RVHS H2 Mathematics Remedial Programme

Topic: Complex Numbers

Basic Mastery Questions

1. **MI Promo 9758/2020/PU2/P1/Q8(a)**
Given that $a=3-\mathrm{i}$ and $b=5+2\mathrm{i}$, find the following complex numbers in the form $x+\mathrm{i}y$,

(i) ab^*, [2]

(ii) $\frac{b}{a^*}$. [3]

Answer: (i) $13-11\mathrm{i}$, (ii) $\frac{1}{10}(17+\mathrm{i})$

(i)	$ab^*=(3-\mathrm{i})(5-2\mathrm{i})$ $=15-6\mathrm{i}-5\mathrm{i}+2\mathrm{i}^2$ $=15-6\mathrm{i}-5\mathrm{i}-2$ $=13-11\mathrm{i}$
(ii)	$\frac{b}{a^*}=\frac{5+2\mathrm{i}}{3+\mathrm{i}}\times\frac{3-\mathrm{i}}{3-\mathrm{i}}$ $=\frac{15-5\mathrm{i}+6\mathrm{i}-2\mathrm{i}^2}{9-\mathrm{i}^2}$ $=\frac{15-5\mathrm{i}+6\mathrm{i}+2}{9+1}$ $=\frac{1}{10}(17+\mathrm{i})$

2. **ASRJC Promo 9758/2020/Q8(a)**
It is given that two complex numbers z and w satisfy the following equations

$$\mathrm{i}w + z = 5$$
$$w^2 + (4\mathrm{i} - 1)z = -11 + 18\mathrm{i}$$

Find z and w. [4]

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Answers: $z = 6 + 2\mathrm{i};\ w = \mathrm{i} - 2$ or $z = 3 + 2\mathrm{i};\ w = -2\mathrm{i} - 2$

$w\mathrm{i} + z = 5 \quad \ldots(1)$

$w^2 + (4\mathrm{i} - 1)z = -11 + 18\mathrm{i} \quad \ldots(2)$

From [1], $z = 5 - w\mathrm{i} \quad \ldots(3)$

Sub [3] into [2], $w^2 + (4\mathrm{i} - 1)(5 - w\mathrm{i}) = -11 + 18\mathrm{i}$

$\Rightarrow w^2 + 20\mathrm{i} + 4w - 5 + \mathrm{i}w = -11 + 18\mathrm{i}$

$\Rightarrow w^2 + (\mathrm{i} + 4)w + (2\mathrm{i} + 6) = 0$

$$\Rightarrow w = \frac{-(\mathrm{i} + 4) \pm \sqrt{(\mathrm{i} + 4)^2 - 4(1)(2\mathrm{i} + 6)}}{2}$$

$$\Rightarrow w = \frac{-\mathrm{i} - 4 \pm \sqrt{\mathrm{i}^2 + 16 + 8\mathrm{i} - 8\mathrm{i} - 24}}{2}$$

$$\Rightarrow w = \frac{-\mathrm{i} - 4 \pm \sqrt{-9}}{2}$$

$$\Rightarrow w = \frac{-\mathrm{i} - 4 \pm 3\mathrm{i}}{2}$$

$\Rightarrow w = \mathrm{i} - 2$ or $-2\mathrm{i} - 2$

Substitute w into [3]:

$z = 5 - (\mathrm{i} - 2)\mathrm{i}$ or $5 - (-2\mathrm{i} - 2)\mathrm{i}$

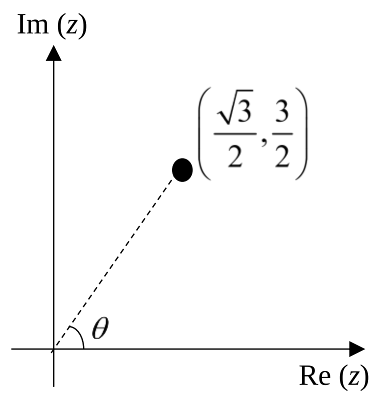
$\Rightarrow z = 6 + 2\mathrm{i}$ or $3 + 2\mathrm{i}$

3. **MI Promo 9758/2020/PU2/P1/Q8(b)**

(i) Express $z = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{i}$ in the form of $r\mathrm{e}^{\mathrm{i}\theta}$. [3]

(ii) Given that the complex number zw has modulus 12 and argument $\frac{2\pi}{3}$, find the exact modulus and argument of complex number w. [3]

Answer: (i) $\sqrt{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$, (ii) $r = 4\sqrt{3},\ \theta = \frac{\pi}{3}$

(i)	$z = \mathrm{e}^{\mathrm{i}\frac{\pi}{6}} + \mathrm{i}$ $= \cos\frac{\pi}{6} + \mathrm{i}\sin\frac{\pi}{6} + \mathrm{i}$ $= \frac{\sqrt{3}}{2} + \frac{3}{2}\mathrm{i}$ Since z is in the first quadrant, $\arg(z) = \theta = \tan^{-1}\frac{\frac{3}{2}}{\frac{\sqrt{3}}{2}}$ $\theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$ $r = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$ $r = \sqrt{3}$ $z = \sqrt{3}\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$ 
(ii)	<u>Method 1</u> $\lvert zw \rvert = \lvert z \rvert \lvert w \rvert$ $12 = \sqrt{3}\lvert w \rvert$ $\lvert w \rvert = \frac{12}{\sqrt{3}} = 4\sqrt{3}$ $\arg(zw) = \arg(z) + \arg(w)$ $\frac{2\pi}{3} = \frac{\pi}{3} + \arg(w)$ $\arg(w) = \frac{\pi}{3}$ ========================= <u>Method 2</u>

	$zw = 12e^{i\frac{2\pi}{3}}$ $\left(\sqrt{3}e^{i\frac{\pi}{3}}\right)w = 12e^{i\frac{2\pi}{3}}$ $w = \dfrac{12e^{i\frac{2\pi}{3}}}{\sqrt{3}e^{i\frac{\pi}{3}}}$ $= 4\sqrt{3}e^{i\frac{2\pi}{3}-i\frac{\pi}{3}}$ $= 4\sqrt{3}e^{i\frac{\pi}{3}}$ $\therefore r = 4\sqrt{3},\ \theta = \dfrac{\pi}{3}$

4. **MI Promo 9758/2020/PU2/P2/Q2(i)**

Do not use a calculator in answering this question.

The roots of the equation $z^2 = -8 - 6\mathrm{i}$ are z_1 and z_2.

Find z_1 and z_2 in cartesian form, $x + \mathrm{i}y$, showing your working. [5]

Click here or scan this to view video example on how to solve such question!

Answer: $-1+3\mathrm{i}$, $1-3\mathrm{i}$

Let $z = x + \mathrm{i}y$. Then, $(x+\mathrm{i}y)^2 = -8 - 6\mathrm{i}$.

$\Rightarrow x^2 - y^2 + 2xy\mathrm{i} = -8 - 6\mathrm{i}$

Comparing real parts, $x^2 - y^2 = -8$. -- (1)

<u>Method 1:</u>

Comparing imaginary parts, $2xy = -6 \Rightarrow x = -\dfrac{3}{y}$ -- (2)

Sub (2) into (1): $\left(-\dfrac{3}{y}\right)^2 - y^2 = -8 \Rightarrow 9 - y^4 = -8y^2$

$y^4 - 8y^2 - 9 = 0$

$(y^2 - 9)(y^2 + 1) = 0$

$y^2 = 9 \Rightarrow y = \pm 3$ $\left(y^2 = -1 \text{ rejected since } y \in \mathbb{R} \Rightarrow y^2 \geq 0\right)$

From (2): $x = \mp 1$

Therefore, the roots of the equation $z^2 = -8 - 6\mathrm{i}$ are

$z_1 = -1 + 3\mathrm{i}$ and $z_2 = 1 - 3\mathrm{i}$.

5. **VJC Prelim 9758/2021/01/Q8(a)(i)**

The complex number w is given by $w = r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $0 \le \theta \le \frac{\pi}{2}$.

Given that $z = \left(1 - \mathrm{i}\sqrt{3}\right)w$, find $|z|$ in terms of r and $\arg(z)$ in terms of θ. [2]

Click here or scan this to view video example on how to solve such question!

Answers: $-\frac{\pi}{3} + \theta$, $2r$

$$\arg(z) = \arg\left(1 - \mathrm{i}\sqrt{3}\right) + \arg(w)$$
$$= -\frac{\pi}{3} + \theta$$

$$|z| = \left|1 - \mathrm{i}\sqrt{3}\right||w|$$
$$= 2r$$

Standard Questions

1. **RI Promo 9758/2020/Q7(a)**
Do not use a calculator in answering this question.
One root of the equation $zz^* + 2\mathrm{i}z = a + 6\mathrm{i}$, where a is real, is $z = 3 - 7\mathrm{i}$. Find the value of a and the other root. [4]

Answer: a = 72, $3 + 9\mathrm{i}$

Given $z = 3 - 7\mathrm{i}$ is a root, substitute it into $zz^* + 2\mathrm{i}z = a + 6\mathrm{i}$,

$$\Rightarrow (3 - 7\mathrm{i})(3 + 7\mathrm{i}) + 2\mathrm{i}(3 - 7\mathrm{i}) = a + 6\mathrm{i}$$

Comparing real part, $3^2 + 7^2 + 14 = a \quad \Rightarrow a = 72$

Let the other root be $x + y\mathrm{i}$.

$x^2 + y^2 + 2\mathrm{i}(x + y\mathrm{i}) = 72 + 6\mathrm{i}$

Comparing imaginary part, $2x = 6 \quad \Rightarrow x = 3$

Comparing real part, $3^2 + y^2 - 2y = 72$

$$y^2 - 2y - 63 = 0$$
$$(y + 7)(y - 9) = 0$$
$$\therefore y = -7 \quad \text{or} \quad y = 9$$

Hence the other root is $3 + 9\mathrm{i}$.

2. **VJC Promo 9758/2020/Q10(a), (e)**

It is given that the complex number $w = -(\sqrt{3}) - \mathrm{i}$.

Find the value of $|w|$. [1]

Find the exact value of arg(w). [1]

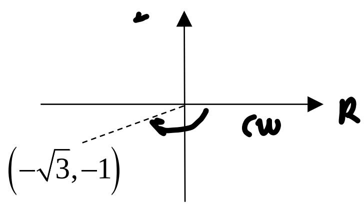
Without using a calculator, find the three smallest positive whole number values of n such that $w^n w^*$ is a real number. [3]

Click here or scan this to view video example on how to solve such question!

Answer: $|w| = 2$, $\arg w = -\frac{5\pi}{6}$, $n = 1, 7, 13$

$|w| = \sqrt{(\sqrt{3})^2 + 1^2} = 2$

$\arg w = -\frac{5\pi}{6}$



$\arg(w^n w^*) = n \arg w + \arg w^* = -\frac{5\pi}{6} n + \frac{5\pi}{6}$

$w^n w^*$ is real $\Rightarrow$ $\arg(w^n w^*) = k\pi$, where $k \in \mathbb{Z}$

$\Rightarrow -\frac{5\pi}{6} n + \frac{5\pi}{6} = k\pi$

$\Rightarrow (-n+1) = \frac{6k}{5}$

$\Rightarrow n = 1 - \frac{6k}{5}$

For the three smallest positive whole number values of n,

we take $k = 0, -5, -10$, $\Rightarrow n = 1, 7, 13$