $\frac{\text{Smallest circumference}}{\text{Circumference of base}}=\frac{1}{2}=\frac{2\pi(\frac{1}{\sqrt{3a}})}{2\pi\left(\sqrt{\frac{1+a(60)^2}{3a}}\right)}$ $\frac{1}{4}=\frac{\frac{1}{3a}}{\frac{1+3600a}{3a}}$ $\frac{1}{4}=\frac{1}{1+3600a}$ $a=\frac{1}{1200}$

4 Functions

<u>Basic Skills</u>

Factorise the following after expressing them in completed square form

(a) x^2+3x+2 (b) $-x^2-2x+3$ (c) $-\frac{x^2}{2}+2x-1$

(d) $\frac{2}{3}x^2+x-\frac{1}{4}$ (e) $-\pi x^2+x+\pi-1$ (f) $\frac{x^2}{2}+\sqrt{6}x+3$

(g) $9x^2+20x+\frac{28}{3}$

(a) x^2+3x+2 $x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2+2$ $=\left(x+\frac{3}{2}\right)^2-\frac{1}{4}$ $=\left(x+\frac{3}{2}\right)^2-\left(\frac{1}{2}\right)^2$ $=\left[\left(x+\frac{3}{2}\right)+\left(\frac{1}{2}\right)\right]\left[\left(x+\frac{3}{2}\right)-\left(\frac{1}{2}\right)\right]$ $=(x+2)(x+1)$	(b) $-x^2-2x+3$ $-\left[x^2+2x+1-1-3\right]$ $=-\left[(x+1)^2-4\right]$ $=(2)^2-(x+1)^2$ $=\left[2+(x+1)\right]\left[2-(x+1)\right]$ $=(x+3)(1-x)$
(c) $-\frac{x^2}{2}+2x-1$ $-\frac{1}{2}\left(x^2-4x+(2)^2-(2)^2+2\right)$ $=-\frac{1}{2}\left[(x-2)^2-2\right]$ $=-\frac{1}{2}\left[(x-2)+\sqrt{2}\right]\left[(x-2)-\sqrt{2}\right]$ $=-\frac{1}{2}\left(x-2+\sqrt{2}\right)\left(x-2-\sqrt{2}\right)$	(d) $\frac{2}{3}x^2+x-\frac{1}{4}$ $\frac{2}{3}\left[x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2-\left(\frac{3}{4}\right)^2-\frac{3}{8}\right]$ $=\frac{2}{3}\left[\left(x+\frac{3}{4}\right)^2-\frac{15}{16}\right]$ $=\frac{2}{3}\left[\left(x+\frac{3}{4}\right)+\frac{\sqrt{15}}{4}\right]\left[\left(x+\frac{3}{4}\right)-\frac{\sqrt{15}}{4}\right]$ $=\frac{2}{3}\left(x+\frac{3+\sqrt{15}}{4}\right)\left(x+\frac{3-\sqrt{15}}{4}\right)$
(e) $-\pi x^2+x+\pi-1$ $-\pi\left(x^2-\frac{1}{\pi}x+\left(\frac{1}{2\pi}\right)^2-\left(\frac{1}{2\pi}\right)^2-1+\frac{1}{\pi}\right)$ $=-\pi\left[\left(x-\frac{1}{2\pi}\right)^2-\frac{1}{4\pi^2}-\frac{4\pi^2}{4\pi^2}+\frac{4\pi}{4\pi^2}\right]$ $=-\pi\left[\left(x-\frac{1}{2\pi}\right)^2-\frac{4\pi^2-4\pi+1}{4\pi^2}\right]$	$-\pi\left[\left(x-\frac{1}{2\pi}\right)^2-\frac{(2\pi-1)^2}{4\pi^2}\right]$ $=-\pi\left[\left(x-\frac{1}{2\pi}\right)^2-\left(\frac{2\pi-1}{2\pi}\right)^2\right]$ $=-\pi\left[\left(x-\frac{1}{2\pi}\right)-\left(\frac{2\pi-1}{2\pi}\right)\right]\left[\left(x-\frac{1}{2\pi}\right)+\left(\frac{2\pi-1}{2\pi}\right)\right]$ $=-\pi(x-1)\left(x+\frac{\pi-1}{\pi}\right)$

(f)	$\dfrac{x^2}{2}+\sqrt{6}x+3$ $=\dfrac{1}{2}\left(x^2+2\sqrt{6}x+6\right)$ $=\dfrac{1}{2}\left(x+\sqrt{6}\right)^2$
(g)	$9x^2+20x+\dfrac{28}{3}=(3x+\dfrac{10}{3})^2+\dfrac{28}{3}-\dfrac{100}{9}$ $=(3x+\dfrac{10}{3})^2-\dfrac{16}{9}$ $=\left(3x+\dfrac{10}{3}-\dfrac{4}{3}\right)\left(3x+\dfrac{10}{3}+\dfrac{4}{3}\right)$ $=(3x+2)\left(3x+\dfrac{14}{3}\right)$ or $\dfrac{1}{3}(3x+2)(9x+14)$

Tutorial Review

Tutorial 4 Questions 2, 7 and 11.

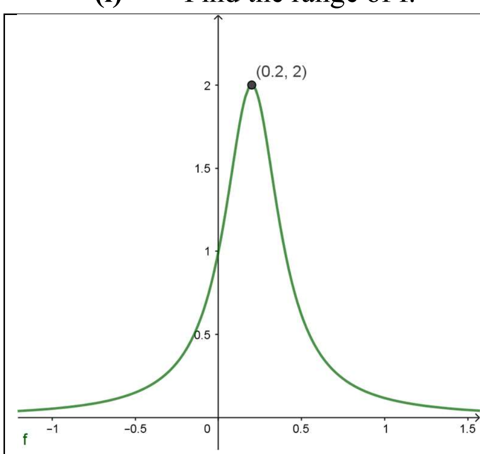
Revision Questions

1. 2011/Prelim/IJC/P1Q9

The function f is defined as follows.

$$\mathrm{f}: x \mapsto \frac{2}{1+(5x-1)^2}, \quad x \in \mathbb{R}.$$

(i) Find the range of f. [2]



$R_\mathrm{f}=(0,2]$

[Note: Do label axes and graph.]

(ii) Give a reason why f does not have an inverse. [1]

The horizontal line $y = 1$ cuts the graph of f at two points. $\therefore$ f is not one-one. Thus f does not have an inverse.

Alt

Since $\mathrm{f}\left(-\dfrac{3}{5}\right)=\mathrm{f}(1)=\dfrac{2}{17}$, f is not one-one. Thus f does not have an inverse.

(iii) If the domain of f is restricted to $x \ge k$, state the least value of k for which the function f^{-1} exists, and find $f^{-1}(x)$ for this domain. [3]

Least value of k = 1/5

Let $y = f(x) = \dfrac{2}{1+(5x-1)^2}, x \ge \dfrac{1}{5}.$

$$1+(5x-1)^2 = \frac{2}{y}$$

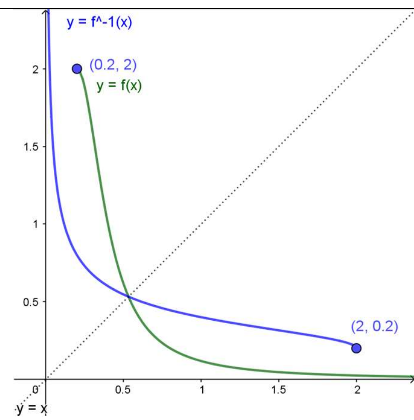
$$5x-1 = \pm\sqrt{\frac{2}{y}-1}$$

Since $x \ge \dfrac{1}{5}, 5x-1 \ge 0.$

$$\therefore\ x = \frac{1}{5}\left(1+\sqrt{\frac{2}{y}-1}\right)$$

$$f^{-1}(x) = \frac{1}{5}\left(1+\sqrt{\frac{2}{x}-1}\right)$$

(iv) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same diagram if the domain of f is restricted to $x \ge k$, where k is the value found in **(iii)**. Your diagram should show clearly the relationship between the two graphs. [2]



(v) Show algebraically that the solution of the equation $f(x) = f^{-1}(x)$ satisfies the equation

$$25x^3 - 10x^2 + 2x - 2 = 0.$$ [2]

$$f(x) = f^{-1}(x)$$

$$\Rightarrow\ f(x) = x$$

$$\Rightarrow\ \frac{2}{1+(5x-1)^2} = x$$

$$\Rightarrow\ x\left(1+25x^2-10x+1\right) = 2$$

$$\Rightarrow\ 25x^3 - 10x^2 + 2x - 2 = 0$$

2. ASRJC JC1 Promo 9758/2019/Q9

(a) The function g is defined by

$$\mathrm{g}: x \mapsto ax+b, \quad x \in \mathbb{R}, x>0,$$

where a and b are positive real numbers.

Show that g^2 exists and hence determine the range of g^2, leaving your answer in terms of a and b. [3]

(b) Function h is defined by

$$\mathrm{h}: x \mapsto \frac{x+7}{x-1}, \quad x \in \mathbb{R}, x \neq 1.$$

(i) Find $\mathrm{h}^{-1}(x)$ and state the domain of h^{-1}. [3]

(ii) Find the exact values of c such that $\mathrm{h}^{2018}(c)=\mathrm{h}^{-1}(c)$. Explain your answers clearly. [3]

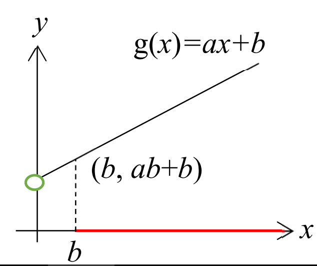
(c) Function f is defined by

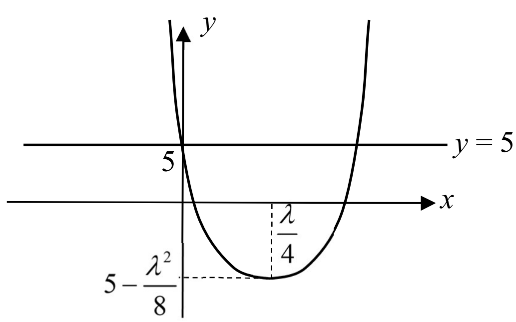
$$\mathrm{f}: x \mapsto 2x^2-\lambda x+5, \quad x \in \mathbb{R},$$

where λ is a non-zero constant.

(i) Give a reason why f^{-1} does not exists. [1]

(ii) For the function f defined above, the range of f is $[-3,\infty)$. If the domain of f is restricted to the set of all positive real numbers, f^{-1} exists. Find the value of λ. [2]

	(a) For g^2 to exist, $\mathrm{R_g} \subseteq \mathrm{D_g}$. From the graph of g, $\mathrm{R_g}=(b,\infty)$ $\mathrm{D_g}=(0,\infty)$ Since $b>0$, $\mathrm{R_g} \subseteq \mathrm{D_g}$. Hence g^2 exists. Consider the graph of g. By restricting the domain of g to the range of g, $\mathrm{R_g}=(b,\infty)$ $\mathrm{R_{g^2}}=(ab+b,\infty)$  	
	(bi) $y=\frac{x+7}{x-1}$ $xy-y=x+7$ $x(y-1)=7+y$ $x=\frac{7+y}{y-1}$ $\mathrm{h}^{-1}(x)=\frac{x+7}{x-1}$ $\mathrm{R_h}=\mathrm{D_{h^{-1}}}=\mathbb{R}\setminus\{-1\}$ **(ii)** $\mathrm{h}^{2018}(c)=\mathrm{h}^{-1}(c)$ $\mathrm{hh}^{2018}(c)=\mathrm{hh}^{-1}(c)$	

	$\underbrace{\mathrm{hh}^{-1}\mathrm{hh}^{-1}\ldots\mathrm{hh}^{-1}}_{2019 \text{ times}}\mathrm{h}(c)=c \quad \left(\because \mathrm{h}=\mathrm{h}^{-1}\right)$ $\mathrm{h}(c)=c$ $\dfrac{c+7}{c-1}=c$ $c+7=c^2-c$ $c^2-2c-7=0$ $c=\dfrac{2\pm\sqrt{4-4(-7)}}{2}$ $c=1\pm2\sqrt{2}$	
	(c)(i) $\mathrm{f}(x)=2x^2-\lambda x+5$ $=2\left(x^2-\dfrac{\lambda}{2}x\right)+5$ $=2\left(x-\dfrac{\lambda}{4}\right)^2+5-\dfrac{\lambda^2}{8}$ **Method 1 (Counter-example)** $\mathrm{f}(0)=5, \quad \mathrm{f}\left(\dfrac{\lambda}{2}\right)=2\left(\dfrac{\lambda}{2}-\dfrac{\lambda}{4}\right)^2+5-\dfrac{\lambda^2}{8}=5$ Since $\mathrm{f}(0)=\mathrm{f}\left(\dfrac{\lambda}{2}\right)$ but $0\neq\dfrac{\lambda}{2}$, $0,\dfrac{\lambda}{2}\in\mathbb{R}$, $\Rightarrow$ f is not 1-1 $\Rightarrow$ inverse function of f does not exists. **Method 2 (Graphical)**  Since the horizontal line $y = 5$ cuts the graph of f at 2 points, f is not 1-1. Hence f^{-1} does not exists. **(ii)** Range of f $=\left[5-\dfrac{\lambda^2}{8},\ \infty\right)=[-3,\ \infty)$ $5-\dfrac{\lambda^2}{8}=-3$ $\lambda^2=8^2$ $\lambda=\pm8$ Since f^{-1} exists when domain of f is restricted to the set of positive real numbers, $\therefore\ \lambda=-8$	

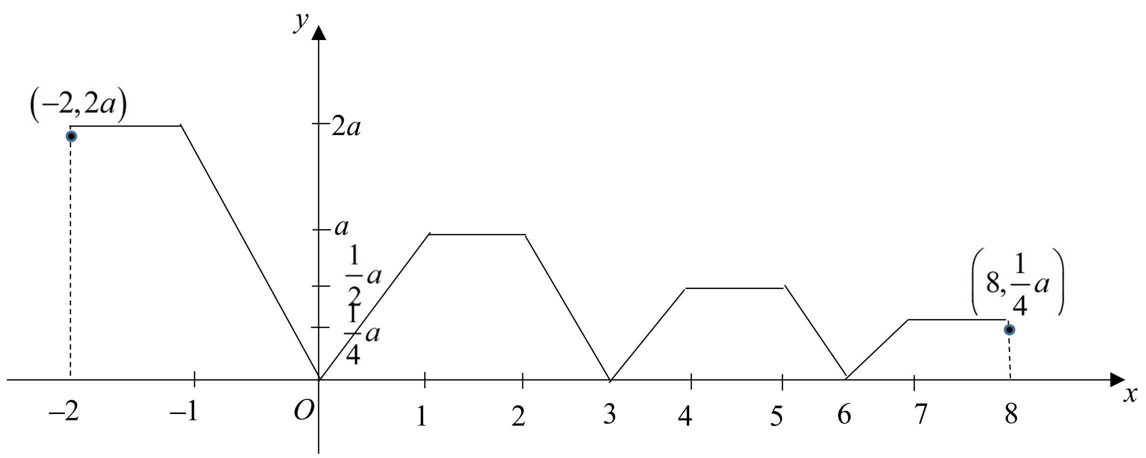
3. 2017/Prelim/PJC/P2/Q3b (modified)

It is given that

$$f(x)=\begin{cases} ax & 0 \le x < 1, \\ a & 1 \le x \le 2, \\ 3a - ax & 2 < x \le 3, \end{cases}$$

and that $f(x+3)=\frac{1}{2}f(x)$, for all real values of x, where a is a positive constant.

(i) Sketch the graph of $y=f(x)$ for $-2 \le x \le 8$. Hence, state the range of f. [4]



$R_f = [0, 2a]$

(ii) Find, in terms of a, $\int_0^3 f(x)\,dx$. [1]

Area of Trapezium $=\dfrac{3+1}{2}a = 2a$ units^2

(iii) Hence, find the value of the constant a for which $\int_0^\infty f(x)\,dx = 16$. [2]

$\int_0^\infty f(x)\,dx = 16$

$2\left(a+\frac{1}{2}a+\frac{1}{4}a+......\right)=16$

$2a\left(\dfrac{1}{1-\frac{1}{2}}\right)=16$

$4a=16$

$a=4$

4. DHS JC1 Promo 9758/2019/Q8

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{2x}{x-2}, \text{ for } x \in \mathbb{R},\ x \neq 2 .$$

(i) Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [2]

(ii) Solve the equation $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$. [1]

The function g is defined as follows.

$$\mathrm{g}: x \mapsto \frac{2}{x-3}, \text{ for } x \in \mathbb{R},\ x \neq 3,\ x \neq 4 .$$

(iii) Find $\mathrm{fg}(x)$. [1]

(iv) Solve the inequality $\mathrm{fg}(x)<x$ for all x in the domain of fg. [3]

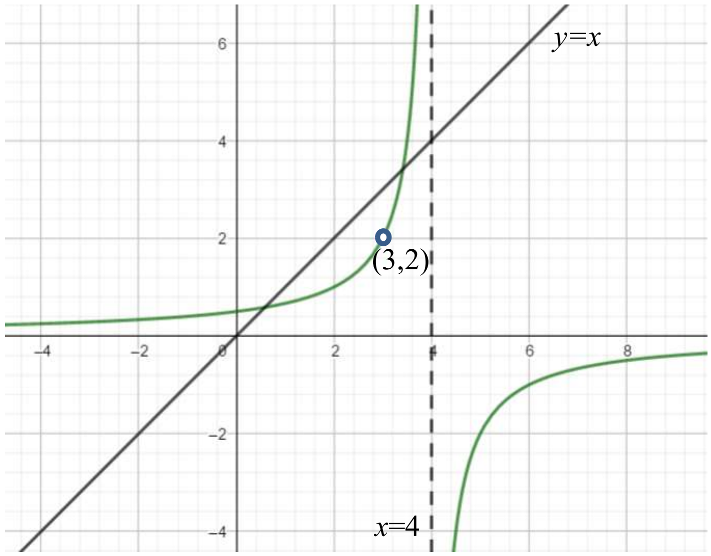
(v) Find the range of fg . [2]

(i)	Let $y=\mathrm{f}(x)=\dfrac{2x}{x-2}$ $xy-2y=2x$ $x(y-2)=2y$ $x=\dfrac{2y}{y-2}$ $\mathrm{f}^{-1}(y)=\dfrac{2y}{y-2}$ $\therefore \mathrm{f}^{-1}(x)=\dfrac{2x}{x-2}$ Domain of f^{-1} = $\mathbb{R}\setminus\{2\}$.
(ii)	$\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Since both graphs intersects at all points in the domain of f, $x \in \mathbb{R}\setminus\{2\}$
(iii)	$\mathrm{fg}(x) = \mathrm{f}\left(\dfrac{2}{x-3}\right)$ $=\dfrac{2\left(\dfrac{2}{x-3}\right)}{\dfrac{2}{x-3}-2}$ $=\dfrac{\dfrac{4}{x-3}}{\dfrac{2-2(x-3)}{x-3}}=\dfrac{4}{8-2x}=\dfrac{2}{4-x}$

(iv)

$\mathrm{fg}(x) < x$

$\dfrac{2}{4-x} < x$



Intersection points between the 2 graphs.
ie $(0.586, 0.586)$ and $(3.41, 3.41)$

For the inequality to be satisfied,
$x > 4$ or $0.586 < x < 3$ or $3 < x < 3.41$
$x > 4$ or $x \in (0.586, 3.41) \setminus \{3\}$

Alternative

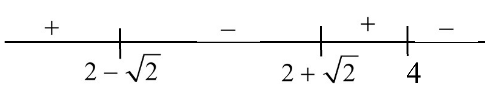
$\dfrac{2}{4-x} < x$

$\dfrac{2}{4-x} - x < 0$

$\dfrac{2-4x+x^2}{4-x} < 0$

$\dfrac{(x-2)^2-2}{4-x} < 0$

$\dfrac{\left(x-2-\sqrt{2}\right)\left(x-2+\sqrt{2}\right)}{4-x} < 0$



$x > 4$ or $2-\sqrt{2} < x < 3$ or $3 < x < 2+\sqrt{2}$
$x > 4$ or $x \in \left(2-\sqrt{2}, 2+\sqrt{2}\right) \setminus \{3\}$

(v)

For the range of fg
$\mathrm{D_g} \xrightarrow{\mathrm{g}} \mathrm{R_g} = \mathbb{R} \setminus \{0, 2\} \xrightarrow{\mathrm{f}} \mathrm{R_{fg}} = \mathbb{R} \setminus \{0, 2\}$

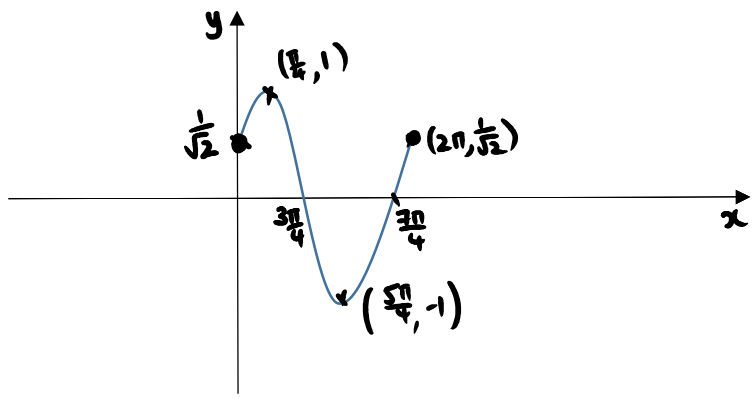
5. 2016/Promo/SAJC/Q7

The functions f and g are defined as follows:

$$\mathrm{f}: x \mapsto \sin\left(x+\frac{\pi}{4}\right), 0 \le x \le 2\pi$$

$$\mathrm{g}: x \mapsto \cos x, 0 \le x \le 2\pi.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$, showing any axial intercepts exactly. Explain clearly why f^{-1} does not exist. [3]



From the graph, we can see that $y=0$ cuts the graph $y=\mathrm{f}(x)$ twice. Thus, f is not 1-1 and f inverse does not exist.

(ii) If the domain of f is further restricted to $[a,b]$ such that f^{-1} exists and the range of f remains the same, state the new domain of f in exact form. [2]

The new restricted domain is $\left[\frac{\pi}{4},\frac{5\pi}{4}\right]$.

For the rest of the question, take the domain of f to be that found in **(ii).**

(iii) Show that the composite function gf^{-1} exists. Find the domain and range of gf^{-1}. [3]

$R_{\mathrm{f}^{-1}} = D_{\mathrm{f}} = \left[\frac{\pi}{4},\frac{5\pi}{4}\right]$ $\qquad D_{\mathrm{g}} = [0,2\pi]$

Thus $R_{\mathrm{f}^{-1}} \subseteq D_{\mathrm{g}}$ and gf^{-1} exists.

$D_{\mathrm{f}^{-1}} \xrightarrow{\mathrm{f}^{-1}} R_{\mathrm{f}^{-1}} = \left[\frac{\pi}{4},\frac{5\pi}{4}\right] \xrightarrow{\mathrm{g}} R_{\mathrm{gf}^{-1}} = [-1,\frac{\sqrt{2}}{2}]$

$D_{\mathrm{gf}^{-1}} = D_{\mathrm{f}^{-1}} = R_{\mathrm{f}} = [-1,1]$

6. MI PU1 Promo 9758/2019/01/Q10

The function f is defined by $\mathrm{f}: x \mapsto x^2 - 6x + 5,\ x \in \mathbb{R}$.

(i) Explain why f does not have an inverse. [2]

(ii) The function f has an inverse if its domain is restricted to $x \le k$. State the largest value of k. [1]

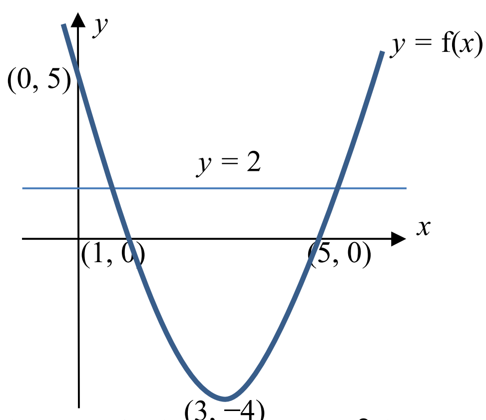
In the rest of the question, the domain of f is restricted to $(-\infty, k]$, with the value of k found in part **(ii)**.

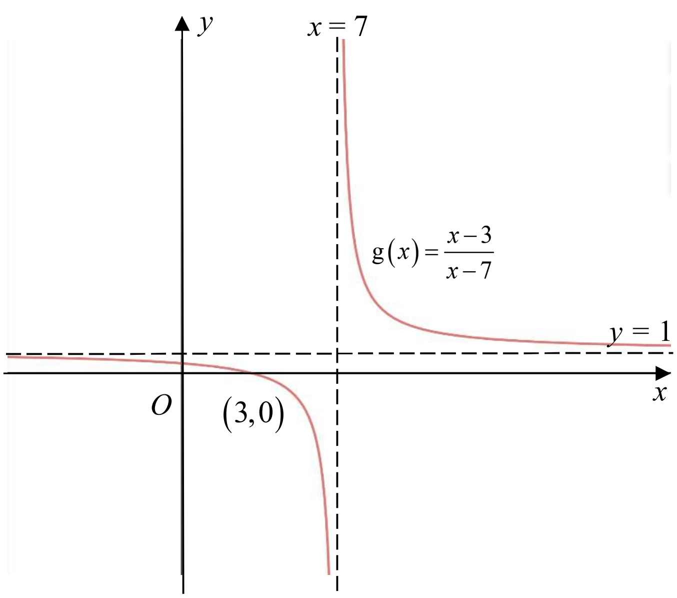
(iii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

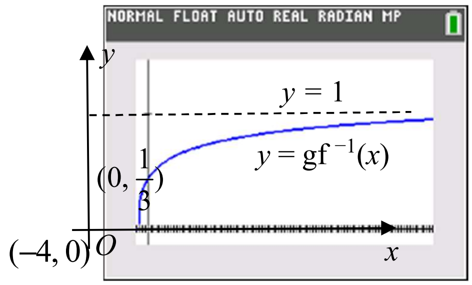
The function g is defined by $\mathrm{g}: x \mapsto \dfrac{x-3}{x-7},\ x \in \mathbb{R}, x \ne 7$.

(iv) Explain why the composite function gf^{-1} exists. [2]

(v) Find an expression for $\mathrm{gf}^{-1}(x)$ and find the range of gf^{-1}. [4]

(i) [2]	<u>Method 1</u>: Horizontal Line Test  Since **the** horizontal line $y=2$ intersects the graph of y = f(x) **more than once**, f is **not one-one**. Hence, f does not have an inverse. <u>Method 2</u>: Counterexample Since there exists $1, 5 \in \mathbb{R}$ such that $1 \ne 5$ and $\mathrm{f}(1) = \mathrm{f}(5) = 0$, f is **not one-one**. Hence, f does not have an inverse.	
(ii)	Largest value of k is 3.	
(iii) [3]	<u>Method 1</u>: Complete the Square Let $y = \mathrm{f}(x)$. $y = x^2 - 6x + 5$ $y = (x-3)^2 - 4$ $x = 3 \pm \sqrt{y+4}$	

	Method 2: Quadratic Formula Let $y=\mathrm{f}\left(x\right)=x^2-6x+5$. $x^2-6x+5-y=0$ $x=\dfrac{-(-6)\pm\sqrt{(-6)^2-4(1)(5-y)}}{2(1)}$ $x=\dfrac{6\pm\sqrt{36-4(5-y)}}{2}$ $x=3\pm\dfrac{1}{2}\sqrt{16+4y}$ $x=3\pm\sqrt{4+y}$ --- Since $x\le 3$, $x=3-\sqrt{y+4}$ (i.e. reject $x=3+\sqrt{y+4}$) $\mathrm{f}^{-1}\left(x\right)=3-\sqrt{x+4}$. Domain of f^{-1} = $[-4,\infty)$.	
(iv) [2]	Range of f^{-1} = Domain of f = $(-\infty,3]$. Domain of g = $\mathbb{R}\setminus\{7\}$. Since Range of f^{-1} $\subseteq$ Domain of g, $\therefore$ gf^{-1} exists.	
(v) [4]	$\mathrm{gf}^{-1}(x)=\mathrm{g}\left(3-\sqrt{x+4}\right)$ $=\dfrac{\left(3-\sqrt{x+4}\right)-3}{\left(3-\sqrt{x+4}\right)-7}$ $=\dfrac{\sqrt{x+4}}{4+\sqrt{x+4}}$.  Method 1: Mapping Method $[-4,\infty)\xrightarrow{\mathrm{f}^{-1}}(-\infty,3]\xrightarrow{\mathrm{g}}[0,1)$.	

	Method 2: Sketch graph of $y = \mathrm{gf}^{-1}(x)$  -- Hence, Range of $\mathrm{gf}^{-1} = [0, 1)$.	

7. NJC JC1 Promo 9758/2019/Q10

Functions f and g are defined by

$$\mathrm{f} : x \mapsto 2 - x + \frac{8}{x+2}, \quad x \in \mathbb{R}, x \neq -2, x > k,$$

$$\mathrm{g} : x \mapsto x^2 - 6x + a\ , \quad x \in \mathbb{R}, x > 0,$$

where a is a constant.

(i) State the least value of k for which the function f^{-1} exists. [1]

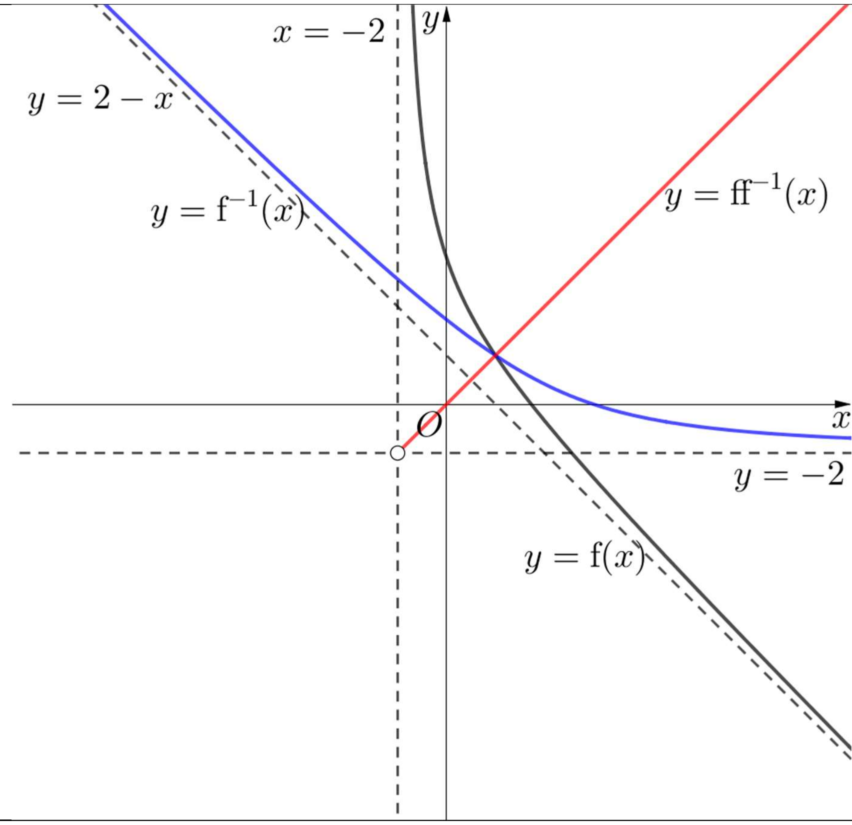
Using this value of k,

(ii) Without finding f^{-1}, sketch, on the same diagram, the graphs of $y = \mathrm{f}(x)$, $y = \mathrm{f}^{-1}(x)$ and $y = \mathrm{f}^{-1}\mathrm{f}(x)$, showing clearly their geometrical relationship. State the equations of any asymptotes. [4]

(iii) Find the smallest integer value of a for which the composite function fg exists and use this value to state the range of fg. [4]

(iv) Given instead that $a = 10$, solve the inequality $\mathrm{fg}(x) + \mathrm{g}(x) \leq 4$ algebraically. [5]

(i)	Minimum value of $k = -2$
(ii)	



(iii)

$\mathrm{g}(x)=x^2-6x+a=(x-3)^2+(a-9)$

$R_\mathrm{g}=[a-9,\infty)\qquad D_\mathrm{f}=(-2,\infty)$

For fg to exist, $R_\mathrm{g}\subseteq D_\mathrm{f}$, $a-9>-2\Rightarrow a>7$

Thus, smallest integer $a=8$

When $a=8$, $R_\mathrm{g}=[-1,\infty)$

$\mathrm{f}(-1)=2-(-1)+\dfrac{8}{-1+2}=3+8=11$

Hence range of fg is $(-\infty,11]$.

(iv)

$$\mathrm{fg}(x)+\mathrm{g}(x)\le 4$$

$$2-\mathrm{g}(x)+\frac{8}{\mathrm{g}(x)+2}+\mathrm{g}(x)\le 4$$

$$2-\frac{8}{\mathrm{g}(x)+2}\ge 0$$

Since $\mathrm{g}(x)+2=x^2-6x+12=(x-3)^2+3>0$ for all real values of x,

$$2(x^2-6x+12)-8\ge 0$$

$$2x^2-12x+16\ge 0$$

$$x^2-6x+8\ge 0$$

$$(x-2)(x-4)\ge 0$$

$$\therefore x\le 2 \quad\text{or}\quad x\ge 4$$

In addition, the domain of g is $x>0$.

Thus, the range of x is $0<x\le 2$ or $x\ge 4$

8. NYJC JC1 Promo 9758/2019/Q8

The functions f, g and h are defined as follows.

$$\begin{aligned} &\mathrm{f}: x \mapsto \left|(x-1)(3-x)\right|, \quad x > k,\ k \in \mathbb{R}, \\ &\mathrm{g}: x \mapsto -mx+1, \qquad x \in \mathbb{R},\ m > 1, \\ &\mathrm{h}: x \mapsto \left|(x-1)(3-x)\right|, \quad x \in \mathbb{R}. \end{aligned}$$

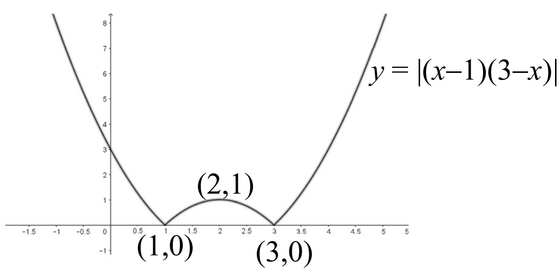
(i) State the least value of k such that f^{-1} exists. [1]

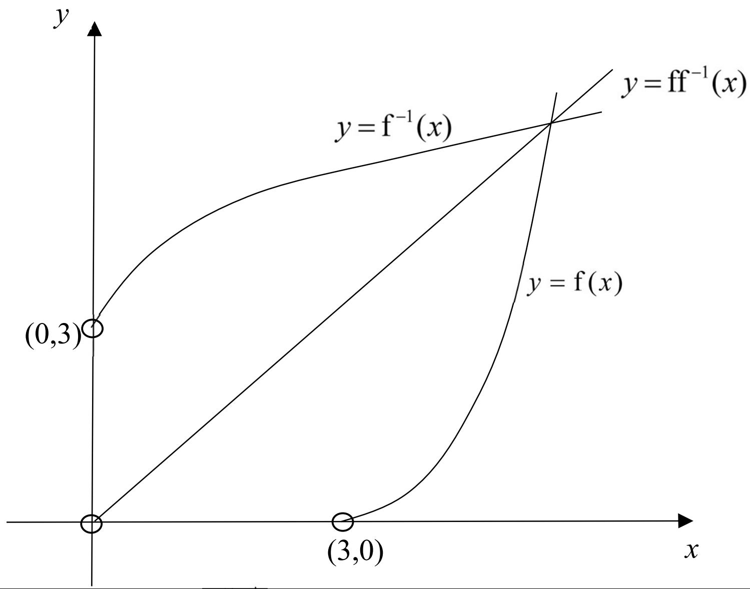
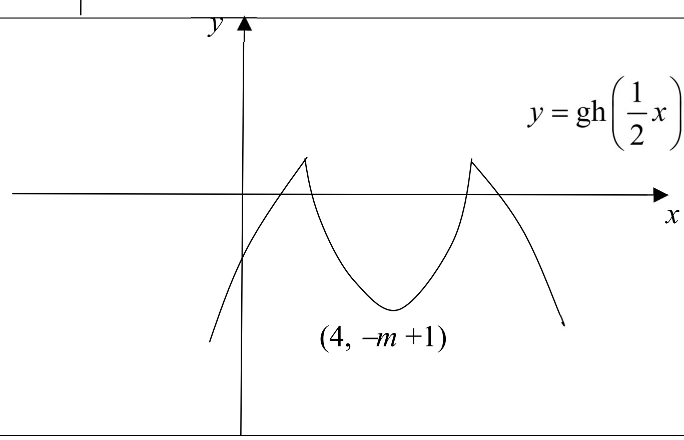
Using the value of k found in part **(i)**,

(ii) find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1} , [4]

(iii) sketch on the same diagram the graphs of $y=\mathrm{f}(x), y=\mathrm{f}^{-1}(x)$ and $y=\mathrm{ff}^{-1}(x)$. [3]

(iv) On a separate diagram, sketch the graph of $y=\mathrm{gh}\left(\frac{1}{2}x\right)$, indicating clearly the coordinates of the turning point. [2]

(i)	For f^{-1} to exist, $k = 3$ 
(ii)	When $x > 3, y = -(x-1)(3-x)$ $y = x^2 - 4x + 3$ $\quad = (x-2)^2 - 1$ $(x-2)^2 = y+1$ $(x-2) = \pm\sqrt{y+1}$ $x = 2 \pm \sqrt{y+1}$ Since $x > 3$ $\mathrm{f}^{-1}(x) = 2 + \sqrt{x+1}$ $\mathrm{D}_{\mathrm{f}^{-1}} = (0, \infty)$

(iii)	
(iv)	

9. 2021/Prelim/EJC/P2/Q3

(a) The function f is given by $\mathrm{f} : x \mapsto \cos\left(\frac{1}{2}x + \frac{1}{6}\pi\right), \quad x \in \mathbb{R},\ 0 \le x \le k.$

(i) State the largest exact value of k for which the function f^{-1} exists. [1]

For the rest of the question, the domain of f is $x \in \mathbb{R},\ 0 \le x \le \frac{4}{3}\pi.$

(ii) Write down the equation of the line in which the graph of $y = \mathrm{f}(x)$ must be reflected in order to obtain the graph of $y = \mathrm{f}^{-1}(x)$. Hence, sketch on the same diagram, the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$, indicating the exact coordinates of the endpoints of both graphs. [3]

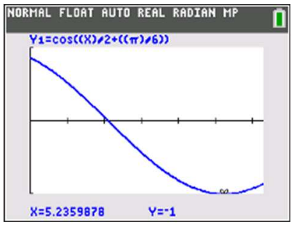
(iii) State the value(s) of x for which $\mathrm{ff}^{-1}(x) = \mathrm{f}^{-1}\mathrm{f}(x)$. [1]

(b) The functions g and h are defined by

$$\begin{aligned} &\mathrm{h} : x \mapsto 2x^2 + 3, && x \in \mathbb{R},\ x \le 0, \\ &\mathrm{hg} : x \mapsto 2x + 3 - 2a, && x > a, \text{ where } a \in \mathbb{R}^+ \end{aligned}$$

Find $\mathrm{g}(x)$ and state the domain of g. [3]

1 (a)(i) From graph on GC, largest k is when $y=-1$

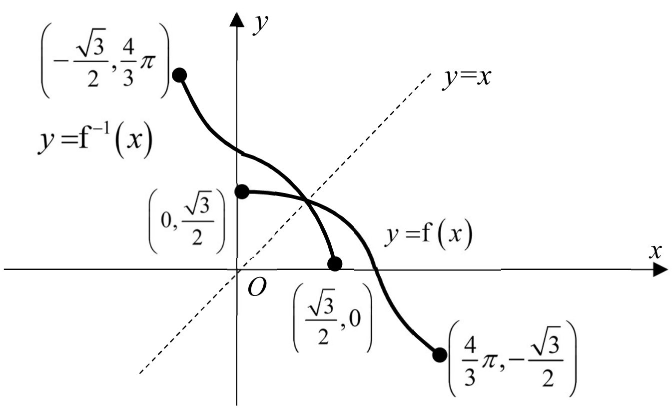


$$\cos\left(\frac{x}{2}+\frac{\pi}{6}\right)=-1$$

i.e. $\Rightarrow \frac{x}{2}+\frac{\pi}{6}=\pi$

$$\Rightarrow x=2\left(\pi-\frac{\pi}{6}\right)=\frac{5\pi}{3}$$

(a)(ii)



(a)(iii) $\mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)=x$

The two functions are equal when the domains intersect.

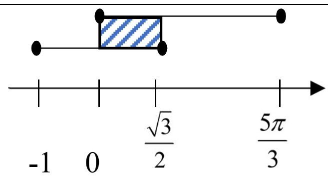
From graph in (a)(ii), $0\le x\le\frac{\sqrt{3}}{2}$.

$$\therefore \mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)=x \text{ for } 0\le x\le\frac{\sqrt{3}}{2}$$

<u>Alternative: (consider domains)</u>

$$D_{\mathrm{f}^{-1}\mathrm{f}}=D_{\mathrm{f}}=\left[0,\frac{5\pi}{3}\right]$$

$$D_{\mathrm{ff}^{-1}}=D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=\left[-1,\frac{\sqrt{3}}{2}\right]$$



$\therefore \mathrm{ff}^{-1}(x)=\mathrm{f}^{-1}\mathrm{f}(x)=x$ for $0\le x\le \dfrac{\sqrt{3}}{2}$

(b) $\mathrm{h}: x \mapsto 2x^2+3, \quad x\in\mathbb{R}, x\le 0,$

$\mathrm{hg}: x \mapsto 2x+3-2a, \quad x\in\mathbb{R}^+, x>a.$

$$\mathrm{h}\left[\mathrm{g}(x)\right]=2x+3-2a$$

$$2\left[\mathrm{g}(x)\right]^2+3=2x+3-2a$$

$$\left[\mathrm{g}(x)\right]^2=x-a$$

$$\mathrm{g}(x)=-\sqrt{x-a} \qquad \because R_{\mathrm{g}}\subseteq D_{\mathrm{h}}=(-\infty,0]$$

$D_{\mathrm{g}}=D_{\mathrm{hg}}=(a,\infty)$

<u>Alternative: find h^{-1}</u>

Let $y=2x^2+3$. Then $x=-\sqrt{\dfrac{y-3}{2}}$ or $\sqrt{\dfrac{y-3}{2}}$ (reject $\because x\le 0$).

So $\mathrm{h}^{-1}(x)=-\sqrt{\dfrac{x-3}{2}}$.

Then $\mathrm{h}^{-1}(\mathrm{hg}(x))=-\sqrt{\dfrac{\mathrm{hg}(x)-3}{2}}$

$\therefore \mathrm{g}(x)=-\sqrt{\dfrac{2x+3-2a-3}{2}}=-\sqrt{x-a}$

$D_{\mathrm{g}}=D_{\mathrm{hg}}=(a,\infty)$

10. 2021/Prelim/RI/P2/Q1

Functions f and g are defined by

$$\mathrm{f}: x\mapsto \mathrm{e}^{(x-1)^2}, \quad x\in\mathbb{R},$$

$$\mathrm{g}: x\mapsto \frac{1}{2-x}, \quad x\in\mathbb{R}, \quad 1\le x<2.$$

(i) Sketch the graph of $y=\mathrm{f}(x)$. [1]

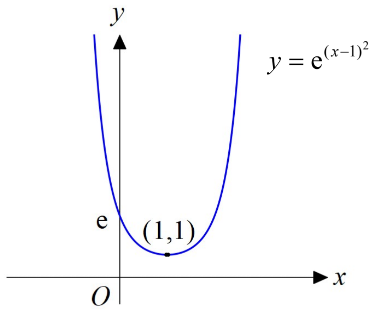
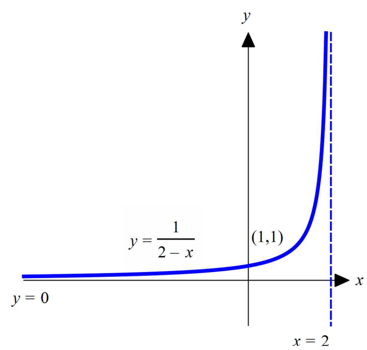
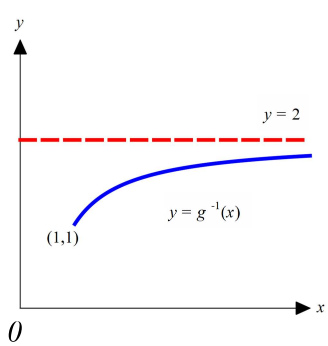
(ii) If the domain of f is restricted to $x\ge k$, state with a reason the least value of k for which the function f^{-1} exists. [2]

In the rest of the question, the domain of f is $x\ge k$, using the value of k found in part **(ii)**.

(iii) Find $\mathrm{g}^{-1}(x)$ and show that the composite function $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists. [4]

(iv) Find the range of $\mathrm{g}^{-1}\mathrm{f}^{-1}$. [1]

RI Prelim 9758/2021/02/Q1

Qn 1: Solutions	
(i) **[1]**	
(ii) **[2]**	For f^{-1} to exist, f must be a one-one function. Least value of $k=1$.
(iii) **[4]**	Let $y=\dfrac{1}{2-x}$ $2-x=\dfrac{1}{y}$ $x=2-\dfrac{1}{y}$ $\mathrm{g}^{-1}(x)=2-\dfrac{1}{x}$  For $\mathrm{g}^{-1}\mathrm{f}^{-1}$ to exist, $\mathrm{R}_{\mathrm{f}^{-1}}\subseteq\mathrm{D}_{\mathrm{g}^{-1}}$. $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=[1,\infty)$ $\mathrm{D}_{\mathrm{g}^{-1}}=\mathrm{R}_{\mathrm{g}}=[1,\infty)$ Since $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{g}^{-1}}$, $\therefore$ $\mathrm{g}^{-1}\mathrm{f}^{-1}$ exists.
(iv) **[1]**	$[1,\infty)\xrightarrow{\mathrm{f}^{-1}}[1,\infty)\xrightarrow{\mathrm{g}^{-1}}[1,2)$ $\mathrm{R}_{\mathrm{g}^{-1}\mathrm{f}^{-1}}=[1,2)$  Note: $\mathrm{D}_{\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{f}}=[1,\infty)$, $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{f}}=[1,\infty)$ **OR** Since $\mathrm{R}_{\mathrm{f}^{-1}}=\mathrm{D}_{\mathrm{g}^{-1}}$, $\mathrm{R}_{\mathrm{g}^{-1}\mathrm{f}^{-1}}=\mathrm{R}_{\mathrm{g}^{-1}}=\mathrm{D}_{\mathrm{g}}=[1,2)$