H2 Mathematics (9758)
Chapter 7 Differentiation
Discussion Solutions

Level 1

1 Differentiate the following with respect to the variable given:

(a) $\sqrt{x}\cot\left(\sqrt{x}\right)$ **(b)** $z\left(\ln(z)\right)^3$ **(c)** $\log_a(1-3x)^3$, where a is a positive constant and $a \neq 1$

(d) $\ln\left(\sin^2\theta\right)$ **(e)** $3^{x\sin x}$ **(f)** $\dfrac{1}{\cos^{-1}2x}$

(g) $\sin^3 x$ **(h)** $\sqrt{\operatorname{cosec}\left(3x-\frac{1}{2}\right)}$ **(i)** $\tan^{-1}\left(1-x^3\right)$

Solutions:

(a)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}\cot\left(\sqrt{x}\right)\right)$ $=\frac{1}{2\sqrt{x}}\cot\left(\sqrt{x}\right)-\frac{1}{2\sqrt{x}}\cdot\sqrt{x}\operatorname{cosec}^2\left(\sqrt{x}\right)$ $=\frac{1}{2\sqrt{x}}\cot\left(\sqrt{x}\right)-\frac{1}{2}\operatorname{cosec}^2\left(\sqrt{x}\right)$
(b)	$\frac{\mathrm{d}}{\mathrm{d}z}\left(z\left(\ln(z)\right)^3\right)$ $=z\left[3\left(\ln(z)\right)^2\times\frac{1}{z}\right]+\left(\ln(z)\right)^3$ $=3\left(\ln(z)\right)^2+\left(\ln(z)\right)^3$ **ALERT:** $\left(\ln(z)\right)^3 \neq 3\ln(z)$
(c)	First, we do a change of base for $\log_a(1-3x)^3$. $\log_a(1-3x)^3$ $=\frac{\log_e(1-3x)^3}{\log_e a}=\frac{\ln(1-3x)^3}{\ln a}=\frac{3\ln(1-3x)}{\ln a}$ Always simplify ln expression before differentiating Note that a is a constant, $a>0, a\neq 1$ $\frac{\mathrm{d}}{\mathrm{d}x}\left(\log_a(1-3x)^3\right)$

	$=3\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln(1-3x)}{\ln a}\right)$ $=\dfrac{3}{\ln a}\cdot\dfrac{-3}{1-3x}$ $=\dfrac{9}{(\ln a)(3x-1)}$
(d)	$\dfrac{\mathrm{d}}{\mathrm{d}\theta}\left(\ln\left(\sin^2\theta\right)\right)$ $=\dfrac{1}{\sin^2\theta}(2\sin\theta\cos\theta)$ $=2\dfrac{\cos\theta}{\sin\theta}$ $=2\cot\theta$
(e)	**Method 1: Using Implicit Differentiation** $3^{x\sin x}$ Let $y=3^{x\sin x}$ $\ln y = x\sin x\ln 3$ Differentiate wrt x, $\dfrac{1}{y}\dfrac{\mathrm{d}y}{\mathrm{d}x}=\left(x\cos x+\sin x\right)\ln 3$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=y(\ln 3)\left(x\cos x+\sin x\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=3^{x\sin x}(\ln 3)\left(x\cos x+\sin x\right)$ **Note:** The power is a variable in x, so we will need to use implicit differentiation here. **ALERT: You cannot do the usual way** $\dfrac{d}{dx}\left(3^{x\sin x}\right)\neq x\sin x\left(3^{x\sin x-1}\right)$ **Replace *y* by** $3^{x\sin x}$ **Method 2: By memorising formula** $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(3^{x\sin x}\right)=3^{x\sin x}(\ln 3)\left(x\cos x+\sin x\right)$ **Lecture Notes pg 4:** $\dfrac{\mathrm{d}}{\mathrm{d}x}a^{\mathrm{f}(x)}=a^{\mathrm{f}(x)}\mathrm{f}'(x)(\ln a)$

(f)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos^{-1}2x}\right)$$

$$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}2x\right)^{-1}$$

$$=-1\left(\cos^{-1}2x\right)^{-2}\times\left(-\frac{1}{\sqrt{1-(2x)^2}}\times 2\right)$$

$$=\frac{2\left(\cos^{-1}2x\right)^{-2}}{\sqrt{1-(2x)^2}}$$

$$=\frac{2}{\left(\cos^{-1}2x\right)^2\sqrt{1-(2x)^2}}$$

$$=\frac{2}{\left(\cos^{-1}2x\right)^2\sqrt{1-4x^2}}$$

ALERT:

$$\cos^{-1}x\neq\frac{1}{\cos x}$$

reciprocal of cosine function

inverse of cosine function

$$\cos^{-1}2x\neq\frac{1}{\cos 2x}$$

$$\frac{1}{\cos^{-1}2x}\neq\cos 2x$$

MF26:

$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$\cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$

Note:
Since this question is to differentiate $\cos^{-1}2x$, remember to chain rule $2x$, i.e. differentiate $2x$ to get 2.

(g)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^3 x\right)$$

$$=3\sin^2 x\left(\cos x\right)$$

(h)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\operatorname{cosec}\left(3x-\frac{1}{2}\right)}\right)$$

$$=\frac{1}{2}\left[\operatorname{cosec}\left(3x-\frac{1}{2}\right)\right]^{-\frac{1}{2}}\left(-\operatorname{cosec}\left(3x-\frac{1}{2}\right)\cot\left(3x-\frac{1}{2}\right)(3)\right)$$

$$=-\frac{3}{2}\cot\left(3x-\frac{1}{2}\right)\sqrt{\operatorname{cosec}\left(3x-\frac{1}{2}\right)}$$

(i)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(1-x^3\right)\right)$$

$$=\frac{1}{1+\left(1-x^3\right)^2}\left(-3x^2\right)$$

$$=\frac{-3x^2}{2-2x^3+x^6}$$

2 2004/I/14 modified

Find the x-coordinates of all the stationary points on the curve

$$y=\frac{x^3}{(x+a)^2}$$

where $a>1$ and state, with reasons, the nature of each point. [7]

2	**Solutions**
	$y=\dfrac{x^3}{(x+a)^2}, \quad a>1$ $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(x+a)^2 3x^2-2(x+a)x^3}{(x+a)^4}$ $=\dfrac{(x+a)(x^2)[3(x+a)-2x]}{(x+a)^4}$ $=\dfrac{x^2(x+3a)}{(x+a)^3}$ Always simplify by factorising, in the case for the numerator, the common factors are $(x+a)(x^2)$. **At stationary points,** $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{x^2(x+3a)}{(x+a)^3}=0$ $x^2(x+3a)=0$ $x^2=0$ or $(x+3a)=0$ $\therefore x=0$ or $x=-3a$ **To determine the nature of the stationary points, we will use the first derivative test. The reason why we do not recommend the use of second derivate test here is because it is very tedious to find** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. Since $a>1$, for $x=(-3a)^-$, we can choose a value like $x=-4a$ and substitute into $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(-4a)^2(-4a+3a)}{(-4a+a)^3}>0$ Since $a>1$, for $x=(-3a)^+$, we can choose a value like $x=-2a$ and substitute into $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{(-2a)^2(-2a+3a)}{(-2a+a)^3}<0$

x	0^-	0	0^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	/	—	/

$\therefore$ At $x=0$, it is a stationary point of inflexion

x	$(-3a)^-$	$-3a$	$(-3a)^+$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	/	—	\

$\therefore$ At $x=-3a$, it is a maximum turning point.

<u>Level 2</u>

3 Prove that $\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \frac{1}{1+x^2}$, where $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$.

Let $y = \tan^{-1} x$ where $-\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2}$.

Differentiating with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan y) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

$$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\tan^2 y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$

4 **2017(9758)/I/5**

When the polynomial $x^3 + ax^2 + bx + c$ is divided by $(x-1)$, $(x-2)$ and $(x-3)$, the remainders are 8, 12 and 25 respectively.

(i) Find the values of a, b and c. [4]

A curve has equation $y = \mathrm{f}(x)$, where $\mathrm{f}(x) = x^3 + ax^2 + bx + c$, with the values of a, b and c found in part **(i)**.

(ii) Show that the gradient of the curve is always positive. Hence explain why the equation $\mathrm{f}(x) = 0$ has only one real root and find this root. [3]

(iii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y = 2x - 3$. [3]

Solutions

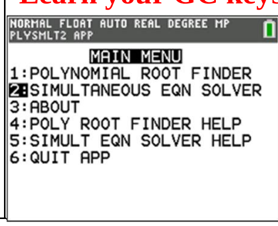
(i)

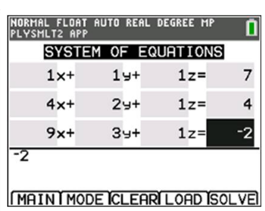
$$a + b + c = 7$$
$$4a + 2b + c = 4$$
$$9a + 3b + c = -2$$

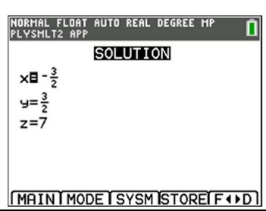
Recall your A math: Reminder and Factor Theorem.
There is **NO NEED** to do any Long Division here!

By GC, $a = -\frac{3}{2}$, $b = \frac{3}{2}$ and $c = 7$.

Learn your GC keystrokes well:







(ii)	$\mathrm{f}(x)=x^3-\frac{3}{2}x^2+\frac{3}{2}x+7$ $\mathrm{f}'(x)=3x^2-3x+\frac{3}{2}=3\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0$ for all $x\in\mathbb{R}$ When trying to show that gradient is positive, you need to use **algebraic method**. Method 1: completing the square since $\mathrm{f}'(x)$ is a quadratic expression. Method 2: Using Discriminant and coefficient of x^2: $D=(-3)^2-4(3)\left(\frac{3}{2}\right)=-9$ and coefficient of $x^2=3>0$, $\mathrm{f}'(x)>0$ for all $x\in\mathbb{R}$. Therefore the gradient of the curve is always positive. (shown) Since there are no turning points and $\mathrm{f}(x)$ is a polynomial, the curve cuts the *x*-axis exactly once. Therefore $\mathrm{f}(x)=0$ has exactly one real root. From GC, the root is -1.33 (3 s.f.).
(iii)	When the tangent to the curve is parallel to the line $y=2x-3$. Gradient of line is **2**. $\mathrm{f}'(x)=2$ $3x^2-3x+\frac{3}{2}=2$ $3x^2-3x-\frac{1}{2}=0$ Using GC, $x=1.15$ or -0.145 (3 s.f)

5 A curve is defined by the equation $x^3 + y^3 + 3xy = 1$. Find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at the point (2, –1).

Solutions	
	$x^3 + y^3 + 3xy = 1$ Apply product rule Differentiate wrt x, $3x^2 + 3y^2\frac{dy}{dx} + 3\left(x\frac{dy}{dx} + y\right) = 0$ (1) Sub. (2, –1) into (1), $3(2)^2 + 3(-1)^2\frac{dy}{dx} + 3\left(2\frac{dy}{dx} - 1\right) = 0$ $9\frac{dy}{dx} = -9$ $\frac{dy}{dx} = -1$ Differentiate (1) wrt x, $6x + 3y^2\frac{d^2y}{dx^2} + \frac{dy}{dx}\left(6y\frac{dy}{dx}\right) + 3\left(x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx}\right) = 0$ $6x + 3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 + 3\left(x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) = 0$ (2) Sub. (2, –1) and $\frac{dy}{dx} = -1$ into (2), $6(2) + 3(-1)^2\frac{d^2y}{dx^2} + 6(-1)(-1)^2 + 3\left(2\frac{d^2y}{dx^2} + 2(-1)\right) = 0$ $9\frac{d^2y}{dx^2} = 0$ $\frac{d^2y}{dx^2} = 0$

6 2013/ACJC/II/3 (modified part)

A curve C has parametric equations

$$x = 2 + t + \frac{2}{t}, \quad y = 2 - t + \frac{2}{t}$$

where $t < 0$.

(a) Find $\frac{dy}{dx}$ in terms of t and hence find the exact value of t for which the tangent to the curve at t is parallel to the y-axis. [4]

(b) Find the value of t for which the distance from the point (1, 0) to the curve is the shortest possible. [2]

(c) By considering $(x-2)^2$ and $(y-2)^2$, find the Cartesian equation of C. [2]

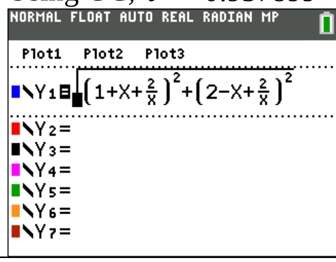
Q6	Solutions
(a)	$x = 2 + t + \frac{2}{t}$ $\frac{dx}{dt} = 1 - \frac{2}{t^2} = \frac{t^2 - 2}{t^2}$ $y = 2 - t + \frac{2}{t}$ $\Rightarrow \frac{dy}{dt} = -1 - \frac{2}{t^2} = \frac{-t^2 - 2}{t^2}$ $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$ $\frac{dy}{dx} = \frac{-t^2 - 2}{t^2} \times \frac{t^2}{t^2 - 2}$ $\frac{dy}{dx} = \frac{-t^2 - 2}{t^2 - 2}$ $\frac{dy}{dx} = \frac{2 + t^2}{2 - t^2}$ When tangent parallel to the y-axis, $\frac{dy}{dx}$ is **undefined.** **(Set the denominator of $\frac{dy}{dx}$ to be zero)** $t^2 - 2 = 0$ $t = \sqrt{2}$ or $t = -\sqrt{2}$ Since $t < 0$, $t = -\sqrt{2}$.

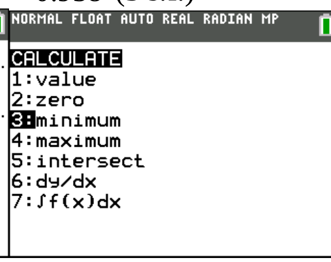
(b) Let distance from (1, 0) to any point on curve = L

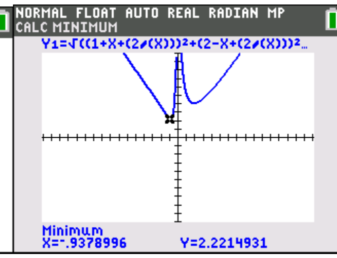
$$L=\sqrt{\left(2+t+\frac{2}{t}-1\right)^2+\left(2-t+\frac{2}{t}-0\right)^2}$$

$$=\sqrt{\left(1+t+\frac{2}{t}\right)^2+\left(2-t+\frac{2}{t}\right)^2}$$

Using GC, $t=-0.937899=-0.938$ (3 s.f.)







(c)

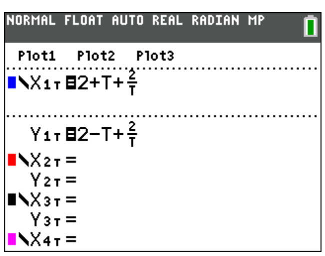
$$x-2=t+\frac{2}{t},\quad y-2=-t+\frac{2}{t}$$

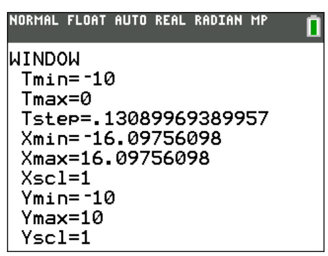
$$\Rightarrow (x-2)^2=t^2+4+\frac{4}{t^2},\quad (y-2)^2=t^2-4+\frac{4}{t^2}$$

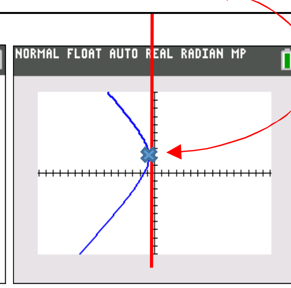
$$\underbrace{(x-2)^2-(y-2)^2=8}, \quad x\le 2-2\sqrt{2}$$

This is hyperbola centered at $(2,2)$.
Since $t<0$, using the GC, you will see only the "left" side of the hyperbola appearing.

Since tangent to the curve is parallel to the y-axis at $t=-\sqrt{2}$, it means that $x=2-2\sqrt{2}$ at the vertex of the hyperbola. Hence the domain of the hyperbola is $x\le 2-2\sqrt{2}$







Alternative Method:

$$x-y=2t\Rightarrow t=\frac{x-y}{2}$$

Subst. $t=\dfrac{x-y}{2}$ into x: $x=2+\dfrac{x-y}{2}+\dfrac{4}{x-y}$

Note: From drawing the parametric graph x is only defined up to the point where tangent is parallel to the y-axis (which is when $t=-\sqrt{2}$)

When $t=-\sqrt{2}$, $x=2-\sqrt{2}+\dfrac{2}{-\sqrt{2}}=2-2\sqrt{2}$

$\therefore$ Cartesian equation of C is $x=2+\dfrac{x-y}{2}+\dfrac{4}{x-y}, x\le 2-2\sqrt{2}$

7 2008(9740)/I/9 Part

It is given that

$$\mathrm{f}(x) = \frac{ax+b}{cx+d},$$

for non–zero constants a, b, c and d.

(i) Given that $ad - bc \neq 0$, show by differentiation that the graph of $y = \mathrm{f}(x)$ has no turning points. [3]

(ii) What can be said about the graph of $y = \mathrm{f}(x)$ when $ad - bc = 0$? [2]

(iii) Deduce from part **(i)** that the graph of

$$y = \frac{3x-7}{2x+1}$$

has a positive gradient at all points of the graph. [1]

7	Solutions
(i)	$\mathrm{f}'(x) = \dfrac{(cx+d)a-(ax+b)c}{(cx+d)^2}$ $= \dfrac{acx+ad-acx-bc}{(cx+d)^2}$ $= \dfrac{ad-bc}{(cx+d)^2}, \quad x \neq -\dfrac{d}{c}$ Since $y = \mathrm{f}(x)$ is given in the quotient form, we should apply quotient rule rather than product rule. Since $ad-bc \neq 0$, $\mathrm{f}'(x) \neq 0$ for all real values of x, $x \neq -\dfrac{d}{c}$, the graph of $y = \mathrm{f}(x)$ has no turning point.
(ii)	If $ad-bc=0$, $\mathrm{f}'(x) = 0$ The gradient of the tangent at any point is 0, thus the graph is a horizontal line. $y = \mathrm{f}(x)$ has turning points is the wrong answer. $\mathrm{f}(x) = k$ for all $x \in \mathbb{R} \setminus \left\{-\dfrac{d}{c}\right\}$ Graph of $y = \mathrm{f}(x)$ is a horizontal line. To find k, let $x = 0$: $\mathrm{f}(0) = k$ $\dfrac{a(0)+b}{c(0)+d} = k$ $k = \dfrac{b}{d}$ $\therefore \mathrm{f}(x) = \dfrac{b}{d}$ The graph is a horizontal line with equation $y = \dfrac{b}{d}$, $x \neq -\dfrac{d}{c}$.

	Alternative $\mathrm{f}(x)=\dfrac{ax+b}{cx+d}$ $=\dfrac{a}{c}+\dfrac{b-\frac{ad}{c}}{cx+d}$ $=\dfrac{a}{c}-\dfrac{ad-bc}{c(cx+d)}$ When $ad-bc=0$, $\mathrm{f}(x)=\dfrac{a}{c}$. The graph is a horizontal line with equation $y=\dfrac{a}{c},\ x\neq-\dfrac{d}{c}$.
(iii)	$\mathrm{f}(x)=\dfrac{3x-7}{2x+1}$, $a=3, b=-7, c=2, d=1$ $ad-bc=(3)(1)-(-7)(2)=17$ $\mathrm{f}'(x)=\dfrac{ad-bc}{(cx+d)^2}=\dfrac{17}{(2x+1)^2}>0$ for all real values of *x*, $x\neq-\dfrac{1}{2}$ $\therefore y=\dfrac{3x-7}{2x+1}$ has a positive gradient at all points of the graph.

8 2012(9740)/I/8

The curve C has equation

$$x-y=(x+y)^2.$$

It is given that C has only one turning point.

(i) Show that $1+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x+2y+1}$. [4]

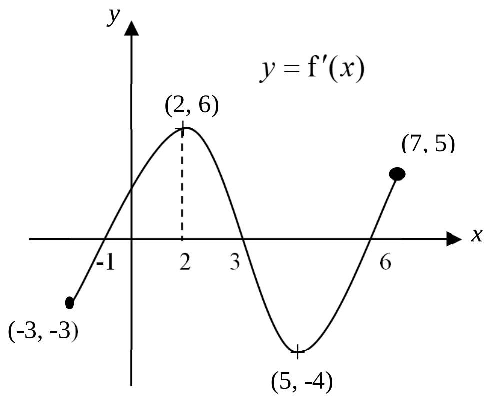
(ii) Hence, or otherwise, show that $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$. [3]

(iii) Hence state, with a reason, whether the turning point is a maximum or minimum. [2]

8	**Solutions**
(i)	$x-y=(x+y)^2$ Differentiate wrt x, $1-\frac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $1-\frac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)+2(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}$ $(2x+2y+1)\frac{\mathrm{d}y}{\mathrm{d}x}=1-2x-2y$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1-2x-2y}{2x+2y+1}$ $\text{LHS}=1+\frac{\mathrm{d}y}{\mathrm{d}x}$ $=1+\frac{1-2x-2y}{2x+2y+1}$ $=\frac{1+2x+2y+1-2x-2y}{2x+2y+1}$ $=\frac{2}{2x+2y+1}=$ RHS (shown) __**Alternative**__ $x-y=(x+y)^2$ Differentiate wrt x, $1-\frac{\mathrm{d}y}{\mathrm{d}x}=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $1=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\frac{\mathrm{d}y}{\mathrm{d}x}$ Add 1 on both sides to create a common factor $1+\frac{\mathrm{d}y}{\mathrm{d}x}$ $1+1=2(x+y)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)+\left(\frac{\mathrm{d}y}{\mathrm{d}x}+1\right)$ Factorise out common factor $1+\frac{\mathrm{d}y}{\mathrm{d}x}$ $2=\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)[2(x+y)+1]$ $1+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x+2y+1}$

(ii)	$1+\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2}{2x+2y+1}$ $1+\frac{\mathrm{d}y}{\mathrm{d}x}=2(2x+2y+1)^{-1}$ Differentiate wrt *x*, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=(2)(-1)(2x+2y+1)^{-2}\left(2+2\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=\left[\frac{-2}{(2x+2y+1)^2}\right](2)\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=-\left(\frac{2}{2x+2y+1}\right)^2\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=-\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ $=-\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$ (shown) "Hence" mean use the result in part **(i)** to do implicit differentiation directly. Replace $\frac{2}{2x+2y+1}$ by $1+\frac{\mathrm{d}y}{\mathrm{d}x}$ that is the result in part **(i)** to prove to the desired result.
(iii)	At turning point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$ $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-(1+0)^3=-1$ Since $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}<0,$ the turning point is a maximum point.

9 The diagram shows the graph of $y = \mathrm{f}'(x)$ on the interval $-3 \le x \le 7$.



State the x-coordinates of all the stationary points on the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 7$. Determine the nature of the stationary points.

Determine the set of values of x for which the graph of $y = \mathrm{f}(x)$ is

(i) strictly increasing,
(ii) strictly decreasing,
(iii) concave upwards,
(iv) concave downwards.

9	**Solutions**

Stationary points at $x = -1$, $x = 3$ and $x = 6$

x	-1^-	-1	-1^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	\	—	/

At $x = -1$, it is a minimum point.

x	3^-	3	3^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	/	—	\

At $x = 3$, it is a maximum point.

x	6^-	6	6^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	\	—	/

At $x = 6$, it is a minimum point.

<u>**Alternative Method**</u> (2nd Derivative Test)
From graph, using 2nd derivative test,

$x = -1$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$	Min point
$x = 3$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$	Max point
$x = 6$	$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$	Min point

(i) $\{x \in \mathbb{R} : -1 \le x \le 3 \text{ or } 6 \le x \le 7\}$

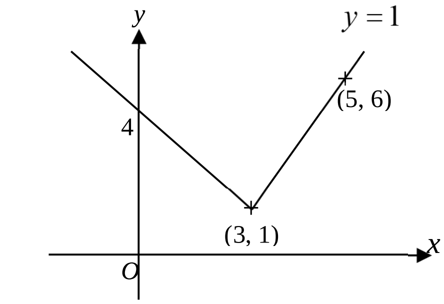
(ii) $\{x \in \mathbb{R} : -3 \le x \le -1 \text{ or } 3 \le x \le 6\}$

(iii) $\{x \in \mathbb{R} : -3 < x < 2 \text{ or } 5 < x < 7\}$

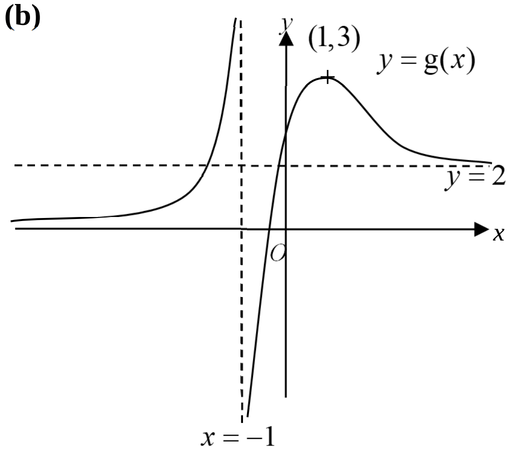
(iv) $\{x \in \mathbb{R} : 2 < x < 5\}$

10 Sketch the graphs of the derivative functions for each of the functions whose graphs are shown below.

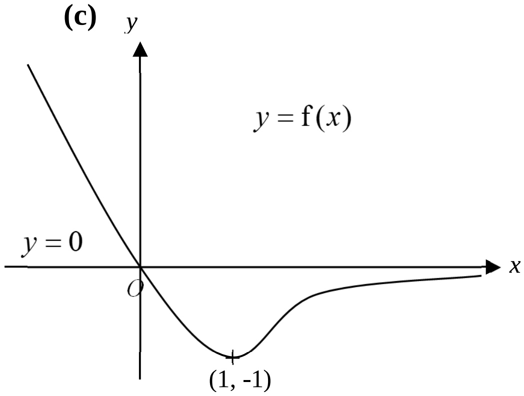
(a)



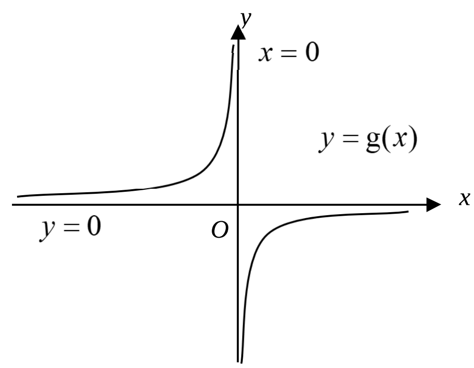
(b)



(c)

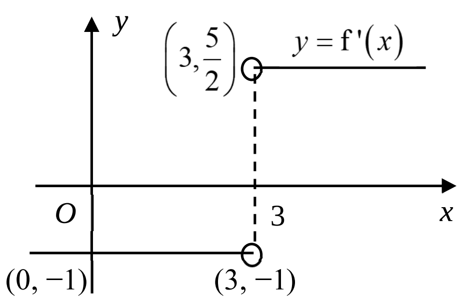


(d)

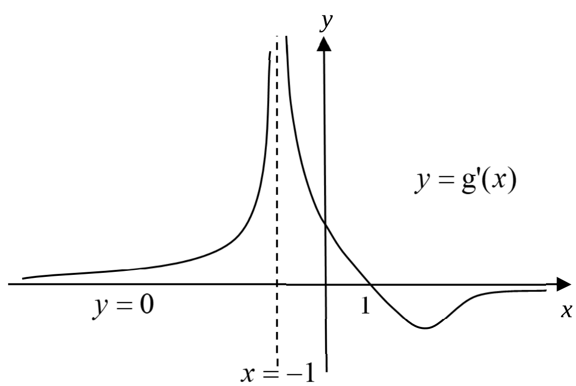


10	**Solutions**

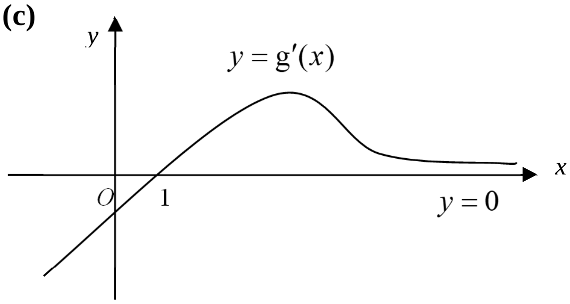
(a)



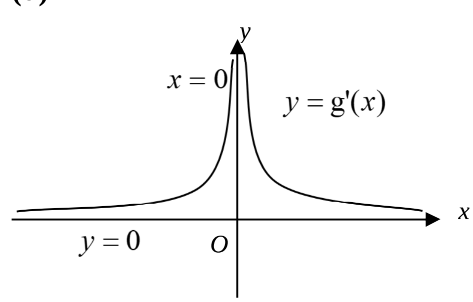
(b)



(c)



(d)



<u>Level 3</u>

11 2015(9740)/I/11 [parts]-modified

A curve C has parametric equations

$$x = \sin^3\theta, \qquad y = 3\sin^2\theta\cos\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cot\theta - \tan\theta$. [3]

(ii) Show that C has a stationary point when $\tan\theta = \sqrt{k}$, where k is an integer to be determined. Find, in non-trigonometric form, the exact coordinates of the stationary point. [5]

The line with equation $y = ax$, where a is a positive constant, meets C at the origin and at the point P.

(iii) Show that $\tan\theta = \dfrac{3}{a}$ at P. Find the exact value of a such that the line passes through the stationary point of C. [3]

Q11	**Solutions**
(i)	$x = \sin^3\theta,\ \ y = 3\sin^2\theta\cos\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = 3\sin^2\theta\cos\theta$ $\dfrac{\mathrm{d}y}{\mathrm{d}\theta} = 3\left[2\sin\theta\cos\theta\cos\theta + \sin^2\theta\left(-\sin\theta\right)\right] = 3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \dfrac{3\left(2\sin\theta\cos^2\theta - \sin^3\theta\right)}{3\sin^2\theta\cos\theta}$ $= \dfrac{2\cos^2\theta - \sin^2\theta}{\sin\theta\cos\theta}$ $= \dfrac{2\cos^2\theta}{\sin\theta\cos\theta} - \dfrac{\sin^2\theta}{\sin\theta\cos\theta}$ $= \dfrac{2\cos\theta}{\sin\theta} - \dfrac{\sin\theta}{\cos\theta}$ $= 2\cot\theta - \tan\theta$ (shown)

(ii)

At stationary point, $\frac{\mathrm{d}y}{\mathrm{d}x}=0$

$$2\cot\theta-\tan\theta=0$$

$$\frac{2}{\tan\theta}-\tan\theta=0$$

$$2-\tan^2\theta=0$$

$$\tan^2\theta=2$$

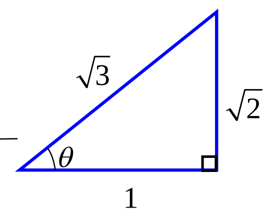
$$\tan\theta=\pm\sqrt{2}$$

$$\tan\theta=\sqrt{2}\left(\because \tan\theta \text{ is positive as } 0\le\theta\le\frac{\pi}{2}\right)\text{ (shown)}$$

$$\therefore k=2$$

Since $\tan\theta=\sqrt{2}$ and $0\le\theta\le\frac{\pi}{2}$,

$$\Rightarrow \cos\theta=\frac{1}{\sqrt{3}} \quad \text{and } \sin\theta=\sqrt{\frac{2}{3}}$$



Good strategy to learn if you want to obtain $\cos\theta$ and $\sin\theta$ from $\tan\theta$, **always draw a right angled triangle and apply trigo ratio.**

$$\therefore x=\left(\sin^3\theta\right)=\left(\sqrt{\frac{2}{3}}\right)^3=\frac{2\sqrt{6}}{9},$$

$$y=3\sin^2\theta\cos\theta=3\left(\sqrt{\frac{2}{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)=\frac{2}{\sqrt{3}}$$

$\therefore$ The coordinates of the stationary point is $\left(\frac{2\sqrt{6}}{9},\frac{2\sqrt{3}}{3}\right)$.

Note: Question wanted **exact coordinates** in **non-trigonometric**

(iii)

$x=\sin^3\theta,\ y=3\sin^2\theta\cos\theta$ ------ **(1)** **Parametric equations**

Since $y=ax$ ------- **(2)** **Cartesian equation**

Substitute (1) into (2) :

$$\overset{y}{3\sin^2\theta\cos\theta}=a\overset{x}{\left(\sin^3\theta\right)}$$

$$3\sin^2\theta\cos\theta=a\left(\sin^3\theta\right)$$

$$\sin^2\theta\left(3\cos\theta-a\sin\theta\right)=0$$

$$\sin\theta=0 \text{ or } \frac{\sin\theta}{\cos\theta}=\frac{3}{a}\Rightarrow\tan\theta=\frac{3}{a}\quad\text{(shown)}$$

(origin)

NOTE:
To solve for the intersection point between **(1)** and **(2)**, we always substitute the **parametric equations** into the **Cartesian equation** and solve for the parameter θ first.

Since $\tan\theta=\sqrt{2}$,

$$\therefore\sqrt{2}=\frac{3}{a}$$

$$a=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

12 2016/MJC/I/9

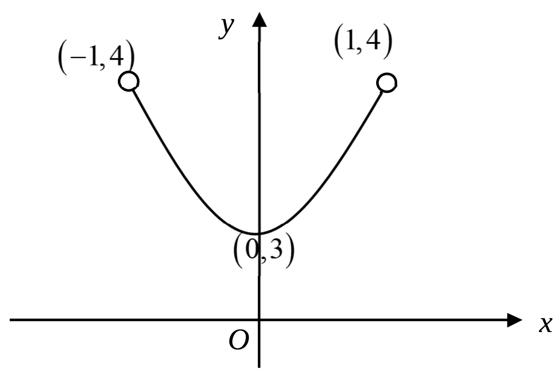
It is given that

$$f(x)=\frac{x}{\sqrt{(1-x^2)}}, \text{ where } -1<x<1.$$

(i) Show by differentiation that f is strictly increasing. [3]

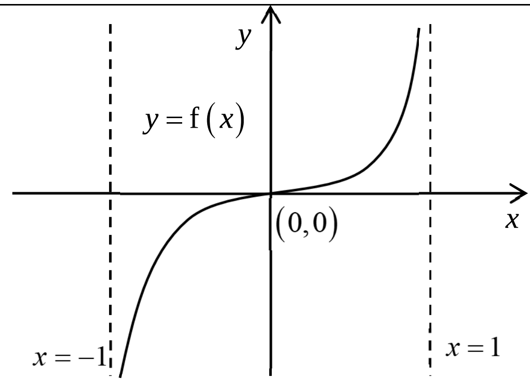
(ii) Sketch the graph of $y=f(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

The diagram below shows the graph of $y=g(x)$, which is continuous and differentiable on $(-1,1)$. It has a minimum turning point at $(0,3)$.



(iii) It is given that $w(x)=g(x)f(x)$, where $-1<x<1$. By finding $w'(x)$ and using your earlier results in parts **(i)** and **(ii)**, determine the number of stationary points on the graph of w. [4]

12	Solutions
(i)	$f(x)=\frac{x}{\sqrt{(1-x^2)}}$ $f'(x)=\frac{\sqrt{(1-x^2)}\cdot 1-x\cdot\frac{1}{2}\cdot(1-x^2)^{-\frac{1}{2}}\cdot(-2x)}{(1-x^2)}$ $=\frac{1}{\sqrt{(1-x^2)}}+\frac{x^2}{(1-x^2)^{\frac{3}{2}}}$ $=\frac{1}{(1-x^2)^{\frac{3}{2}}}>0\left(\because -1<x<1, \therefore (1-x^2)^{\frac{3}{2}}>0\right)$ Since $f'(x)>0$ for $-1<x<1$, f is strictly increasing.

(ii)	
(iii)	$\mathrm{w}'(x) = \mathrm{g}(x)\mathrm{f}'(x) + \mathrm{g}'(x)\mathrm{f}(x)$ From **(i)**, $\mathrm{f}'(x) > 0$ for $-1 < x < 1$. From **(ii)**, $\mathrm{f}(x) = \begin{cases} < 0 & \text{for } -1 < x < 0 \\ = 0 & \text{for } x = 0 \\ > 0 & \text{for } 0 < x < 1 \end{cases}$ From the given graph, $\mathrm{g}(x) > 0$ for $-1 < x < 1$ and $\mathrm{g}'(x) = \begin{cases} < 0 & \text{for } -1 < x < 0 \\ = 0 & \text{for } x = 0 \\ > 0 & \text{for } 0 < x < 1 \end{cases}$ Therefore, When $-1 < x < 0$, $\mathrm{w}'(x) > 0$. . When $x = 0$, $\mathrm{w}'(x) > 0$. When $0 < x < 1$, $\mathrm{w}'(x) > 0$. Since $\mathrm{w}'(x) > 0 \neq 0$ for $-1 < x < 1$, there are no stationary points in the graph of w.