National Junior College
2016 – 2017 H2 Further Mathematics
Topic F6: Numerical Methods (Lecture Notes)

Key Questions to Answer:

- What is a continuous function?
- How to locate the roots of an equation by means of simple graphical or numerical methods?
- How to use the method of linear interpolation and Newton Raphson method to find an approximation to a root of an equation?
    - What are the cases where the method fails to converge?
- How to use iterations involving recurrence relations of the form $x_{n+1} = \mathrm{F}(x_n)$ to determine a root to a prescribed degree of accuracy?
    - What are the cases where the method fails to converge?
- How to approximate the integral of a function using the trapezium rule and Simpson's rule?
- How to approximate solutions of first order differential equations using Euler method and improved Euler method respectively?

Background

In this topic, we will learn to use various numerical methods to approximate the roots of an equation which cannot be solved exactly., for example solving $x = \cos x$. For many engineering and design problems, we cannot have an analytic solution but there is no real need for one. In such cases, all that is needed an accurate approximation to the actual solution. We term this a *numerical solution* to the problem, and its associated method a *numerical method*.

§1 Continous Function

A function f defined on $[a,b]$ is continuous if for all points $x_0 \in [a,b]$, for each $\varepsilon > 0$. there exists $\delta > 0$ such that whenever $|x-a| < \delta$, we have $|\mathrm{f}(x) - \mathrm{f}(a)| < \varepsilon$.

Informally to say, a function f defined on $[a,b]$ is continuous if it can be sketched from one end to the other with one stroke of the pen.

Can you list some functions which are continuous?

A function can fail to be continuous at a point $x = a$ for any one of the following three reasons.

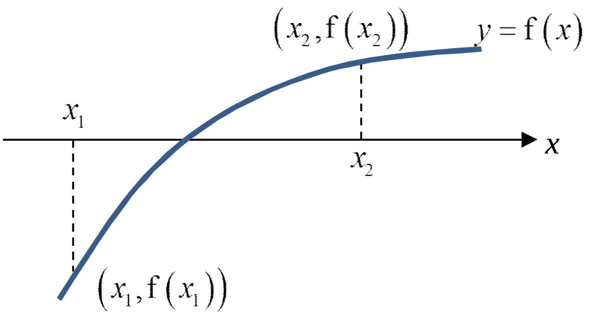
- $\mathrm{f}(a)$ does not exist
- $\lim_{x \to a} \mathrm{f}(x)$ does not exist
- $\lim_{x \to a} \mathrm{f}(x) \neq \mathrm{f}(a)$

§2 Location of Roots

Let $y = \mathrm{f}(x)$ be a function of x. The number α is said to be a solution (or root) of the equation $\mathrm{f}(x) = 0$ when α satisfies $\mathrm{f}(\alpha) = 0$.

Consider a continuous curve $y = \mathrm{f}(x)$.

If $\mathrm{f}(x_1)$ and $\mathrm{f}(x_2)$ have opposite signs, the curve must cross the $x-$axis between x_1 and x_2.



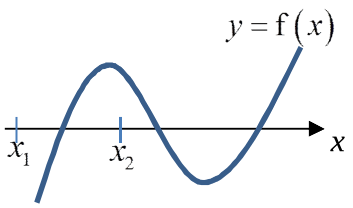
$\mathrm{f}(x_1) < 0$

$\mathrm{f}(x_2) > 0$

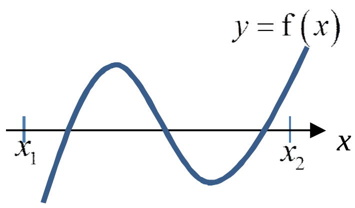
$\therefore$ There is a root between x_1 and x_2.

In general, if $\mathrm{f}(x)$ is continuous and

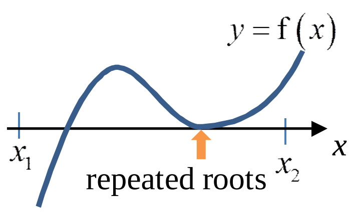
(i) $\mathrm{f}(x_1)\mathrm{f}(x_2) < 0$ (i.e. $\mathrm{f}(x_1)$ and $\mathrm{f}(x_2)$ have **opposite** signs), then the equation $\mathrm{f}(x) = 0$ has an **<u>ODD</u>** number of real roots between x_1 and x_2 (Some of which may be repeated).



1 root in $[x_1, x_2]$

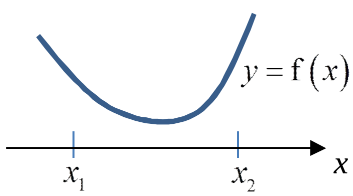


3 root in $[x_1, x_2]$

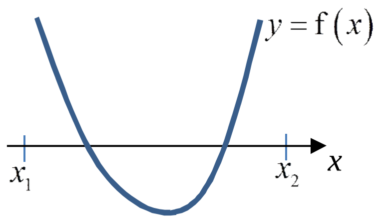


3 root in $[x_1, x_2]$

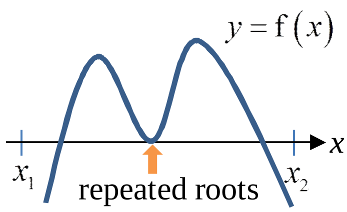
(ii) $\mathrm{f}(x_1)\mathrm{f}(x_2) > 0$ (i.e. $\mathrm{f}(x_1)$ and $\mathrm{f}(x_2)$ have the **same** sign), then the equation $\mathrm{f}(x) = 0$ has either **<u>NO</u>** real roots or an **<u>EVEN</u>** number of real roots between x_1 and x_2 (Some of which may be repeated).



No real roots in $[x_1, x_2]$



2 real roots in $[x_1, x_2]$



4 roots in $[x_1, x_2]$

Example 2.0.1

Explore the possibility of real roots for $\mathrm{f}(x)=2x^3-x^2+1$.

For $x\in(0,1]$, $1-x^2\geq 0$ and $2x^3>0$.
$\therefore\ \mathrm{f}(x)>0\ \forall x\in(0,1]$.
For $x>1$, $2x^3-x^2>0$
$\therefore\ \mathrm{f}(x)>0\ \forall x\in(1,\infty)$
Thus, $\mathrm{f}(x)>0\ \forall x\in(0,\infty)$
$\Rightarrow \mathrm{f}(x)$ has no real roots for all $x>0$.

For $x\in(-\infty,-1)$, $2x^3<0$ and $1-x^2<0$
$\therefore\ \mathrm{f}(x)<0\ \forall x\in(-\infty,-1)$
$\Rightarrow \mathrm{f}(x)$ has no real roots for all $x<-1$.
Now, $\mathrm{f}(-1)=-2<0$ and $\mathrm{f}(0)=1>0$
$\therefore\ \mathrm{f}(x)$ has either 1 or 3 real roots in $[-1,0]$

§3 Approximate Roots of Equations using Graphical Method

The real roots of an equation $\mathrm{f}(x)=0$ can be located and their approximate values found by sketching (plotting) $y=\mathrm{f}(x)$ and locating its $x-$ intercepts on graph paper.

It is often better to rearrange $\mathrm{f}(x)=0$ into the form $\mathrm{f}_1(x)=\mathrm{f}_2(x)$, where $\mathrm{f}_1(x)$ and $\mathrm{f}_2(x)$ are standard functions whose graphs can be easily sketched or plotted on graph paper. The x values of the points of intersection of the two graphs then give the roots of the equation $\mathrm{f}(x)=0$.

While graphical methods are quick, it is nevertheless a visual method that relies heavily on graphing programs. Is there a computational method that can produce such numerical approximations? Preferably, one that can even be carried out by hand (if needed).

§4 Approximate Roots of Equations Using Numerical Methods

This is the process of successive approximations whereby each approximation is used as a base for the next approximation. We must first find an interval in which the root lies (either by graphical method or by the investigation of the sign of $\mathrm{f}(x)$).

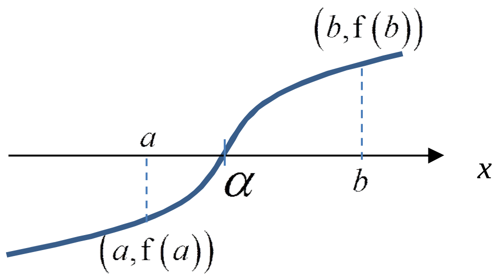
Note:
- Such methods can be used only if $\mathrm{f}(x)$ is **continuous on this interval.**
- There can be **only one** root within this interval.

In this section, we will learn the following numerical methods.
(a) Interval Bisection
(b) Linear Interpolation
(c) Newton-Raphson
(d) Iterations involving recurrence relations of the form $x_{n+1}=\mathrm{F}(x_n)$

Convergence to the roots for (a) and (b) are guaranteed. However, this is not so for (c) and (d).

4.1 Interval Bisection

Suppose the equation $\mathrm{f}(x)=0$ has only one root α in the interval (a,b). Then $\mathrm{f}(a)$ and $\mathrm{f}(b)$ must be of opposite signs.



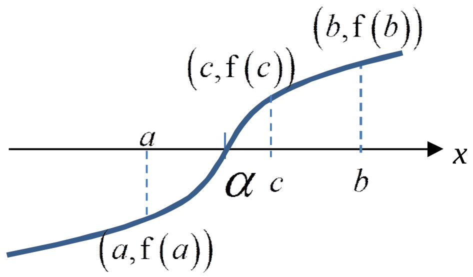
$\mathrm{f}(a)<0$

$\mathrm{f}(b)>0$

$\therefore$ There is a root between a and b.

Let $c=\dfrac{a+b}{2}$.

Case (I): If $\mathrm{f}(a)$ and $\mathrm{f}(c)$ are of opposite signs, then the root $\alpha\in(a,c)$.

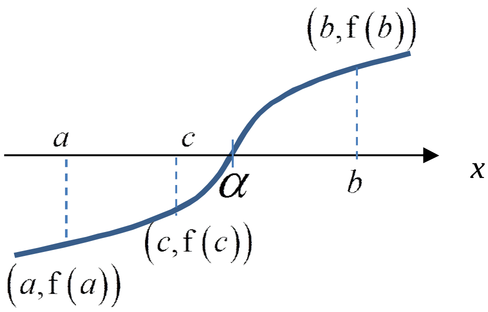


$\mathrm{f}(a)<0$

$\mathrm{f}(b)>0$

$\mathrm{f}(c)>0$

Case (II): If $\mathrm{f}(c)$ and $\mathrm{f}(b)$ are of opposite signs, then the root $\alpha\in(c,b)$.



$\mathrm{f}(a)<0$

$\mathrm{f}(b)>0$

$\mathrm{f}(c)<0$

This process of bisection is repeated until the desired degree of accuracy is obtained.

Eample 4.1.1

Find the larger root of $e^x - 3x = 0$ correct to 1 decimal place.

Solution:

Let $f(x) = e^x - 3x$

$f'(x) = e^x - 3$. $f'(x) = 0 \Rightarrow x = \ln 3 \simeq 1.1$

$f''(x) = e^x > 0 \ \forall x$

$\therefore \ x = \ln 3$ gives a minimum point.

$\Rightarrow \beta$ is the largest root.

$$\left.\begin{array}{l}\text{Now } f(1) = e - 3 < 0 \\ f(2) = e^2 - 6 > 0\end{array}\right\} \therefore \beta \in (1,2)$$

To find β :

Estimate of root	f (x)	Interval containing root
1	< 0	
2	> 0	$(1, 2)$
1.5	< 0	$(1.5, 2)$
1.75	> 0	$(1.5, 1.75)$
1.625	> 0	$(1.5, 1.625)$
1.5625	> 0	$(1.5, 1.5625)$
1.53125	> 0	$(1.5, 1.53125)$

Since $1.5 < \beta < 1.53125$, $\beta = 1.5$ (1.d.p)

Note:

(i) Convergence to the root is **guaranteed**.

(ii) This method gives a **slow convergence**, i.e., many iterations are required.

4.2 Linear Interpolation

Let $\alpha \in (a,b)$ be the solution of $f(x) = 0$. This method gives a better approximation to α by reducing the interval where α lies.

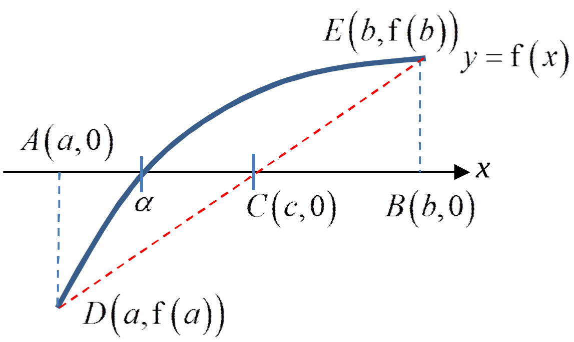
Let $A = (a,0)$

$B = (b,0)$

$C = (c,0)$

$D = (a, f(a))$

$E = (b, f(b))$



Join D to E. We see that c gives a better approximation to α.

Let $p = |\mathrm{f}(a)|, q = |\mathrm{f}(b)|$. [The aim is to find an expression for c.]

Considering triangles ACD and BCE, by similar triangles,

$$\frac{AC}{BC} = \frac{AD}{BE}$$

$$\frac{c-a}{b-c} = \frac{p}{q}$$

$$pb - pc = cq - aq$$

$$c(q+p) = pb + aq$$

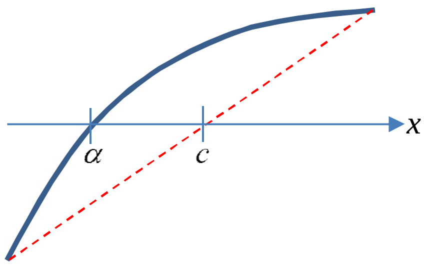
$$c = \frac{pb+aq}{q+p}$$

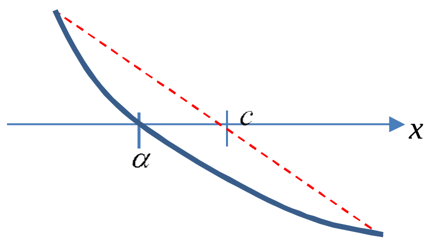
$$\text{i.e. } c = \frac{b|\mathrm{f}(a)| + a|\mathrm{f}(b)|}{|\mathrm{f}(a)| + |\mathrm{f}(b)|}.$$

Next, compute $\mathrm{f}(c)$. The sign of $\mathrm{f}(c)$ determines if α lies in (a,c) or (c,b). With this new interval, we apply the same method until the desired accuracy is obtained.

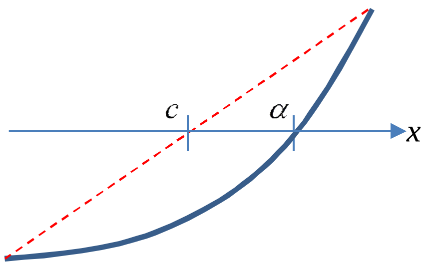
Any error in the approximation depends on the shape (**gradient and concavity**) of the curve in the interval.

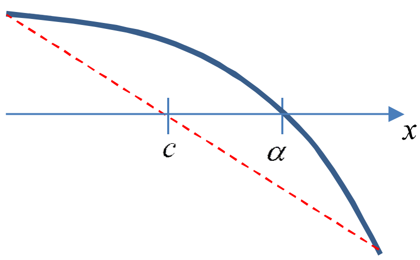
<table>
<tr><th></th><th>Positive gradient $(\mathrm{f}'(x) > 0)$</th><th>Negative gradient $(\mathrm{f}'(x) < 0)$</th></tr>
<tr><td>Overestimating

$(\mathrm{f}'(x)\mathrm{f}''(x) < 0)$</td><td>

Curve concave downwards</td><td>

Curve concave upwards</td></tr>
<tr><td>Underestimating

$(\mathrm{f}'(x)\mathrm{f}''(x) > 0)$</td><td>

Curve concave upwards</td><td>

Curve concave downwards</td></tr>
</table>

Example 4.2.1

Use linear interpolation to fnd the larger root of the equation $\ln x = x-2$ correct to 3 decimal places.

Solution:

Let $\text{f}\left(x\right)=\ln x-x+2$

$\text{f}'\left(x\right)=\frac{1}{x}-1$. When $\text{f}'\left(x\right)=0,\ x=1$.

$\text{f}''\left(x\right)=-\frac{1}{x^2}<0\ \forall x\in\mathbb{R}\setminus\left\{0\right\}$.

$\therefore\ x=1$ gives a maximum point.

Let α be the larger root $\Rightarrow \alpha>1$. To find α :

Estimate of root	f (x)	Interval containing α
3	>0	
4	<0	$\left(3,4\right)$
3.5	<0	$\left(3,3.5\right)$
$\frac{3\left\|\text{f}\left(3.5\right)\right\|+3.5\left\|\text{f}\left(3\right)\right\|}{\left\|\text{f}\left(3.5\right)\right\|+\left\|\text{f}\left(3\right)\right\|}=3.14256$ (5.d.p)	>0	$\left(3.14256,3.5\right)$
$\frac{3.14256\left\|\text{f}\left(3.5\right)\right\|+3.5\left\|\text{f}\left(3.14256\right)\right\|}{\left\|\text{f}\left(3.5\right)\right\|+\left\|\text{f}\left(3.14256\right)\right\|}=3.14611$ (5.d.p)	>0	$\left(3.14611,3.5\right)$
$\frac{3.14611\left\|\text{f}\left(3.5\right)\right\|+3.5\left\|\text{f}\left(3.14611\right)\right\|}{\left\|\text{f}\left(3.5\right)\right\|+\left\|\text{f}\left(3.14611\right)\right\|}=3.14619$ (5.d.p)	>0	$\left(3.14619,3.5\right)$

[Check if $\alpha=3.146$]

$$\left.\begin{aligned}\text{f}\left(3.1455\right)>0\\ \text{f}\left(3.1465\right)<0\end{aligned}\right\}\therefore\alpha\in\left(3.1455,3.1465\right)$$

So, $\alpha=3.146\ \left(3.\text{d.p}\right)$

Example 4.2.2

Show that the equation $x^3-5x-3=0$ has only one root between 2 and 2.5. Use linear interpolation to find an approximation to this root correct to 2 decimal places.

Solution:

Let $\mathrm{f}(x)=x^3-5x-3$.

$\left.\begin{aligned}\mathrm{f}(2)&=-5<0\\ \mathrm{f}(2.5)&=0.125>0\end{aligned}\right\}\therefore$ There is at least 1 root, α , in $(2,2.5)$

Now, $\mathrm{f}'(x)=3x^2-5>0\ \forall x\in(2,2.5)$

$\Rightarrow \mathrm{f}(x)$ is increasing in $(2,2.5)$

$\Rightarrow \mathrm{f}(x)$ cuts the $x-$axis only once in $(2,2.5)$.

$\therefore \mathrm{f}(x)$ has only 1 root in $(2,2.5)$.

Estimate of root	$\mathrm{f}(x)$	Interval containing root
2	<0	
2.5	>0	$(2,2.5)$
$\dfrac{2\lvert\mathrm{f}(2.5)\rvert+2.5\lvert\mathrm{f}(2)\rvert}{\lvert\mathrm{f}(2.5)\rvert+\lvert\mathrm{f}(2)\rvert}=2.4878$ (4.d.p)	<0	$(2.4878,2.5)$
$\dfrac{2.4878\lvert\mathrm{f}(2.5)\rvert+2.5\lvert\mathrm{f}(2.4878)\rvert}{\lvert\mathrm{f}(2.5)\rvert+\lvert\mathrm{f}(2.4878)\rvert}=2.4908$ (4.d.p)	<0	$(2.4908,2.5)$

[Check if $\alpha=2.49$]

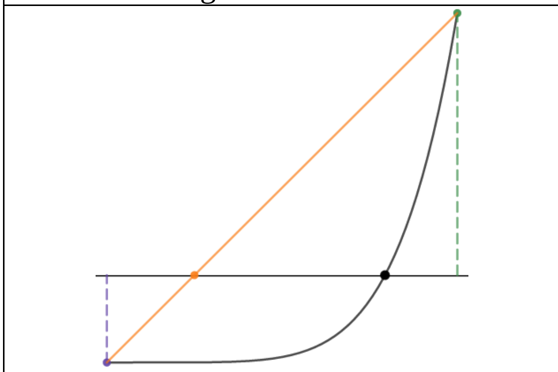
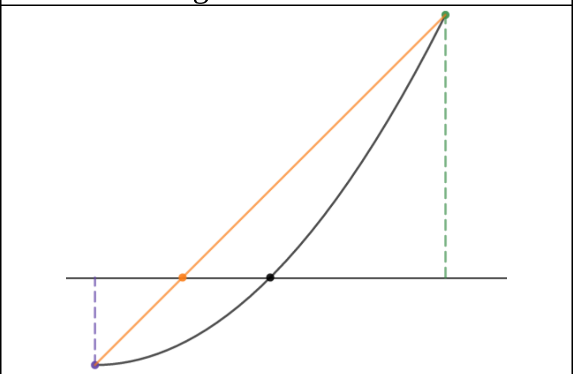
$\mathrm{f}(2.485)<0$ and $\mathrm{f}(2.494)<0 \therefore \alpha\in(2.485,2.494)$

So, $\alpha=2.49$ (2.d.p)

Note:

(i) Convergence to the root is **guaranteed**.

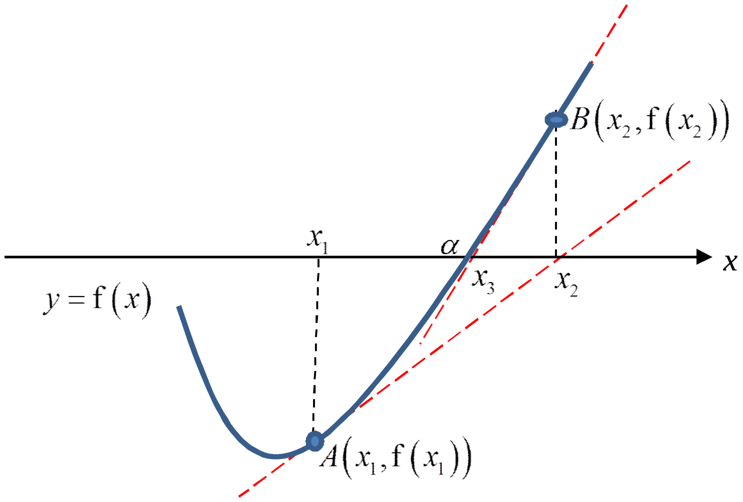
(ii) The **rate of convergence** to the root depends on how the **gradient** of the curve **changes** in (a,b) (i.e. the rate of change of $\mathrm{f}'(x)$ in (a,b)).

If $\mathrm{f}'(x)$ changes considerably, then the rate of convergence is slow.	If $\mathrm{f}'(x)$ does not change much, then the rate of convergence is fast.
	

4.3 Newton–Raphson's Method

Newton's method is the best-known procedure for finding the roots of an equation. Due to its simplicity in formula and fast convergence, it is often the first choice to approximate a solution to an equation.

Let α be the root of the equation $\mathrm{f}(x)=0$ and x_1 the first approximation to α.



Geometrically, we extrapolate the tangent line to the curve at x_1 to cut the $x-$axis at x_2. x_2 is usually closer to α than x_1.

Similarly, the tangent to the curve at x_2 cuts the $x-axis$ at x_3, which is closer to α than x_2. Repeat these iterations until x_{n+1} to achieve the desired degree of accuracy.

Equation of tangent at A: $\dfrac{y-\mathrm{f}(x_1)}{x-x_1}=\mathrm{f}'(x_1)$

i.e. $y-\mathrm{f}(x_1)=\mathrm{f}'(x_1)(x-x_1)$.

When this tangent cuts the x-axis, $y=0, x=x_2$.

$$0-\mathrm{f}(x_1)=\mathrm{f}'(x_1)(x_2-x_1)$$

$$x_2-x_1=-\frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$$

$$x_2=x_1-\frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)}$$

Repeating the same process, $x_3=x_2-\dfrac{\mathrm{f}(x_2)}{\mathrm{f}'(x_2)}$.

Thus we have the **Newton–Raphson's formula:**

$$x_{r+1}=x_r-\frac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}.$$

(This formula is given in the formula sheet MF26)

Example 4.3.1

Using the Newton–Raphson method, find the root of the equation $\sec x + x = 2$, correct to 2 decimal places, starting with the approximate value $x = 1$.

Solution:

Let $\mathrm{f}(x) = \sec x + x - 2$
$\mathrm{f}'(x) = \sec x \tan x + 1$

Let $x_1 = 1$ and α be the equation's root. By Newton – Raphson's method:

$$x_2 = x_1 - \frac{\mathrm{f}(x_1)}{\mathrm{f}'(x_1)} = 1 - \frac{\mathrm{f}(1)}{\mathrm{f}'(1)} = 0.7809 \text{ (4.d.p)}$$

$$x_3 = 0.7809 - \frac{\mathrm{f}(0.7809)}{\mathrm{f}'(0.7809)} = 0.7021 \text{ (4.d.p)}$$

$$x_4 = 0.7021 - \frac{\mathrm{f}(0.7021)}{\mathrm{f}'(0.7021)} = 0.6965 \text{ (4.d.p)}$$

[Check if $\alpha = 0.70$]

$$\left.\begin{aligned} \mathrm{f}(0.695) < 0 \\ \mathrm{f}(0.705) > 0 \end{aligned}\right\} \therefore \alpha \in (0.695, 0.705)$$

$\therefore \alpha = 0.70$ (2.d.p)

Example 4.3.2

Use the Newton – Raphson's method to obtain an approximation to the p^{th} root of any number a. Hence determine $\sqrt{30}$ to 3 decimal places.

Solution:

Let α be the p^{th} root of a, i.e. $\alpha = a^{\frac{1}{p}}$, or $\alpha^p = a$.
Let $\mathrm{f}(x) = x^p - a \Rightarrow \mathrm{f}'(x) = px^{p-1}$. By Newton – Raphson's method,

$$\begin{aligned} x_{r+1} &= x_r - \frac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)} \\ &= x_r - \frac{x_r^p - a}{px_r^{p-1}} \\ &= \frac{1}{px_r^{p-1}}\left[px_r^p - x_r^p + a\right] \\ &= \frac{1}{px_r^{p-1}}\left[a + (p-1)x_r^p\right]. \end{aligned}$$

To determine $\sqrt{30}$, let $\alpha = 30, p = 2$.

Then $x_{r+1} = \frac{1}{2x_r}[30 + x_r^2]$.

Observe that $25 < 30 < 36 \Rightarrow 5 < \sqrt{30} < 6$.

Let $x_1 = 5.5$ be the first approximation.

$$x_2 = \frac{1}{2x_1}\left(30 + x_1^2\right) = \frac{1}{2(5.5)}\left(30 + 5.5^2\right) = 5.47727 \text{ (5.d.p)}$$

$$x_3 = \frac{30 + 5.47727^2}{2(5.47727)} = 5.47723 \text{ (5.d.p)}$$

[Check if $\sqrt{30} = 5.477$]

$$\left.\begin{array}{l} \mathrm{f}(5.4765) < 0 \\ \mathrm{f}(5.4775) > 0 \end{array}\right\} \therefore \sqrt{30} \in (5.4765, 5.4775)$$

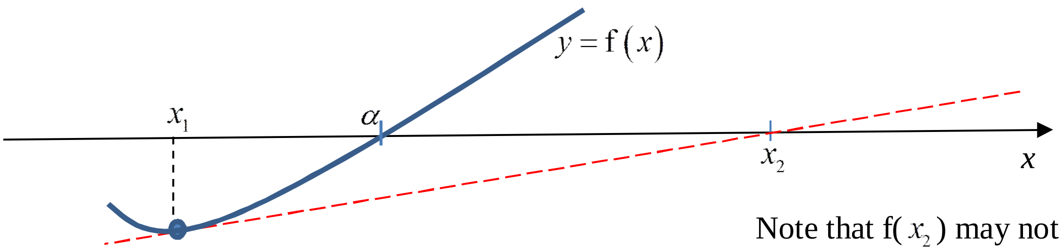
$\therefore \sqrt{30} = 5.477$ (3.d.p)

Newton-Raphson Method Can FAIL!!

The rate of convergence using Newton-Raphson's method depends on the first approximation and the shape of the curve in the neighbourhood of the root.

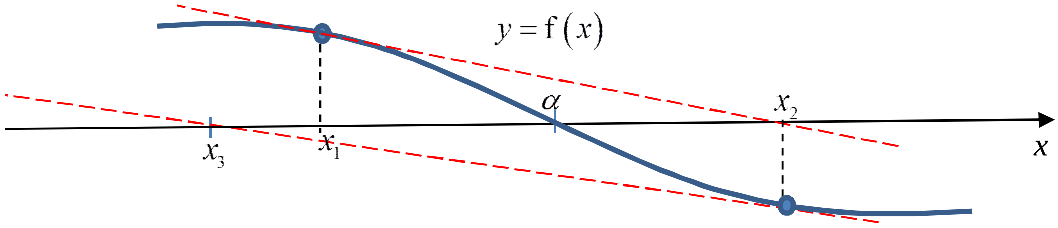
Newton-Raphson's method fails when

(I) $\mathrm{f}'(x_1)$ is almost zero, i.e., gradient at x_1 is too gentle.

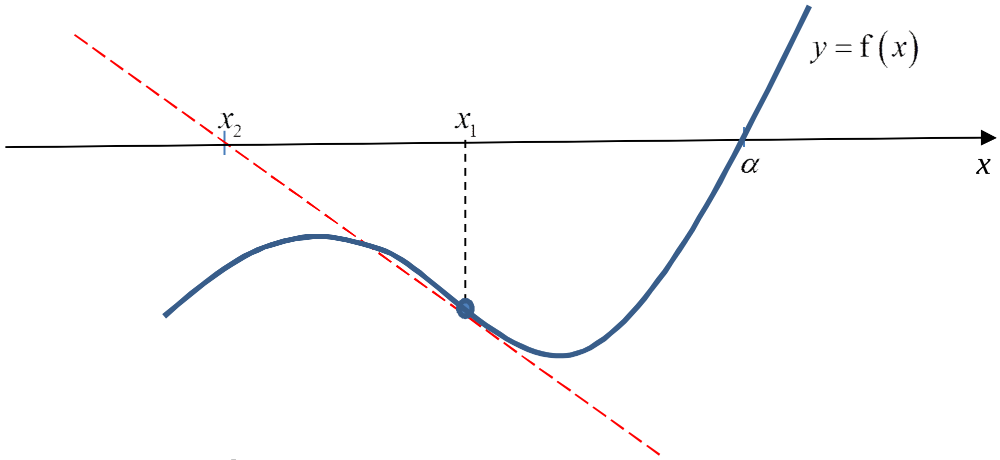


Note that f(x_2) may not be defined or on the same piece

(II) There is a non-stationary/stationary point of inflexion near α.

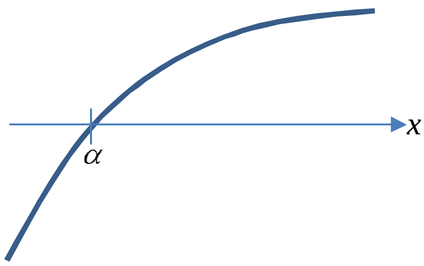
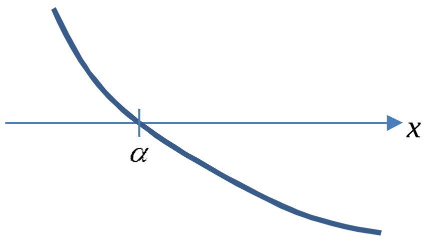
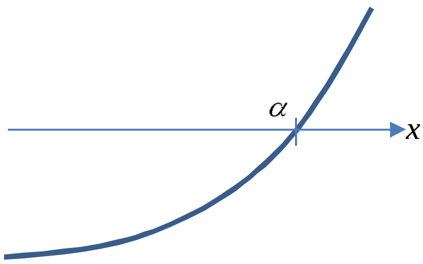
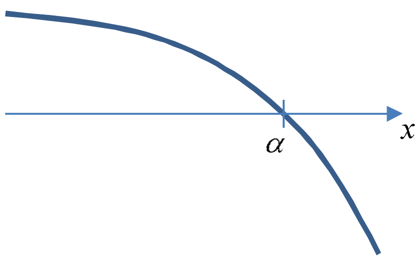


(III) x_1 is too far from α (or there is a turning point between x_1 and α).



(IV) It converges to another root.

Any error in this method again depends on the shape **(gradient and concavity)** of the curve.

	Positive gradient $(\mathrm{f}'(x)>0)$	Negative gradient $(\mathrm{f}'(x)<0)$
Underestimating $(\mathrm{f}'(x)\mathrm{f}''(x)<0)$	 Curve concave downwards	 Curve concave upwards
Overestimating $(\mathrm{f}'(x)\mathrm{f}''(x)>0)$	 Curve concave upwards	 Curve concave downwards

Example 4.3.3

Let $\mathrm{f}\left(x\right)=-\mathrm{e}^{-3x}+x-1$.

(i) Sketch the appropriate graphs to show that the equation $\mathrm{f}\left(x\right)=0$ has exactly one real root α. Find the interval $\left(n,n+1\right)$, where n is an integer, in which α lies.

(ii) Use the method of linear interpolation once on this interval to obtain an estimate c, of the root α to 5 decimal places.

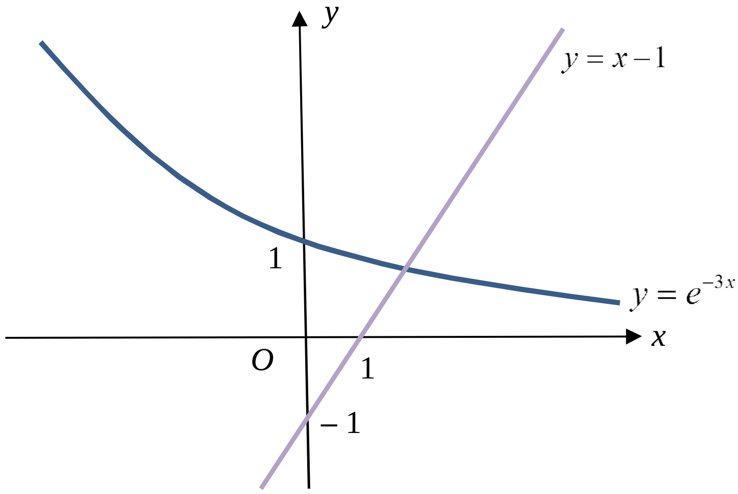
(iii) Taking c as the initial value, apply the Newton-Raphson method to estimate the value of α correct to 3 decimal places.

(iv) Find $\mathrm{f}'\left(x\right)$ and $\mathrm{f}''\left(x\right)$ in terms of x. By considering the signs of $\mathrm{f}'\left(x\right)$ and $\mathrm{f}''\left(x\right)$ for all real values of x, explain why the linear interpolation method gives an overestimate of α.

Does the Newton-Raphson method also give an overestimate? Explain.

Solution:

(i) Sketch $y=\mathrm{e}^{-3x}$ and $y=x-1$.



From sketch, $\mathrm{f}\left(x\right)=0$ has only 1 real root α.

$$\left.\begin{array}{l}\mathrm{f}\left(1\right)=-0.0497871<0\\ \mathrm{f}\left(2\right)=0.9975212>0\end{array}\right\}\therefore \alpha\in\left(1,2\right),\text{ so } n=1.$$

(ii) Using linear interpolation on $\left(1,2\right)$,

$$c=\frac{\left|\mathrm{f}\left(2\right)\right|+2\left|\mathrm{f}\left(1\right)\right|}{\left|\mathrm{f}\left(2\right)\right|+\left|\mathrm{f}\left(1\right)\right|}=1.04754\ \text{(5.d.p)}$$

(iii) By Newton-Raphson's method, $\mathrm{f}'\left(x\right)=3\mathrm{e}^{-3x}+1$.

By GC,

$x_1 = 1.04754$

$x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)} = 1.04367$ (5.d.p)

$x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)} = 1.04367$ (5.d.p)

[Check if $\alpha = 1.044$]

$\left.\begin{array}{l} f(1.0435) < 0 \\ f(1.0445) > 0 \end{array}\right\} \therefore \alpha \in (1.0435, 1.0445)$

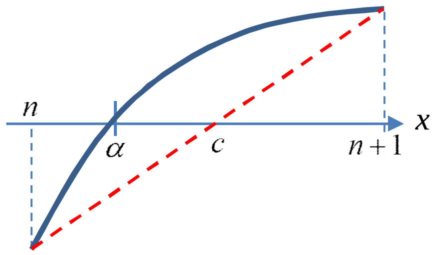
So, $\alpha = 1.044$ (3.d.p)

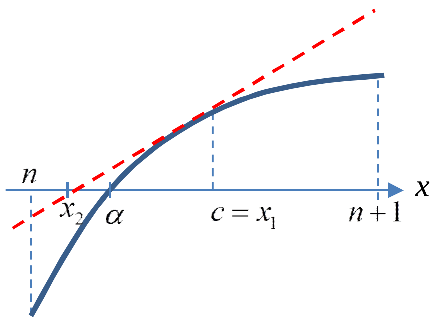
(iv) $f'(x) = 3e^{-3x} + 1,\ f''(x) = -9e^{-3x}$.

Now, $f'(x) > 0$ for all $x \Rightarrow f(x)$ is an increasing function

$f''(x) < 0$ for all $x \Rightarrow f(x)$ concave downwards.

$\therefore$ linear interpolation gives an overestimation.



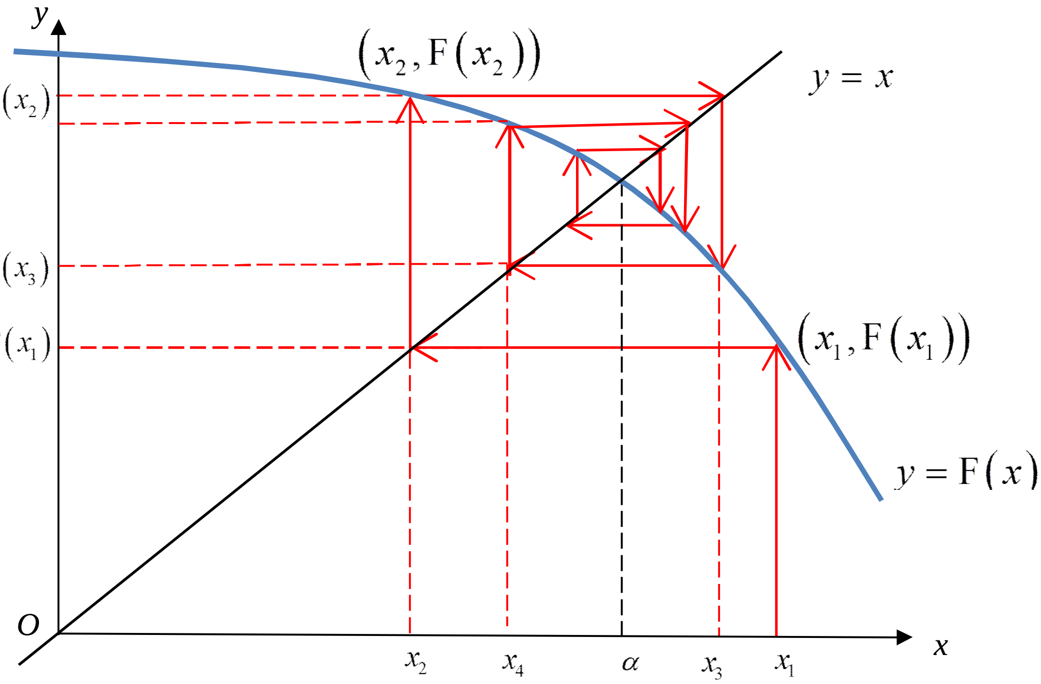


Note that tangent at $x = c$ cuts the x-axis on the left of α.

$\Rightarrow$ Newton-Raphson method gives an underestimation

4.4 Iterative formula of the recurrence relations of the form: $x_{n+1} = \mathrm{F}(x_n)$

Let $\alpha \in (a,b)$ be the root of the equation $\mathrm{f}(x) = 0$. The equation can be re-written in the form $x = \mathrm{F}(x)$ where $\mathrm{F}(x)$ is another expression of x. Then α is the value of x at the point of intersection of $y = x$ and $y = \mathrm{F}(x)$.



Let x_1 be the first approximation to the root α.

Using the iterative formula $x_{n+1} = \mathrm{F}(x_n)$, we obtain $x_2 = \mathrm{F}(x_1)$.

From the above diagram, we see that x_2 is closer to α than is x_1. i.e. x_2 is a better approximation to α.

Applying the iterative formula again, we obtain $x_3 = \mathrm{F}(x_2)$, where x_3 is a better approximation to α than is x_2 .We can repeat the above process until the required degree of accuracy is achieved.

Example 4.4.1

Show that the equation $x = \ln(8 - x)$ has a root between 1 and 2. Calculate this root correct to 3 decimal places using the formula $x_{n+1} = \ln(8 - x_n)$.

Solution:

Let $\text{f}(x) = \ln(8-x) - x.$

$$\left.\begin{aligned}\text{f}(1) &= \ln 7 - 1 \simeq 0.946 > 0 \\ \text{f}(2) &= \ln 6 - 2 \simeq -0.208 < 0\end{aligned}\right\} \therefore \text{ f}(x) \text{ has at least 1 root in } (1,2).$$

Now, $\text{f}'(x) = \dfrac{-1}{8-x} - 1 < 0 \ \forall x \in (1,2)$

$\Rightarrow \text{f}(x)$ is decreasing in $(1,2)$

$\Rightarrow \text{f}(x)$ cuts x-axis only once in $(1,2)$

Let α be the equation's root and $x_1 = 1$ be the first approximation to α. By GC,

$x_1 = 1$

$x_2 = \text{F}(x_1) = \text{F}(1) = \ln 7 = 1.94591$ (5.d.p)

$x_3 = \ln(8 - 1.94591) = 1.80073$ (5.d.p)

$x_4 = \ln(8 - 1.80073) = 1.82443$ (5.d.p)

$x_5 = \ln(8 - 1.82443) = 1.82060$ (5.d.p)

$x_6 = \ln(8 - 1.82060) = 1.82122$ (5.d.p)

[Check if $\alpha = 1.821$]

$$\left.\begin{aligned}\text{f}(1.8205) > 0 \\ \text{f}(1.8215) < 0\end{aligned}\right\} \therefore \alpha \in (1.8205, 1.8215)$$

So, $\alpha = 1.821$ (3.d.p)

Example 4.4.2

Use the iterative formula $x_{n+1} = \frac{1}{5}\left(1 - x_n^2\right)$, where $x_1 = 0$, to find the positive root of $x^2 + 5x - 1 = 0$, correct to 4 decimal places. Discuss, with the aid of a graph, the behaviours of the sequence $x_1, x_2, x_3, \ldots$ when **(i)** $x_1 = -6$ **(ii)** $x_1 = -5$.

Solution:

$$\left.\begin{aligned}x^2 + 5x - 1 &= 0 \\ 5x &= 1 - x^2 \\ x &= \frac{1}{5}\left(1 - x^2\right)\end{aligned}\right\} \therefore \text{ Iterative formula is } x_{n+1} = \frac{1}{5}\left(1 - x_n^2\right) \quad \left[\text{F}(x)\right]$$

Let $\text{f}(x) = x^2 + 5x - 1$ and α be the equation's root.

$x_1 = 0$

$x_2 = \mathrm{F}(0) = \frac{1}{5}(1) = 0.2$

$x_3 = \frac{1}{5}\left(1 - 0.2^2\right) = 0.192$

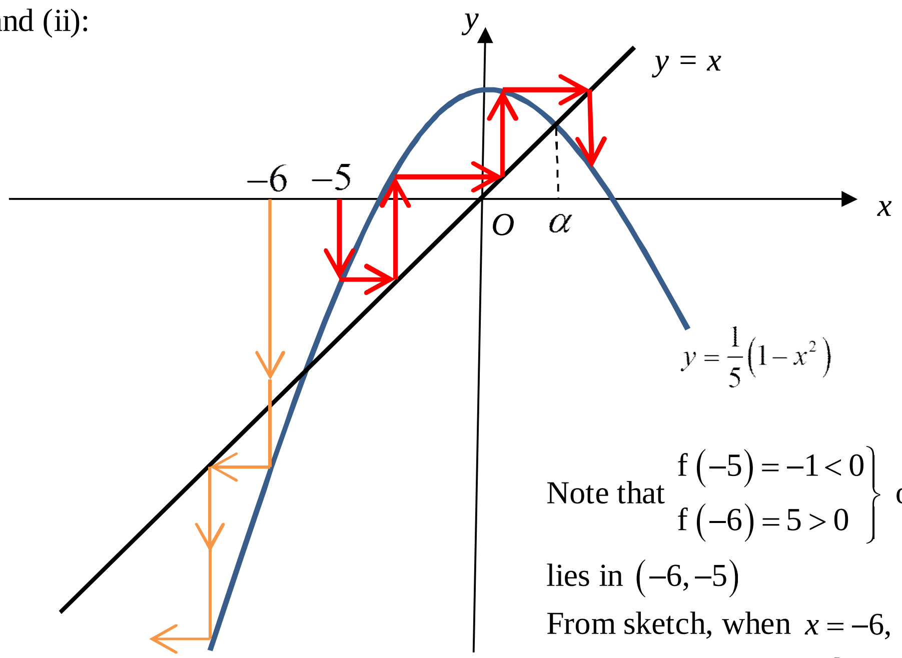
$x_4 = \frac{1}{5}\left(1 - 0.192^2\right) = 0.1926272$

$x_5 = \frac{1}{5}\left(1 - 0.1926272^2\right) = 0.192579$ (6.d.p)

[Check if $\alpha = 0.1926$]

$$\left.\begin{aligned} \mathrm{f}(0.19255) < 0 \\ \mathrm{f}(0.19265) > 0 \end{aligned}\right\} \therefore \alpha \in (0.19255, 0.19265) \Rightarrow \alpha = 0.1926$$

(i) and (ii):



Note that $\left.\begin{aligned} \mathrm{f}(-5) = -1 < 0 \\ \mathrm{f}(-6) = 5 > 0 \end{aligned}\right\}$ other root lies in $(-6, -5)$

From sketch, when $x = -6$, the sequence $x_1, x_2, x_3, \ldots$ diverges from the 2 roots.

When $x = -5$, the sequence $x_1, x_2, x_3, \ldots$ converges to the positive root α.

Note:

(i) **Convergence** using the simple iterative formula $x_{n+1} = \mathrm{F}(x_n)$ is **NOT guaranteed**.

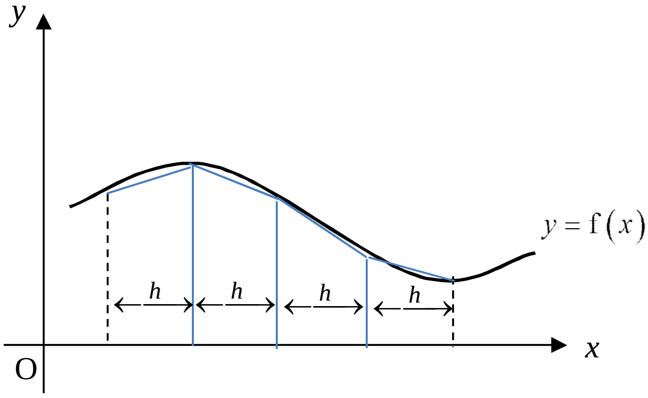
(ii) A given equation $\mathrm{f}(x) = 0$ may give rise to **several different arrangements** of the form $x = \mathrm{F}(x)$, but not all the corresponding iterative processes will converge to the required root.

(iii) **In general, the iteration $x_{n+1} = \mathrm{F}(x_n)$ may converge to a root α if** $\left|\mathrm{F}'(\alpha)\right| < 1$

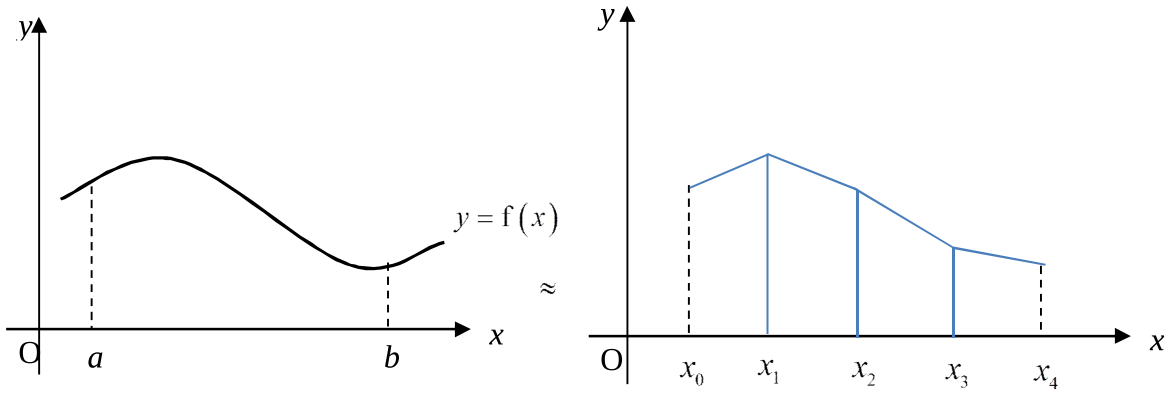
§5 Approximattion of Integral of a Continuous Function Using Trapezium Rule and Simpson's Rule

In modern times, we look at the calculation of an area bounded by non-straight lines using definite integrals. When the going gets tough for finding the antiderivative of the integrand, usually a numerical method is called for to estimate the value of the definite integral. We will now look at two methods for finding the approximate value of an area bounded partly by a curve and the *x*-axis.

5.1 Trapezium Rule



If the area represented by $\int_a^b f(x)\ dx$ is divided into strips, each of width *h*, as shown above, then each such strip is approximately a trapezium. Using the sum of the areas of these strips as an approximation for the actual value of the area, we have



i.e.

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(x_0)+f(x_1)\right)+\frac{h}{2}\left(f(x_1)+f(x_2)\right)+\frac{h}{2}\left(f(x_2)+f(x_3)\right)+\frac{h}{2}\left(f(x_3)+f(x_4)\right)$$

$$\simeq \frac{h}{2}\left[f(x_0)+2\left(f(x_1)+f(x_2)+f(x_3)\right)+f(x_4)\right]$$

This method is known as trapezium rule and here we used it with five ordinates.

Note that the ordinates must be evenly spread out (i.e. the width of all strips must be the same).

Generalising, used with *n* ordinates and let the closed interval $\left[a,b\right]$ be partitioned into equal intervals of width $h=\dfrac{b-a}{n-1}$ as follows:

$$a=x_0<x_1<\ldots<x_{n-1}=b\,.$$

The trapezium rule is

$$\int_a^b \mathrm{f}\left(x\right)\,\mathrm{d}x \approx \frac{h}{2}\sum_{k=0}^{n-1}\left(\mathrm{f}\left(x_k\right)+\mathrm{f}\left(x_{k-1}\right)\right)$$
$$=\frac{h}{2}\left[\mathrm{f}\left(x_0\right)+2\left\{\mathrm{f}\left(x_1\right)+\mathrm{f}\left(x_2\right)+\ldots+\mathrm{f}\left(x_{n-2}\right)\right\}+\mathrm{f}\left(x_{n-1}\right)\right]$$

Example 5.1.1

Use the trapezium rule, with five ordinates, to evaluate $\int_0^{0.8} \mathrm{e}^{x^2}\,\mathrm{d}x$.

Solution:

Let $\mathrm{f}\left(x\right)=\mathrm{e}^{x^2}$.

We need to partition the closed interval $\left[0,0.8\right]$ into equal intervals of width $h=\dfrac{0.8-0}{5-1}=0.2$

.

Thus $\mathrm{f}\left(x_0\right)=\mathrm{f}\left(0\right)=1,$

$\mathrm{f}\left(x_1\right)=\mathrm{f}\left(0.2\right)=\mathrm{e}^{0.2^2}=\mathrm{e}^{0.04},$

$\mathrm{f}\left(x_2\right)=\mathrm{f}\left(0.4\right)=\mathrm{e}^{0.4^2}=\mathrm{e}^{0.16},$

$\mathrm{f}\left(x_3\right)=\mathrm{f}\left(0.6\right)=\mathrm{e}^{0.6^2}=\mathrm{e}^{0.36},$

$\mathrm{f}\left(x_4\right)=\mathrm{f}\left(0.8\right)=\mathrm{e}^{0.8^2}=\mathrm{e}^{0.64}.$

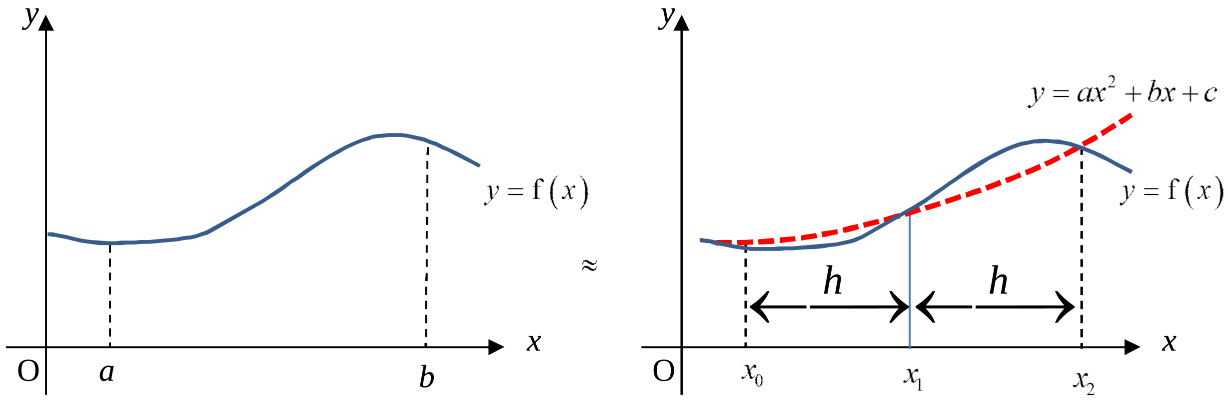
Using trapezium rule,

$$\therefore \int_0^{0.8} \mathrm{e}^{x^2}\,\mathrm{d}x \approx \frac{0.2}{2}\left[\mathrm{f}\left(x_0\right)+2\left\{\mathrm{f}\left(x_1\right)+\mathrm{f}\left(x_2\right)+\mathrm{f}\left(x_3\right)\right\}+\mathrm{f}\left(x_4\right)\right]$$
$$=0.1\left[1+2\left\{\mathrm{e}^{0.04}+\mathrm{e}^{0.16}+\mathrm{e}^{0.36}\right\}+\mathrm{e}^{0.64}\right]$$
$$=1.0191783$$
$$\approx 1.02\ \left(\text{to 3 significant figures}\right)$$

Can you sketch the region represented by $\int_0^{0.8} \mathrm{e}^{x^2}\,\mathrm{d}x$ and its approximated area using the trapezium rule on the same diagram?

5.2 Simpson's Rule

Suppose that the area represented by $\int_a^b \mathrm{f}(x)\ \mathrm{d}x$ is divided by the ordinates $\mathrm{f}(x_0)$, $\mathrm{f}(x_1)$, $\mathrm{f}(x_2)$ into two strips each of width h as shown below. A particular parabola, $y = ax^2 + bx + c$, can be found passing through the three points with the same x-coordinates. Simpson's rule uses the area under that parabola as an approximation for the value of the area under the curve $y = \mathrm{f}(x)$.



$$\text{i.e. } \int_a^b \mathrm{f}(x)\ \mathrm{d}x \approx \int_{x_1-h}^{x_1+h} ax^2 + bx + c\ \mathrm{d}x \qquad \text{---}(1)$$

If $y = ax^2 + bx + c$ is the parabola through the ordinates as shown above, then $(x_1 - h, \mathrm{f}(x_0))$, $(x_1, \mathrm{f}(x_1))$, $(x_1 + h, \mathrm{f}(x_2))$ are on this parabola.,

i.e.

$$\mathrm{f}(x_0) = a(x_1 - h)^2 + b(x_1 - h) + c \qquad \text{---}(2)$$

$$\mathrm{f}(x_1) = a(x_1)^2 + b(x_1) + c \qquad \text{---}(3)$$

$$\mathrm{f}(x_2) = a(x_1 + h)^2 + b(x_1 + h) + c \qquad \text{---}(4)$$

Now the area represented by $\int_{x_1-h}^{x_1+h} ax^2 + bx + c\ \mathrm{d}x$

$$= \frac{a}{3}\left[(x_1+h)^3 - (x_1-h)^3\right] + \frac{b}{2}\left[(x_1+h)^2 - (x_1-h)^2\right] + c\left[(x_1+h) - (x_1-h)\right]$$

which simplifies to

$$\frac{h}{3}\left[2a\{(x_1+h)^2 + (x_1+h)(x_1-h) + (x_1-h)^2\} + 3b(2x_1) + 6c\right]$$

Then using (2), (3) and (4) we find that

$$\int_{x_1-h}^{x_1+h} ax^2 + bx + c\ \mathrm{d}x = \frac{h}{3}\left(\mathrm{f}(x_0) + 4\mathrm{f}(x_1) + \mathrm{f}(x_2)\right).$$

From (1),

$$\int_a^b \mathrm{f}(x)\ \mathrm{d}x \approx \frac{h}{3}\left(\mathrm{f}(x_0) + 4\mathrm{f}(x_1) + \mathrm{f}(x_2)\right).$$

This argument can be extended to cover any even number of strips, i.e. any odd number of ordinates.

Hence Simpson's Rule, with $(2n+1)$ ordinates, is

$$\int_a^b \mathrm{f}(x)\ \mathrm{d}x \approx \frac{h}{3}\left(\mathrm{f}(x_0)+4\mathrm{f}(x_1)+2\mathrm{f}(x_2)+4\mathrm{f}(x_3)+2\mathrm{f}(x_4)+\cdots++4\mathrm{f}(x_{2n-1})+\mathrm{f}(x_{2n})\right)$$
.

Note that the use of Simpson's Rule requires an odd number of ordinates. For ease of computation, the ordinates used can be arranged in the form

$$\frac{h}{3}\left(1\text{st}+\text{last}+4\left(2\text{nd}+4\text{th}+\ldots\right)+2\left(3\text{rd}+5\text{th}+\ldots\right)\right)$$

$$\text{or}\quad \frac{h}{3}\left(\mathrm{f}(x_0)+\mathrm{f}(x_{2n})+4\sum_{r=1}^{2n}\mathrm{f}(x_{2r-1})+2\sum_{r=1}^{2n-2}\mathrm{f}(x_{2r})\right).$$

Example 5.2.1

Use Simpson's Rule with five ordinates to find an approximate value for

$$\int_0^{\pi} \sqrt{\sin\theta}\ \mathrm{d}\theta .$$

Solution:

Taking five ordinates from $\theta = 0$ to $\theta = \pi$ gives four strips each of width $\frac{\pi}{4}$.

Thus

$$\mathrm{f}(x_0)=\mathrm{f}(0)=\sqrt{\sin 0}=0,$$

$$\mathrm{f}(x_1)=\mathrm{f}\left(\frac{\pi}{4}\right)=\sqrt{\sin\frac{\pi}{4}}=2^{-1/4},$$

$$\mathrm{f}(x_2)=\mathrm{f}\left(\frac{\pi}{2}\right)=\sqrt{\sin\frac{\pi}{2}}=1,$$

$$\mathrm{f}(x_3)=\mathrm{f}\left(\frac{3\pi}{4}\right)=\sqrt{\sin\frac{3\pi}{4}}=2^{-1/4},$$

$$\mathrm{f}(x_4)=\mathrm{f}(\pi)=\sqrt{\sin\pi}=0.$$

Hence, using Simpson's Rule,

$$\begin{aligned}\int_0^{\pi} \sqrt{\sin\theta}\ \mathrm{d}\theta &\approx \left(\frac{1}{3}\right)\left(\frac{\pi}{4}\right)\left[\mathrm{f}(x_0)+\mathrm{f}(x_4)+4\left(\mathrm{f}(x_1)+\mathrm{f}(x_3)\right)+2\mathrm{f}(x_2)\right]\\ &=\frac{\pi}{12}\left[0+0+4\left(2^{-1/4}+2^{-1/4}\right)+2(1)\right]\\ &=2.284768109\\ &\approx 2.28\ \ (3\text{ significant figures})\end{aligned}$$

§6 Approximation of Solutions of First Order Differential Equations Using Euler's Method

While it is usually desirable to be able to obtain exact or analytic solutions to a differentiation equation, it is not always possible to do so. In fact, there are many differentiation equations (related to important problems in the real world) which cannot be solved exactly.

<u>So what do we do when faced with a differential equation that we cannot solve?</u>
The answer depends on what you are looking for. If you are only looking for long-term behaviour of a solution, you can always sketch a slope field. The problem with this approach is that it is only really good for getting general trends in solutions. There are times when we will need something more. For instance, we may need to determine how a specific solution behaves or to find some values that the solution will take.

In these cases we resort to *numerical methods* that will allow us to approximate solutions to differential equations. There are many different methods that can be used to approximate solutions to a differential equation. In this section, we are going to look at one of the oldest and easiest method, and apply it to initial value problems. This method is called the Euler's method, named after the Swiss mathematician Leonhard Euler. In addition, we will look at the improved Euler's method, which is essentially an improvement from the Euler's method.

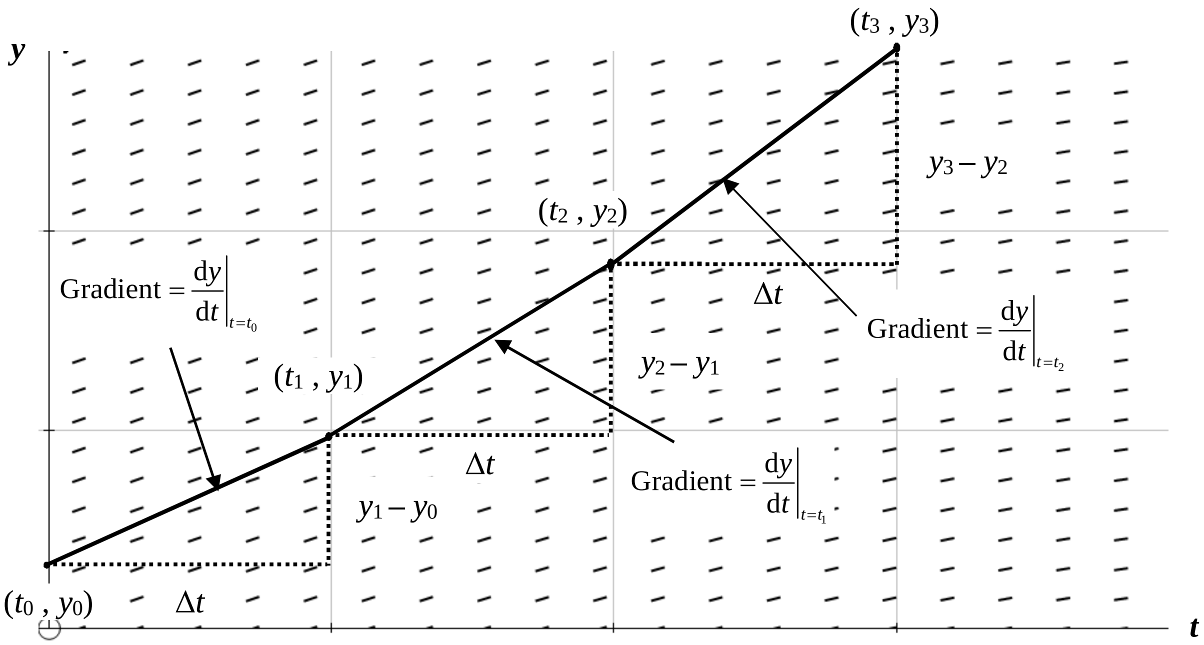
6.1 Principle of the Euler's Method

The key principle in Euler's method is the use of a linear approximation for the tangent to the solution curve. This is closely related to the concept of slope fields for differential equations.

Given an initial value problem (i.e. we know the values of the starting point / initial conditions),

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{f}(t,\ y), \qquad y(t_0) = y_0$$

we start at (t_0, y_0) and take small steps along the t-axis, guided by the tangents to the slope field. This is illustrated in Figure 7.1 in which the "small step" is of value Δt units.



We calculate the value of the slope of the slope field at the point (t_0, y_0) denoted by $\left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t_0}$ first.

We then draw a line segment with this slope extending from the point (t_0, y_0) to the point (t_1, y_1), where $t_1 = t_0 + \Delta t$. We get

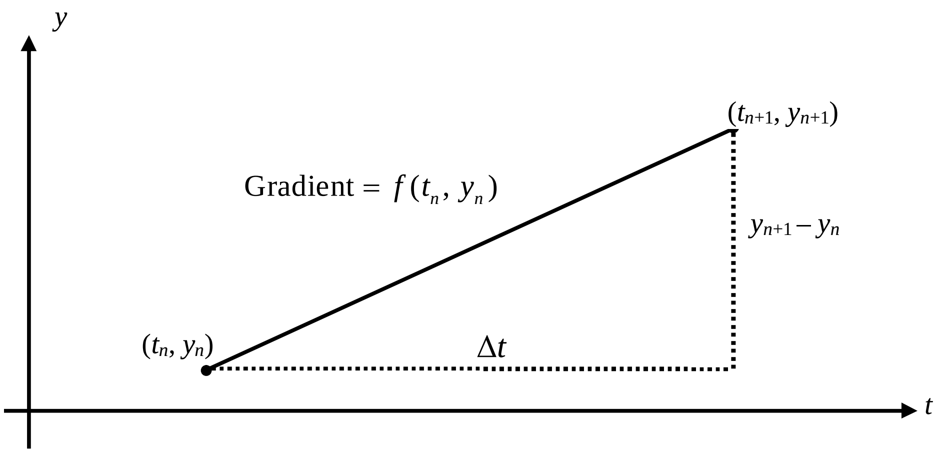
$$\frac{y_1 - y_0}{t_1 - t_0} = \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=t_0} \Rightarrow \frac{y_1 - y_0}{\Delta t} = \mathrm{f}\left(t_0, y_0\right) \Rightarrow y_1 = y_0 + \Delta t \cdot \mathrm{f}\left(t_0, y_0\right)$$

The value of y_1 calculated here **estimates** the actual value of y_1.

We can repeat the procedure to find the estimate of y_2 which is another Δt interval away. We calculate the value of the slope of the slope field at the point (t_1, y_1) and then draw a line segment with this slope extending from point (t_1, y_1) to the point (t_2, y_2), where $t_2 = t_1 + \Delta t$. We get

$$\frac{y_2 - y_1}{t_2 - t_1} = \left.\frac{\mathrm{d}y}{\mathrm{d}t}\right|_{t=1} \Rightarrow \frac{y_2 - y_1}{\Delta t} = \mathrm{f}\left(t_1, y_1\right) \Rightarrow y_2 = y_1 + \Delta t \cdot \mathrm{f}\left(t_1, y_1\right)$$

We can continue in a similar fashion to find estimates of y at intervals of Δt apart, or for a pre-determined value of t, i.e. $y_3 = y_2 + \Delta t \cdot \mathrm{f}\left(t_2, y_2\right)$, $y_4 = y_3 + \Delta t \cdot \mathrm{f}\left(t_3, y_3\right)$, etc.



Straight line segment joining (t_n, y_n) and (t_{n+1}, y_{n+1})

In general, if we choose a step size of Δt, then $\Delta t = t_{n+1} - t_n$. Referring to Figure 7.2, suppose that the value of y at t_n is known or already determined, we can find the value of y at the next time level (i.e. at t_{n+1}) by assuming that the line joining (t_n, y_n) and (t_{n+1}, y_{n+1}) has the same gradient as the line segment of the slope field at (t_n, y_n). Thus, we have

$$\frac{y_{n+1} - y_n}{t_{n+1} - t_n} = \frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{f}\,(t_n, y_n),$$

which gives the relationship

$$\boxed{y_{n+1} = y_n + \Delta t \cdot \mathrm{f}\,(t_n, y_n)}$$

which is the result for the Euler's method.

We call Δt the ***step size*** of the method. Each computation of an estimate is termed an *iteration*, and the estimates found are called *iterates*. The value of the step size will have an impact on the number of iterations required to reach the desired value of t. We shall discuss this using Example 6.1.1.

(The formula for Euler Method with step size h is given in MF26.)

Example 6.1.1

Consider the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 1,\ \ y(0) = 1. \text{ -------} \quad (1)$$

Apply Euler's method to obtain an approximation for $y(1)$ using step sizes $\Delta t = 0.2$ and $\Delta t = 0.1$. Plot all intermediate approximations on a graph and compare them with the exact solution.

Here $\mathrm{f}(t, y) = 2y - 1$, so Euler's method's formula will take the form

$$y_{n+1} = y_n + \Delta t \cdot (2y_n - 1)$$

When $\Delta t = 0.2$, and starting with $t_0 = 0$, $y_0 = 1$, the first three approximate values are

$$y_1 = 1 + (0.2) \cdot [2(1) - 1] = 1.2\,,$$
$$y_2 = 1.2 + (0.2) \cdot [2(1.2) - 1] = 1.48\,,$$
$$y_3 = 1.48 + (0.2) \cdot [2(1.48) - 1] = 1.872\,.$$

The rest of the iterations are represented in Table 1.

Thus, using a step size of 0.2, the approximate value for $y(1) = 3.1891$.

n	t_n	$y_{n,\,approx}$
0	0	1
1	0.2	1.2000
2	0.4	1.4800
3	0.6	1.8720
4	0.8	2.4208
5	1.0	3.1891

Table 1

When $\Delta t = 0.1$, and starting with $t_0 = 0$, $y_0 = 1$, the first three approximate values are

$$y_1 = 1 + (0.1) \cdot [2(1) - 1] = 1.1,$$
$$y_2 = 1.1 + (0.1) \cdot [2(1.1) - 1] = 1.22,$$
$$y_3 = 1.22 + (0.1) \cdot [2(1.22) - 1] = 1.364.$$

The rest of the iterations are represented in Table 2.

Thus, using a step size of 0.1, the approximate value for $y(1) = 3.5959$.

(Notice that when we half the step size, we double the number of iterations required.)

n	t_n	$y_{n,\,approx}$
0	0	1
1	0.1	1.1000
2	0.2	1.2200
3	0.3	1.3640
4	0.4	1.5368
5	0.5	1.7442
6	0.6	1.9930
7	0.7	2.2916
8	0.8	2.6499
9	0.9	3.0799
10	1.0	3.5959

Table 2

The "variable separable" method can be used to solve (1) exactly.

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 1 \quad \Rightarrow \quad \int \frac{1}{2y-1}\,\mathrm{d}y = \int 1\ \mathrm{d}t \quad \Rightarrow \quad \frac{1}{2}\ln|2y-1| = t + C \quad \Rightarrow \quad y = \frac{Be^{2t}+1}{2}$$

When $t = 0$, $y = 1$, we get $B = 1$,

$$\therefore\ y = \frac{\mathrm{e}^{2t}+1}{2}$$

Tables 3 and 4 show the values of the estimates compared with the actual values of y for both step sizes of 0.2 and 0.1 respectively. The error between the estimated and actual value are also calculated where

Absolute Error = $\left|y_{n,actual} - y_{n,approx}\right|$ and Percentage Error = $\dfrac{|y_{n,actual} - y_{n,approx}|}{y_{n,actual}} \times 100\%$

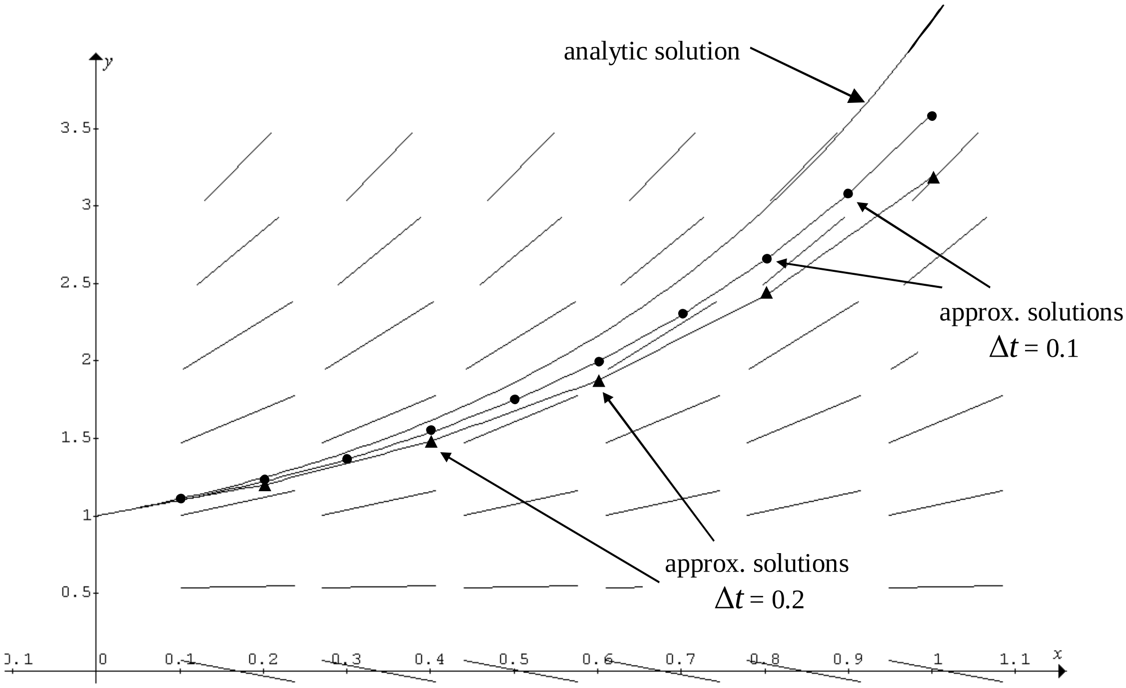
n	t_n	$y_{n,\ approx}$	$y_{n,\ actual}$	$\lvert y_{n,\ actual} - y_{n,\ approx}\rvert$	**Percentage Error**
0	0	1	1	0	0%
1	0.2	1.2000	1.2459	0.0459	3.69%
2	0.4	1.4800	1.6128	0.1328	8.23%
3	0.6	1.8720	2.1601	0.2881	13.34%
4	0.8	2.4208	2.9765	0.5557	18.67%
5	1.0	3.1891	4.1945	1.0054	23.97%

Table 3: Calculation of errors involved when $\Delta t = 0.2$

n	t_n	$y_{n,\ approx}$	$y_{n,\ actual}$	$\lvert y_{n,\ actual} - y_{n,\ approx}\rvert$	**Percentage Error**
0	0	1	1	0	0%
1	0.1	1.1000	1.1107	0.0107	0.96%
2	0.2	1.2200	1.2459	0.0259	2.08%
3	0.3	1.3640	1.4111	0.0471	3.34%
4	0.4	1.5368	1.6128	0.0760	4.71%
5	0.5	1.7442	1.8591	0.1150	6.18%
6	0.6	1.9930	2.1601	0.1671	7.73%
7	0.7	2.2916	2.5276	0.2360	9.34%
8	0.8	2.6499	2.9765	0.3266	10.97%
9	0.9	3.0799	3.5248	0.4449	12.62%
10	1.0	3.5959	4.1945	0.5987	14.27%

Table 4: Calculation of errors involved when $\Delta t = 0.1$

The figure below shows what the approximations would look like, and how much they deviate from the actual graph on a slope-field diagram.



Graphical Representation of Analytic vs Approximation Solutions

From Table 3, we observe that the error in y_n increases as n increases i.e. as t_n gets further and further away form the starting point t_0. The same observation can be made from Table 4.

Comparing the actual value of $y(1)$ with the approximations obtained in Table 3 and Table 4, we realise that the approximation obtained with a step size of 0.1 is better than that obtained with a step size of 0.2. This corresponds to a smaller absolute error value for the smaller step size of 0.1. In general, as the step size used gets smaller, better approximations will be obtained. However, we note that as the step size is decreased, the number of iterations (and hence computational time) will escalate. There is thus a trade-off between accuracy and speed (or complexity) in deciding the step size to use.

6.2 Use of Technology for the Euler's Method

In real-life uses of Euler's method, hundreds of steps would be required as small step size would be used to improve accuracy. This would result in the estimation of the solution using this method by hand prohibitive. A computer programme will be useful in performing these tedious computations. In this course, we will introduce two means of implementing the Euler's method, one using Microsoft Excel, and the other using the TI-84+ Graphic Calculator.

<u>Using Microsoft Excel</u>

Basic knowledge of how to use equations in Excel will allow us to easily implement the Euler's method.

<u>Using the TI-84+ Graphic Calculator</u>

The TI-84+ Graphic Calculator can be programmed to implement the Euler's method. The programming code (adapted from code developed by Dr. K.C. Ang) is provided below:

Description:

This program implements Euler's method for the initial value problem,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{f}\left(t, y\right), \qquad y(t_0) = y_0$$

to obtain an estimate for the value of y at a user-defined value of t.

Usage: D = a real number representing step size
Y = a real number, initial value of y, ie., $y(0)$
T = a real number, value of t at which $y(t)$ is to be approximated

Instructions:

(i) Enter $Y_1 = \mathrm{f}(x, y)$ in the 'Y=' editor. (The variable x is used to substitute t.)

(ii) Turn on Stat Plot 1, select scatter plot or line plot, and set XList to $\mathbf{L_1}$ **and YList to** $\mathbf{L_2}$**.**

(iii) Create a new program by pressing [PRGM] and choosing NEW.

(iv) Give the program a name (EULER) and key in the codes below.

Codes	Remarks
:ClrAllLists	ClrAllLists is in [MEM]
:ClrHome	ClrHome is in [PRGM] under I/O
:Prompt D	Prompt is in the [PRGM] under I/O
:Prompt Y	
:Prompt T	
:0 → X	→ is the [STO>] key; X is the [X,T,θ,*n*] key
:X → L_1(1)	
:Y → L_2(1)	
:round(T/D) → N	round is in [MATH] under NUM
:For(I, 1, N)	For is in [PRGM] under CTL
:Y+D*Y_1 → Y	Y_1 is in [VARS] under Y-VARS, Function
:X+D → X	
:Disp Y	Disp is in [PRGM] under I/O
:X → L_1(I+1)	
:Y → L_2(I+1)	
:End	End is in [PRGM] under CTL
:FnOff	FnOff is in [VARS] under Y-VARS, On/Off
:Pause	Pause is in [PRGM] under CTL

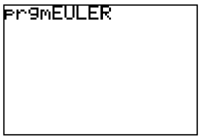
:$L_1(1) \rightarrow$ Xmin	Xmin is in VARS under Windows
:$L_1(N+1) \rightarrow$ Xmax	Xmax is in VARS under Windows
:min(L_2) $\rightarrow$ Ymin	Ymin is in VARS under Windows
:max(L_2) $\rightarrow$ Ymax	Ymax is in VARS under Windows
:DispGraph	DispGraph is in PRGM under I/O

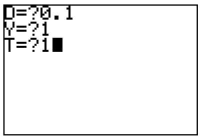
(v) End your editing by pressing QUIT .

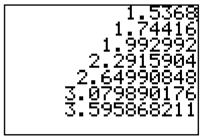
(iv) Use program by pressing PRGM and selecting the program you wish to run.

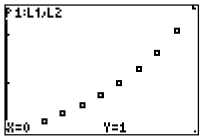
With the computer programme, you can now achieve a higher degree of accuracy with less tedious computations by hand.

Try running the programme for Example 7.1.1 where $\Delta t = 0.1$, and you should get the following screens:









Note that you will ***NOT*** *be able to use this programme in the examinations as re-setting your GC will clear the programme.*

6.3 Improved Euler's Method

Euler's method can be improved. The improved Euler's method (also known as the Heun's Method) works as follows:

We first apply Euler's method to find an approximation to the next y value and denote it as y_{n+1}^*. We then apply Euler's method again, but now we use a linear line segment whose slope is the **average** of $\mathrm{f}\left(t_n,\, y_n\right)$ and $\mathrm{f}\left(t_{n+1},\, y_{n+1}^*\right)$. That is, the 'new' line segment will have a gradient of $\frac{1}{2}\left[\mathrm{f}\left(t_n,\, y_n\right)+\mathrm{f}\left(t_{n+1},\, y_{n+1}^*\right)\right]$. This is illustrated in the figure below

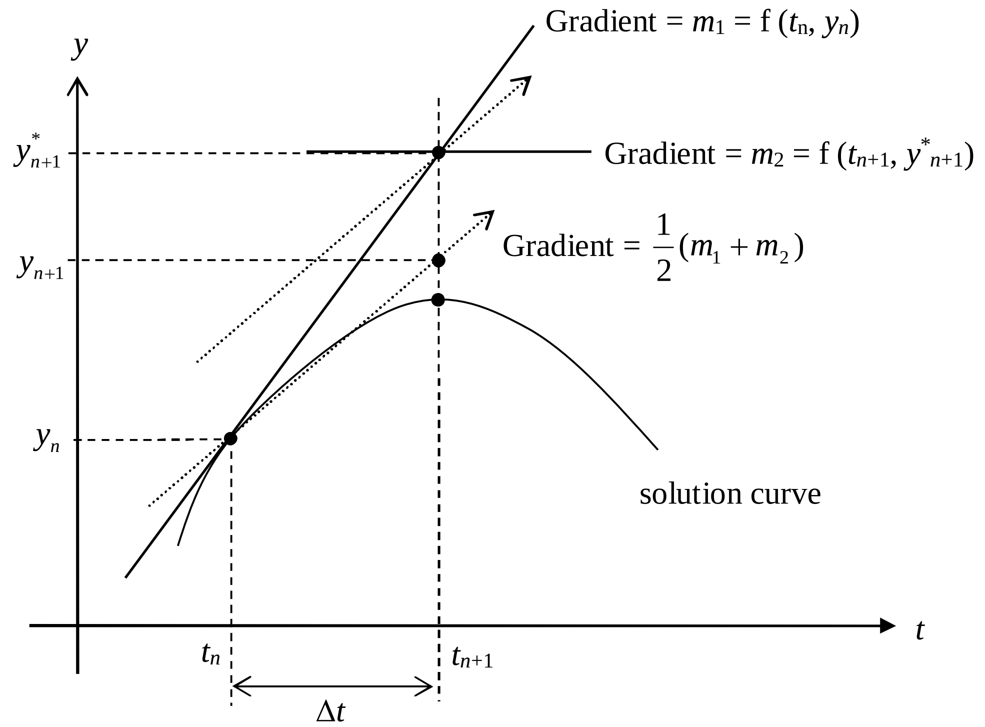


Illustration of improved Euler's method

As shown in the figure, a line with slope m_1 = f (t_n, y_n) is first drawn from (t_n, y_n). From this line, we first find an estimate y_{n+1}^*. This is essentially Euler's method. We then compute m_2 = f (t_{n+1}, y^*_{n+1}), and then determine the average of m_1 and m_2. We then apply Euler's method again, but using this average value as the gradient of the straight line to be drawn from point (t_n, y_n) to obtain the actual estimate of y_{n+1}.

Each evaluation of the improved Euler's method consists of two steps:

$$y_{n+1}^* = y_n + \Delta t \cdot \mathrm{f}\left(t_n, y_n\right)$$
$$y_{n+1} = y_n + \Delta t \cdot \left[\frac{\mathrm{f}\left(t_n, y_n\right)+\mathrm{f}\left(t_{n+1}, y_{n+1}^*\right)}{2}\right]$$

(The formula for Improved Euler Method with step size h is given in MF26.)

Example 6.3.1 (Revisiting 6.1.1)

Consider again the initial value problem

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2y - 1,\ \ y(0) = 1\text{. -------}\quad (1)$$

Apply the improved Euler's method to obtain an approximation for $y(1)$ using step sizes $\Delta t = 0.2$ and $\Delta t = 0.1$. Compare with results obtained using Euler's method.

Solution:

Here $f(t, y) = 2y - 1$, so improved Euler's method's formula will take the form

$$y^*_{n+1} = y_n + \Delta t \cdot (2y_n - 1)$$
$$y_{n+1} = y_n + \Delta t \cdot \left[\frac{(2y_n - 1) + (2y^*_{n+1} - 1)}{2} \right]$$

Beginning with $t_0 = 0$, $y_0 = 1$, $\Delta t = 0.2$, the first two approximate values are

$$y_1^* = 1 + (0.2) \cdot \left[2(1) - 1\right] = 1.2$$
$$y_1 = 1 + (0.2) \cdot \left[\frac{[2(1) - 1] + [2(1.2) - 1]}{2} \right] = 1.24$$
$$y_2^* = 1.24 + (0.2) \cdot \left[2(1.24) - 1\right] = 1.536$$
$$y_2 = 1.24 + (0.2) \cdot \left[\frac{[2(1.24) - 1] + [2(1.536) - 1]}{2} \right] = 1.5952$$

n	t_n	y^*_n	y_n, approx
0	0.0	---	1.0000
1	0.2	1.2000	1.2400
2	0.4	1.5360	1.5952
3	0.6	2.0333	2.1209
4	0.8	2.7693	2.8989
5	1.0	3.8585	4.0504

Table 5

The rest of the iterations are represented in Table 5.

Thus, using a step size of 0.2, the approximate value for $y(1) = 4.0504$.

Beginning with $t_0 = 0$, $y_0 = 1$, $\Delta t = 0.1$, the first two approximate values are

$$y_1^* = 1 + (0.1) \cdot \left[2(1) - 1\right] = 1.1$$
$$y_1 = 1 + (0.1) \cdot \left[\frac{[2(1) - 1] + [2(1.1) - 1]}{2} \right] = 1.11$$
$$y_2^* = 1.11 + (0.1) \cdot \left[2(1.11) - 1\right] = 1.232$$
$$y_2 = 1.11 + (0.1) \cdot \left[\frac{[2(1.1100) - 1] + [2(1.232) - 1]}{2} \right] = 1.2442$$

The rest of the iterations are represented in Table 6.

Thus, using a step size of 0.1, the approximate value for $y(1)$ = 4.1523.

n	t_n	y^*_n	y_n, approx
0	0.0	---	1.0000
1	0.1	1.1000	1.1100
2	0.2	1.2320	1.2442
3	0.3	1.3930	1.4079
4	0.4	1.5895	1.6077
5	0.5	1.8292	1.8514
6	0.6	2.1216	2.1487
7	0.7	2.4784	2.5114
8	0.8	2.9136	2.9539
9	0.9	3.4446	3.4937
10	1.0	4.0924	4.1523

Table 6

Table 7 and Table 8 show the comparison of the estimated y_n values and absolute error values obtained from the Euler's and improved Euler's method when Δt = 0.2 and Δt = 0.1 respectively.

n	t_n	$y_{n,\,approx}$		$y_{n,\,actual}$	$\lvert y_{n,\,\text{actual}} - y_{n,\,\text{approx}}\rvert$	
		(Euler)	**(iEuler)**		**(Euler)**	**(iEuler)**
0	0.0	1.0000	1.0000	1.0000	0.0000	0.0000
1	0.2	1.2000	1.2400	1.2459	0.0459	0.0059
2	0.4	1.4800	1.5952	1.6128	0.1328	0.0176
3	0.6	1.8720	2.1209	2.1601	0.2881	0.0392
4	0.8	2.4208	2.8989	2.9765	0.5557	0.0776
5	1.0	3.1891	4.0504	4.1945	1.0054	0.1441

Table 7: Comparison of Euler's and improved Euler's Method when Δt = 0.2

n	t_n	$y_{n,\,approx}$		$y_{n,\,actual}$	$\lvert y_{n,\,\text{actual}} - y_{n,\,\text{approx}}\rvert$	
		(Euler)	**(iEuler)**		**(Euler)**	**(iEuler)**
0	0.0	1.0000	1.0000	1.0000	0.0000	0.0000
1	0.1	1.1000	1.1100	1.1107	0.0107	0.0007
2	0.2	1.2200	1.2442	1.2459	0.0259	0.0017
3	0.3	1.3640	1.4079	1.4111	0.0471	0.0031
4	0.4	1.5368	1.6077	1.6128	0.0760	0.0051
5	0.5	1.7442	1.8514	1.8591	0.1150	0.0078
6	0.6	1.9930	2.1487	2.1601	0.1671	0.0114
7	0.7	2.2916	2.5114	2.5276	0.2360	0.0162
8	0.8	2.6499	2.9539	2.9765	0.3266	0.0227
9	0.9	3.0799	3.4937	3.5248	0.4449	0.0311
10	1.0	3.5959	4.1523	4.1945	0.5987	0.0422

Table 8: Comparison of Euler's and improved Euler's Method when Δt = 0.1

From Table 7, it can be seen that the improved Euler's method results in a better approximation to the actual value of $y(1)$ when Δt = 0.2. The absolute error is also much smaller. A similar observation can be found in Table 8. Clearly, the improved Euler's method is superior to the Euler's method. Once again, we note that the approximation improves with decreasing step size, whether the Euler's or improved Euler's method is used.

Note that the improved Euler's Method actually requires *two* evaluations of f at each step while the Euler's method requires only one. This is an important consideration as most of the computational time in each step is spent on calculating f, thus contributing significantly to the overall computing effort.

Note: It is **INCORRECT** to compute all the values of y^*_{n+1} first using the Euler's method and then use these values to find all the values of y_{n+1}. Can you see why?

Efficiency Analysis

In the example discussed above, at Δt = 0.2, Euler's method requires 5 evaluations of f to reach t = 1.0 while the improved Euler's method requires 10 evaluations. However, at Δt = 0.1, the Euler's method also requires 10 evaluations of f to reach t = 1.0. We can see from the tables that the improved Euler's method with Δt = 0.2 yields a better set of results then the Euler's method with Δt = 0.1 even though the same number of evaluations of f is involved. Thus, the improved Euler's method is clearly more efficient and gives significantly better results.