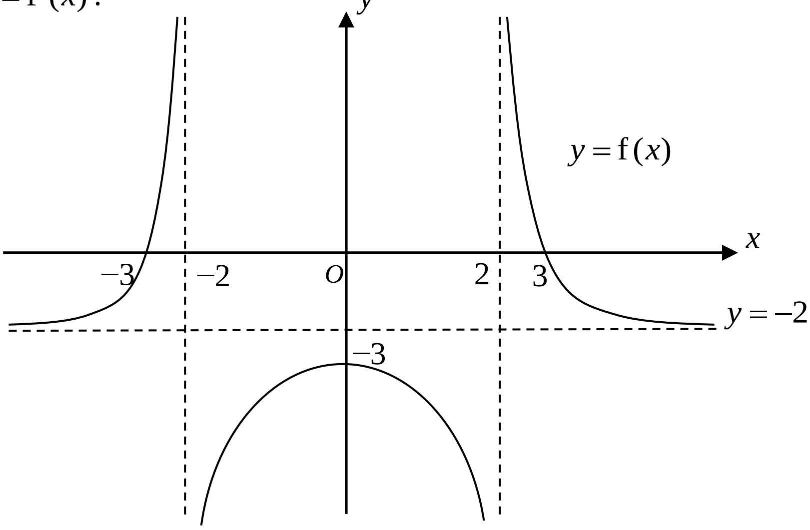
National Junior College
2016 – 2017 H2 Mathematics
Differentiation Applications **Tutorial**

Basic Mastery Questions

Geometrical Results of the Gradient Function

1. The diagram below shows the graph of $y = \mathrm{f}(x)$. Sketch, on a separate diagram, the graph of $y = \mathrm{f}'(x)$.



Tangents and Normals

2. The equation of a curve is $2x^2 - 3xy + y^2 = 5$. Find the equation of the tangent and normal to the curve at the point (4, 3).

3. A curve has equation $x^2 - 8x + y^2 - 4y + 6xy + 4 = 0$.
   (i) Find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.
   (ii) Find the coordinates of the point(s) on the curve at which the tangent is parallel to the x–axis.

4. It is given that a is a positive constant. A curve has parametric equations

$$x = 5a\sec\theta,\quad y = 3a\tan\theta,\ \text{where } -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

   Find the coordinates of the point on the curve at which the normal is parallel to $y = x$.

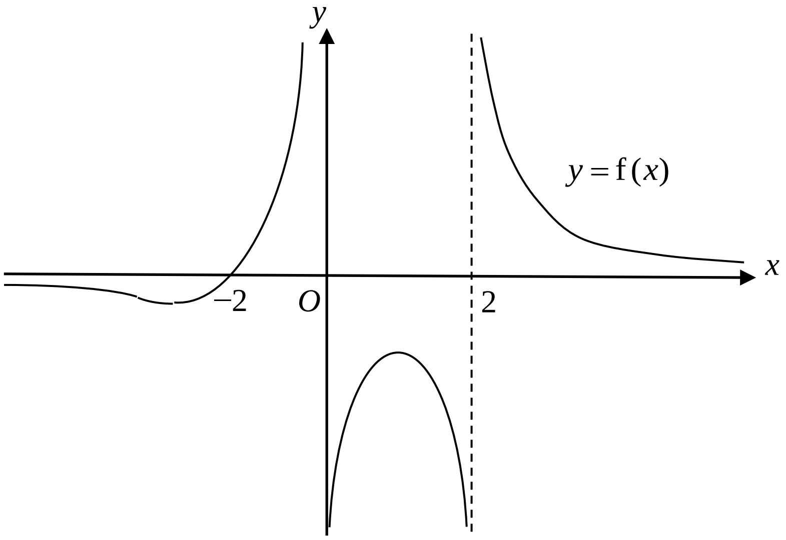
Practical Problems involving Differentiation

5. A metal sheet of area 100 cm^2 is used to manufacture a closed cylinder can. Find, to two significant figures, the largest possible volume of the can.

6. A viscous liquid is poured onto the top of a table and a circular patch is formed that increases in area at a constant rate of $\frac{5\pi}{4}$ cm^{2}s^{-1}. Find the rate at which the radius r is increasing at the instant when r = 20 cm.

Practice Questions

1. The diagram below shows the sketch of the graph of $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = \dfrac{x+2}{x(x-2)}$.



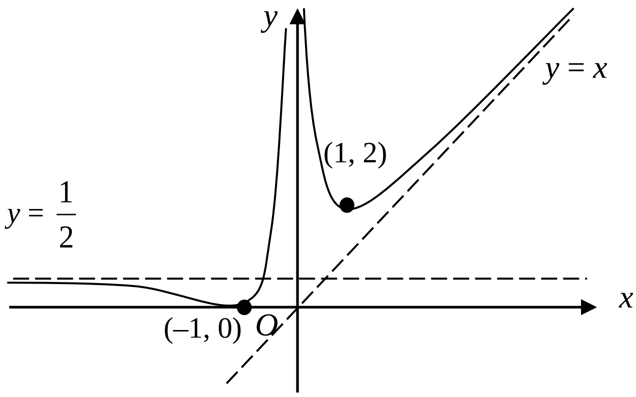
(i) Give the equations of the asymptotes.

(ii) Using an algebraic method, find the exact range of f.

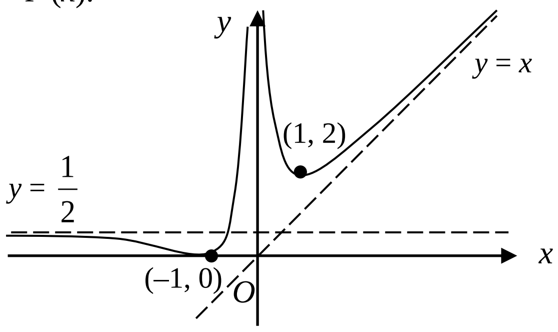
(iii) Sketch the graph of $y = \mathrm{f}'(x)$, giving the coordinates of stationary point(s).

(iv) Deduce the range of values of x for which $\mathrm{f}''(x) > 0$.

2. The diagram below shows the graph of $y = \mathrm{f}'(x)$. Deduce the range of values of x where the graph of $y = \mathrm{f}(x)$ is concave upwards.

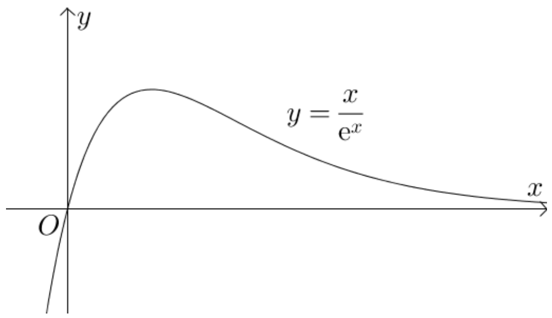


3. The diagram shows the graph of $y = \mathrm{f}(x)$. On a separate clearly labelled diagram, sketch the graph of $y = \mathrm{f}'(x)$.



[08/HCI /JC1/Promo]

4. The diagram shows a sketch of the curve $y=\dfrac{x}{\mathrm{e}^x}$.



(i) Find, by differentiation, the exact maximum value of y.

(ii) Hence show that $\ln x \le x-1$ for all positive values of x.

(iii) Determine the range of values of x for which the graph of $y=\dfrac{x}{\mathrm{e}^x}$ is concave upwards. [08/DHS/JC2/Prelim]

5. The equation of a curve C is $x^3+xy+2y^3=k$, where k is a constant. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y.

It is given that C has a tangent which is parallel to the y–axis. Show that the y-coordinate of the point of contact of the tangent with C must satisfy $216y^6+4y^3+k=0$.

Hence show that $k \le \dfrac{1}{54}$.

Find the possible values of k in the case where the line $x = -6$ is a tangent to C.

6. A curve is defined by the parametric equations, $x = t^2$, $y = t^3$. Prove that the equation of the tangent at the point with parameter t is $2y-3tx+t^3=0$.

(i) This tangent passes through a fixed point (X, Y). Give a brief argument to show that there cannot be more than 3 tangents passing through (X, Y).

(ii) The tangent at the point where $t = 2$ meets the curve again at the point where $t = u$. Find the value of u.

7. A curve is given parametrically by the equations $x=2t-1,\ y=\dfrac{1}{2t+1}$, where $t\in\mathbb{R}$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t. Deduce that the curve shows a decreasing function.

(ii) Show that there is no tangent to the curve parallel to the chord joining the points where $t=-1$ and $t=\dfrac{5}{8}$.

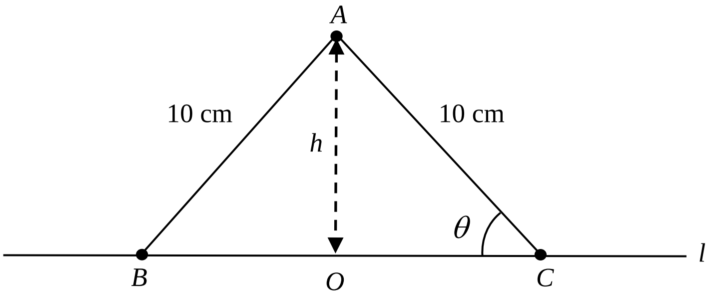
(iii) The curve passes through the point Q when $t=0.5$. Find the coordinates of the point at which the normal to the curve at Q meets the curve again.

(iv) The origin is denoted by the point O. The normal to the curve at Q intersects the x-axis at the point P. Find the exact area of the triangle OPQ.

8. A metal tank with square base is expanding due to heating. After t seconds, the tank has dimensions x cm by x cm by $10x$ cm. Given that the area of the horizontal cross-section is increasing at 0.032 cm^2 s^{-1} when $x = 8$, find, at this instant,

   (i) the rate of increase of the side of the cross-section, and
   (ii) the rate of increase of the volume. [2006/SRJC/P1]

9. The diagram below shows an isosceles triangle ABC with fixed lengths AB and AC of 10 cm each. A is a variable point which is at a height h cm directly above the point O while B and C are variable points which move horizontally along the line l.

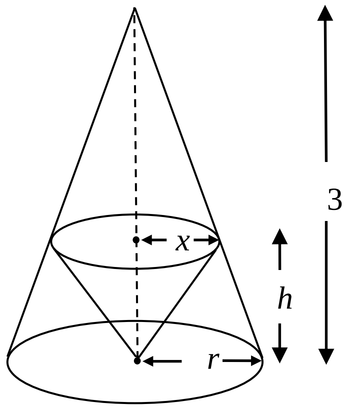


   Given that A descends vertically towards the point O such that the area of triangle ABC is decreasing at a constant rate of 0.7 cm^2/s, determine at the instant when A is 6 cm above O,
   (i) the rate of change of θ,
   (ii) the rate at which C is moving away from O.

10. A piece of wire of length 8 cm is cut into 2 pieces, one of length x cm, the other $(8-x)$ cm. The piece of length x cm is bent to form a circle with circumference x cm. The other piece is bent to form a square with perimeter $(8-x)$ cm. Show that, as x varies, the sum of the areas enclosed by these 2 pieces of wire is a minimum when the radius of the circle is $\dfrac{4}{4+\pi}$ cm.

11. The perimeter of an isosceles triangle of base length x cm and two sides of equal length y cm each is 12 cm. Show that its area can be expressed as $\dfrac{x}{2}\sqrt{36-6x}$.
   Hence show that the area is maximum when the triangle is equilateral.
   [2008/SAJC/Promo]

12. An art sculptor has conceptualised a glass sculpture design as shown below.



The sculpture consists of a right conical shell of height 3 metres and base radius r metres. Inside the cone holds an inverted solid glass cone of height h metres and base radius x metres (see above diagram). Let V denotes the volume of the inverted cone.

(i) Show that $V = \frac{1}{3}\pi r^2 \left(h - \frac{2}{3}h^2 + \frac{h^3}{9} \right)$.

(ii) The sculptor decides to cast the inverted cone in gold to increase the value of his sculpture. Find the height h of the inverted cone that would maximise the value of the sculpture, justifying your answer.

13. A curve C has parametric equations

$$x = \sin^3\theta, \; y = 3\sin^2\theta\cos\theta, \; 0 \le \theta \le \frac{\pi}{2}.$$

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cot\theta - \tan\theta$.

(ii) Show that C has a turning point when $\tan\theta = \sqrt{k}$, where k is an integer to be determined. Find, in non-trigonometric form, the exact coordinates of the turning point and explain why it is a maximum.

The line with equation $y = ax$, where a is a positive constant, meets C at the origin and at the point P.

(iii) Show that $\tan\theta = \frac{3}{a}$ at P. Find the exact value of a such that the line passes through the maximum point of C.

[N2015/I/11]

Further Practice Questions

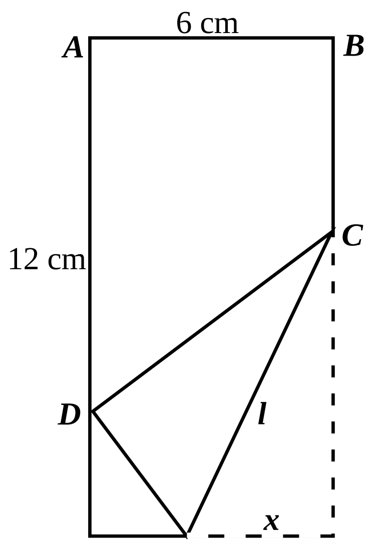
1. A curve C, given by $xy + 4e^y = 4x$, cuts the x-axis at point P. The line L is the tangent to C at P. Show that the equation of L is $5y = 4x - 4$.

   (a) Find the exact coordinates of the point R on L such that the distance OR is the shortest, where O is the origin.

   (b) A point Q moves along L with its x-coordinate decreasing at a rate of 4 units/s. Find the rate of change of the area of triangle OPQ.

[2012/AJC/Promo]

2(a)



A rectangular piece of paper measures 12cm by 6cm. For an origami fold, the lower right-hand corner is folded over so as to reach the leftmost edge of the paper as shown in the diagram above.

The length of the resulting crease is denoted by l and x is the horizontal length being folded over.

By considering the area of trapezium $ABCD$ and the total area of the paper, show that

$$l^2 = x^2 + \frac{3x^2}{x-3}.$$

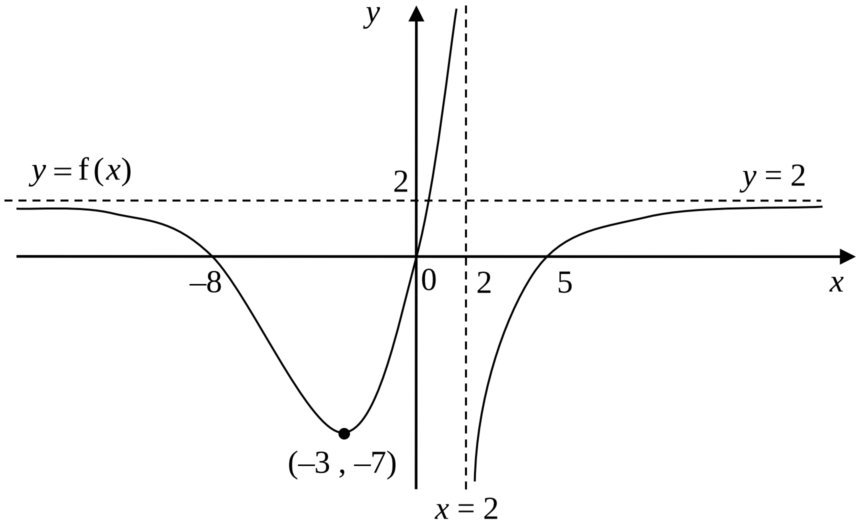
Hence, find the exact minimum value of l.

(b) Sand is poured onto a growing conical pile of sand, at the constant rate of $2\text{m}^3\text{s}^{-1}$. Given that the side of the cone always makes an angle of 60° to the cone's base, find the rate of change of the exposed surface area of the sand pile when the volume of the cone is 200 m^3.

[The curved surface area of a cone of radius r and slant height l is πrl.]

[2012/NYJC/Promo]

3. The graph of $y = \mathrm{f}(x)$ has asymptotes $x = 2$, $y = 2$ and a minimum point at $(-3, -7)$ as shown in the diagram. It cuts the x-axis at the points with coordinates $(-8, 0)$, $(0, 0)$ and $(5, 0)$.

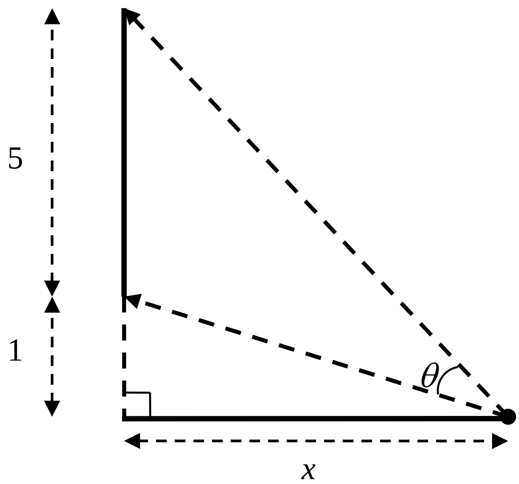


Sketch, on separate diagram, the graph of $y = \mathrm{f}'(x)$, indicating clearly the asymptotes, turning points and intercepts on the axes whenever applicable.

[2012/RI/Promo]

Challenging Questions

1. A movie theatre screen which is 5 m high, has its lower edge 1 m above an observer's eye. The visual angle θ of the observer seated x m away is as shown in the diagram below.

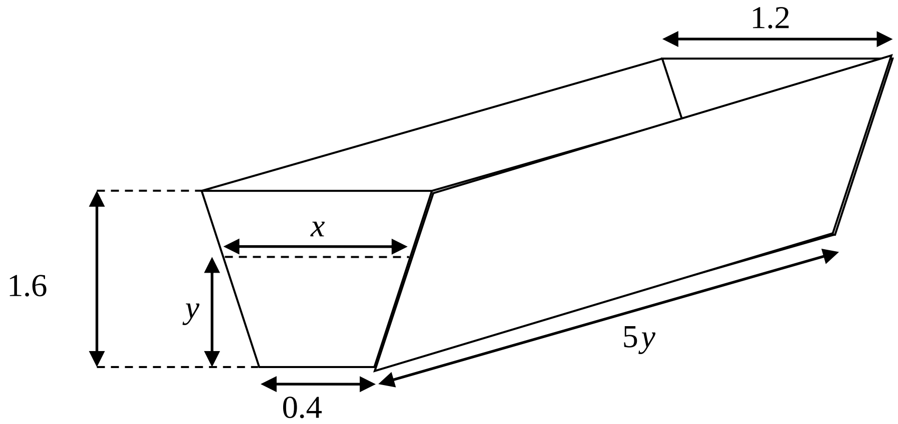


(i) Show that $\theta = \tan^{-1}\left(\dfrac{6}{x}\right) - \tan^{-1}\left(\dfrac{1}{x}\right)$.

(ii) Find the exact distance the observer should sit to obtain the largest visual angle. [You need not establish that the distance gives the largest visual angle.]

(iii) Suppose that the observer is situated between 2 m and 15 m from the screen. Find, to the nearest degree, the smallest visual angle.

2. The diagram below shows a tank with dimensions $5y$ m, 1.2 m, 0.4 m and 1.6 m as indicated. A liquid is poured into the tank at a rate of $0.05\text{ m}^3/\text{s}$. At time t seconds, the liquid in the tank has a depth of y m and surface width of x m.



(i) Using similar triangles, show that $x = \frac{y}{2} + 0.4$. Deduce that $V = \frac{y^2}{4}(5y+8)$, where V is the volume of liquid in the tank at time t.

(ii) Initially, the tank is filled with 0.5 m^3 of liquid. Find the rate of change of the liquid depth at time $t = 55s$.

(iii) Hence find the rate of change of the surface width of the liquid at time $t = 55s$.

<u>Answers to Differentiation and its Applications (Tutorial 2)</u>

Basic Mastery Questions

2. $y = \frac{7}{6}x - \frac{5}{3}$; $y = -\frac{6}{7}x + \frac{45}{7}$ 3. $(1, 1)$, $\left(-\frac{1}{2}, \frac{3}{2}\right)$ 4. $\left(\frac{25}{4}a, -\frac{9}{4}a\right)$

5. 77 cm^3 6. $\frac{1}{32}$ cm/s

Practice Questions

1(i) $y = 0$, $x = 0$, $x = 2$ (ii) $\mathrm{f}(x) \geq \sqrt{2} - \frac{3}{2}$ or $\mathrm{f}(x) \leq -\sqrt{2} - \frac{3}{2}$ (iii) $-7.69 < x < 0$ or $x > 2$

2. $-1 < x < 0$; $x > 1$ 4 (i) $y = \frac{1}{\mathrm{e}}$ (iii) $x > 2$ 5. $k = -212$ or -220

6. (ii) $u = -1$ 7(ii) $\left(-\frac{17}{8}, -8\right)$ (iv) $\frac{1}{64}$ sq units

8(i) 0.002 cm/s (ii) $3.84\text{ cm}^3/\text{s}$ 10(i) -0.025 rad/s (ii) 0.15 cm/s

12(ii) $h = 1$ m 13 (ii) $k = 2$; $\left(\frac{2\sqrt{6}}{9}, \frac{2\sqrt{3}}{3}\right)$ (iii) $\frac{3\sqrt{2}}{2}$

Further Practice Questions

1(a) $R = \left(\frac{16}{41}, -\frac{20}{41}\right)$ (b) $-\frac{8}{5}\text{ units}^2/\text{s}$ 2(a) $l = \left(\frac{9}{2}\right)\sqrt{3}$ cm (b) $0.963\text{ m}^2/\text{sec}$

Challenging Questions

1(ii) $\sqrt{6}$ m (iii) 18° 2(ii) $0.00645\,\text{ms}^{-1}$ (iii) 0.00323