7. Integration and its Applications

1 **CJC/2010Promo/10**

Find

(a) $\int \frac{x}{3+x}\,\mathrm{d}x$ [2]

(b) $\int x^2 \ln x\,\mathrm{d}x$ [3]

(c) $\int \frac{x+3}{x^2+4x+7}\,\mathrm{d}x$ [4]

(d) $\int \frac{2x-1}{\sqrt{4-x^2}}\,\mathrm{d}x$ by using the substitution $x = 2\sin\theta$. [4]

2 **DHS/2009Promo/9**

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\sin 2x)$. [1]

(ii) Find $\int \frac{\sin x + \cos x}{(\cos x - \sin x)^2}\,\mathrm{d}x$. [2]

(iii) Hence find the exact value of $\int_0^{\frac{\pi}{6}} \frac{\sin x \sin 2x + \cos x \sin 2x}{(\cos x - \sin x)^2}\,\mathrm{d}x$. [4]

3 **ACJC/2010Promo/9**

Find the following integrals.

(a) $\int \frac{\cos 3x - \cos ec^2 3x}{\sin 3x + \cot 3x}\,dx$ [2]

(b) $\int \frac{1-x}{\sqrt{1-16x^2}}\,dx$ [4]

(c) $\int (1-x)^{-2} \ln x\,dx$ [4]

4 **TJC/2009Promo/2**

Integrate the following:

(a) $\int \frac{1}{x \ln x^2}\,\mathrm{d}x$, where $x > 0$. [2]

(b) $\int \frac{\mathrm{e}^{\sqrt{2x-1}}}{\sqrt{2x-1}}\,\mathrm{d}x$, where $x > \frac{1}{2}$. [2]

5 Using the property $\sec^4\theta = \left(\sec^2\theta\right)\left(\sec^2\theta\right)$, find $\int \sec^4\theta \; \mathrm{d}\theta$. [3]

6 JJC/2010Promo/12

(a) Write down constants A and B such that, for all values of x,
$x+4 = A(2x+6)+B$. [2]

Hence find $\int \frac{x+4}{x^2+6x+13} \, \mathrm{d}x$. [4]

(b) Using the substitution $x = \frac{1}{u}$, show that $\int_2^4 \frac{1}{x^3} \mathrm{e}^{\frac{1}{x}} \, \mathrm{d}x = \int_{\frac{1}{4}}^{\frac{1}{2}} u\mathrm{e}^u \, \mathrm{d}u$. [2]

Hence evaluate $\int_2^4 \frac{1}{x^3} \mathrm{e}^{\frac{1}{x}} \, \mathrm{d}x$, giving your answer in exact form. [3]

(c) Find the exact value of $\int_{-3}^{0} \left|x+2\right|^3 \, \mathrm{d}x$. [3]

7 RVHS/2010Promo/11 (modified)

(a) Using the substitution $u = \mathrm{e}^x$, find $\int \frac{1}{\mathrm{e}^x + 2\mathrm{e}^{-x}} \, \mathrm{d}x$. [4]

(b) Show that

$\int_0^1 \frac{4x-5}{\sqrt{3+2x-x^2}} \, \mathrm{d}x = \frac{a\sqrt{3}+b-\pi}{6}$, where a and b are constants to be found. [6]

8 VJC/2013Promo/5

(a) **(i)** Prove that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{x^2+1}\right) = \frac{2}{\left(x^2+1\right)^2} - \frac{1}{x^2+1}$. [2]

(ii) Find the exact value of $\int_0^1 \frac{1}{\left(x^2+1\right)^2} \, \mathrm{d}x$. [3]

(b) Find the constant A such that $\frac{1}{1-\mathrm{e}^{2x}} = A + \frac{\mathrm{e}^{2x}}{1-\mathrm{e}^{2x}}$. Hence find $\int \frac{1}{1-\mathrm{e}^{2x}} \, \mathrm{d}x$. [3]

9 HCI/2020Prelim/I/6

Find

(a) $\displaystyle\int \frac{3\mathrm{e}^x}{5-0.3\mathrm{e}^x}\,\mathrm{d}x$, [2]

(b) $\displaystyle\int \cos(\ln x)\,\mathrm{d}x$, where $x>0$, [3]

(c) the exact value of $\displaystyle\int_0^3 \left|2\mathrm{e}^x-5\right|\,\mathrm{d}x$. [3]

10 NJC/2020Promo/6

(i) Find $\displaystyle\int \frac{x}{\sqrt{1-k^2x^2}}\,\mathrm{d}x$, where k is a positive constant. [2]

(ii) Hence, find $\displaystyle\int \left(\sin^{-1} kx\right)\frac{x}{\sqrt{1-k^2x^2}}\,\mathrm{d}x$. [3]

(iii) Evaluate $\displaystyle\int_0^{\frac{1}{\sqrt{2}}} \left(\sin^{-1} x\right)\frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$, giving your answer in the form $\displaystyle\frac{1}{\sqrt{a}}\left(1-\frac{\pi}{b}\right)$, where a and b are integers to be determined. [2]

(iv) Deduce the exact value of

$$\int_m^{\frac{1}{\sqrt{2}}+m} \left[\sin^{-1}(x-m)\right]\frac{x-m}{\sqrt{1-(x-m)^2}}\,\mathrm{d}x$$

for any constant m. Explain your answer. [2]

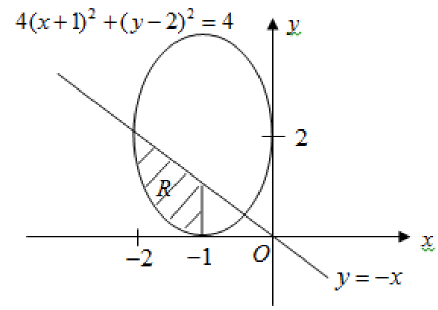
11 TJC/2014Promo/6

(a) Using an algebraic method, find the exact value of $\displaystyle\int_1^4 \frac{|x-2|}{x}\,\mathrm{d}x$. [3]

(b) Sketch and shade the finite region bounded by the curve $y = x^2 + 2$, the lines $y = x$ and $x = 1$, and the y-axis. Find the exact volume of the solid formed when the region is rotated 2π radians about the y-axis. [4]

12 ACJC/2012Promo/15

The shaded region R in the diagram below is bounded by the curve $4(x+1)^2+(y-2)^2=4$ and the lines $y=-x$ and $x=-1$.



(i) Using the substitution $x=\cos\theta-1$, show that

$$\int_{-2}^{-1}\sqrt{-x^2-2x}\ \mathrm{d}x=\frac{\pi}{4}.$$ [5]

(ii) Hence, or otherwise, find the exact area of R. [3]

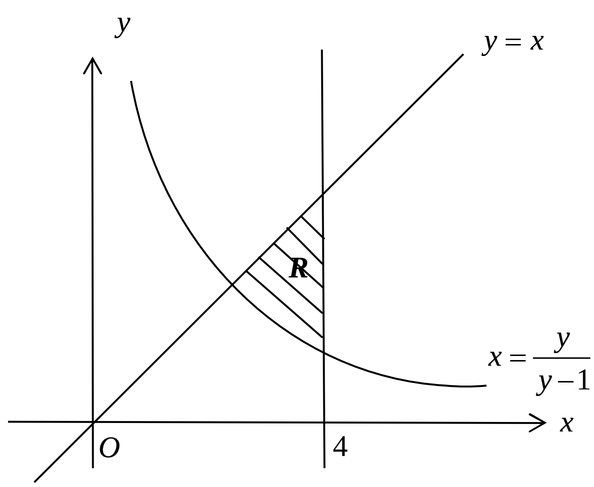
(iii) By translating both the curve and the line $y=-x$ one unit in the positive x-direction, or otherwise, find the volume of the solid formed when R is rotated through 2π radians about the line $x=-1$. [3]

13 NJC/2009Promo/10

(a) **(i)** Find the derivative of e^{x^2+2x}. [1]

(ii) Hence, find $\int(x+1)^3\,\mathrm{e}^{x^2+2x}\ \mathrm{d}x$. [3]

(b)



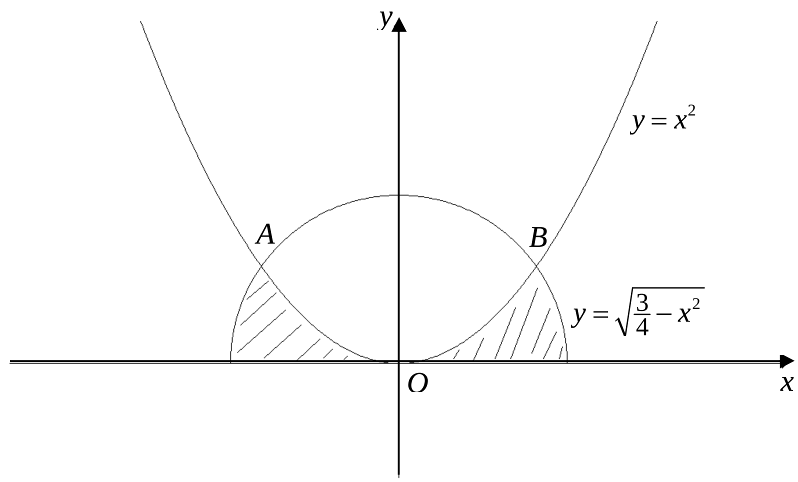
The diagram above shows part of the graph of a curve C given by $x=\dfrac{y}{y-1}$.

The region R is bounded by C and the lines $y=x$ and $x=4$.

Write down the equation of the curve obtained when C is translated by 4 units in the negative x-direction.

Hence, or otherwise, write down also an expression for the volume V of the solid formed when R is rotated 2π radians about the line $x=4$ and find its numerical value, giving your answer correct to 3 decimal places. [5]

14 RI/2009Prelim/I/9



The diagram above shows the graphs of $y = \sqrt{\frac{3}{4} - x^2}$ and $y = x^2$. Find, in exact form, the coordinates of the points of intersection A and B. [2]

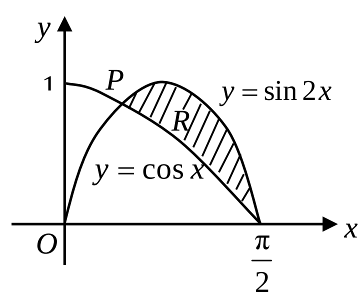
The shaded region R is bounded by the two curves and the x-axis.

(i) Find the area of R, giving your answers correct to two decimal places. [3]

(ii) Find the exact volume of the solid of revolution formed when R is rotated through π radians about the y-axis. [3]

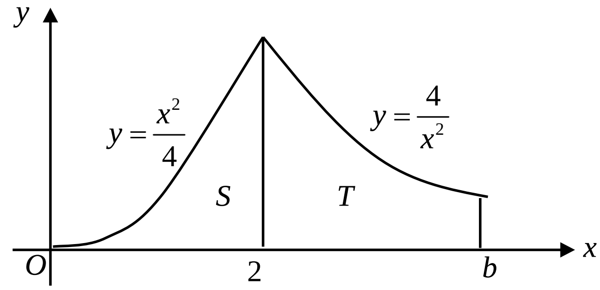
15 DHS/2010Promo/11

(a) The shaded region R in the diagram below is bounded by the curves $y = \sin 2x$ and $y = \cos x$.



Find the x-coordinate of the point of intersection P of the two curves. Hence, by integration, find the exact area of R. [5]

(b) The diagram below (not drawn to scale) shows two regions S and T. The region S is bounded by part of the curve $y=\frac{x^2}{4}$, $x=2$ and the x-axis. The region T is bounded by part of the curve $y=\frac{4}{x^2}$, $x=2$, $x=b$ and the x-axis.



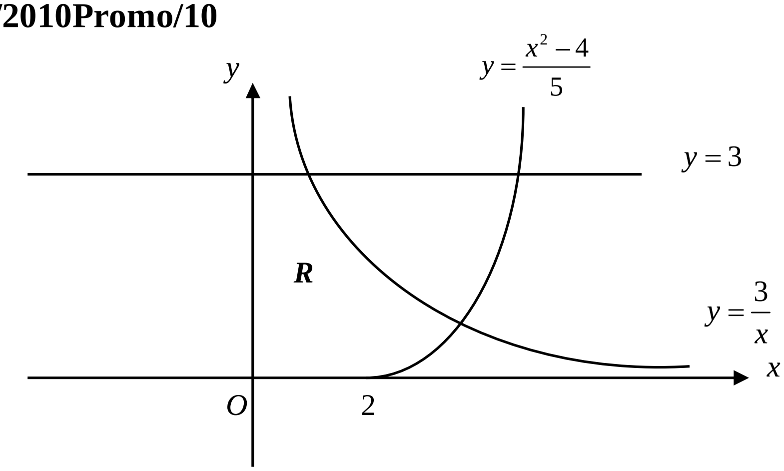
Find the value of b if the area of region S is equal to the area of region T. [4]

Let V_S and V_T be the volume of the solid of revolution formed when the region S and region T are rotated through 4 right angles about the x-axis respectively.

Let W_S be the volume of the solid of revolution formed when region S is rotated through 4 right angles about the y-axis.

Find the value of b if $V_S+V_T=\frac{1}{2}W_S$. [5]

16 JJC/2010Promo/10



Region $\boldsymbol{R}$ is bounded by the curves $y=\frac{3}{x}$, $y=\frac{x^2-4}{5}$, $y=3$, and the axes as shown in the diagram above.

(i) Verify that the curves $y=\frac{3}{x}$ and $y=\frac{x^2-4}{5}$ intersect at $(3,1)$. [1]

(ii) Find the area of the region $\boldsymbol{R}$. [3]

(iii) Find the volume of the solid generated when $\boldsymbol{R}$ is rotated 2π radians about the x–axis. [3]

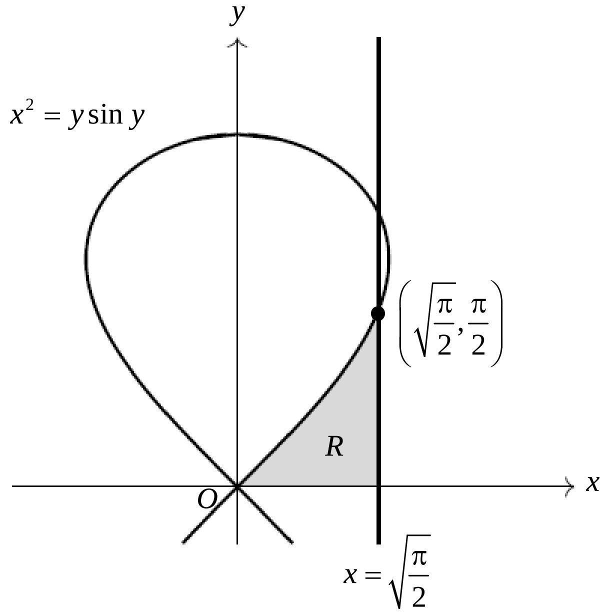
(iv) Find the volume of the solid generated when $\boldsymbol{R}$ is rotated 2π radians about the y–axis, giving your answer in exact form. [4]

17 RVHS/2020Promo/8

(a) Use the substitution $u = e^x$ to evaluate $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^{3x}}{e^{2x}+1}\,dx$ exactly. [4]

(b) The diagram below shows a graph of $x^2 = y\sin y$ with the line $x = \sqrt{\frac{\pi}{2}}$.

It is given that curve intersects the line at $\left(\sqrt{\frac{\pi}{2}}, \frac{\pi}{2}\right)$.



The region R in the first quadrant, is bounded by the curve $x^2 = y\sin y$, the line $x = \sqrt{\frac{\pi}{2}}$ and x-axis.

(i) Find $\displaystyle\int_0^a x\sin x\,dx$ in terms of a, where $0 < a \le \pi$. [2]

(ii) The region R is rotated through 2π radians about the y-axis.
Using the result from part (b)(i), find the exact volume of the solid obtained. [3]

18 NJC/2010Promo/12

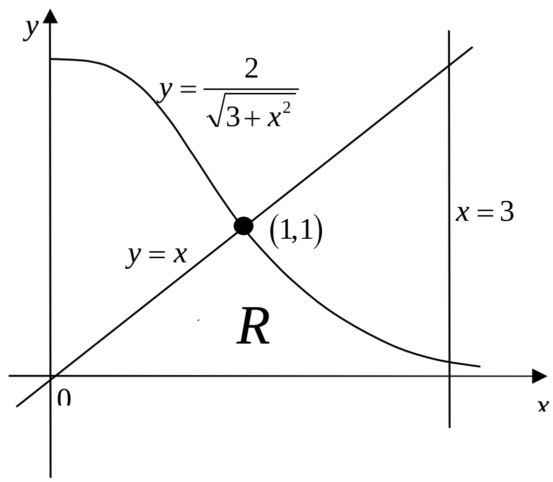
(a) By using the substitution $x = \sin t$ or otherwise, show that

$$\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}}\, dx = \int e^t \sin t \, dt . \qquad [2]$$

Hence, find the integral $\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}}\, dx$. [4]

(b)

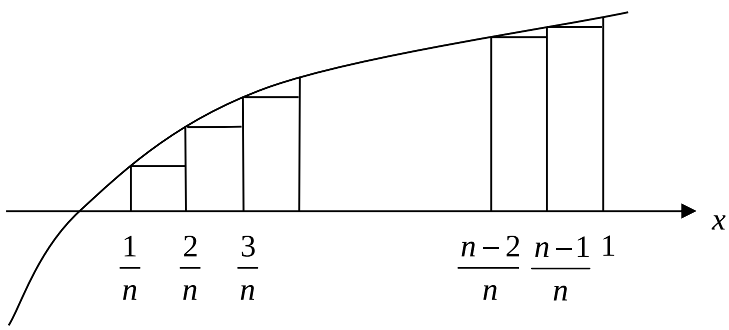
The diagram above shows part of the graph of a curve C given by $y = \frac{2}{\sqrt{3+x^2}}$. C intersects the line $y = x$ at $(1,1)$. The region R is bounded by C, the lines $y = x$ and $x = 3$, and the x-axis. Find V, the volume of the solid formed when R is rotated through 2π radians about the x-axis. [3]



19 JJC/2009Promo/12

(i) Find the exact value of $\int_0^1 \ln(x+1)\,dx$. [4]

(ii)



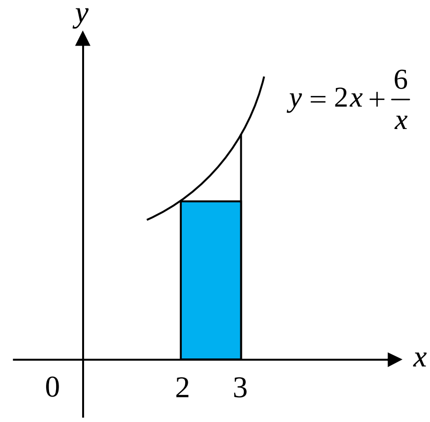
The diagram shows the graph of $y = \ln(x+1)$ with $n-1$ rectangles approximating the area under the curve between $x = 0$ and $x = 1$. Each rectangle has width $\frac{1}{n}$. Show that the total area of these rectangles, A, can be expressed as $\frac{1}{n}\sum_{r=1}^{n-1} \ln\left[\frac{r+n}{n}\right]$. [1]

(iii) By letting $n = 100$, find A correct to 5 decimal places.
Hence, find an approximate value for $\ln 2$. [3]

20 ACJC/2008Promo/11

The diagram shows a sketch of part of the graph of $y = 2x + \frac{6}{x}$. By considering the shaded rectangle and the area of the region between the graph and the x-axis for $2 \le x \le 3$, show that

$$\int_2^3 \left(2x + \frac{6}{x}\right) dx > 7. \qquad [1]$$



Show also that $\int_2^3 \left(2x + \frac{6}{x}\right) dx < 8$. [1]

Hence deduce that $\frac{1}{p} < \ln 1.5 < \frac{1}{q}$, where p and q are positive integers to be determined.

Answers

1	(a) $x - 3\ln\|3 + x\| + c$ (b) $\left[\frac{x^3}{3}\ln x\right] - \frac{x^3}{9} + C$ (c) $\frac{1}{2}\left[\ln\|x^2 + 4x + 7\| + \frac{2}{\sqrt{3}}\tan^{-1}\frac{x+2}{\sqrt{3}}\right] + C$ (d) $-2\sqrt{4 - x^2} - \sin^{-1}\frac{x}{2} + c$
2	(i) $2\cos 2x$ (ii) $\frac{1}{(\cos x - \sin x)} + C$ (iii) $\frac{3\sqrt{3}}{2} - \frac{3}{2}$
3	(a) $\frac{1}{3}\ln\|\sin 3x + \cot 3x\| + c$ (b) $\frac{\sin^{-1}(4x)}{4} + \frac{1}{16}\sqrt{1 - 16x^2} + C$ (c) $\frac{\ln x}{1 - x} + \ln\|1 - x\| - \ln\|x\| + C$
4	(a) $\frac{1}{2}\ln\|\ln x\| + C$ (b) $e^{\sqrt{2x-1}} + C$
5	(a) $\tan\theta + \frac{1}{3}\tan^3\theta + C$
6	(a) $\frac{1}{2}\ln\|x^2 + 6x + 13\| + \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c$ (b) $\frac{3}{4}e^{\frac{1}{4}} - \frac{1}{2}e^{\frac{1}{2}}$ (c) $\frac{17}{4}$
7	(a) $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{e^x}{\sqrt{2}}\right) + c$ (b) $\frac{24\sqrt{3} - 48 - \pi}{6}$
8	(a) (i) $\frac{2}{(x^2 + 1)^2} - \frac{1}{x^2 + 1}$ (ii) $\frac{1}{4} + \frac{\pi}{8}$ (b) $x - \frac{1}{2}\ln\|1 - e^{2x}\| + C$
9	(a) $-10\ln\|5 - 0.3e^x\| + C$ (b) $\frac{x}{2}\left[\cos(\ln x) + \sin(\ln x)\right] + C$ (c) $10\ln 2.5 + 2e^3 - 23$

10	(i) $\frac{-1}{k^2}\left(1-k^2x^2\right)^{\frac{1}{2}}+C$ (ii) $\frac{-\left(\sin^{-1}kx\right)\left(1-k^2x^2\right)^{\frac{1}{2}}}{k^2}+\frac{x}{k}+D$ (iii) $\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{4}\right)$
11	(a) 1 (b) $\frac{11}{6}\pi$
12	(i) $\frac{\pi}{4}$ (ii) $\frac{\pi}{2}-\frac{1}{2}$ (iii) 3.14
13	(i) $2(x+1)e^{x^2+2x}$ (ii) $\frac{1}{2}(x+1)^2 e^{x^2+2x}-\frac{1}{2}e^{x^2+2x}+C$ (b) $x=\frac{y}{y-1}-4$, 12.802 units^3
14	$\left(-\frac{1}{\sqrt{2}},\frac{1}{2}\right)$; $\left(\frac{1}{\sqrt{2}},\frac{1}{2}\right)$, (i) 0.34 units^2, (ii) $\frac{5}{24}\pi$ unit^3.
15	(a) $\frac{\pi}{6}$; $\frac{1}{4}$ units^2 (b)(i) $b=3$ (b)(ii) $b=4.31$
16	(ii) 5.83 (iii) 46.2 (iv) $\frac{25}{2}\pi$
17	(a) $\sqrt{3}-\frac{\pi}{12}-1$ (b) (i) $\sin a - a\cos a$ (ii) $\frac{\pi^3}{4}-\pi$
18	$\int \frac{xe^{\sin^{-1}x}}{\sqrt{1-x^2}}\,dx=\frac{1}{2}e^{\sin^{-1}x}\left(x-\sqrt{1-x^2}\right)+c$, 4.85
19	(i) $2\ln 2-1$ (iii) $A=0.38282$; $\ln 2\approx 0.691$
20	$p=3,\ q=2$