A2: EQUATIONS AND INEQUALITIES

- Conditions for a quadratic equation to have:
  - (i) two real roots
  - (ii) two equal roots
  - (iii) no real roots

  and related conditions for a given line to:
  - (i) intersect a given curve
  - (ii) be a tangent to a given curve
  - (iii) not intersect a given curve
- Solving simultaneous equations in two variables by substitution, with one of the equations being linear equation
- Solving quadratic inequalities, and representing the solution on the number line

1. Solve the simultaneous equations $x^2 - xy + y^2 - 7 = 0$ $y - 3x + 7 = 0$	[4]
2. Find the set of values of the constant k for which the curve $y = x^2 + 12x - 4k + 41$ lies completely above the line $y = kx + \frac{9}{4}k$.	[4]
3. A line has equation $y = 1 - 2x$ and a curve has equation $y = 3x^2 + x + 5$. Determine, with reasons, whether the line intersects, is a tangent to, or does not intersect the curve.	[3]
4. Find the coordinates of the points of intersection of the curve $x^2 + xy + 5 = 0$ and the line $3x + y = 3$.	[4]
5. Find the range of values of k such that the line $y = 2k + x$ intersects the curve $y^2 - xy - x^2 = 5$ at 2 distinct points.	[5]
6. Find the range of values of p given that $y = x^2 + (p - 1)x + 4$ is always positive.	[3]
7. The equation of a curve is $y = x^2 - 3x - 1$. Find the range of values of x for which $y + 3 > 0$.	[3]
8. Solve the simultaneous equations. $2x + y - 1 = 0$ $3x^2 + 5xy - y^2 + 3 = 0$	[5]

9. Find the set of values for which the curve $y = (3k - 2)x^2 + 6kx + (3k + 4)$ lies entirely above or below the x-axis.	[3]
10. Find the range of values of x for which $-2x^2 + x + 3$ is positive.	[3]
11. A line has equation $y = x + 2$ and the equation of a circle is $(x - 2)^2 + y^2 = 2$. Determine whether the line intersects, is a tangent to, or does not intersect the circle. Give a reason for your answer.	[3]
12. Solve the simultaneous equations $xy + x^2 = 26$ $2y - x = 1$	[4]
13. Given that these simultaneous equations $x^2 + y^2 - 16 = 0$ $x - y = k$ have exactly one pair of solutions, show that $k = \pm p\sqrt{2}$, where p is an integer.	[6]
14. Solve the inequality $2x^2 + 6x + 13 < 3x(x - 2)$.	[3]
15. A line has equation $y = x - 7$ and a curve has equation $y = 3x^2 - 5x + 1$. Determine whether the line intersects, is a tangent to, or does not intersect the curve. Give a reason for your answer.	[3]
16. Find the coordinates of the points of intersection between the line $x + y = 4$ and the curve $2xy - y + 5x^2 = 8$.	[5]
17. The line $y = 3x + 2$ intersects the curve $xy = 2 + y$ at the points P and Q. Find the midpoint of the line PQ.	[5]
18. Find the smallest prime value of k for which the line $y = 2x + k$ cuts the curve $y = 3x^2 + 5x + 7$ at two distinct points.	[4]
19. Find the coordinates of the points of intersection of the line $y = 3x + 1$ and the curve $y = \frac{1}{x} - 1$.	[5]
20. Find the range of values of x for which $6x^2 > 47x - 52$.	[3]
21. Find the possible values of k for which the line $y = 2x + k$ will be a tangent to the curve $x^2 + y^2 = 5$.	[4]

A2: EQUATIONS AND INEQUALITIES (MARKING SCHEME)

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1. Solve the simultaneous equations

$x^2 - xy + y^2 - 7 = 0$

$y - 3x + 7 = 0$

$x^2 - xy + y^2 - 7 = 0$ --- (1)

$y = 3x - 7$ --- (2)

Sub(stitute) (2) to (1)

$x^2 - x(3x - 7) + (3x - 7)^2 - 7 = 0$

$x^2 - 3x^2 + 7x + 9x^2 - 42x + 49 - 7 = 0$

$7x^2 - 35x + 42 = 0$

$7\left(x^2 - 5x + 6\right) = 0$

$7(x - 3)(x - 2) = 0$

$x = 3$ or $x = 2$

$y = 2$ or $y = -1$

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2. Find the set of values of the constant k for which the curve

$y = x^2 + 12x - 4k + 41$ lies completely above the line $y = kx + \frac{9}{4}k$.

$x^2 + 12x - 4k + 41 = kx + \frac{9}{4}k$ ★ equate two expressions

$x^2 + 12x - kx - 4k - \frac{9}{4}k + 41 = 0$ ★ equate to 0

$x^2 + (12 - k)x - \frac{25}{4}k + 41 = 0$

Discriminant < 0, $b^2 - 4ac < 0$

$(12 - k)^2 - 4(1)\left(-\frac{25}{4}k + 41\right) < 0$ ★ $b^2 - 4ac < 0$

$144 - 24k + k^2 + 25k - 164 < 0$

$k^2 + k - 20 < 0$

$(k - 4)(k + 5) < 0$ ★ factorise

$-5 < k < 4$

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3. A line has equation $y = 1 - 2x$ and a curve has equation $y = 3x^2 + x + 5$. Determine, with reasons, whether the line intersects, is a tangent to, or does not intersect the curve. $1 - 2x = 3x^2 + x + 5$ ★ equate two expressions $3x^2 + 3x + 4 = 0$ ★ equate to 0 $b^2 - 4ac$ ★ determine relation of the curve and line through the discriminant $= (3)^2 - 4(3)(4)$ $=- 39$ Since discriminant $(b^2 - 4ac) < 0$, the line **does not intersect the curve**.	[3]
4. Find the coordinates of the points of intersection of the curve $x^2 + xy + 5 = 0$ and the line $3x + y = 3$. ★ Apply simultaneous equations $x^2 + xy + 5 = 0$ --- (1) $y = 3 - 3x$ --- (2) Sub(stitute) (2) to (1): $x^2 + x(3 - 3x) + 5 = 0$ $x^2 + 3x - 3x^2 + 5 = 0$ $- 2x^2 + 3x + 5 = 0$ $(- 2x + 5)(x + 1) = 0$ ★ Factorise $- 2x =- 5$ or $x =- 1$ $x = \frac{5}{2}$ or $x =- 1$ $y =- \frac{9}{2}$ or $y = 6$ Therefore, the coordinates are $(- 1, 6)$ and $\left(\frac{5}{2}, - \frac{9}{2}\right)$	[4]

5. Find the range of values of k such that the line $y = 2k + x$ intersects the curve $y^2 - xy - x^2 = 5$ at 2 distinct points. ★ Apply simultaneous equations $y = 2k + x$ --- (1) $y^2 - xy - x^2 = 5$ --- (2) Sub(stitute) (1) to (2): $(2k + x)^2 - x(2k + x) - x^2 = 5$ $4k^2 + 4kx + x^2 - 2kx - x^2 - x^2 = 5$ $-x^2 + 2kx + 4k^2 - 5 = 0$ ★ For the line to intersect the curve at 2 distinct points, $b^2 - 4ac > 0$ $(2k)^2 - 4(-1)\left(4k^2 - 5\right) > 0$ $4k^2 + 16k^2 - 20 > 0$ $20k^2 - 20 > 0$ $k^2 - 1 > 0$ $(k - 1)(k + 1) > 0$ $k < -1$ or $k > 1$ Hence, $k < -1$ or $k > 1$	[5]
6. Find the range of values of p given that $y = x^2 + (p - 1)x + 4$ is always positive. ★ $b^2 - 4ac < 0$ (always positive) $(p - 1)^2 - 4(1)(4) < 0$ $(p - 1)^2 - 16 < 0$ $(p - 1 - 4)(p - 1 + 4) < 0$ $(p - 5)(p + 3) < 0$ $-3 < p < 5$ Hence, $-3 < p < 5$	[3]

7. The equation of a curve is $y = x^2 - 3x - 1$. Find the range of values of x for which $y + 3 > 0$. $y + 3 > 0$ $x^2 - 3x - 1 + 3 > 0$ $x^2 - 3x + 2 > 0$ $(x - 2)(x - 1) > 0$ $x < 1$ or $x > 2$	[3]
8. Solve the simultaneous equations. $2x + y - 1 = 0$ $3x^2 + 5xy - y^2 + 3 = 0$ $y = 1 - 2x$ --- (1) $3x^2 + 5xy - y^2 + 3 = 0$ --- (2) Sub(stitute) (1) to (2): $3x^2 + 5x(1 - 2x) - (1 - 2x)^2 + 3 = 0$ $3x^2 + 5x - 10x^2 - \left(1 - 4x + 4x^2\right) + 3 = 0$ $-7x^2 + 5x - 1 + 4x - 4x^2 + 3 = 0$ $-11x^2 + 9x + 2 = 0$ $(11x + 2)(-x + 1) = 0$ $x = -\frac{2}{11}$ or $x = 1$ $y = \frac{15}{11}$ or $y = -1$	[5]

9. Find the set of values for which the curve $y = (3k - 2)x^2 + 6kx + (3k + 4)$ lies entirely above or below the x-axis. For a curve to be entirely above or below the x-axis, $b^2 - 4ac < 0$ $(6k)^2 - 4(3k - 2)(3k + 4) < 0$ $36k^2 - 4\left(9k^2 + 12k - 6k - 8\right) < 0$ $36k^2 - 4\left(9k^2 + 6k - 8\right) < 0$ $36k^2 - 36k^2 - 24k + 32 < 0$ $-24k + 32 < 0$ $32 < 24k$ $\frac{32}{24} < k$ $k > \frac{4}{3}$	[3]
10. Find the range of values of x for which $-2x^2 + x + 3$ is positive. $-2x^2 + x + 3 > 0$ ★ Ensure the coefficient of x^2 is positive $2x^2 - x - 3 < 0$ $(2x - 3)(x + 1) < 0$ $-1 < x < \frac{3}{2}$	[3]
11. A line has equation $y = x + 2$ and the equation of a circle is $(x - 2)^2 + y^2 = 2$. Determine whether the line intersects, is a tangent to, or does not intersect the circle. Give a reason for your answer. $(x - 2)^2 + y^2 = 2$ --- (1) $y = x + 2$ --- (2) Sub(stitute) (2) to (1): $(x - 2)^2 + (x + 2)^2 = 2$ $x^2 - 4x + 4 + x^2 + 4x + 4 = 2$ $2x^2 + 6 = 0$ $b^2 - 4ac$ $= (0)^2 - 4(2)(6)$ $= -48$ (< 0) $b^2 - 4ac < 0$, line does not intersect the curve	[3]

12. Solve the simultaneous equations $xy + x^2 = 26$ $2y - x = 1$ $xy + x^2 = 26$ --- (1) $2y - 1 = x$ --- (2) Sub (2) to (1): $(2y - 1)(y) + (2y - 1)^2 = 26$ $2y^2 - y + 4y^2 - 4y + 1 = 26$ $6y^2 - 5y + 1 - 26 = 0$ $6y^2 - 5y - 25 = 0$ $(2y - 5)(3y + 5) = 0$ $y = \frac{5}{2}$ or $y = -\frac{5}{3}$ $x = 4$ or $x = -\frac{13}{3}$ When $y = \frac{5}{2}$, $x = 4$ When $y = -\frac{5}{3}$, $x = -\frac{13}{3}$	[4]

13. Given that these simultaneous equations [6]

$x^2 + y^2 - 16 = 0$

$x - y = k$

have exactly one pair of solutions, show that $k = \pm p\sqrt{2}$, where p is an integer.

$x^2 + y^2 - 16 = 0$ --- (1)

$x - k = y$ --- (2)

$x^2 + (x - k)^2 - 16 = 0$

$x^2 + x^2 - 2kx + k^2 - 16 = 0$

$2x^2 - 2kx + k^2 - 16 = 0$

$b^2 - 4ac = 0$

$(-2k)^2 - 4(2)\left(k^2 - 16\right) = 0$

$4k^2 - 8k^2 + 128 = 0$

$-4k^2 + 128 = 0$

$4k^2 = 128$

$k^2 = 32$

$k = \pm\sqrt{2 \times 16}$

$k = \pm 4\sqrt{2}$

14. Solve the inequality $2x^2 + 6x + 13 < 3x(x - 2)$. [3]

$2x^2 + 6x + 13 < 3x^2 - 6x$

$-x^2 + 12x + 13 < 0$

$x^2 - 12x - 13 > 0$ ★ Ensure the coefficient of x^2 is positive

$(x - 13)(x + 1) > 0$ ★ Factorise

$x < -1$ or $x > 13$

15. A line has equation $y = x - 7$ and a curve has equation $y = 3x^2 - 5x + 1$. Determine whether the line intersects, is a tangent to, or does not intersect the curve. Give a reason for your answer. $3x^2 - 5x + 1 = x - 7$ ★ equate two expressions $3x^2 - 6x + 8 = 0$ ★ equate to 0 $b^2 - 4ac$ ★ determine relation of the curve and line through the discriminant $= (-6)^2 - 4(3)(8)$ $= -60\ (< 0)$ Since the discriminant is less than 0, the line and the curve do not meet.	[3]
16. Find the coordinates of the points of intersection between the line $x + y = 4$ and the curve $2xy - y + 5x^2 = 8$. $y = 4 - x$ --- (1) $2xy - y + 5x^2 = 8$ --- (2) Sub (1) to (2): $2x(4 - x) - (4 - x) + 5x^2 = 8$ $8x - 2x^2 - 4 + x + 5x^2 = 8$ $3x^2 + 9x - 12 = 0$ $x^2 + 3x - 4 = 0$ $(x + 4)(x - 1) = 0$ $x = -4$ or $x = 1$ $y = 8$ or $y = 3$ The points of intersection are $(-4, 8)$ and $(1, 3)$	[5]

17. The line $y = 3x + 2$ intersects the curve $xy = 2 + y$ at the points P and Q. Find the midpoint of the line PQ. $y = 3x + 2$ --- (1) $xy = 2 + y$ --- (2) Sub (1) to (2) $x(3x + 2) = 2 + 3x + 2$ $3x^2 + 2x = 3x + 4$ $3x^2 - x - 4 = 0$ $(3x - 4)(x + 1) = 0$ $x = \frac{4}{3}$ or $x = -1$ $y = 6$ or $y = -1$ $\left(\frac{4}{3}, 6\right)$ and $(-1, -1)$ Midpoint of PQ $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ Midpoint of PQ $= \left(\frac{\frac{4}{3}-1}{2}, \frac{6-1}{2}\right)$ Midpoint of PQ $= \left(\frac{1}{6}, 2\frac{1}{2}\right)$	[5]
18. Find the smallest prime value of k for which the line $y = 2x + k$ cuts the curve $y = 3x^2 + 5x + 7$ at two distinct points. $3x^2 + 5x + 7 = 2x + k$ $3x^2 - 3x + 7 - k = 0$ ★ For the line to intersect the curve at 2 distinct points, $b^2 - 4ac > 0$ $(-3)^2 - 4(3)(7 - k) > 0$ $9 - 84 + 12k > 0$ $12k - 75 > 0$ $k > 6\frac{1}{4}$ The smallest prime value of $k = 7$	[4]

20. Find the range of values of x for which $6x^2 > 47x - 52$. $6x^2 - 47x + 52 > 0$ $(2x - 13)(3x - 4) > 0$ $x < 1\frac{1}{3}$ or $x > 6\frac{1}{2}$	[3]
21. Find the possible values of k for which the line $y = 2x + k$ will be a tangent to the curve $x^2 + y^2 = 5$. $x^2 + y^2 = 5$ --- (1) $y = 2x + k$ --- (2) Sub (2) to (1) $x^2 + (2x + k)^2 = 5$ $x^2 + 4x^2 + 4kx + k^2 = 5$ $5x^2 + 4kx + k^2 - 5 = 0$ For one real root, $b^2 - 4ac = 0$ $(4k)^2 - 4(5)\left(k^2 - 5\right) = 0$ $16k^2 - 20k^2 + 100 = 0$ $-4k^2 + 100 = 0$ $k^2 = 25$ $k = \pm 5$	[4]