Solutions to Tutorial 6B: Applications of Integration

Basic Mastery Questions

1. Evaluate exactly the following integrals by using the given substitutions (and use a GC to check your answers):

(a) $\int_{\frac{2}{\sqrt{3}}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$, using $u=\frac{1}{x}$

(b) $\int_0^1 \frac{2x}{\sqrt{1+x^4}}\,dx$, using $\theta=\tan^{-1}\left(x^2\right)$

Solution:

1. a) $\int_{2/\sqrt{3}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx$

Using $u=\frac{1}{x} \Rightarrow x=\frac{1}{u}$

$\frac{dx}{du}=-\frac{1}{u^2}$

When $x=2/\sqrt{3}$, $u=\frac{\sqrt{3}}{2}$

When $x=2$, $u=\frac{1}{2}$

$= \int_{\sqrt{3}/2}^{1/2} \frac{1}{\frac{1}{u}\sqrt{\frac{1}{u^2}-1}}\cdot\left(-\frac{1}{u^2}\right)du$

$= \int_{\sqrt{3}/2}^{1/2} -\frac{1}{\sqrt{1-u^2}}\,du$

$= \int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-u^2}}\,du$

$= \left[\sin^{-1}u\right]_{1/2}^{\sqrt{3}/2}$

Note: $\sin\frac{\pi}{6}=\frac{1}{2}$

$\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$

$= \frac{\pi}{3}-\frac{\pi}{6} = \frac{\pi}{6}$ #

b) $\int_0^1 \frac{2x}{\sqrt{1+x^4}}\,dx$

Using $\theta=\tan^{-1}(x^2)$

$\tan\theta = x^2$

Diff wrt x:

$\sec^2\theta\,\frac{d\theta}{dx}=2x$

$\Rightarrow \sec^2\theta\,d\theta = 2x\,dx$

When $x=0$, $\theta=\tan^{-1}0=0$

$x=1$, $\theta=\tan^{-1}(1)=\frac{\pi}{4}$

$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,d\theta$

$= \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$

$= \left[\ln(\sec\theta+\tan\theta)\right]_0^{\frac{\pi}{4}}$

$= \ln\left(\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right)-\ln(\sec 0+\tan 0)$

$= \ln(\sqrt{2}+1)-\ln(1+0)$

$= \ln(\sqrt{2}+1)$

2. Evaluate exactly the following integrals using integration by parts substitutions (and use a GC to check your answers):

(a) $\int_0^{\infty} x^2 e^{-3x}\,dx$

(b) $\int_1^{e} (\ln x)^2\,dx$

Solution for 2(a):

Consider

$\int x^2 e^{-3x}\,dx$

$u = x^2$ $\quad \frac{dv}{dx} = e^{-3x}$

$\frac{du}{dx} = 2x$ $\quad v = -\frac{1}{3}e^{-3x}$

$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\int xe^{-3x}\,dx$

$u = x$ $\quad \frac{dv}{dx} = e^{-3x}$

$\frac{du}{dx} = 1$ $\quad v = -\frac{1}{3}e^{-3x}$

$= -\frac{1}{3}x^2e^{-3x} + \frac{2}{3}\left[-\frac{1}{3}xe^{-3x} + \frac{1}{3}\int e^{-3x}\,dx\right]$

$= -\frac{1}{3}x^2e^{-3x} - \frac{2}{9}xe^{-3x} + \frac{2}{9}\int e^{-3x}\,dx$

$\therefore \int_0^{\infty} x^2e^{-3x}\,dx$

$= \left[-\frac{1}{3}x^2e^{-3x}\right]_0^{\infty} - \left[\frac{2}{9}xe^{-3x}\right]_0^{\infty} - \frac{2}{27}\int_0^{\infty} -3e^{-3x}\,dx$

$= 0 - 0 - \frac{2}{27}\left[e^{-3x}\right]_0^{\infty}$

$= -\frac{2}{27}(0-1) = \frac{2}{27}$ #

Solution for 2(b):

Consider

$u = (\ln x)^2$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{2}{x}\ln x$ $\quad v = x$

$\int (\ln x)^2\,dx = x(\ln x)^2 - 2\int \ln x\,dx$

$= x(\ln x)^2 - 2x\ln x + 2\int 1\,dx$

$u = \ln x$ $\quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x}$ $\quad v = x$

$\therefore \int_1^{e} (\ln x)^2\,dx$

$= \left[x(\ln x)^2\right]_1^{e} - \left[2x\ln x\right]_1^{e} + 2\left[x\right]_1^{e}$

$= e - 2e + 2e - 2$

$= e - 2$ #

3. (a) Evaluate $\int_{-\pi}^{\pi} \sin x \, dx$.

(b) Find the area of the region bounded by the curve $y = \sin x$, the x-axis and the lines $x = 0$ and $x = \pi$.

(c) Find the area of the region bounded by the curve $y = \sin x$, the x-axis and the lines $x = -\pi$ and $x = 0$. Compare with the answer in (b).

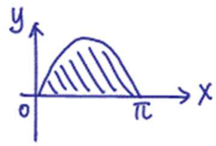
(d) Deduce the area bounded by the curve $y = \sin x$, the x-axis and the lines $x = -\pi$ and $x = \pi$. Compare with the answer in (a).

Solution:

3a) $\int_{-\pi}^{\pi} \sin x \, dx = \left[-\cos x\right]_{-\pi}^{\pi}$

$= 1 + (-1) = 0$ #

b)

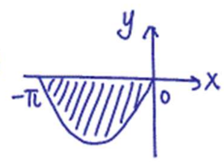


$\text{Area} = \int_0^{\pi} \sin x \, dx$

$= \left[-\cos x\right]_0^{\pi} = -\cos\pi + \cos 0$

$= 1 + 1 = 2 \text{ units}^2$ #

c)



$\text{Area} = \left| \int_{-\pi}^{0} \sin x \, dx \right|$

$= \left| \left[-\cos x\right]_{-\pi}^{0} \right|$

$= \left| -\cos 0 + \cos\pi \right| = \left| -1 - 1 \right| = 2 \text{ units}^2$ #

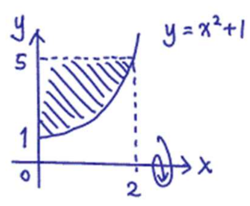
d) Total area $= 2 + 2 = 4 \text{ units}^2$ #

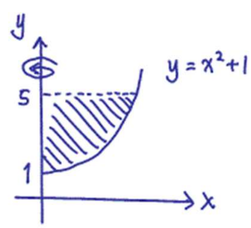
We need to take the positive value when evaluating area which falls below the x-axis.

4. Find the volume generated by revolving the region in the first quadrant, bounded by the curve $y = x^2 + 1$, the y-axis and the line $y = 5$, about (i) the x-axis, (ii) the y-axis through 4 right angles.

Solution:

4. i)





Volume generated

$= \pi(5)^2(2) - \pi\int_0^2 (x^2+1)^2 \, dx$

$= 50\pi - \pi\int_0^2 x^4 + 2x^2 + 1 \, dx$

$= 50\pi - \pi\left[\frac{x^5}{5} + \frac{2x^3}{3} + x\right]_0^2$

$= \frac{544}{15}\pi \text{ units}^3$ #

Volume generated

$= \pi\int_1^5 x^2 \, dy$

$= \pi\int_1^5 y - 1 \, dy$

$= \pi\left[\frac{1}{2}y^2 - y\right]_1^5 = 8\pi \text{ units}^3$ #