RAFFLES INSTITUTION
H2 Mathematics (9758)
2016 Year 5

Chapter 3C: Vectors III
Equations of Planes

SYLLABUS INCLUDE

- Vector and Cartesian equations of planes.
- Finding the foot of the perpendicular and distance from a point to a plane.
- Finding the angle between a line and a plane, or between two planes.
- Relationships between
  - (i) a line and a plane,
  - (ii) two planes,
- Finding the intersections of lines and planes.

CONTENT

INTRODUCTION

In this chapter, we shall extend the use of vectors to represent equations of planes in three dimensional spaces and to apply knowledge we acquired in Vectors I and II to solve problems that involves planes in three-dimensional geometry.

1 EQUATION OF A PLANE

1.1 Equation of a Plane in Vector, Scalar Product and Cartesian Forms

1.1.1 Vector Equation of a Plane

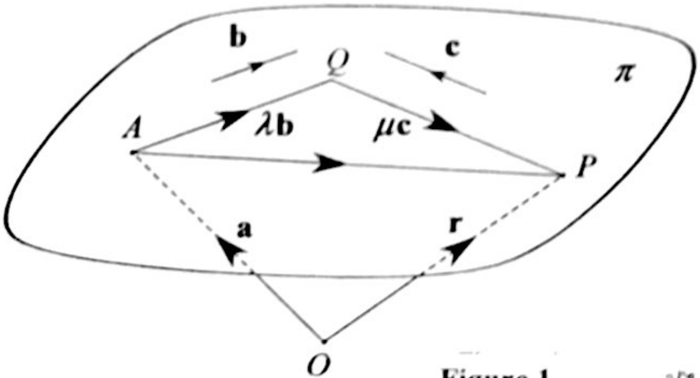


Figure 1

To find the vector equation of a straight line, we need a point and a direction to define the line.

For a plane π we need a point and two directions. Suppose π contains the point A with position vector $\mathbf{a}$ and π is parallel to two non-zero and non-parallel vectors $\mathbf{b}$ and $\mathbf{c}$ as shown in Figure 1. Then for any point P on π, since $\overrightarrow{AP} = \overrightarrow{AQ} + \overrightarrow{QP}$ with $\overrightarrow{AQ} // \mathbf{b}$ and $\overrightarrow{QP} // \mathbf{c}$, the vector equation of π is given by

one point
2 direction
parallel to
the plane

$$\pi : \underset{\sim}{r} = \underset{\sim}{a} + \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}, \quad \lambda, \mu \in \mathbb{R}, \text{ where } \underset{\sim}{r} = \overrightarrow{OP}$$

Remark: The vector equation of a plane is not unique since we can choose any point on the plane and any two non-parallel directions that are parallel to the plane. It is an extension to the vector equation of a line.

direction vectors $\underset{\sim}{b}$ and $\underset{\sim}{c}$ are free vectors (need not lie on the plane)

Example 1

Find a vector equation for each of the following planes.

(i) The plane containing the line $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$, $\lambda \in \mathbb{R}$, and parallel to the vector $\begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$.

(ii) The plane that passes through the origin O and the points $A(-3,0,1)$ and $B(1,1,-7)$.

(iii) The plane that contains the point $C(-2,1,-2)$ and the line $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Solution

(i) [Note that the plane passes through $(1,-1,-4)$ and is parallel to $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$.]

The plane has vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -1 \\ -4 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 1 \\ -6 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

(ii) [Note that the plane contains the origin and is parallel to $\overset{OA}{\begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$ and $\overset{OB}{\begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}}$.]

The plane has vector equation $\lambda \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -7 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

(iii) The plane is parallel to $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$.

The plane has vector equation $\underset{\sim}{r} = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$, $\lambda, \mu \in \mathbb{R}$.

1.1.2 Equation of a Plane in Scalar Product Form

Given a plane π, we can find a vector $\mathbf{n}$ that is perpendicular to π, so $\mathbf{n}$ is perpendicular to any vector parallel to π. Vector $\mathbf{n}$ is called a normal vector of π.

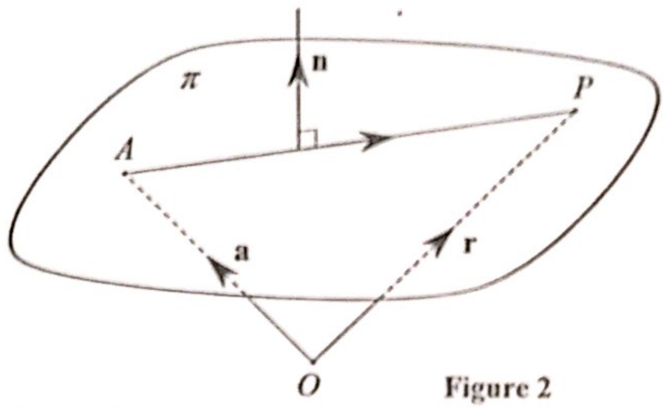


Figure 2

Suppose π contains point A with position vector $\mathbf{a}$ and has a normal vector $\mathbf{n}$, as shown in Figure 2. Let P be any point on π and $\mathbf{r} = \overrightarrow{OP}$. Since AP lies on π, AP is perpendicular to $\mathbf{n}$, so

$(\vec{OP} \cdot \vec{OA}) \cdot \underset{\sim}{n} = 0$

$$\overrightarrow{AP} \cdot \mathbf{n} = 0 \;\Rightarrow\; (\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \;\Rightarrow\; \mathbf{r} \cdot \mathbf{n} = \underbrace{\mathbf{a} \cdot \mathbf{n}}_{\text{constant}}.$$

r represents a general point on the plane

Hence the equation of π in scalar product form is

$\pi : \underset{\sim}{r} \cdot \underset{\sim}{n} = d$, where $d = \underset{\sim}{a} \cdot \underset{\sim}{n}$ is a scalar

1 point and 1 normal vector help you define a plane

Remark: The plane with equation $\mathbf{r} \cdot \mathbf{n} = 0$ passes through the origin O. Why?

does $\begin{pmatrix}0\\0\\0\end{pmatrix}$ satisfy the plane equation? $\begin{pmatrix}0\\0\\0\end{pmatrix} \cdot \underset{\sim}{n} = 0$ i.e. $\pi : \underset{\sim}{r} = \lambda \underset{\sim}{b} + \mu \underset{\sim}{c}$

Exercise: The plane π has equation $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, $\lambda, \mu \in \mathbb{R}$. Show that $\mathbf{n} \,//\, (\mathbf{b} \times \mathbf{c})$.

Answer: The plane π is parallel to $\underset{\sim}{b}$ and $\underset{\sim}{c}$.

$\underset{\sim}{n}$ is perpendicular to ~~both~~ the plane π

Thus $\mathbf{n}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$,

($\underset{\sim}{b} \times \underset{\sim}{c}$ is perpendicular to both $\underset{\sim}{b}$ and $\underset{\sim}{c}$)

so $\mathbf{n}$ is parallel to $\mathbf{b} \times \mathbf{c}$.

Example 2

Find the equation of each of the following planes in scalar product form.

(i) The plane that passes through $A(1,2,-2)$ and is perpendicular to the vector $\begin{pmatrix}2\\0\\-1\end{pmatrix}$.

normal vector

point

(ii) The plane that passes through $B(-3,0,1)$ and is parallel to the plane $\mathbf{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=\frac{\sqrt{3}}{2}$.

the planes share the normal vector

point direction point direction

(iii) The plane that contains the lines $\mathbf{r}=\begin{pmatrix}1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}3\\2\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$ and $\mathbf{r}=\begin{pmatrix}0\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\3\\-1\end{pmatrix}$, $\mu\in\mathbb{R}$.

(iv) The xy-plane.

Solution

(i) An equation of the plane is $\pi : \underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} = \begin{pmatrix}1\\2\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} = 4$,

i.e. $\pi : \underset{\sim}{r}\cdot\begin{pmatrix}2\\0\\-1\end{pmatrix} = 4$.

(ii) An equation of the plane is $\underset{\sim}{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}-3\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix} = -3+0-2 = -5$,

i.e. $\underset{\sim}{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix} = -5$.

(iii) The plane is perpendicular to $\begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}1\\3\\-1\end{pmatrix}=\begin{pmatrix}-2-3\\-(-3-1)\\9-2\end{pmatrix}=\begin{pmatrix}-5\\4\\7\end{pmatrix}$.

find normal vector first

An equation of the plane is $\underset{\sim}{r}\cdot\begin{pmatrix}-5\\4\\7\end{pmatrix} = \begin{pmatrix}1\\-2\\0\end{pmatrix}\cdot\begin{pmatrix}-5\\4\\7\end{pmatrix} = -5-8$,

i.e. $\underset{\sim}{r}\cdot\begin{pmatrix}-5\\4\\7\end{pmatrix} = -13$.

$\Rightarrow$ check whether correct by subbing $\begin{pmatrix}1\\-2\\0\end{pmatrix}$ and $\begin{pmatrix}0\\2\\-3\end{pmatrix}$ and seeing if they satisfy

point $\begin{pmatrix}0\\0\\0\end{pmatrix}$ normal vector $\begin{pmatrix}0\\0\\1\end{pmatrix}$

(iv) An equation of the plane is $\underset{\sim}{r}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = 0$.

normal vector: $\begin{pmatrix}0\\0\\1\end{pmatrix}$

1.1.3 Cartesian Equation of a Plane

Let $P(x, y, z)$ be any point on the plane π, which passes through a fixed point A with position vector $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and is perpendicular to the normal vector $\mathbf{n} = n_1\mathbf{i} + n_2\mathbf{j} + n_3\mathbf{k}$. From 1.1.2, the equation of π in scalar product form is $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} = d$. Hence

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = d$$

$$\Rightarrow \quad n_1x + n_2y + n_3z = a_1n_1 + a_2n_2 + a_3n_3 = d.$$

So the cartesian equation of π is expand scalar product

π : $\underset{\sim}{r}$: $n_1x + n_2y + n_3z = d$ where $d = a_1n_1 + a_2n_2 + a_3n_3$

a linear equation in three unknowns

Example 3

The equation of the plane π is given by $\mathbf{r} \cdot \begin{pmatrix} -1 \\ 2 \\ -5 \end{pmatrix} = 4.$

Write down the cartesian equation of π and find a vector equation of π.

Solution

The cartesian equation of π is $-x + 2y - 5z = 4$.

→ make 1 unknown the subject
→

So $x = -4 + 2y - 5z$, and

$r = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4+2y-5z \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$

seperate

Letting $\lambda = y$ and $\mu = z$, a vector equation of the plane π is

$\pi : \underset{\sim}{r} = \begin{pmatrix} -4 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \mu\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$

1.2 Calculations Involving a Line and a Plane

1.2.1 Angle between a Line and a Plane

In Figure 3, the acute angle between line $l: \mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and plane $\pi: \mathbf{r}.\mathbf{n} = d$ is denoted by θ.

The angle θ can be found by first finding ϕ, the acute angle between l and the normal to π, which is also the angle between $\mathbf{b}$ and $\mathbf{n}$. Then the angle between l and π is $\theta = 90° - \phi$.

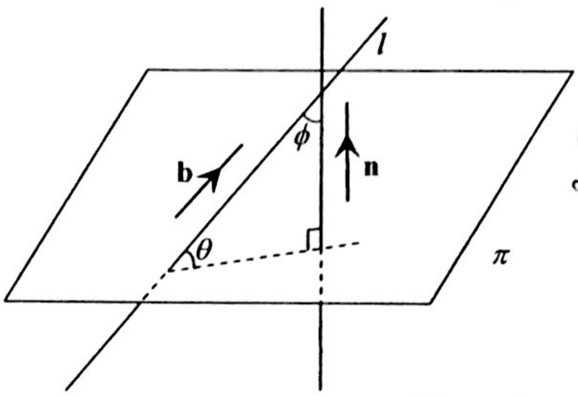


Figure 3

recall: $\cos\phi = \frac{\underset{\sim}{b} \cdot \underset{\sim}{n}}{|\underset{\sim}{b}||\underset{\sim}{n}|}$

Example 4

Find the acute angle between the plane π with equation $\mathbf{r}\cdot(2\mathbf{i}-\mathbf{j}+2\mathbf{k}) = 4$ and

(i) the line l: $\mathbf{r} = 4\mathbf{i}-6\mathbf{j}-2\mathbf{k}+\lambda(\mathbf{i}-2\mathbf{j})$, $\lambda \in \mathbb{R}$, **(ii)** the y-axis.

Solution

(i) Let ϕ be the angle between $\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}1\\-2\\0\end{pmatrix}$.

$$\cos\phi = \frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\0\end{pmatrix}}{\sqrt{9}\sqrt{5}}$$

$$= \frac{4}{3\sqrt{5}}$$

$\phi = 53.395°$ (3 d.p.)

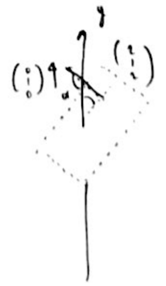
$\theta = 90° - 53.395°$

$= 36.405°$

$= 36.4°$ (1d.p.)

Hence the acute angle between π and l is 36.4°.

(ii) The direction vector of the y-axis is $\begin{pmatrix}0\\1\\0\end{pmatrix}$.



Let α be the angle between $\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$.

$$\cos\alpha = \frac{\begin{pmatrix}2\\-1\\2\end{pmatrix}\cdot\begin{pmatrix}0\\1\\0\end{pmatrix}}{\sqrt{9}\sqrt{1}}$$

$= -\frac{1}{3}$

$\alpha = \cos^{-1} -\frac{1}{3}$

$= 109.471°$ (3d.p.)

~~$\theta = 180° - 109.471°$~~

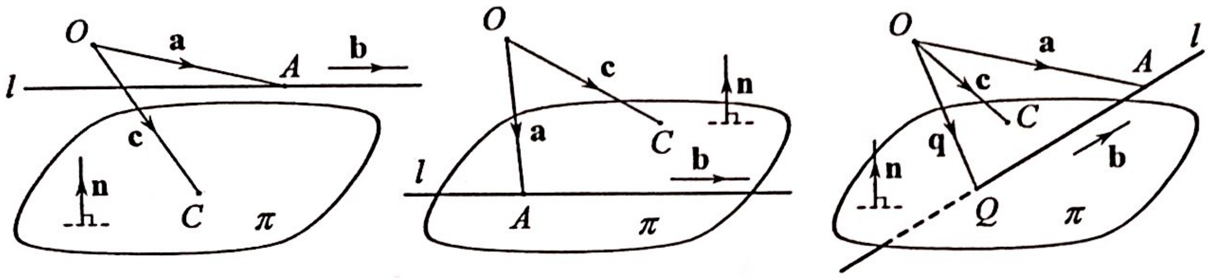
~~$= 70.5°$ (…)~~

$\theta = 90° - (180° - 109.5°) = 19.5°$

Hence the acute angle between π and the y-axis is 70.5°.

1.2.2 Point of Intersection between a Line and a Plane

Given a line l: $\mathbf{r}=\mathbf{a}+\lambda\mathbf{b}$, $\lambda\in\mathbb{R}$ and a plane π: $\mathbf{r}\cdot\mathbf{n}=\mathbf{c}\cdot\mathbf{n}$, there are three possible scenarios, as shown in Figure 4.



(1) l and π do not intersect

(2) l is on π

(3) l and π intersect at Q

$l // \pi$ $l // \pi$ $l \not\parallel \pi$

Figure 4

We proceed as follows to determine the relation between l and π:

(i) If $\mathbf{b}\cdot\mathbf{n}=0$, then l and π are parallel. Why? $\Rightarrow$ $\underset{\sim}{b} \perp \underset{\sim}{n}$ but $\underset{\sim}{n}$ is perpendicular to the plane, so b is parallel to the plane.

Moreover if $\mathbf{a}\cdot\mathbf{n}=\mathbf{c}\cdot\mathbf{n}$, then l lies on π as in scenario (2). Otherwise l is not on π as in scenario (1).

Note: Example 5(i) illustrates how we could show a line is on a plane in one step.

(ii) If $\mathbf{b}\cdot\mathbf{n}\neq 0$, then l and π intersect at exactly one point as in scenario (3).

The intersecting point can be found by solving simultaneous equations of l and π using the substitution method as in Example 5(ii) below.

Example 5

The line l has equation $\mathbf{r} = 5\mathbf{i} + 5\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j})$, $\lambda \in \mathbb{R}$.

(i) Show that l lies on the plane π_1 with equation $\mathbf{r} \cdot (-\mathbf{i} + 2\mathbf{j}) = -5$.

(ii) Find the position vector of the point P where l meets the plane π_2 with equation $\mathbf{r} \cdot (\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) = 11$.

Solution

(i) Any point on l has position vector $\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

$\left[\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$

$= -5 + 0 + 0 + \lambda(-2 + 2 + 0)$

$= -5$

So any point on l is also on π_1.

Thus l lies on π_1.

(ii) When l: $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ intersects π_2: $\mathbf{r} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 11$,

$\left[\begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right] \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 11$ sub it in

$5 + 10 + 2\lambda + 2\lambda = 11$

$4\lambda = -4$

$\Rightarrow \lambda = -1$

Hence $\overrightarrow{OP} = \begin{pmatrix} 5 \\ 0 \\ 5 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$.

1.3 Calculations Involving a Point and a Plane

1.3.1 Foot of Perpendicular from a Point to a Plane

In Figure 5a, the foot of perpendicular from point P to plane π is the point F, where the line through P and perpendicular to π meets π. Point F is also called the projection of point P onto π.
The distance PF is the perpendicular distance from P to π.

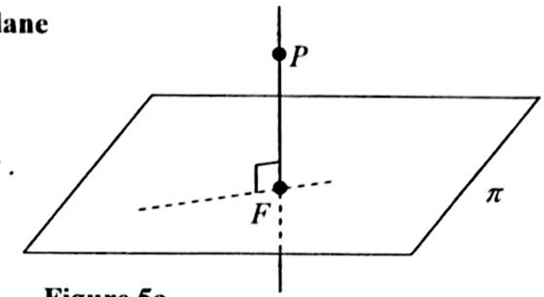


Figure 5a

1. line PF ⊥ π
2. point F lies on π

Example 6
The plane π has equation $\mathbf{r}\cdot(\mathbf{i}+\mathbf{j}-3\mathbf{k})=1$ and the points A and B have position vectors $\mathbf{i}$ and $\mathbf{i}+2\mathbf{j}$ respectively, with respect to the origin O.

(i) Show that π contains A but does not pass through B.

(ii) The foot of perpendicular from B to π is P. Find the position vector of P and determine the perpendicular distance from B to π.

(iii) Find the position vector of Q, the image of B after reflection in π.

Solution
(i) $\begin{pmatrix}1\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=1$, so π contains A.

$\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=3\neq 1$, so π does not pass through B.

(ii) Where $\ell_{BP}:\ \mathbf{r}=\underbrace{\begin{pmatrix}1\\2\\0\end{pmatrix}}_{\overrightarrow{OB}}+\lambda\underbrace{\begin{pmatrix}1\\1\\-3\end{pmatrix}}_{\tilde{n}}$ intersects π: $\mathbf{r}\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=1$,

point P
$\left[\begin{pmatrix}1\\2\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\-3\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\-3\end{pmatrix}=1$
$1+2+\lambda+\lambda+9\lambda=1$
$11\lambda=-2$
$\lambda=\frac{-2}{11}$ //

$$11\lambda=-2 \Rightarrow \lambda=-\frac{2}{11}$$

$$\overrightarrow{OP}=\begin{pmatrix}1\\2\\0\end{pmatrix}-\frac{2}{11}\begin{pmatrix}1\\1\\-3\end{pmatrix}=\frac{1}{11}\begin{pmatrix}9\\20\\6\end{pmatrix}.$$

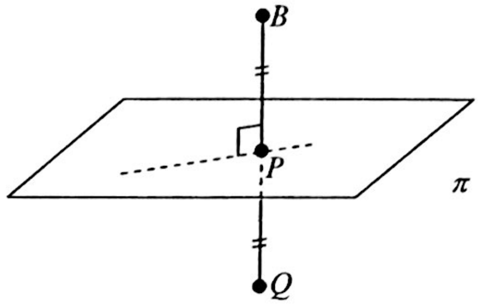
The perpendicular distance from B to π is the distance BP.

$$\overrightarrow{BP}=-\frac{2}{11}\begin{pmatrix}1\\1\\-3\end{pmatrix}$$

$\overrightarrow{OP}-\overrightarrow{OB}$

$$BP=\left|-\frac{2}{11}\begin{pmatrix}1\\1\\-3\end{pmatrix}\right|$$

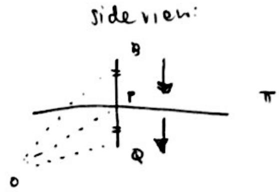
$$=\frac{2}{11}\sqrt{1^2+1^2+3^2}=\frac{2}{\sqrt{11}}$$



(iii) $\overrightarrow{OQ}=\overrightarrow{OB}+2\overrightarrow{BP}$

$=\begin{pmatrix}1\\2\\0\end{pmatrix}-\frac{4}{11}\begin{pmatrix}1\\1\\-3\end{pmatrix}$

$=\frac{1}{11}\begin{pmatrix}7\\18\\12\end{pmatrix}$



1.3.2 Distance from a Point to a Plane

Suppose the plane π passes through a fixed point A and has a normal vector $\mathbf{n}$, and the foot of perpendicular from point P to π is the point F. The distance (or shortest distance or perpendicular distance) from point P to π is the distance PF.

The distance PF is also the length of projection of $\overrightarrow{AP}$ onto $\mathbf{n}$, which is $\left|\overrightarrow{AP}\cdot\hat{\mathbf{n}}\right|$.

onto

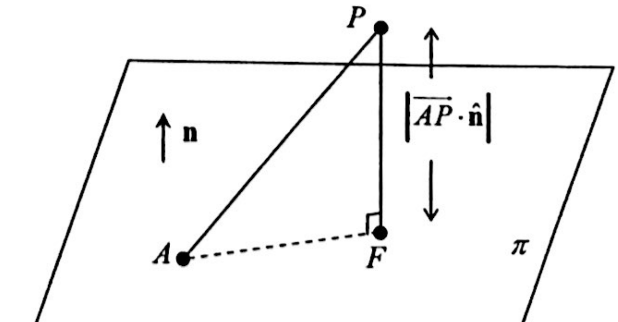
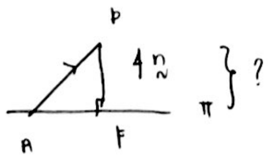


Figure 5b



Example 7

Find the distance from point $B\ (1, 7, -10)$ to plane π with equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}-2\mathbf{k})=13$.

Solution

Let $\mathbf{n}=\mathbf{i}+2\mathbf{j}-2\mathbf{k}$.

$\underset{\sim}{r}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=13$ $\quad\begin{pmatrix}13\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}=13$

Method 1: Let A be the point on π with coordinates (13, 0, 0).
The required distance is

$$\left|\overrightarrow{AB}\cdot\hat{\mathbf{n}}\right| = \frac{\left|\begin{pmatrix}-12\\7\\-10\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}\right|}{3}$$

$$\frac{|AB\cdot \underset{\sim}{n}|}{|\underset{\sim}{n}|} = \frac{\left|\begin{pmatrix}-12\\7\\-10\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}\right|}{3} = \frac{22}{3}$$

Method 2: Let P be any point on π. The required distance is

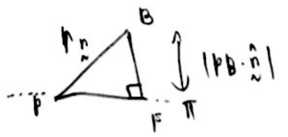
$$\left|\overrightarrow{PB}\cdot\hat{\mathbf{n}}\right| = \frac{\left|\left(\overrightarrow{OB}-\overrightarrow{OP}\right)\cdot\mathbf{n}\right|}{|\mathbf{n}|}$$

$$= \frac{\left|\overrightarrow{OB}\cdot\mathbf{n}-\overrightarrow{OP}\cdot\mathbf{n}\right|}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{\left|\begin{pmatrix}1\\7\\-10\end{pmatrix}\cdot\begin{pmatrix}1\\2\\-2\end{pmatrix}-13\right|}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{|1+14+20-13|}{3} = \frac{22}{3}$$

Note: Any point P on plane with equation $\mathbf{r}\cdot\mathbf{n}=13$ satisfies $\overrightarrow{OP}\cdot\mathbf{n}=13$



1.4 Calculations Involving Two Planes

1.4.1 Angle between Two Planes

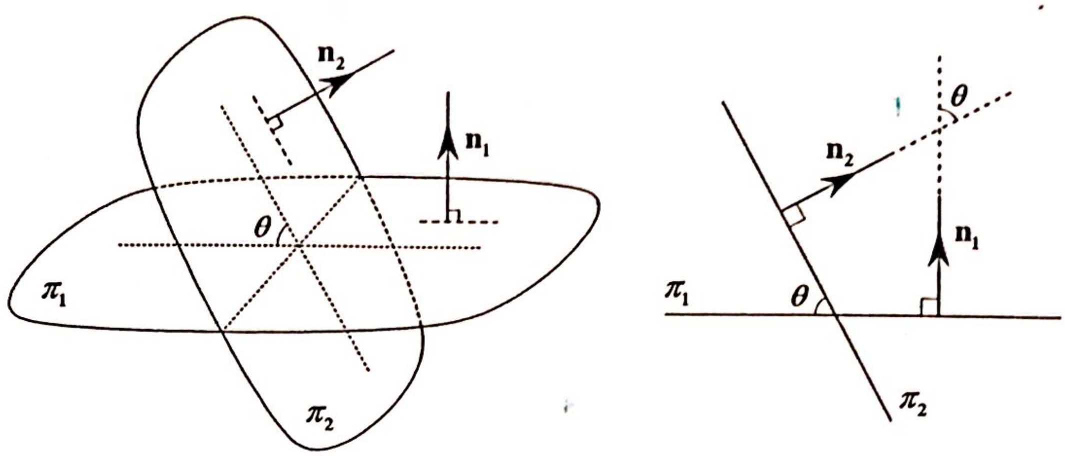


Figure 6

Figure 6 shows two intersecting planes. The angle between two planes is the angle between their normals.

Remark: Two planes are perpendicular if their normal vectors are perpendicular.
The planes are parallel if their normal vectors are parallel.

Example 8

Find the acute angle between the planes with equations $\mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+2\mathbf{k})=4$ and $\mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-\mathbf{k})=4$.

Solution

Let θ be the angle between their normal vectors. Then

$$\cos\theta = \frac{\begin{pmatrix}1\\3\\2\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}}{\left|\begin{pmatrix}1\\3\\2\end{pmatrix}\right|\left|\begin{pmatrix}1\\-1\\-1\end{pmatrix}\right|}$$

$$= \frac{-4}{\sqrt{14}\sqrt{3}}$$

$\theta = 128.1°$ (nearest 0.1°)

The acute angle between the planes is $180° - 128.1° = 51.9°$.

1.4.2 Intersection of Two Planes

Given two distinct planes, there are two possible cases as shown in Figure 7.

(1) The two planes do not intersect, so they are parallel.

In Figure 7, π_1 and π_3 do not intersect and are parallel, so $\mathbf{n}_1 = \lambda \mathbf{n}_3$ for some $\lambda \in \mathbb{R}$.

(2) The two planes intersect in a line.

In Figure 7, π_1 and π_2 intersect in the line l.

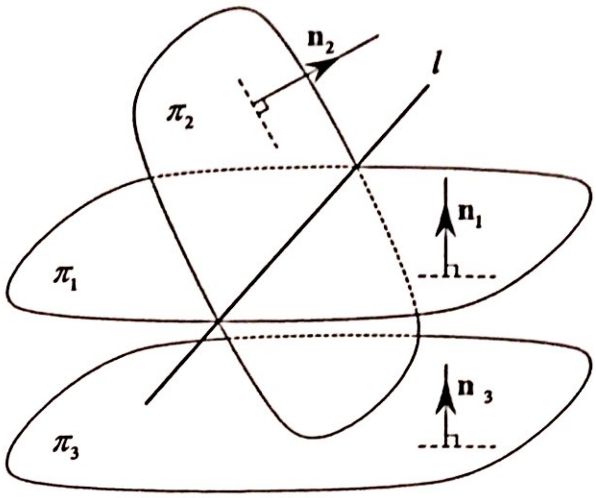


Figure 7

In the case where two planes π_1: $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and π_2: $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ intersect, where $\mathbf{n}_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{n}_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, there are three methods to determine the line of intersection l.

Method 1 (See Example 9a Method 1)
Solve the cartesian equations of π_1 and π_2 given by

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \end{cases},$$

a system of two linear equations with three unknowns x, y and z. We can solve for two variables in terms of the third, which serves as a parameter. This gives the equation of the line of intersection in parametric form.

$\hookrightarrow \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

line (parametric form)

Method 2 (See Example 9a Method 2)
Since l lies on both planes, it is perpendicular to both $\mathbf{n}_1$ and $\mathbf{n}_2$, so l is parallel to $\mathbf{n}_1 \times \mathbf{n}_2$.
It remains to find a point A with position vector $\mathbf{a}$ on l. This point can be obtained by observation or by solving the cartesian equations for π_1 and π_2 with one extra condition, like $x = 0$, $y = 0$ or $z = 0$.
The vector equation of l is then given by

$$l: \ \mathbf{r} = \mathbf{a} + \lambda(\mathbf{n}_1 \times \mathbf{n}_2), \ \lambda \in \mathbb{R}.$$

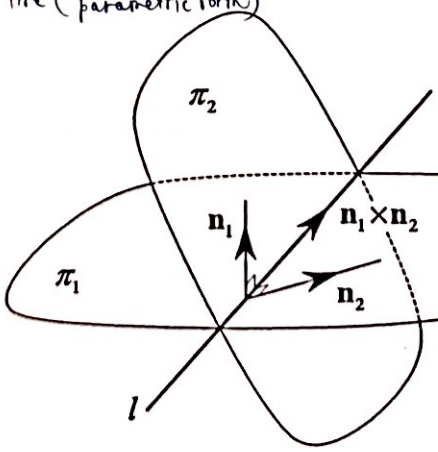


Figure 8

Method 3
We can use the trick in **Method 2** to first find two points A and B on l, then use the points to find the vector equation of l.

Example 9

Determine if the following pairs of planes are parallel or intersecting and find a vector equation of the intersecting line wherever applicable.

(a) $\pi_1: \ \mathbf{r}\cdot(\mathbf{i}+3\mathbf{j}+2\mathbf{k})=4$ and $\pi_2: \ \mathbf{r}\cdot(\mathbf{i}-\mathbf{j}-\mathbf{k})=4$.

(b) $\pi_3: \ \mathbf{r}=(2\mathbf{i}+\mathbf{j})+\alpha(-2\mathbf{i}+\mathbf{j})+\beta(\mathbf{i}-\mathbf{j}+\mathbf{k}), \ \alpha, \beta\in\mathbb{R}$ and

$\pi_4: \ \mathbf{r}=(3\mathbf{i}+4\mathbf{j}-\mathbf{k})+\lambda(-2\mathbf{i}+\mathbf{j})+\mu(\mathbf{j}-2\mathbf{k}), \ \lambda, \mu\in\mathbb{R}$.

Solution

$\underset{\sim}{n_1}$ $\underset{\sim}{n_2}$

(a) Since $\begin{pmatrix}1\\3\\2\end{pmatrix}\neq\lambda\begin{pmatrix}1\\-1\\-1\end{pmatrix}$ for any $\lambda\in\mathbb{R}$, the normal vectors are not parallel, and so neither are π_1 and π_2. Hence the two planes are intersecting.

To find intersecting line:

Method 1

$\pi_1: \ x+3y+2z=4$ $\quad \underset{\sim}{r}\cdot\begin{pmatrix}1\\3\\2\end{pmatrix}=4$

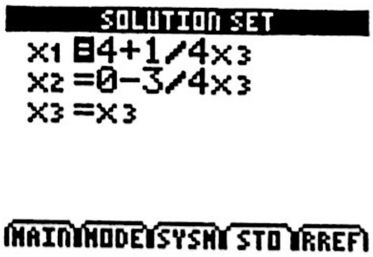
$\pi_2: \ x-y-z=4$ $\quad \underset{\sim}{r}\cdot\begin{pmatrix}1\\-1\\-1\end{pmatrix}=4$

This system of equations can be solved with the GC **(APPS PlySmlt2)**.

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4+\frac{1}{4}z\\-\frac{3}{4}z\\z\end{pmatrix}$$

$$=\begin{pmatrix}4\\0\\0\end{pmatrix}+\underbrace{\tfrac{1}{4}z}\begin{pmatrix}1\\-3\\1\end{pmatrix}$$

$$=\begin{pmatrix}4\\0\\0\end{pmatrix}+t\begin{pmatrix}1\\-3\\1\end{pmatrix}, \ t\in\mathbb{R}$$



An equation of the intersecting line is

Method 2

$n_1 \times n_2$ direction vector

The intersecting line is parallel to $\begin{pmatrix}1\\3\\2\end{pmatrix} \times \begin{pmatrix}1\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\3\\-4\end{pmatrix}$.

To find a point on the line, we only need to find any point that satisfies

$$\begin{cases} x+3y+2z=4 & \to \pi_1 \\ x-y-z=4 & \to \pi_2 \end{cases}$$

When $z=0$ $\begin{cases} x+3y=4 \\ x-y=4 \end{cases} \Rightarrow y=0,\ x=4$

Hence one point on the line is $(4,0,0)$.

so an equation of the line is $\mathbf{r} = \begin{pmatrix}4\\0\\0\end{pmatrix} + \lambda \begin{pmatrix}-1\\3\\-4\end{pmatrix},\ \lambda \in \mathbb{R}$.

point direction

(b) Let $\mathbf{n}_3$ and $\mathbf{n}_4$ be vectors normal to π_3 and π_4 respectively.

$$\mathbf{n}_3 = \begin{pmatrix}-2\\1\\0\end{pmatrix} \times \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}1-0\\-(-2-0)\\2-1\end{pmatrix} = \begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\mathbf{n}_4 = \begin{pmatrix}-2\\1\\0\end{pmatrix} \times \begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}2-0\\-4\\-2\end{pmatrix} = \begin{pmatrix}-2\\-4\\-2\end{pmatrix} = -2\mathbf{n}_3$$

Since $\mathbf{n}_3$ and $\mathbf{n}_4$ are parallel, so are π_3 and π_4.

Example 10 [PJC Prelim 9740/2008/01/Q10 (modified)]

The line l_1 and plane π_1 have vector equations

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} - 4\mathbf{k} + \lambda(3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}), \qquad \lambda \in \mathbb{R}$$

$$\mathbf{r} = \mathbf{i} + 5\mathbf{j} - 4\mathbf{k} + s(2\mathbf{i} + \mathbf{j} + 4\mathbf{k}) + t(3\mathbf{i} + 2\mathbf{j} + 7\mathbf{k}), \quad s, t \in \mathbb{R}$$

respectively.

(i) State the coordinates of a point of intersection between l_1 and π_1, and hence give a geometrical interpretation about l_1 and π_1. **[2]**

(ii) The plane π_2 passes through the point A with coordinates (3, 6, 1) and contains the line l_1. Find, in the form $\mathbf{r} \cdot \mathbf{n} = p$, the equation of π_2. **[2]**

(iii) The plane π_3 has equation $x + 2y - z = 0$. Determine, with a reason, if the planes π_1, π_2 and π_3 have a common point of intersection. **[3]**

(iv) Find the perpendicular distance between π_1 and π_3, giving your answer in the surd form. **[2]**

Solution

(i) From the equations of l_1 and π_1, they have a common point $(1, 5, -4)$ and are parallel. Hence l_1 lies on π_1.

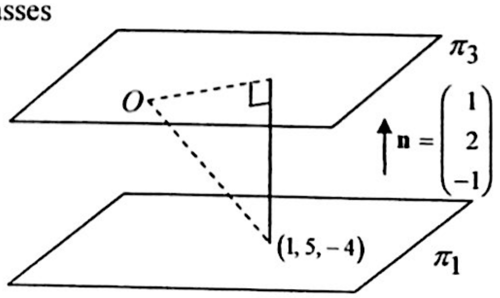
(ii) A vector parallel to π_2 is $\begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$, and a normal vector of π_2 is $\begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$.

π_2 has equation $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, i.e. $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix} = -2$.

(iii) A normal of π_1 is $\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \times \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$, and π_1 has equation $\mathbf{r} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = -15$ or $x + 2y - z = 15$. The equations of π_1 and π_3 show they are two parallel planes. Thus, the 3 planes have no common point of intersection.

(iv) From the equation of π_3, observe that π_3 passes through the origin. Hence the distance is

$$\left| \begin{pmatrix} 1 \\ 5 \\ -4 \end{pmatrix} \cdot \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right| = \frac{15}{\sqrt{6}}.$$



Example 11 [HCI Prelim 9740/2008/01/Q12(b)]

Referring to the origin O, two planes π_1 and π_2 are given by

$$\pi_1:\ \mathbf{r}\cdot\begin{pmatrix}1\\2\\-4\end{pmatrix}=13 \quad \text{and} \quad \pi_2:\ \mathbf{r}\cdot\begin{pmatrix}1\\3\\3\end{pmatrix}=-8.$$

(i) Given that the point $A\ (1,\ 7,\ -10)$ lies on π_2, find $\overrightarrow{OB}$ where B is the image of A when reflected in the plane π_1. [4]

(ii) Write down the Cartesian equations of both π_1 and π_2. [1]

Find a vector equation of the line of intersection of π_1 and π_2. [1]

(iii) Find a vector equation of the plane which is the image of π_2 when π_2 is reflected in π_1. [3]

Solution

(i) The line through A and normal to π_1 is $\mathbf{r}=\begin{pmatrix}1\\7\\-10\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-4\end{pmatrix}$, $\lambda\in\mathbb{R}$. Suppose the line meets π_1 at F, then

$$\left[\begin{pmatrix}1\\7\\-10\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\-4\end{pmatrix}\right]\cdot\begin{pmatrix}1\\2\\-4\end{pmatrix}=13$$

$$(1+14+40)+\lambda(1+4+16)=13\Rightarrow\lambda=-2$$

$\overrightarrow{OF}=\begin{pmatrix}1\\7\\-10\end{pmatrix}-2\begin{pmatrix}1\\2\\-4\end{pmatrix}=\begin{pmatrix}-1\\3\\-2\end{pmatrix}$. Since F is the midpoint of AB, $\overrightarrow{OF}=\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OB}\right)$.

Hence $\overrightarrow{OB}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}-1\\3\\-2\end{pmatrix}-\begin{pmatrix}1\\7\\-10\end{pmatrix}=\begin{pmatrix}-3\\-1\\6\end{pmatrix}$

(ii) π_1: $x+2y-4z=13 \quad \ldots(1)$

π_2: $x+3y+3z=-8 \quad \ldots(2)$

Using GC to solve, the equation of the line of intersection is $r=\begin{pmatrix}55\\-21\\0\end{pmatrix}+\mu\begin{pmatrix}18\\-7\\1\end{pmatrix}$, $\mu\in\mathbb{R}$.

(iii) Since B and l lie on the image plane of π_2, a vector parallel to π_2 is

$$\begin{pmatrix}55\\-21\\0\end{pmatrix}-\begin{pmatrix}-3\\-1\\6\end{pmatrix}=2\begin{pmatrix}29\\-10\\-3\end{pmatrix}.$$

The equation of image plane is $\mathbf{r}=\begin{pmatrix}-3\\-1\\6\end{pmatrix}+s\begin{pmatrix}18\\-7\\1\end{pmatrix}+t\begin{pmatrix}29\\-10\\-3\end{pmatrix}$ where $s,\ t\in\mathbb{R}$.

Example 12 [NYJC Prelim 9740/2008/01/Q11]

Two planes π_1 and π_2 have equations

$$\mathbf{r}=(3+\lambda-2\mu)\mathbf{i}+(2-3\lambda+\mu)\mathbf{j}+(1-\mu)\mathbf{k} \text{ and } \mathbf{r}\cdot\begin{pmatrix}a\\-1\\b\end{pmatrix}=5 \text{ respectively.}$$

(i) Show that the Cartesian equation of π_1 is $3x+y-5z=6$. [2]

(ii) Given that the point $A\ (3,\ 2,\ 1)$ lies on π_2 and that the two planes are perpendicular to each other, find the values of a and of b. [3]

(iii) Find a vector equation of l, the line of intersection between the planes π_1 and π_2. [2]

(iv) The point B has position vector $-6\mathbf{i}+3\mathbf{j}+11\mathbf{k}$. Find the coordinates of the point C on π_2 such that $\angle ACB=90^\circ$. [3]

(v) The line L is the reflection of the line AB in the plane π_2. Find an equation for the line L and determine the acute angle between the lines l and L. [4]

Solution

(i) $\mathbf{r}=\begin{pmatrix}3+\lambda-2\mu\\2-3\lambda+\mu\\1-\mu\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-3\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\1\\-1\end{pmatrix}$

A normal vector of plane π_1 is $\begin{pmatrix}1\\-3\\0\end{pmatrix}\times\begin{pmatrix}-2\\1\\-1\end{pmatrix}=\begin{pmatrix}3\\1\\-5\end{pmatrix}$.

Hence equation of π_1 is $\mathbf{r}\cdot\begin{pmatrix}3\\1\\-5\end{pmatrix}=\begin{pmatrix}3\\2\\1\end{pmatrix}\cdot\begin{pmatrix}3\\1\\-5\end{pmatrix}$, so cartesian equation of π_1 is $3x+y-5z=6$.

(ii) The point A lies on π_2, so $\begin{pmatrix}3\\2\\1\end{pmatrix}\cdot\begin{pmatrix}a\\-1\\b\end{pmatrix}=5 \Rightarrow 3a+b=7$ ---(1)

Planes π_1 and π_2 are perpendicular, so $\begin{pmatrix}3\\1\\-5\end{pmatrix}\cdot\begin{pmatrix}a\\-1\\b\end{pmatrix}=0 \Rightarrow 3a-5b=1$ ---(2)

Solving (1) & (2), $a=2$, $b=1$.

(iii) $\begin{pmatrix}3\\1\\-5\end{pmatrix}\times\begin{pmatrix}2\\-1\\1\end{pmatrix}=\begin{pmatrix}-4\\-13\\-5\end{pmatrix}$

$l:\ \mathbf{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\lambda\begin{pmatrix}4\\13\\5\end{pmatrix},\ \lambda\in\mathbb{R}.$

(iv) Point C is where the line BC with equation $\mathbf{r}=\begin{pmatrix}-6\\3\\11\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}$, $\mu\in\mathbb{R}$ meets π_2:

$$\left[\begin{pmatrix}-6\\3\\11\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-1\\1\end{pmatrix}=5$$

$$(-12-3+11)+\mu(4+1+1)=5$$

$$\mu=\frac{3}{2}$$

$\overrightarrow{OC}=\begin{pmatrix}-6\\3\\11\end{pmatrix}+\frac{3}{2}\begin{pmatrix}2\\-1\\1\end{pmatrix}$, so C has coordinates $\left(-3,\frac{3}{2},\frac{25}{2}\right)$.

(v) Let B' be the reflection of B in π_2.

$\overrightarrow{OC}=\dfrac{\overrightarrow{OB}+\overrightarrow{OB'}}{2}\quad\Rightarrow\quad\overrightarrow{OB'}=2\overrightarrow{OC}-\overrightarrow{OB}=14\mathbf{k}$

$$\overrightarrow{AB'}=\begin{pmatrix}0\\0\\14\end{pmatrix}-\begin{pmatrix}3\\2\\1\end{pmatrix}=\begin{pmatrix}-3\\-2\\13\end{pmatrix}$$

Hence an equation for the line L is $\mathbf{r}=\begin{pmatrix}3\\2\\1\end{pmatrix}+\alpha\begin{pmatrix}-3\\-2\\13\end{pmatrix}$, $\alpha\in\mathbb{R}$. Let θ be the angle between $\begin{pmatrix}4\\13\\5\end{pmatrix}$ and $\begin{pmatrix}-3\\-2\\13\end{pmatrix}$. Then $\cos\theta=\dfrac{\begin{pmatrix}4\\13\\5\end{pmatrix}\cdot\begin{pmatrix}-3\\-2\\13\end{pmatrix}}{\left|\begin{pmatrix}4\\13\\5\end{pmatrix}\right|\left|\begin{pmatrix}-3\\-2\\13\end{pmatrix}\right|}=\dfrac{-12+13\times3}{\sqrt{4^2+13^2+5^2}\sqrt{3^2+2^2+13^2}}=\dfrac{27}{\sqrt{210}\sqrt{182}}$

$$\theta=82.1^\circ\ \text{(nearest }0.1^\circ\text{)}$$

Hence the required acute angle is 82.1°.

CONCLUSION

With the introduction of vector product and equations of planes, many new and interesting problems in three-dimensional geometry are created and this makes the study of vectors more complete.

VECTORS 3C

SUMMARY