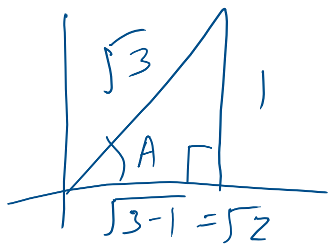
Student solution and comments 2022 Prelim Y6 HL Math P1

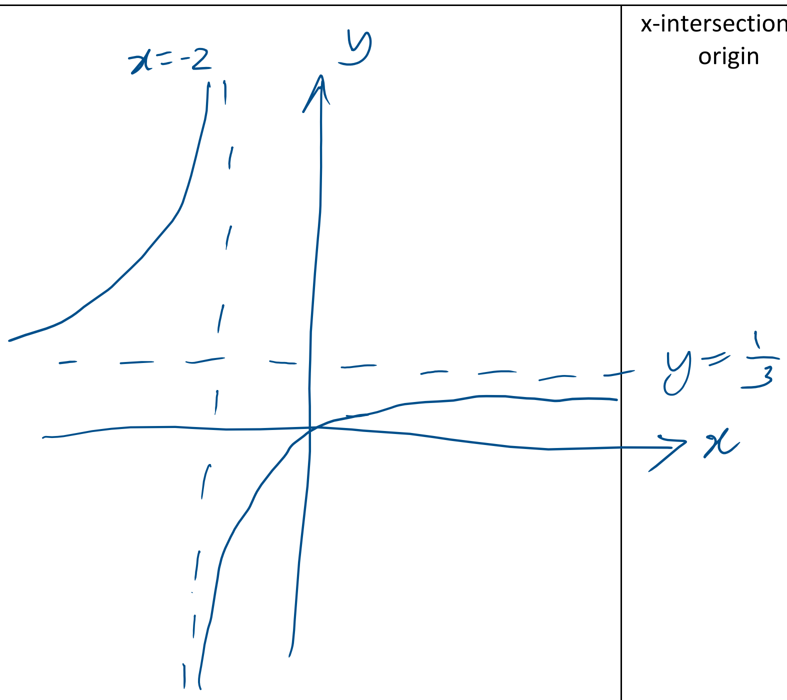
Section A

1.		Comments
(a)(i)	$\sin A = \frac{1}{\sqrt{3}}$  $\tan A = \frac{1}{\sqrt{2}}$	Some students forgot to square root the side 2
(ii)	$\tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$ Let $x = \tan\frac{A}{2}$: $\Rightarrow \frac{1}{\sqrt{2}} = \frac{2x}{1-x^2}$ $1 - x^2 = 2\sqrt{2}x$ $x^2 + 2\sqrt{2}x - 1 = 0$ $x = \frac{-2\sqrt{2} \pm \sqrt{\left(2\sqrt{2}\right)^2 - 4(-1)}}{2}$ $x = \frac{-2\sqrt{2} \pm \sqrt{12}}{2} = \frac{-2\sqrt{2} \pm 2\sqrt{3}}{2}$ $x = -\sqrt{2} + \sqrt{3} \Rightarrow \tan\frac{A}{2} = \sqrt{3} - \sqrt{2}$	Students did not realize that the quadratic equations can be solved using formula method, instead tried to factorize.

(b)	$2\sin x = \tan x$ $2\sin x = \dfrac{\sin x}{\cos x}$ $2\sin x\cos x - \sin x = 0$ $\sin x\,(2\cos x - 1) = 0$ $\sin x = 0 \Rightarrow x = 0$ $2\cos x - 1 = 0 \Rightarrow x = \frac{1}{2} \Rightarrow x = \frac{\pi}{3}$ or $x = -\frac{\pi}{3}$	$\sin x$ was deleted on both side, thus missing a root which is $\sin x = 0$; do not cancel out on both sides of the equation, instead shift everything to LHS and factorise.

2.		Comments
(a)	$n = 10$	Generally well done; remember number of terms in binomial expansion = n+1
(b)	$r^{th} term = \binom{10}{r}(x^2)^{10-r}\left(\frac{k}{x}\right)^r = \binom{10}{r}(x)^{20-3r}k^r$ $20 - 3r = 14 \Rightarrow r = 2$ $\therefore \binom{10}{2}k^2 = 180$ where $\binom{10}{2} = \frac{10!}{8!\,2!} = \frac{9\times10}{2} = 45$ $45k^2 = 180$ $k^2 = 4 \Rightarrow k = 2$	Generally well done

3.		Comments
(a)	$c(-2) + 6 = 0 \Rightarrow c = 3$	Generally well done
(b)	$y = \dfrac{1}{3}$	Generally well done

(c)		x-intersection at origin
(d)	$\frac{x}{3x+6} = 2 \Rightarrow x = -\frac{12}{5}$ Hence $0 < \frac{x}{3x+6} < 2$: $x < -\frac{12}{5}$ or $x > 0$	Since "hence" is used, attempt to find intersection must be seen.

4.		Comments
(a)	$\frac{u_6}{u_3} = \frac{u_3}{u_2}$	Generally well done
	$\frac{1+5d}{1+2d} = \frac{1+2d}{1+d}$	
	$(1+2d)^2 = (1+5d)(1+d)$ $1+4d+4d^2 = 1+d+5d+d^2$	
	$d^2 + 2d = 0$	
	$d(d+2) = 0$	
	$d = 0$ (rej.) $d = -2$	
(b)	$u_n = 1+(n-1)(-2) = -17$	Generally well done
	$-2n = -20$ $n = 10$	
	$\sum_{r=1}^{10} u_r = 1+(-1)+(-3)+...+(-17)$	Generally well done; some

	$\sum_{r=1}^{10} u_r = \frac{10}{2}\left(1+(-17)\right)$	students did not read the question carefully and tried to find sum of GP
	$\sum_{r=1}^{10} u_r = -80$	

5.		Comments
(a)	Let P_n be the statement that $6^n - 1$ is divisible by 5 for $n \geq 2$ and $n \in \mathbb{Z}^+$.	Generally well done
	OR	
	Let P_n be the statement that $6^n - 1 = 5K, K \in \mathbb{Z}^+$, for $n \geq 2$ and $n \in \mathbb{Z}^+$.	
	$P_2 : 6^2 - 1 = 35 = 5 \times 7$ $\therefore P_2$ is true.	Some students accidentally wrote P1 instead
	Assume P_k is true for some integer k $6^k - 1 = 5Q, Q \in \mathbb{Z}^+$	
	P_{k+1} : $6^{k+1} - 1 = 6\left(6^k\right) - 1$	
	$= 6\left(5Q+1\right) - 1$ $= 30Q + 5$ $= 5(6Q) + 5$ $= 5(6Q+1)$ $= 5R, R \in \mathbb{Z}^+$	Some students assume LHS = 5A and manipulate from the result. Always start from one side and show to the other. Conventionally start from LHS.
	Since P_2 is true, P_k is true $\Rightarrow P_{k+1}$ is true, by MI, P_n is true for $n \geq 2$ and $n \in \mathbb{Z}^+$. * Make sure initial step is true (P_2 is true); inductive step (P_k is true $\Rightarrow P_{k+1}$ is true) is true, these two are a pair with "if.. then..." relationship.	Some students used ambiguous phrasing.

6.		Comments
(a)	Assume $a + b < 2\sqrt{ab}$ for $a, b \in \mathbb{Z}^+$	Some students did not write down the correct negation. Conditions should be kept the same (a, b are positive integers and not equal), while the therefore statement is

		negated. (you can seen < as the complement set of $\geq$)
	$(a+b)^2 < 4ab$ $a^2+2ab+b^2 < 4ab$ $a^2-2ab+b^2 < 0$	
	$(a-b)^2 < 0$	
	However, $(a-b)^2 \geq 0$ and since $a \neq b$ So $(a-b)^2 > 0$ (contradiction) By contradiction, $a+b \geq 2\sqrt{ab}$	The contradiction should not allow some cases to be true. Eg. Some students assumed $a+b \leq 2\sqrt{ab} \Rightarrow (a-b)^2 \leq 0$ and contradiction is $(a-b)^2 \geq 0$, allowing $(a-b)^2 = 0$ to be true.

7.		Comments
(a)	Point of inflexion occurs when $f''(a)=0$ and $f''(x)$ changes sign at $x=a$. Minimal point of $f'(x)$ satisfy the condition, therefore	Well done
	$x=3$ is the point of inflexion.	
(b)	$\int_0^1 f'(x)dx = f(1)-f(0)$	Well done
	$= f(1)-1=1$	
	$\Rightarrow f(1)=2$	
	$\int_1^4 f'(x)dx = f(4)-f(1) = -5$	Quite a number of students forgot the integral result is the signed area under graph.
	$f(4)=f(1)-5$	
	$f(4)=2-5=-3$	

8.		Comments
(a)	Method 1: using geometry formula rectangle area + trapezium area =1	Well done
	$k+\frac{1}{2}(k+2k)(1)=1$	
	$k+\frac{3}{2}k=1$ $\frac{5}{2}k=1$	

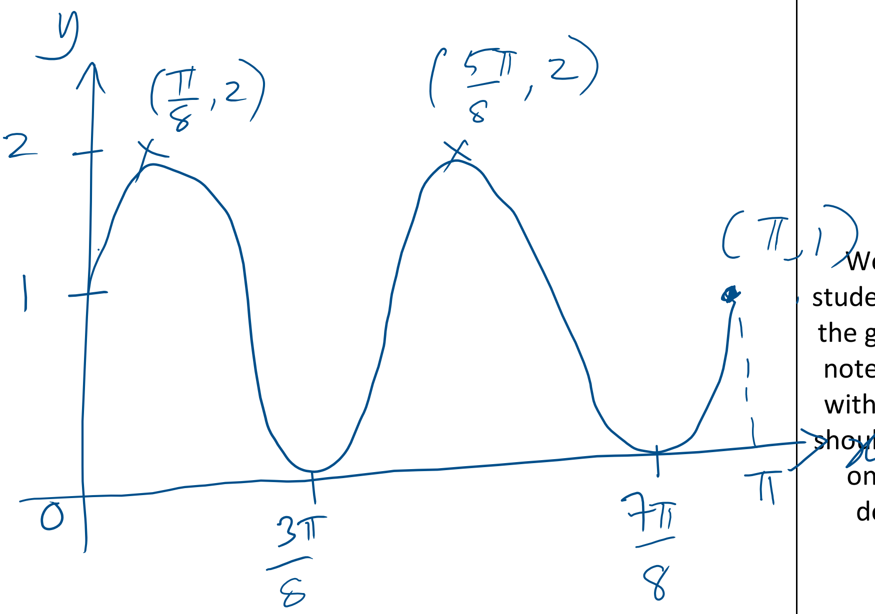
	$k = \frac{2}{5}$	
	Method 2: using integration	
	$\int_2^3 k dx + \int_3^4 k(x-2)dx = 1$	
	$[kx]_2^3 + k\left[\frac{x^2}{2} - 2x\right]_3^4 = 1$ $3k - 2k + k\left(8 - 8 - \left(\frac{9}{2} - 6\right)\right) = 1$ $k + \frac{3}{2}k = 1$ $\frac{5}{2}k = 1$	
	$k = \frac{2}{5}$	
(b)	Area of rectangle = $\int_2^3 k dx = \frac{2}{5} < \frac{1}{2}$	
	$\frac{2}{5} + \int_3^m \frac{2}{5}(x-2)dx = \frac{1}{2}$	Some students forgot to add the area under graph from x=2 to 3 to the 0.5 probability
	$\frac{2}{5}\left[\frac{x^2}{2} - 2x\right]_3^m = \frac{1}{10}$	
	$\frac{m^2}{2} - 2m - \left(\frac{9}{2} - 6\right) = \frac{1}{4}$	
	$2m^2 - 8m + 6 = 1$	
	$2m^2 - 8m + 5 = 0$	
	$m = \frac{8 \pm \sqrt{64-40}}{4}$	Some students forgot to reject one answer
	$m = \frac{8 \pm \sqrt{24}}{4} = \frac{4 \pm \sqrt{6}}{2}$	
	$m = \frac{4+\sqrt{6}}{2}$	
	since $m > 3$	

9.		Comments
(a)	$\int x^3 e^x dx = x^3 e^x - \int 3x^2 e^x dx$	Well done. A few students did not show any working for second and third integration by parts. Do be reminded that all steps should be
	$= x^3 e^x - 3\int x^2 e^x dx$	
	$= x^3 e^x - 3\left[x^2 e^x - \int 2xe^x dx\right]$	
	$= x^3 e^x - 3x^2 e^x + 6\int xe^x dx$	

		shown for show question.
	$=x^3e^x-3x^2e^x+6\left[xe^x-\int e^x dx\right]$ $=x^3e^x-3x^2e^x+6xe^x-6e^x+C$	
	$=e^x(x^3-3x^2+6x-6)+C$	
(b)	$x=\sqrt{t}\Rightarrow x^2=t$	
	$2xdx=dt$	
	$\int_0^2 te^{\sqrt{t}}dt=\int_0^{\sqrt{2}} x^2e^x 2xdx=2\int_0^{\sqrt{2}} x^3e^x dx$	Quite a number of students forgot to substitute the limits while t is replaced by x
	$=2[e^x(x^3-3x^2+6x-6)]_0^{\sqrt{2}}$	
	$=2[e^{\sqrt{2}}(2\sqrt{2}-6+6\sqrt{2}-6)-(-6)]$	Quite a number of students did not realize that the last 6 does not have a factor of $e^{\sqrt{2}}$.
	$=2e^{\sqrt{2}}(8\sqrt{2}-12)+12$	

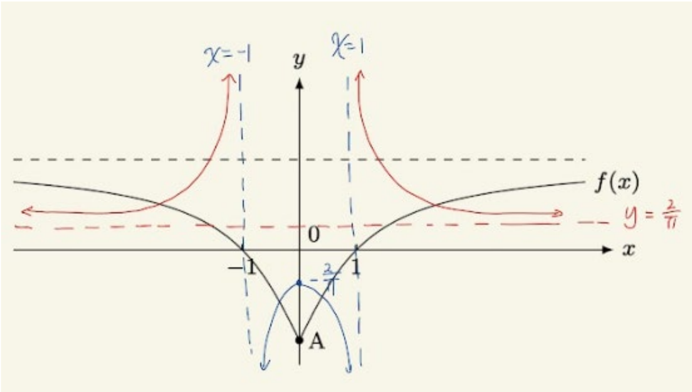
Section B

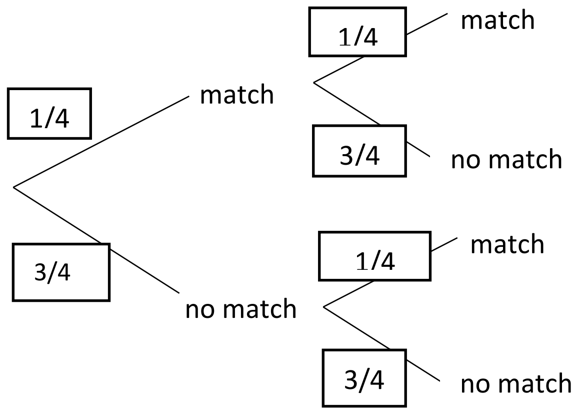
11.		Comments
(a)	$h(x)$ $=(\sin(2x)+\cos(2x))^2$ $=(\sin(2x))^2+2\sin(2x)\cos(2x)+(\cos(2x))^2$ $=1+\sin(4x)$	Well done
(b)(i)	$\sin x\to\sin(4x)$: stretch the graph of $y=\sin x$ by factor of ¼ units parallel to the x-axis. $\sin(4x)\to\sin(4x)+1$: translate the graph of $y=\sin(4x)+1$ by 1 unit in the positive y-axis.	Well done

(ii)		Well done, a few students draw outside the given domain. Do note that only graph within given domain should be shown, else one mark will be deducted in IB.
(b)(i)	Period = $\frac{\pi}{2}$	Well done
(ii)	Range = $[0, 2]$	Well done
(c)	Area bounded by curve, x-axis and y-axis $= \int_0^{\frac{3\pi}{8}} (\sin(4x) + 1)\, dx$ $= \left[\frac{-\cos(4x)}{4} + x\right]_0^{\frac{3\pi}{8}}$ $= \left[\frac{-\cos(\frac{3\pi}{2})}{4} + \frac{3\pi}{8} - \left(\frac{-\cos(0)}{4}\right)\right]$ $= \frac{3\pi}{8} + \frac{1}{4}$	A lot of students did not identify the correct region under graph. Note that if y-axis is specified, it must line one of the sides of the area. All the sides specified must appear in the region you identify.

11.		Comments
(a)	$f(-x) = \arcsin\left(\frac{(-x)^2 - 1}{(-x)^2 + 1}\right)$	
	$= \arcsin\left(\frac{x^2 - 1}{x^2 + 1}\right) = f(x)$	
	$f(-x) = f(x)$	

	$f(x)$ is an even function	
	OR graph is *symmetrical about the y-axis* as shown in the diagram,	
	So $f(-x)=f(x)$,	
	function must be even.	Generally well done.
(b)	$\lim\limits_{x\to\infty}\frac{x^2-1}{x^2+1}=\lim\limits_{x\to\infty}\frac{1-\frac{1}{x^2}}{1+\frac{1}{x^2}}=1$	
	Or apply L'hopital's rule $\lim\limits_{x\to\infty}\frac{x^2-1}{x^2+1}=\frac{\infty}{\infty}\overset{H}{=}\lim\limits_{x\to\infty}\frac{2x}{2x}\overset{H}{=}\lim\limits_{x\to\infty}\frac{2}{2}=1$	
	$\arcsin(1)=\frac{\pi}{2}$	
	So horizontal asymptote $y=\frac{\pi}{2}$	Students missed crucial steps or wrote mathematically incorrect notation like $\frac{1}{\infty}$
(c)	(i) $f'(x)=\frac{1}{\sqrt{1-\left(\frac{x^2-1}{x^2+1}\right)^2}}\frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2}$	
	Take out $\frac{1}{(x^2+1)^2}$ from inside the square root in the denominator $f'(x)=\frac{1}{\sqrt{(x^2+1)^2-(x^2-1)^2}}\frac{4x}{(x^2+1)}$ $f'(x)=\frac{1}{\sqrt{4x^2}}\frac{4x}{(x^2+1)}$	
	$f'(x)=\frac{2x}{\sqrt{x^2}(x^2-1)}=\frac{2x}{\lvert x\rvert(x^2-1)}$	Students who were unable to do the question did not apply chain rule correctly.
	(ii) $\sqrt{x^2}=\lvert x\rvert=2$	
	$f'(-2)=\frac{-4}{2(4+1)}=-\frac{2}{5}$	
	Since $f'(x)<0$, f is decreasing at $x=-2$.	Generally well done
(d)	$f(0)=\arcsin\left(\frac{0-1}{0+1}\right)=\arcsin(-1)=-\frac{\pi}{2}$	

		Generally well done
(e)	(i)let $y = g(x)$, try to express x as the subject	
	$y = \arcsin\left(\frac{x^2-1}{x^2+1}\right)$	
	$\sin y = \frac{x^2-1}{x^2+1}$	
	$x^2 \sin y + \sin y = x^2 - 1$ $x^2(\sin y - 1) = -1 - \sin y$ $x^2 = \frac{1+\sin y}{1-\sin y}$ or $x^2 = \frac{-1-\sin y}{\sin y - 1}$	
	$x = \pm\sqrt{\frac{1+\sin y}{1-\sin y}}$	
	Since $x \geq 0$, $x = \sqrt{\frac{1+\sin y}{1-\sin y}}$	
	$g^{-1}(x) = \sqrt{\frac{1+\sin x}{1-\sin x}}$	Generally well done.

12.		Comments
(a)		Generally well done
(b)	P(at least one match)= 1- P(no match in 4 rolls)	
	$=1-\left(\frac{3}{4}\right)^4$	
	$=1-\frac{81}{256}=\frac{175}{256}$	The easiest method was to take complement. Students who computed the answer by dividing into cases found difficulty in getting the answers.
(c)	P $(X=0)=\left(\frac{n-1}{n}\right)^n$	Generally well done
(d)	$P(X\geq 1)\ =1-P(X=0)=1-\left(\frac{n-1}{n}\right)^n$	Generally well done
	$=1-\left(1-\frac{1}{n}\right)^n$	
(e)	$e=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+...$ from formula booklet	
	$\frac{1}{e}=e^{-1}$	
	Substitute $x=-1$ $\frac{1}{e}=1+(-1)+\frac{(-1)^2}{2!}+\frac{(-1)^3}{3!}+\frac{(-1)^4}{4!}+\frac{(-1)^5}{5!}+...$	Generally well done for those who were aware of the standard formula.
	$\frac{1}{e}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+...$	

(f) (i)	$\left(1-\frac{1}{n}\right)^{n}$ $=1-\binom{n}{1}\left(\frac{1}{n}\right)+\binom{n}{2}\left(\frac{1}{n}\right)^{2}-\binom{n}{3}\left(\frac{1}{n}\right)^{3}+\binom{n}{4}\left(\frac{1}{n}\right)^{4}-\binom{n}{5}$	
	$=\binom{n}{1}\left(\frac{1}{n}\right)-\binom{n}{2}\left(\frac{1}{n}\right)^{2}+\binom{n}{3}\left(\frac{1}{n}\right)^{3}-\binom{n}{4}\left(\frac{1}{n}\right)^{4}+\binom{n}{5}($	
	$=\frac{n}{1}\left(\frac{1}{n}\right)-\frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^{2}+\frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^{3}$ $-\frac{n(n-1)(n-2)(n-3)}{3!}\left(\frac{1}{n}\right)^{4}+\ldots$	
	Factorize out an n from each factor in the numerator $=\frac{n^{2}\left(1-\frac{1}{n}\right)}{2!}\left(\frac{1}{n^{2}}\right)+\frac{n^{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}\left(\frac{1}{n^{3}}\right)$ $-\frac{n^{4}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{3!}\left(\frac{1}{n^{4}}\right)+\ldots$	Tedious working. Students may not have had time to complete the working for this question.
(f) (ii)	$\lim_{n\to\infty}P(X\geq 1)=\lim_{n\to\infty}\left[1-\left(1-\frac{1}{n}\right)^{n}\right]$	
	$\lim_{n\to\infty}\left[1-\left(1-\frac{1}{n}\right)^{n}\right]$ $=\lim_{n\to\infty}1-\frac{n^{2}\left(1-\frac{1}{n}\right)}{2!}\left(\frac{1}{n^{2}}\right)+\frac{n^{3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{3!}\left(\frac{1}{n^{3}}\right)-$ $\frac{n^{4}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)}{3!}\left(\frac{1}{n^{4}}\right)+\ldots$	
	And cancel out the identical powers of n in each term $=\lim_{n\to\infty}1-\frac{\cancel{n^{2}}(1-0)}{2!}\left(\frac{1}{\cancel{n^{2}}}\right)+\frac{\cancel{n^{3}}(1-0)(1-0)}{3!}\left(\frac{1}{\cancel{n^{3}}}\right)$ $-\frac{\cancel{n^{4}}(1-0)(1-0)(1-0)}{3!}\left(\frac{1}{\cancel{n^{4}}}\right)+\ldots$ $=\lim_{n\to\infty}\left[1-\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\ldots\right)\right]$	Generally well done. Students missed out crucial steps resulting in loss of marks.
	$=1-\frac{1}{e}$	