(TOPICAL REVISION)
NUMERICAL METHODS (PART 1)
APPROXIMATION OF ROOTS OF EQUATIONS

*Note: Questions with * may require concepts and techniques you may not have learnt yet.*

1 Show that the equation $x^3 - 5x + 1 = 0$, has exactly one root in (0,1). Use two iterations of linear interpolation between $x = 0$ and $x = 1$ to yield a fractional approximation of this root. [4]

Three possible rearrangements of the given equation in the form $x = F(x)$ are

$$x = \sqrt[3]{5x-1}, \qquad x = \frac{1}{5}(x^3+1), \qquad x = x^3 - 4x + 1.$$

Only one of these rearrangements will provide an iterative method, of the form $x_{n+1} = F(x_n)$, $x_0 = 2$, which converges to the **root between 2 and 3**. Use this rearrangement to find this root correct to 3 significant figures. [4]

It is known the correct rearrangement above provides a convergent iterative method by analysing the derivative of F. For this purpose, show that $0 < F'(x) < 1$, whenever $x \geq 2$. [2]

[EJC/FM/2018/P1/9]

2 **(a)** The curve with equation $y = 2e^{x^2} + x - 3$ has exactly one stationary point in the interval $[-1, 0]$. Use the Newton-Raphson method to find the x-coordinate of the stationary point, correct to 4 decimal places. [5]

(b) **(i)** Show that the equation $x^3 - 7x + 2 = 0$ has a root, α, in the interval [0, 1]. [1]

(ii) Student A uses the recurrence relation $x_{n+1} = \frac{1}{7}\left(14x_n - x_n{}^3 - 2\right)$ for finding α. Explain why Student A will fail to find α. [2]

(iii) Student B uses the recurrence relation $x_{n+1} = \frac{1}{7}\left(x_n{}^3 + 2\right)$ for finding α. Find the approximate value of α to 5 decimal places. [2]

[NJC/FM/2019/P1/8]

3 Let $f(x) = \sec^2 2x - e^x$. The equation $f(x) = 0$ has a root α in the interval [0.1, 0.3].

(a) Use Newton-Raphson method with initial approximation $x_0 = 0.1$ to find the first four approximations x_1, x_2, x_3, x_4 to α. Deduce the behaviour of x_n for large n and use a graph of the function f to help you explain why the sequence is not converging to α. [5]

(b) Determine, with explanation, which one of these iterative formulae is more suitable in approximating the root α.

(I) $x_{n+1} = \ln\left(\sec^2(2x_n)\right)$ **(II)** $x_{n+1} = \frac{1}{2}\cos^{-1}\left(e^{-\frac{x_n}{2}}\right)$

Using an initial approximation of $x_0 = 0.3$ and the chosen iterative formula, find the approximation to the root α, correct to three decimal places. [4]

[RVHS/FM/2018/P1/Q10]

4 **(i)** The function f is such that f(a)f(b) < 0, where $a < b$. A student concludes that the equation f(x) = 0 has exactly one root in the interval (a, b).
Illustrate with a sketch, two possible scenarios in which the student could be wrong. [2]

Given now that the equation $\sqrt{x-3}-\dfrac{2}{x^2}=0$ has exactly one root α in the interval (3,4).

(ii) Derive the iterative formula $x_{n+1}=3+\dfrac{4}{x_n^4}$ using the fixed point iteration method.

Using 3 as the initial value, apply this iterative formula to find an approximation for α, correct to 3 decimal places. You are required to check the accuracy of your answer in this question. [3]

(iii) By using linear interpolation once, obtain, correct to 3 decimal places, a first approximation α_1 to α. Using α_1 as the initial value, apply the Newton-Raphson method once to obtain α_2, leaving your answer to 3 decimal places. [3]

(iv) Explain why the Newton-Raphson method in this case fails to give an approximation to α. [1]

(v) Illustrate, on a single diagram, how α_1 and α_2 are obtained. [2]

[CJC/FM/2018/P1/Q5]

5 **(i)** By sketching the graphs of $y=\tan^{-1}\left(\dfrac{x}{5}\right)$ and $y=\sin^{-1}\left(\dfrac{x}{6}\right)$ on the same diagram,

show that the equation $\tan^{-1}\left(\dfrac{x}{5}\right)=\sin^{-1}\left(\dfrac{x}{6}\right)$ has a positive real root, α. [2]

(ii) Find the integer N such that $N<\alpha<N+1$. [2]

(iii) Illustrate graphically, using $x_0=N$, why the iterative procedure

$$x_{n+1}=5\tan\left(\sin^{-1}\left(\frac{x_n}{6}\right)\right)$$

will not give a good approximate value of α. [2]

(iv) Taking the initial approximation to be N, use the Newton-Raphson method to find α to 2 decimal places. [3]

[RI/FM/2017/P2/Q1]

6 **(i)** Show, with the aid of a sketch graph, that the equation

$$x+k\ln x=0$$

has exactly one real root, α, in the interval $\dfrac{1}{2}<x<1$ if $k>\dfrac{1}{2\ln 2}$. [3]

(ii) In the case $k=1$, a student tries to use fixed point iteration in the form of $x=\text{F}(x)$ to find the value of α.

(a) In his first attempt, the student uses $\text{F}(x)=-\ln x$ and $x_0=0.5$. Calculate the value of x_1 and x_2, correct to 4 decimal places. Explain why this method will fail to find the value of α. [2]

(b) Suggest a possible $\mathrm{F}(x)$ for the student and using $x_0 = 0.5$, find the value of α, correct to 3 decimal places. [2]

(c) Use a diagram to explain how the iteration in **(b)** converges to α, showing clearly the position of x_0, x_1 and x_2. [2]

[HCI/FM/2017/P2/Q2]

7 **(i)** Show algebraically that the equation $x^3 - x^2 - 6 = 0$ has exactly one real root in the interval $(2,\ 3)$. [2]

(ii) Find an estimate of the root using 2 iterations of linear interpolation, correct to two decimal places. [2]

(iii) Determine algebraically if the estimate in **(ii)** is an underestimate or overestimate of the root. [3]

The equation $x^3 - x^2 - 6 = 0$, can be rearranged in the following ways (you are not required to verify):

(A) $x = \left(x^2 + 6\right)^{\frac{1}{3}}$ (B) $x = x^2 - \dfrac{6}{x}$ (C) $x = \left(x + \dfrac{6}{x}\right)^{\frac{1}{2}}$

(iv) Determine with reasons which of the above expressions (A), (B) and (C) converges under the fixed-point iteration method using the initial value $x_0 = 2$. If it converges, use a graph to demonstrate 3 iterations of its convergence to the root, labelling the points x_0, x_1 and x_2 clearly. [5]

[TJC/FM/2019/P1/7]

8 By considering the graphs of $y = \tan^2 x$ and $y = x^3$, show that the equation $\tan^2 x - x^3 = 0$ has exactly two roots in the interval $\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$. Denoting the smaller root by α, where $1 < \alpha < 2$, use linear interpolation once on the interval $[1,\ 2]$ to estimate the value of α, giving your answer correct to 3 decimal places. Comment on the suitability of the method used in this case. [3]

Taking $x_1 = 1.8$, where x_1 is the initial approximation of α, use the Newton-Raphson method to obtain a sequence of approximations for α, giving your answer correct to 3 decimal places. You should demonstrate that the root is found to the required degree of accuracy. [3]

With the aid of a sketch, explain why any initial approximation x_1 such that $\dfrac{\pi}{2} < x_1 < \alpha$ will produce a sequence converging to α whereas some approximations x_1 such that $\alpha < x_1 < \dfrac{3\pi}{2}$ will not converge to α. [3]

[HCI et al/FM/2018/P1/3]

9 The function f is such that $\mathrm{f}(x) = 2\cos x - \mathrm{e}^{-x}$.

(i) Matthias concludes that the equation $\mathrm{f}(x) = 0$ has no roots in the interval $[1, 5]$. Explain how Matthias may have arrived at this conclusion and draw a sketch to illustrate why he is wrong. [2]

The equation $2\cos x - \mathrm{e}^{-x} = 0$ has a root α in the interval $[1, 2]$.

(ii) Alice uses linear interpolation twice on the interval $[1, 2]$ to find an approximation to α. Find the approximation to α given by this method, giving your answer to 2 decimal places. Without any further interpolation, demonstrate how Alice can verify its correctness to 2 decimal places. [4]

(iii) Tabitha decides to use the Newton-Raphson method to find α with an initial approximation of $\alpha_1 = 1$, terminating the process when she has found two successive iterates that are equal when rounded to 5 decimal places. State the value of each of the iterates calculated correct to 5 decimal places. [3]

Buoyed by her success, Tabitha decided to try and find other roots of the equation by using other starting points. Knowing that there is another root β in the interval $[3, 7]$, she decided to apply the Newton-Raphson method to find β with an initial approximation of $\beta_1 = 3$. Describe what she will discover and with an aid of a sketch, explain her discovery briefly. [2]

(iv) Henry decided to use the following recurrence relation to find the roots of $y = \mathrm{f}(x)$:

$$x_{n+1} = -\ln(2\cos x_n) \text{ for } n \geq 1.$$

Using $x_1 = 1.5$ as the starting point, calculate his obtained output for x_2 and explain why he will fail when attempting to calculate x_3. [2]

Using $x_1 = 1$ as the starting point, state the output correct to 2 decimal places. With an aid of a sketch, explain briefly the recursive process that led to this output. [2]

[ACJC/FM/2019/P1/9]

10* Show that the equation $2x^5 - \sqrt{3}x^4 + 32x - 16\sqrt{3} = 0$ has a root α between 0 and 1. [1]

Use the Newton-Raphson method with $\mathrm{f}(x) = 2x^5 - \sqrt{3}x^4 + 32x - 16\sqrt{3}$, to find α correct to 4 decimal places. [3]

Suggest an integer k such that α can be expressed in the form $\frac{\sqrt{k}}{2}$. Without the use of a calculator, find the other roots of the equation $\mathrm{f}(x) = 0$. Leave your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \leq \pi$. [4]

On an Argand diagram, the point P_0 represents the root α and the points P_k $(k = 1, 2, 3, 4)$, taken in the anti-clockwise direction starting from P_0, represent the remaining roots.
Find the exact area of the pentagon $P_0P_1P_2P_3P_4$. [3]

[VJC/FM/2018/P2/4]

Answers

1 $\frac{16}{79}$; 2.13

2(a) -0.2364 (b) (iii) $\alpha = 0.28917$

3(a) $x_1 \approx -0.146$, $x_2 \approx -0.0418$, $x_3 \approx -0.00480$, $x_4 \approx -0.00008$ (b) $\alpha \approx 0.240$

4(ii) $\alpha = 3.046$ (iii) 2.995

5(ii) 3 (iv) 3.32

6(ii) (a) $x_1 = 0.6931$, $x_2 = 0.3665$ (4 d.p.)

(b) $\mathrm{F}(x) = \mathrm{e}^{-x}$; $\alpha = 0.567$ (3 d.p.)

7(ii) 2.19 (iii) underestimate

8 1.306; $\alpha = 1.928$ (to 3 d.p.)

9(ii) 1.45 **(iii)** $\alpha_4 = 1.45367, \alpha_5 = 1.45367$ (iv) $x_2 = 1.95564$ (failed); -0.54 (2 d.p)

10 $\alpha = 0.8660$ (4 dp.); k (omitted), $x = 2\mathrm{e}^{-\mathrm{i}\frac{3\pi}{4}}, 2\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}, 2\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}, 2\mathrm{e}^{\mathrm{i}\frac{3\pi}{4}}$; area = $6 + \frac{\sqrt{6}}{2}$ units^2