Chapter 5 (Pure Mathematics) Techniques of Differentiation

Objectives

At the end of the chapter, you should be able to:

(a) understand the derivative of $\mathrm{f}(x)$ as the gradient of the tangent to the graph of $y=\mathrm{f}(x)$ at a point;

(b) use of standard notations $\mathrm{f}'(x)$ and $\frac{\mathrm{d}y}{\mathrm{d}x}$;

(c) differentiate x^n for any rational n, e^x, $\ln x$, together with constant multiples, sums and differences;

(d) use the chain rule to differentiate a composition of 2 or 3 functions;

(e) find the approximate value of a derivative at a given point using a graphing calculator

Content

References

New Syllabus Additional Mathematics (8th Edition), Shinglee Publishers Pte Ltd.

Relevant Resources

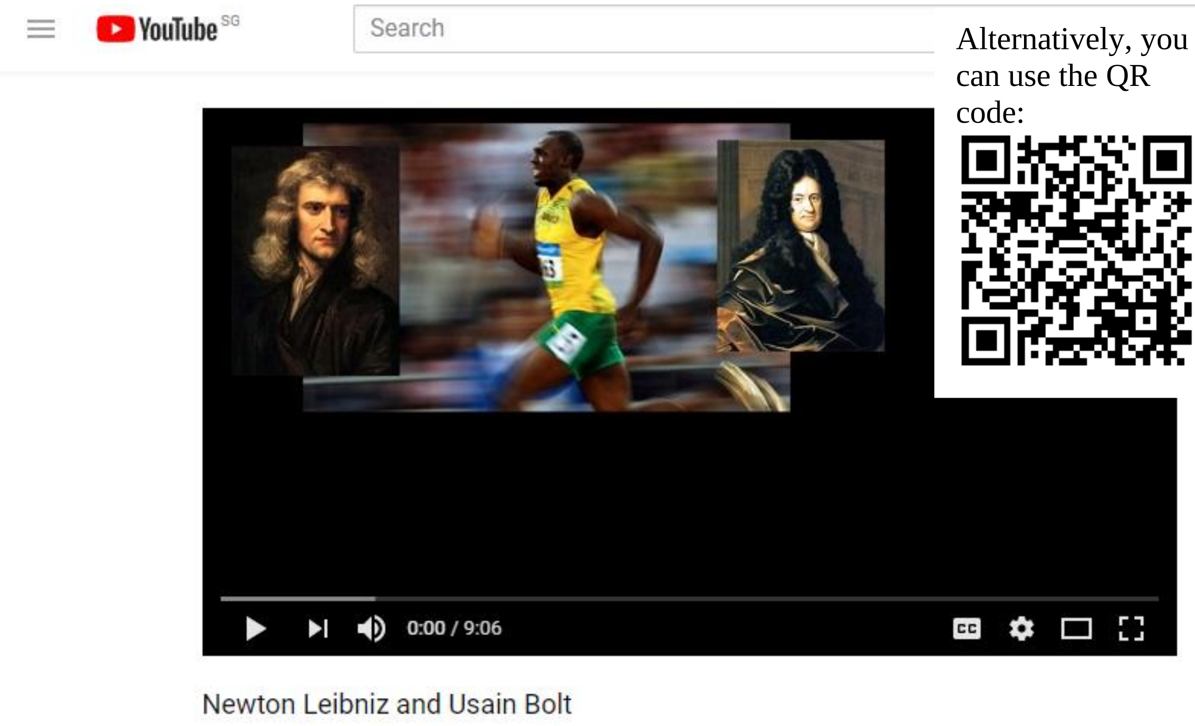
- http://mathinsite.bmth.ac.uk/applet/diffchords/diffchords.html (Applet to explore graphically the differentiation of a quadratic curve of $y=x^2$
- http://www.slu.edu/classes/maymk/GeoGebra/TangentToDerivatives.html (Applet to explore differentiation of being the slope of the line tangent to a curve)

<u>5.1 Introduction</u>

View the following video

1. <u>https://www.youtube.com/watch?time_continue=15&v=EKvHQc3QEow or</u>
2. Scan the QR code on the right.

<u>You should see this page:</u>



Alternatively, you can use the QR code:

Note:

1. The process of finding the derivative of a function is called <u>differentiation</u>.

   $\frac{\mathrm{d}y}{\mathrm{d}x}$ means we differentiate the function y with respect to x.

2. If $y=\mathrm{f}(x)$, we may write $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)=\mathrm{f}'(x)$

   $\mathrm{f}'(x)$ is known as the first derivative of $\mathrm{f}(x)$.

3. For a curve $y=\mathrm{f}(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ denotes gradient of the tangent to the curve $y=\mathrm{f}(x)$ at the point $(x,\mathrm{f}(x))$.

4. $\frac{\mathrm{d}y}{\mathrm{d}x}$ also denotes the rate of change of the function y with respect to x.

5.2 Derivatives of Basic Functions

Basic Functions	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
Algebraic (for any rational n)	x^n	nx^{n-1}
Exponential	e^x	e^x
Logarithmic	$\ln x$	$\frac{1}{x}$

Example 1 (Differentiate algebraic functions)
Differentiate the following with respect to x:

(a) $y = x^6$ $y = x^6$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5$	(b) $y = x^{-4}$ $y = x^{-4}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = -4x^{-5}$ $= -\frac{4}{x^5}$
(c) $y = \sqrt{x}$ $y = \sqrt{x} = x^{\frac{1}{2}}$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$ $= \frac{1}{2x^{\frac{1}{2}}}$ $= \frac{1}{2\sqrt{x}}$	(d) $y = 6$ $y = 6$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$

5.3 Rules of Differentiation

Suppose $\mathrm{f}\left(x\right)$ and $\mathrm{g}\left(x\right)$ are functions of *x* and *a* and *b* are constants.

Rules	Example
1 Constant Multiple of Function $\frac{\mathrm{d}}{\mathrm{d}x}\left[a\,\mathrm{f}(x)\right]=a\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$ $=a\,\mathrm{f}\,'\left(x\right)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[20x^2\right]=20(2x)=40x$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[-5e^x\right]=-5(e^x)=-5e^x$ $\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln x}{3}\right]=\frac{1}{3}\left(\frac{1}{x}\right)=\frac{1}{3x}$
2 Sum and Differences of Functions $\frac{\mathrm{d}}{\mathrm{d}x}(a\,\mathrm{f}(x)\pm b\,\mathrm{g}(x))=a\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)\pm b\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{g}(x)$ $=a\mathrm{f}\,'(x)+b\mathrm{g}'(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[20x^2+3\ln x\right]=40x+3\left(\frac{1}{x}\right)$ $=40x+\frac{3}{x}$
3 Chain Rule $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$ where $y=\mathrm{f}(u)$ and $u=\mathrm{g}(x)$	See Example 3
$\frac{\mathrm{d}\,y}{\mathrm{d}\,x}=\frac{\mathrm{d}\,y}{\mathrm{d}\,u}\times\frac{\mathrm{d}\,u}{\mathrm{d}\,v}\times\frac{\mathrm{d}\,v}{\mathrm{d}\,x}$ where $y=\mathrm{f}(u)\,,u=g\left(v\right)$ and $v=h\left(x\right)$	See Example 4(e)

Example 2

Differentiate the following with respect to x:

(a) 3 (b) $2x$ (c) $2x^3$ (d) $4\sqrt[3]{x}$

(e) $\frac{1}{4x^2}$ (f) $5e^x$ (g) $\frac{1}{5}\ln x$ (h) $6x^2+7x$

(i) $7e^x-2x^{-\frac{1}{3}}+\ln 5$ (j) $\frac{\ln x}{2}+x^2$ (k) $\frac{x^5-7x+5}{\sqrt{x}}$ (l) $\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)^2$

Solution:

(a) $\frac{d}{dx}(3)=0$

(b) $\frac{d}{dx}(2x)=2\frac{d}{dx}(x)$
$=2(1)=2$

(c) $\frac{d}{dx}(2x^3)=2\frac{d}{dx}(x^3)$
$=2(3x^2)$
$=6x^2$

(d) $\frac{d}{dx}(4\sqrt[3]{x})=4\frac{d}{dx}\left(x^{\frac{1}{3}}\right)$
$=4\left(\frac{1}{3}x^{-\frac{2}{3}}\right)$
$=\frac{4}{3x^{\frac{2}{3}}}=\frac{4}{3\sqrt[3]{x^2}}$

(e) $\frac{d}{dx}\left(\frac{1}{4x^2}\right)=\frac{1}{4}\frac{d}{dx}(x^{-2})$
$=\frac{1}{4}(-2x^{-3})$
$=-\frac{1}{2x^3}$

(f) $\frac{d}{dx}(5e^x)=5\frac{d}{dx}(e^x)$
$=5e^x$

(g) $\frac{d}{dx}\left(\frac{1}{5}\ln x\right)=\frac{1}{5}\frac{d}{dx}(\ln x)$
$=\frac{1}{5}\left(\frac{1}{x}\right)$
$=\frac{1}{5x}$

(h)
$\frac{d}{dx}(6x^2+7x)=6(2x)+7(1)$
$=12x+7$

(i)
$\frac{d}{dx}\left(7e^x-2x^{-\frac{1}{3}}+\ln 5\right)$
$=7(e^x)-2\left(-\frac{1}{3}x^{-\frac{4}{3}}\right)+0$
$=7e^x+\frac{2}{3\sqrt[3]{x^4}}$

(j)
$\frac{d}{dx}\left(\frac{\ln x}{2}+x^2\right)=\frac{1}{2}\left(\frac{1}{x}\right)+2x$
$=\frac{1}{2x}+2x$

(k) $\frac{d}{dx}\left(\frac{x^5-7x+5}{\sqrt{x}}\right)$
$=\frac{d}{dx}\left(\frac{x^5}{\sqrt{x}}-\frac{7x}{\sqrt{x}}+\frac{5}{\sqrt{x}}\right)$
$=\frac{d}{dx}\left(x^{\frac{9}{2}}-7x^{\frac{1}{2}}+5x^{-\frac{1}{2}}\right)$
$=\frac{9}{2}x^{\frac{7}{2}}-\frac{7}{2}x^{-\frac{1}{2}}-\frac{5}{2}x^{-\frac{3}{2}}$
$=\frac{9}{2}x^{\frac{7}{2}}-\frac{7}{2\sqrt{x}}-\frac{5}{2\sqrt{x^3}}$

(l) $\frac{d}{dx}\left(\sqrt{x}-\frac{3}{\sqrt{x}}\right)^2$
$=\frac{d}{dx}\left(x-6+\frac{9}{x}\right)$
$=1+9(-x^{-2})$
$=1-\frac{9}{x^2}$

Note:

The use of the product rule and the quotient rule is out of the H1 Mathematics syllabus. You may use them if you wish to.

5.3.1 Extension of Derivatives of Basic Functions Using Chain Rule

Type of Function	Basic		General	
	y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
Algebraic	x^n	nx^{n-1}	$\left[\mathrm{f}(x)\right]^n$	$n\left[\mathrm{f}(x)\right]^{n-1}\mathrm{f}'(x)$
Exponential	e^x	e^x	$e^{\mathrm{f}(x)}$	$e^{\mathrm{f}(x)}\mathrm{f}'(x)$
Logarithmic	$\ln x$	$\frac{1}{x}$	$\ln\left[\mathrm{f}(x)\right]$	$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$

Example 3 [Chain Rule]

Differentiate the following with respect to x:

(a) $y=\left(6x^3+8\right)^2$

(b) $y=\left(\ln x\right)^4$

(c) $y=\frac{1}{\sqrt{e^x-6x}}$

(d) $y=\ln\left(x^2+1\right)$

Solution:

(a) Let $u=6x^3+8 \Rightarrow y=u^2$

$\frac{du}{dx}=\frac{d}{dx}(6x^3+8)=18x^2$ and $\frac{dy}{du}=\frac{d}{du}\left(u^2\right)=2u$

By Chain Rule, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=\left(2u\right)\left(18x^2\right)$

$=2(6x^3+8)\left(18x^2\right)$

$=216x^5+288x^2$

Shorter working:

$y=\left(6x^3+8\right)^2$

$\frac{dy}{dx}=2(6x^3+8)\left(18x^2\right)$

$=216x^5+288x^2$

(b) Let $u=\ln x \Rightarrow y=u^4$

$\frac{du}{dx}=\frac{1}{x}$ and $\frac{dy}{du}=4u^3$

By Chain Rule, $\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx}$

$=\left(4u^3\right)\left(\frac{1}{x}\right)$

$=4\left(\ln x\right)^3\left(\frac{1}{x}\right)$

$=\frac{4\left(\ln x\right)^3}{x}$

Shorter working:

$y=\left(\ln x\right)^4$

$\frac{dy}{dx}=4\left(\ln x\right)^3\left(\frac{1}{x}\right)=\frac{4\left(\ln x\right)^3}{x}$

Is this correct?

$y=\left(\ln x\right)^4=4\ln x$

$\frac{dy}{dx}=\frac{4}{x}$

(c) Let $u = e^x - 6x \quad \Rightarrow y = u^{-\frac{1}{2}}$

$\frac{\mathrm{d}u}{\mathrm{d}x} = e^x - 6$ and $\frac{\mathrm{d}y}{\mathrm{d}u} = \left(-\frac{1}{2}\right)u^{-\frac{3}{2}}$

By Chain Rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

$$= \left(-\frac{1}{2}\right)u^{-\frac{3}{2}}\left(e^x - 6\right)$$

$$= -\frac{1}{2}\left(e^x - 6\right)\left(e^x - 6x\right)^{-\frac{3}{2}}$$

Shorter working:

$$y = \frac{1}{\sqrt{e^x - 6x}}$$

$$y = \left(e^x - 6x\right)^{-\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \left[-\frac{1}{2}\left(e^x - 6x\right)^{-\frac{3}{2}}\right]\left(e^x - 6\right)$$

$$= -\frac{1}{2}\left(e^x - 6\right)\left(e^x - 6x\right)^{-\frac{3}{2}}$$

(d) Let $u = (x^2 + 1) \quad \Rightarrow y = \ln u$

$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 1\right) = 2x$ and $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{\mathrm{d}}{\mathrm{d}u}\left(\ln u\right) = \frac{1}{u}$

By Chain Rule, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$

$$= \left(\frac{1}{u}\right)(2x)$$

$$= \left(\frac{2x}{x^2 + 1}\right)$$

Shorter working:

Let $y = \ln(x^2 + 1)$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{(x^2 + 1)} \frac{\mathrm{d}}{\mathrm{d}x}\left(x^2 + 1\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{x^2 + 1}$$

Example 4

Differentiate the following with respect to x:

(a) $y=\sqrt[4]{\frac{x^2}{3}-\frac{5}{x^2}+2x}$, (b) $y=4e^{x^2+3x-5}$,

(c) $y=\ln\left(ex^2-2\mathrm{e}^{-4x}\right)$, (d) $y=\ln\left(\frac{5x^2-2}{3}\right)^4$.

(e) $y=\sqrt{\mathrm{e}^{x^2-7x}}$

Solution:

(a) $y=\sqrt[4]{\frac{x^2}{3}-\frac{5}{x^2}+2x}$

$$=\left(\frac{x^2}{3}-\frac{5}{x^2}+2x\right)^{\frac{1}{4}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{4}\left(\frac{x^2}{3}-\frac{5}{x^2}+2x\right)^{-\frac{3}{4}}\left[\frac{2x}{3}-5\left(-2x^{-3}\right)+2\right]$$

$$=\frac{1}{4}\left(\frac{x^2}{3}-\frac{5}{x^2}+2x\right)^{-\frac{3}{4}}\left(\frac{2x}{3}+\frac{10}{x^3}+2\right)$$

(c) $y=\ln\left(ex^2-2\mathrm{e}^{-4x}\right)$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2ex-2\left(-4\mathrm{e}^{-4x}\right)}{ex^2-2\mathrm{e}^{-4x}}$$

$$=\frac{2ex+8\mathrm{e}^{-4x}}{ex^2-2\mathrm{e}^{-4x}}$$

(b) $y=4e^{x^2+3x-5}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4e^{x^2+3x-5}\left(2x+3\right)$$

$$=4\left(2x+3\right)e^{x^2+3x-5}$$

(d) $y=\ln\left(\frac{5x^2-2}{3}\right)^4$

$$=4\left[\ln\left(5x^2-2\right)-\ln 3\right]$$

$$=4\ln\left(5x^2-2\right)-4\ln 3$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4\left(\frac{10x}{5x^2-2}\right)$$

$$=\frac{40x}{5x^2-2}$$

(e) Let $u=\mathrm{e}^{x^2-7x}$ $\Rightarrow$ $y=\sqrt{u}$

$$\frac{\mathrm{d}y}{\mathrm{d}u}=\frac{1}{2}u^{-\frac{1}{2}}=\frac{1}{2\sqrt{u}}$$

Let $v=x^2-7x$ $\Rightarrow$ $u=\mathrm{e}^v$

$$\frac{\mathrm{d}u}{\mathrm{d}v}=\mathrm{e}^v$$

$$\frac{\mathrm{d}v}{\mathrm{d}x}=2x-7$$

By Chain Rule, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}v}\times\dfrac{\mathrm{d}v}{\mathrm{d}x}$

$$=\left(\frac{1}{2\sqrt{u}}\right)\left(\mathrm{e}^{v}\right)\left(2x-7\right)$$

$$=\frac{1}{2\sqrt{\mathrm{e}^{x^2-7x}}}\left(\mathrm{e}^{x^2-7x}\right)\left(2x-7\right)$$

$$=\frac{1}{2}\left(2x-7\right)\sqrt{\mathrm{e}^{x^2-7x}}$$

Note:
For Example 4(d), if we had not simplified our expression for y before performing differentiation, we would have the following solution:

$$y=\ln\left(\frac{5x^2-2}{3}\right)^4$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{4\left(\frac{5x^2-2}{3}\right)^3\left(\frac{10x}{3}\right)}{\left(\frac{5x^2-2}{3}\right)^4}=\frac{40x}{5x^2-2}$$

It is usually easier to simplify the expression before performing differentiation.

Exercise 1

Differentiate the following with respect to x:

(a) $y=3x^4+2x+5$	(b) $y=(3x^2+4x-1)^8$	(c) $y=\sqrt{x^2-7}$
(d) $y=2\left(x^2+5\right)^3$	(e) $y=\left(x^3+2x+1\right)^{-1}$	(f) $y=\dfrac{1}{\sqrt{x^4-1}}$
(g) $y=\mathrm{e}^{x^5+3}$	(h) $y=5e^{e^x}$	(i) $y=\left(\mathrm{e}^x-\mathrm{e}^{-x}\right)^2$
(j) $y=5e^{2x+1}$	(k) $y=3\ln\left(x^2\right)$	(l) $y=(\ln 2x)^3$
(m) $y=\ln\left(\ln x\right)$	(n) $y=\ln\sqrt{x^6-4}$	(o) $y=\ln\left(\ln x\right)^3$

Solution:

(a) $y=3x^4+2x+5$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3\left(4x^3\right)+2+0$ $=12x^3+2$	(b) $y=(3x^2+4x-1)^8$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=8\left(3x^2+4x-1\right)^7\left[3\left(2x\right)+4\right]$ $=8\left(6x+4\right)\left(3x^2+4x-1\right)^7$ $=16\left(3x+2\right)\left(3x^2+4x-1\right)^7$	(c) $y=\sqrt{x^2-7}$ $=\left(x^2-7\right)^{\frac{1}{2}}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{2}\left(x^2-7\right)^{-\frac{1}{2}}\left(2x\right)$ $=\dfrac{x}{\sqrt{x^2-7}}$

(d) $y=2\left(x^2+5\right)^3$ $\frac{dy}{dx}=2.3\left(x^2+5\right)^2.2x$ $=12x\left(x^2+5\right)^2$	(e) $y=\left(x^3+2x+1\right)^{-1}$ $\frac{dy}{dx}=-1\left(x^3+2x+1\right)^{-2}\left(3x^2+2\right)$ $=-\frac{3x^2+2}{\left(x^3+2x+1\right)^2}$	(f) $y=\frac{1}{\sqrt{x^4-1}}=\left(x^4-1\right)^{-\frac{1}{2}}$ $\frac{dy}{dx}=-\frac{1}{2}\left(x^4-1\right)^{-\frac{3}{2}}\left(4x^3\right)$ $=-\frac{2x^3}{\sqrt{\left(x^4-1\right)^3}}$
(g) $y=e^{x^5+3}$ $\frac{dy}{dx}=e^{x^5+3}\left(5x^4\right)$ $=5x^4e^{x^5+3}$	(h) $y=5e^{e^x}$ $\frac{dy}{dx}=5e^{e^x}e^x$ $=5e^{e^x+x}$	(i) $y=\left(e^x-e^{-x}\right)^2$ $\frac{dy}{dx}=2\left(e^x-e^{-x}\right)\left(e^x+e^{-x}\right)$ $=2\left(e^{2x}-e^{-2x}\right)$
(j) $y=5e^{2x+1}$ $\frac{dy}{dx}=5e^{2x+1}.2$ $=10e^{2x+1}$	(k) $y=3\ln x^2$ $=3\left(2\ln x\right)$ $\frac{dy}{dx}=\frac{6}{x}$	(l) $y=\left(\ln 2x\right)^3$ $\frac{dy}{dx}=3\left(\ln 2x\right)^2\frac{1}{2x}2$ $=\frac{3\left(\ln 2x\right)^2}{x}$
(m) $y=\ln\left(\ln x\right)$ $\frac{dy}{dx}=\frac{1}{\ln x}\frac{d}{dx}\left[\ln x\right]$ $=\frac{1}{\ln x}\frac{1}{x}=\frac{1}{x\ln x}$	(n) $y=\ln\sqrt{x^6-4}$ $=\ln\left(x^6-4\right)^{\frac{1}{2}}$ $=\frac{1}{2}\ln\left(x^6-4\right)$ $\frac{dy}{dx}=\frac{1}{2}\frac{6x^5}{x^6-4}=\frac{3x^5}{x^6-4}$	(o) $y=\ln\left(\ln x\right)^3$ $=3\ln\left(\ln x\right)$ $\frac{dy}{dx}=\frac{3}{\ln x}\frac{1}{x}=\frac{3}{x\ln x}$

Answer:

(a) $12x^3+2$	(b) $16\left(3x+2\right)\left(3x^2+4x-1\right)^7$	(c) $\frac{x}{\sqrt{x^2-7}}$
(d) $12x\left(x^2+5\right)^2$	(e) $-\frac{3x^2+2}{\left(x^3+2x+1\right)^2}$	(f) $-\frac{2x^3}{\sqrt{\left(x^4-1\right)^3}}$
(g) $5x^4e^{x^5+3}$	(h) $5e^{e^x+x}$	(i) $2\left(e^{2x}-e^{-2x}\right)$
(j) $10e^{2x+1}$	(k) $\frac{6}{x}$	(l) $\frac{3\left(\ln 2x\right)^2}{x}$
(m) $\frac{1}{x\ln x}$	(n) $\frac{3x^5}{x^6-4}$	(o) $\frac{3}{x\ln x}$

5.4 Finding Numerical Values of Derivatives for Given Values of *x* by GC

You can use TI-84 graphic calculator to find numerical values of derivatives for given values of *x*.

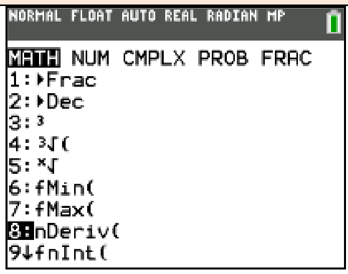
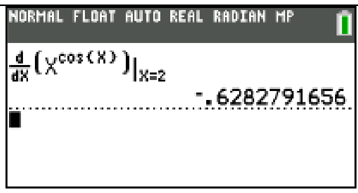
Example 6

Find the numerical value of the derivative of $x^{\cos x}$ when $x = 2$.

Solution:

From GC, $\frac{\mathrm{d}}{\mathrm{d}x} x^{\cos x} \Big|_{x=2} = -0.628$ (3 s.f.)

<table>
<tr><th>Steps</th><th>Screenshot</th><th>Remarks</th></tr>
<tr><td>On the Home screen, press m and select 8: nDeriv(.

Key in the variable that you are going to differentiate with respect to, the function, and the value of x that you want to evaluate, as shown on the right.</td><td>
</td><td>Or you may use the shortcut key F2 by pressing a @ on the Home screen and select 3: nDeriv(.</td></tr>
<tr><td>Press e to obtain the answer.</td><td>
</td><td></td></tr>
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Exercise 2

Find the numerical value of the derivative of

(a) $\tan x$ when $x = \frac{\pi}{6}$.

(b) e^{x^2} when $x = \frac{1}{2}$.

Solution:

(a) From GC, $\frac{\mathrm{d}}{\mathrm{d}x} \tan x \Big|_{x=\frac{\pi}{6}} = 1.33$ (3 s.f.)

(b) From GC, $\frac{\mathrm{d}}{\mathrm{d}x} e^{x^2} \Big|_{x=\frac{1}{2}} = 1.28$ (3 s.f.)

Answer: 1.33; 1.28

5.5 Higher Order Derivatives (Out of Syllabus)

Suppose $y = \mathrm{f}(x)$ is a function of *x*.
When *y* is differentiated with respect to *x*, the resultant function is referred to as the first derivative of *y* with respect to *x*, denoted by $\frac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$.
When $\frac{\mathrm{d}y}{\mathrm{d}x}$ is differentiated again with respect to $x\left(\text{i.e.} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\right)$, the resulting function is referred to as the second derivative of *y* with respect to *x*, denoted by $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ or $\mathrm{f}''(x)$.
Further differentiation of *y* will produce even higher order derivatives.
In general, the n^{th} derivative of *y* is denoted by $\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \mathrm{f}^{(n)}(x)$, $n \in \mathbb{Z}^+$.

For example, $y = 4x^4 + 5x - 1$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^4 + 5x - 1\right)$$
$$= 16x^3 + 5$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$
$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(16x^3 + 5\right)$$
$$= 48x^2$$

Note:

1. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) \neq \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$

2. $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ is the rate of change of $\frac{\mathrm{d}y}{\mathrm{d}x}$ with respect to *x*.

<u>Example 5</u>
Find the first and second derivatives of the following functions with respect to *x*:
(a) $y = 3x^4 + 5x^3 + 3x - 6$
(b) $y = \ln x$
(c) $y = \sqrt{3x - 4}$
Solution:

(a) $y = 3x^4 + 5x^3 + 3x - 6$

$$\frac{dy}{dx} = 12x^3 + 15x^2 + 3$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 36x^2 + 30x$$

(b) $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x} = x^{-1}$$

$$\frac{d^2y}{dx^2} = -x^{-2}$$

$$= -\frac{1}{x^2}$$

(c) $y = \sqrt{3x-4}$

$$= (3x-4)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}(3x-4)^{-\frac{1}{2}}(3)$$

$$= \frac{3}{2\sqrt{3x-4}}$$

$$\frac{d^2y}{dx^2} = \frac{3}{2}\left(-\frac{1}{2}\right)(3x-4)^{-\frac{3}{2}}(3)$$

$$= -\frac{9}{4\sqrt{(3x-4)^3}}$$

Exercise 3

Find the first and second derivatives of the following functions with respect to x:

(a) $y = (2x+1)^5$

(b) $y = 4x^3 + \frac{1}{x^2} - \ln x$

Solution:

(a) $y = (2x+1)^5$

$$\frac{dy}{dx} = 5(2x+1)^4 .2 = 10(2x+1)^4$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 40(2x+1)^3 .2 = 80(2x+1)^3$$

(b) $y = 4x^3 + \frac{1}{x^2} - \ln x$

$$\frac{dy}{dx} = 12x^2 - \frac{2}{x^3} - \frac{1}{x}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= 24x + \frac{6}{x^4} + \frac{1}{x^2}$$

Answers: $80(2x+1)^3$; $24x + \frac{6}{x^4} + \frac{1}{x^2}$

Practice Yourself

1. [HCI/2017/Prelim/Q2]

Differentiate the following with respect to *x*.

(a) $\left(x+\ln x\right)^2$,

(b) $\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}$.

[**Ans:** (a) $\frac{2}{x}\left(x+\ln x\right)\left(x+1\right)$ (b) $\frac{1}{2\left(2-x\right)^{\frac{3}{2}}}\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}$]

Solution:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x+\ln x\right)^2 = 2\left(x+\ln x\right)\left(1+\frac{1}{x}\right)$$
$$= \frac{2}{x}\left(x+\ln x\right)\left(x+1\right)$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)} = \frac{1}{2\left(2-x\right)^{\frac{3}{2}}}\mathrm{e}^{\left(\frac{1}{\sqrt{2-x}}\right)}$$

2. [NJC/2017/Prelim/Q4]

Differentiate the following with respect to *x*.

(i) $\ln\left(\frac{4}{\sqrt{12+3x^2}}\right)$,

(ii) $\frac{1}{\sqrt{2-3x}}$.

[**Ans:** (a)(i) $-\frac{x}{4+x^2}$ (a)(ii) $\frac{3}{2}\left(2-3x\right)^{-\frac{3}{2}}$ or $\frac{3}{2\left(\sqrt{2-3x}\right)^3}$]

Solution:
(a)(i)

Let $y = \ln\left(\dfrac{4}{\sqrt{12+3x^2}}\right) = \ln 4 - \dfrac{1}{2}\ln(12+3x^2)$

$\dfrac{dy}{dx} = -\dfrac{1}{2}\left(\dfrac{6x}{12+3x^2}\right) = -\dfrac{x}{4+x^2}$

(ii)

$\dfrac{d}{dx}\left(\dfrac{1}{\sqrt{2-3x}}\right) = \dfrac{d}{dx}\left((2-3x)^{-\frac{1}{2}}\right)$

$= -\dfrac{1}{2}(2-3x)^{-\frac{3}{2}}(-3)$

$= \dfrac{3}{2}(2-3x)^{-\frac{3}{2}}$ or $\dfrac{3}{2\left(\sqrt{2-3x}\right)^3}$

3. [RI/2017/Prelim/Q1]

Differentiate

(a) $\ln\{7\sqrt{(1+6x^2)}\}$,

(b) $\dfrac{1}{6(1-7x)^2}$.

[**Ans:** (a) $\dfrac{6x}{1+6x^2}$ (b) $\dfrac{7}{3(1-7x)^3}$]

Solution:

(a) $\dfrac{d}{dx}\ln\{7\sqrt{(1+6x^2)}\} = \dfrac{d}{dx}[\ln 7 + \dfrac{1}{2}\ln(1+6x^2)] = \dfrac{1}{2}\,\dfrac{12x}{1+6x^2} = \dfrac{6x}{1+6x^2}$

(b) $\dfrac{d}{dx}\dfrac{1}{6(1-7x)^2} = \dfrac{1}{6}\dfrac{d}{dx}(1-7x)^{-2} = \dfrac{1}{6}\times[-2(1-7x)^{-3}(-7)] = \dfrac{7}{3(1-7x)^3}$

4. [SAJC/2017/Prelim/Q3(a)]

Differentiate with respect of x, simplifying your answers,

(i) $\dfrac{1}{\sqrt{x^2-3x+1}}$,

(ii) $\ln\left(\dfrac{x^2+3x+2}{x^2+4x+3}\right)$.

[**Ans:** (i) $-\dfrac{2x-3}{2\sqrt{\left(x^2-3x+1\right)^3}}$ (ii) $\dfrac{1}{(x+2)(x+3)}$]

Solution:

(a) (i)

Let $y=\dfrac{1}{\sqrt{x^2-3x+1}}=\left(x^2-3x+1\right)^{-\frac{1}{2}}$

$$\frac{dy}{dx}=-\frac{1}{2}\left(x^2-3x+1\right)^{-\frac{3}{2}}(2x-3)$$

$$=-\frac{2x-3}{2\sqrt{\left(x^2-3x+1\right)^3}}$$

(ii)

Let

$$y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln\frac{(x+1)(x+2)}{(x+1)(x+3)}=\ln\frac{x+2}{x+3}$$

$$=\ln(x+2)-\ln(x+3)$$

$$\frac{dy}{dx}=\frac{1}{x+2}-\frac{1}{x+3}$$

$$=\frac{1}{(x+2)(x+3)}$$

Alternatively,

$$y=\ln\left(\frac{x^2+3x+2}{x^2+4x+3}\right)=\ln(x^2+3x+2)-\ln(x^2+4x+3)$$

$$\frac{dy}{dx}=\frac{2x+3}{x^2+3x+2}-\frac{2x+4}{x^2+4x+3}$$

$$=\frac{2x+3}{(x+1)(x+2)}-\frac{2x+4}{(x+1)(x+3)}$$

$$=\frac{(2x+3)(x+3)-(2x+4)(x+2)}{(x+1)(x+2)(x+3)}$$

$$=\frac{x+1}{(x+1)(x+2)(x+3)}$$

$$=\frac{1}{(x+2)(x+3)}$$

5. [SRJC/2017/Prelim/Q1(a)]

Differentiate each of the following functions with respect to x, simplifying your answers:

(i) $\left(\frac{2}{3}x+1\right)^{-6}$

(ii) $5e^{1-2x}+\frac{1}{8x^3}$

[**Ans:** (a)(i) $-\dfrac{4}{\left(\dfrac{2}{3}x+1\right)^7}$; (a)(ii) $-10e^{1-2x}-\dfrac{3}{8x^4}$]

Solution:

(a) (i)

Let $y=\left(\dfrac{2}{3}x+1\right)^{-6}$

$$\frac{dy}{dx}=(-6)\left(\frac{2}{3}x+1\right)^{-7}\left(\frac{2}{3}\right)$$

$$=-\frac{4}{\left(\frac{2}{3}x+1\right)^7}$$

(ii)

Let $y=5e^{1-2x}+\dfrac{1}{8x^3}$

$$\frac{dy}{dx}=5(-2)e^{1-2x}+\frac{1}{8}(-3)x^{-4}$$

$$=-10e^{1-2x}-\frac{3}{8x^4}$$

6. [TJC/2017/Prelim/Q2(a)]

Differentiate $\dfrac{e^x-1}{e^{2x}}$.

[**Ans:** $2e^{-2x}-e^{-x}$]

Solution:

$$\frac{d}{dx}\frac{e^x-1}{e^{2x}}=\frac{d}{dx}\left[\frac{e^x}{e^{2x}}-\frac{1}{e^{2x}}\right]$$

$$=\frac{d}{dx}\left[e^{-x}-e^{-2x}\right]$$

$$=-e^{-x}+2e^{-2x}$$

$$=2e^{-2x}-e^{-x}$$

7. [VJC/2017/Prelim/Q3(a)]

Differentiate $\ln\left(\frac{5x+2}{4x^2}\right)$.

[**Ans:** $\frac{5}{5x+2}-\frac{2}{x}$]

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{5x+2}{4x^2}\right)$$
$$=\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln(5x+2)-\ln 4x^2\right]$$
$$=\frac{5}{5x+2}-\frac{8x}{4x^2}$$
$$=\frac{5}{5x+2}-\frac{2}{x}$$

8. [YJC/2017/Prelim/Q1]

Differentiate with respect to *x*,

(a) $5\ln\left(1-3x^2\right)$,

(b) $\frac{1}{(2x+3)^2}$.

[**Ans:** (a) $-\frac{30x}{1-3x^2}$ (b) $-4(2x+3)^{-3}$]

Solution:

(a) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(5\ln\left(1-3x^2\right)\right)=5\left(\frac{1}{1-3x^2}\right)(-6x)$$
$$=-\frac{30x}{1-3x^2}$$

(b) $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{(2x+3)^2}\right)=\frac{\mathrm{d}}{\mathrm{d}x}(2x+3)^{-2}$$
$$=-2(2x+3)^{-3}(2)$$
$$=-4(2x+3)^{-3}$$

Summary

1. For a curve $y=\mathrm{f}(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ denotes gradient of the tangent to the curve $y=\mathrm{f}(x)$ at the point $\left(x,\mathrm{f}(x)\right)$.
2. $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ also denotes the rate of change of the function y with respect to x.
3. The first derivative of y is denoted by $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ or $\mathrm{f}'(x)$
4. The second derivative of y is denoted by $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=\mathrm{f}''(x)$.
5. The n^{th} derivative of y is denoted by $\dfrac{\mathrm{d}^ny}{\mathrm{d}x^n}=\mathrm{f}^{(n)}(x)$.
6. Rules of Differentiation

Rules	Example
Constant Multiple of Function $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[a\,\mathrm{f}(x)\right]=a\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$ $=a\,\mathrm{f}'(x)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[20x^2\right]=20(2x)=40x$ $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[-5e^x\right]=-5(e^x)=-5e^x$ $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{\ln x}{3}\right]=\dfrac{1}{3}\left(\dfrac{1}{x}\right)=\dfrac{1}{3x}$
Linear Combinations of Functions $\dfrac{\mathrm{d}}{\mathrm{d}x}(a\,\mathrm{f}(x)\pm b\,\mathrm{g}(x))=a\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)\pm b\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{g}(x)$ $=a\mathrm{f}'(x)+b\mathrm{g}'(x)$	$\dfrac{\mathrm{d}}{\mathrm{d}x}\left[20x^2+3\ln x)\right]=40x+3\left(\dfrac{1}{x}\right)$ $=40x+\dfrac{3}{x}$
Chain Rule $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}x}$ where $y=\mathrm{f}(u)$ and $u=\mathrm{g}(x)$	$y=\left(6x^3+8\right)^2$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=2(6x^3+8)\left(18x^2\right)$ $=216x^5+288x^2$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}u}\times\dfrac{\mathrm{d}u}{\mathrm{d}v}\times\dfrac{\mathrm{d}v}{\mathrm{d}x}$ where $y=\mathrm{f}(u)$, $u=g(v)$ and $v=h(x)$	$y=\left[\ln\left(x^2+1\right)\right]^3$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3\left[\ln(x^2+1)\right]^2\left(\dfrac{2x}{x^2+1}\right)$ $=\left[\ln(x^2+1)\right]^2\left(\dfrac{6x}{x^2+1}\right)$

7. Derivatives of Functions

Type of Function	Basic		General	
	y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
Algebraic	a (constant term)	0	a (constant term)	0
	$x^n,\ n \neq 0$	nx^{n-1}	$\left[f(x)\right]^n$	$n\left[f(x)\right]^{n-1} f'(x)$
Exponential	e^x	e^x	$e^{f(x)}$	$e^{f(x)} f'(x)$
Logarithmic	$\ln x$	$\frac{1}{x}$	$\ln\left[f(x)\right]$	$\frac{f'(x)}{f(x)}$

Checklist

I am able to:

☐ The differentiate x^n for any rational n, e^x, and $\ln x$, together with constant multiples, sums and differences.

☐ Use the chain rule to differentiate a composition of 2 or 3 functions.