Tampines Meridian Junior College
2024 H2 Mathematics (9758)
Chapter 7 Differentiation
Learning Package

Resources

- ☐ Core Concept Notes
- ☐ Discussion Questions
- ☐ Extra Practice Questions

SLS Resources

- ☐ Recordings on Core Concepts
- ☐ Quick Concept Checks

Reflection or Summary Page

H2 Mathematics (9758)
Chapter 7 Differentiation
Core Concept Notes

Success Criteria:

Surface Learning	Deep Learning	Transfer Learning
☐ Interpret the derivative $\frac{dy}{dx}$ as the gradient of a curve ☐ Use a graphing calculator to find the approximate value of a derivative at a given point ☐ Differentiate algebraic, trigonometric, logarithmic, exponential and inverse trigo functions ☐ Determine the nature of the stationary points (local maximum and minimum points and stationary point of inflexion) analytically, in simple cases, using the first derivative test or second derivative test ☐ Locate the maximum and minimum points using a graphing calculator	☐ Use chain rule, product rule and quotient rule ☐ Interpret graphically • $f'(x) > 0$ (strictly increasing) • $f'(x) = 0$ (stationary) • $f'(x) < 0$ (strictly decreasing) • $f''(x) > 0$ (concave upwards) • $f''(x) < 0$ (concave downwards) ☐ Use a graphing calculator to investigate the relationship between the graphs of $y = f(x)$ and $y = f'(x)$	☐ Differentiate functions defined implicitly ☐ Find higher derivatives ☐ Differentiate functions defined parametrically ☐ Relate the graph of $y = f'(x)$ to the graph of $y = f(x)$

§1 Differentiation

In mathematics, differentiation is a subfield of calculus concerned with the study of the rates at which quantities change. It is one of the two traditional divisions of calculus, the other being integration, the study of the area under a curve.

The primary objects of study in differentiation are the derivative of a function, related notions such as the differential, and their applications. The process of finding a derivative is called differentiation. Geometrically, the derivative at a point is the slope of the tangent line to the graph of the function at that point, provided that the derivative exists and is defined at that point.

The modern development of calculus is usually credited to Isaac Newton (1643–1727) and Gottfried Wilhelm Leibniz (1646–1716), who provided independent and unified approaches to differentiation and derivatives. The key insight, however, that earned them this credit, was the fundamental theorem of calculus relating differentiation and integration.

Note: For $\frac{\mathrm{d}}{\mathrm{d}x}y = \frac{\mathrm{d}y}{\mathrm{d}x}$

- $\frac{\mathrm{d}}{\mathrm{d}x}$: an **operator** to **differentiate a function** with respect to x
- $\frac{\mathrm{d}y}{\mathrm{d}x}$: a **function** which is the **result** of **differentiating *y*** with respect to x

1.1 Basic Rules of Differentiation

Let n and c be constants and u and v be differentiable functions of x i.e. $u = \mathrm{g}(x)$ and $v = \mathrm{h}(x)$

Basic Rules	**Example**
(1) $\frac{\mathrm{d}}{\mathrm{d}x}(c) = 0$ **(2)** $\frac{\mathrm{d}}{\mathrm{d}x}(cx^n) = c\frac{\mathrm{d}}{\mathrm{d}x}(x^n) = cnx^{n-1}$ **(3)** $\frac{\mathrm{d}}{\mathrm{d}x}(u \pm v) = \frac{\mathrm{d}}{\mathrm{d}x}u \pm \frac{\mathrm{d}}{\mathrm{d}x}v = \frac{\mathrm{d}u}{\mathrm{d}x} \pm \frac{\mathrm{d}v}{\mathrm{d}x}$	$\frac{\mathrm{d}}{\mathrm{d}x}(2x^3+3)$ $= 2\frac{\mathrm{d}}{\mathrm{d}x}(x^3) + \frac{\mathrm{d}}{\mathrm{d}x}(3)$ $= 2(3x^2) + 0$ $= 6x^2$

Note: Since $u = \mathrm{g}(x)$, the derivative $\frac{\mathrm{d}u}{\mathrm{d}x}$ may also be written as $\mathrm{g}'(x)$.

1.2 Chain Rule, Product and Quotient Rule of Two Functions

Let y, u and v be differentiable functions of x.

		Examples
Chain Rule	$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}$	$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(2x+7)^6 &= 6(2x+7)^5\frac{\mathrm{d}}{\mathrm{d}x}(2x+7)\\ &= 6(2x+7)^5(2)\\ &= 12(2x+7)^5\end{aligned}$
Product Rule	$\frac{\mathrm{d}}{\mathrm{d}x}(uv)=u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$	$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[3x(x+7)^6\right] &= 3x\cdot\frac{\mathrm{d}}{\mathrm{d}x}(x+7)^6+(x+7)^6\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x)\\ &= 3x\left[6(x+7)^5(1)\right]+(x+7)^6(3)\\ &= (x+7)^5\left[18x+3(x+7)\right]\\ &= 21(x+7)^5(x+1)\end{aligned}$
Quotient Rule	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right)=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$	$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2x-1}\right) &= \frac{(2x-1)\frac{\mathrm{d}}{\mathrm{d}x}(x^2)-(x^2)\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)}{(2x-1)^2}\\ &= \frac{(2x-1)(2x)-(x^2)(2)}{(2x-1)^2}\\ &= \frac{2x^2-2x}{(2x-1)^2}=\frac{2x(x-1)}{(2x-1)^2}\end{aligned}$

Note: Do not invent your own product rule or quotient rule!!

E.g. $\frac{\mathrm{d}}{\mathrm{d}x}3x(x+7)^6\neq\frac{\mathrm{d}}{\mathrm{d}x}(3x)\frac{\mathrm{d}}{\mathrm{d}x}(x+7)^6$ and $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{2x-1}\right)\neq\frac{\frac{\mathrm{d}}{\mathrm{d}x}(x^2)}{\frac{\mathrm{d}}{\mathrm{d}x}(2x-1)}$

§2 Differentiation of Logarithmic and Exponential Functions

Basic	**General (Using Chain Rule)**	**Example**
$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x}\right)=\mathrm{e}^{x}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\mathrm{f}(x)}\right)=\mathrm{e}^{\mathrm{f}(x)}\mathrm{f}'(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{3x^2}\right)=\mathrm{e}^{3x^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^2\right)$ $=6x\mathrm{e}^{3x^2}$
$\frac{\mathrm{d}}{\mathrm{d}x}\left(a^{x}\right)=a^{x}\ln a,$ where $a>0$	$\frac{\mathrm{d}}{\mathrm{d}x}a^{\mathrm{f}(x)}=a^{\mathrm{f}(x)}\mathrm{f}'(x)(\ln a)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(2^{x^2}\right)=2^{x^2}(\ln 2)(2x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)=\frac{1}{x}$	$\frac{\mathrm{d}}{\mathrm{d}x}(\ln \mathrm{f}(x))=\frac{1}{\mathrm{f}(x)}\mathrm{f}'(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\ln\left(x^3+1\right)\right]=\frac{1}{x^3+1}\left(3x^2\right)$ $=\frac{3x^2}{x^3+1}$
$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a x)=\frac{1}{\ln a}\cdot\frac{1}{x},$ where $a>0,\ a\neq 1.$	$\frac{\mathrm{d}}{\mathrm{d}x}(\log_a \mathrm{f}(x))$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln \mathrm{f}(x)}{\ln a}\right)$ $=\frac{1}{\ln a}\frac{\mathrm{d}}{\mathrm{d}x}(\ln \mathrm{f}(x))$ $=\frac{1}{\ln a}\cdot\frac{1}{\mathrm{f}(x)}\mathrm{f}'(x)$	$\frac{\mathrm{d}}{\mathrm{d}x}\left[\log_{10}\left(1+x^2\right)\right]$ $=\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\ln\left(1+x^2\right)}{\ln 10}\right]$ $=\frac{1}{\ln 10}\cdot\frac{2x}{1+x^2}$

Recall: If *a*, *b* and *c* are positive numbers and $a\neq 1, c\neq 1,$ then $\log_a b=\frac{\log_c b}{\log_c a}$.

Note:
Given a function which involves logarithmic functions, try to **simplify** using the laws of logarithms before carrying out differentiation.

Example 1 Differentiate $\ln\left(\frac{x^3}{1+x^2}\right)$ with respect to *x*.

Solution:

$$\ln\left(\frac{x^3}{1+x^2}\right)=\ln\left(x^3\right)-\ln\left(1+x^2\right)$$
$$=3\ln x-\ln\left(1+x^2\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(\frac{x^3}{1+x^2}\right)\right)=\frac{\mathrm{d}}{\mathrm{d}x}\left(3\ln x-\ln\left(1+x^2\right)\right)$$
$$=\frac{3}{x}-\frac{2x}{1+x^2}$$

Laws of Natural Logarithm

1) $\ln xy=\ln x+\ln y$
2) $\ln\frac{x}{y}=\ln x-\ln y$
3) $\ln x^y=y\ln x$

§3 Differentiation of Trigonometric Functions

Basic	**General (Using Chain Rule)**
$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)=\cos x$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin \mathrm{f}(x)\right)=\mathrm{f}'(x)\cos \mathrm{f}(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\cos x)=-\sin x$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos \mathrm{f}(x)\right)=\mathrm{f}'(x)\left(-\sin \mathrm{f}(x)\right)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\tan x)=\sec^2 x$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan \mathrm{f}(x)\right)=\mathrm{f}'(x)\sec^2 \mathrm{f}(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\cot x)=-\mathrm{cosec}^2\, x$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot \mathrm{f}(x)\right)=\mathrm{f}'(x)\left(-\mathrm{cosec}^2\, \mathrm{f}(x)\right)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x)=\sec x\tan x$ (MF27)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec \mathrm{f}(x)\right)=\mathrm{f}'(x)\sec \mathrm{f}(x)\tan \mathrm{f}(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{cosec}\, x)=-\mathrm{cosec}\, x\cot x$ (MF27)	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{cosec}\, \mathrm{f}(x)\right)=\mathrm{f}'(x)\left(-\mathrm{cosec}\, \mathrm{f}(x)\cot \mathrm{f}(x)\right)$

Note: For trigonometric functions above, x is in **radians.**
Refer to Annex A for the proof of derivative of $\sec x$.

Example 2 Differentiate

(a) $x^3 \sin 2x$ with respect to x,

(b) $\sin \theta^2 + \cos^2 \theta$ with respect to θ.

Solution:

(a)

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 \sin 2x\right)$$
$$= x^3 (\cos 2x)(2) + (\sin 2x)\left(3x^2\right)$$
$$= 2x^3 \cos 2x + 3x^2 \sin 2x$$
$$= x^2 (2x \cos 2x + 3 \sin 2x)$$

(b)

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin \theta^2 + \cos^2 \theta\right)$$
$$= 2\theta \cos \theta^2 - 2\cos \theta \sin \theta$$

§4 Inverse Trigonometric Functions

4.1 Definition

For the three trigonometric functions, their inverse functions are defined for the respective **principal range** as shown in the table below.

Inverse Trigonometric Function	Principal Domain	Principal Range
$y=\sin^{-1} x$	$[-1,1]$ i.e. $-1 \le x \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ i.e. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y=\cos^{-1} x$	$[-1,1]$	$[0,\pi]$
$y=\tan^{-1} x$	$\mathbb{R}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Note: $\sin^{-1} x \ne \frac{1}{\sin x}$. Hence $\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x \ne \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\sin x}\right)$

Remember that $\sin^{-1} x$ is the inverse function and **not the same** as $(\sin x)^{-1}$, which is the reciprocal function.

4.2 Differentiation of Inverse Trigonometric Functions

Basic (MF27)	General (Using Chain Rule)
$\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x)=\frac{1}{\sqrt{1-x^2}}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}\left(\mathrm{f}(x)\right)\right)=\frac{1}{\sqrt{1-\left(\mathrm{f}(x)\right)^2}}\cdot \mathrm{f}'(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x)=-\frac{1}{\sqrt{1-x^2}}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1}\left(\mathrm{f}(x)\right)\right)=-\frac{1}{\sqrt{1-\left(\mathrm{f}(x)\right)^2}}\cdot \mathrm{f}'(x)$
$\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x)=\frac{1}{1+x^2}$	$\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(\mathrm{f}(x)\right)\right)=\frac{1}{1+\left(\mathrm{f}(x)\right)^2}\cdot \mathrm{f}'(x)$

Refer to Annex A for the proof of derivative of Inverse Trigonometric Functions.

Example 3

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if

(a) $y=\sin^{-1}(3x)$

(b) $y=\tan^{-1}\left(5x^2\right)$

Solution:

(a)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sqrt{1-(3x)^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(3x)$$

$$=\frac{3}{\sqrt{1-9x^2}}$$

(b)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{1+\left(5x^2\right)^2}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2\right)$$

$$=\frac{10x}{1+25x^4}$$

§5 **Implicit Differentiation**

Consider the following two equations relating two variables, *x* and *y*.

(1) $y = 2x^2 + 3x$ where *y* can be expressed explicitly as a function of *x* and we can find $\frac{\mathrm{d}y}{\mathrm{d}x}$ directly.

(2) $y + y^2 = 2xy + 3x^2$ where it is not easy to express *y* explicitly in terms of *x*. This is an example of an **implicit function**.

The rules of differentiation that we have learnt can also be applied to implicit functions. In particular, if *y* is a function of *x* and $\mathrm{f}(y)$ a function of *y*, then by chain rule,

$$\boxed{\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(y) &= \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{f}(y)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \mathrm{f}'(y)\frac{\mathrm{d}y}{\mathrm{d}x}\end{aligned}}$$

Example 4 Differentiate $y + y^2 = 2xy + 3x^2$ with respect to *x* and find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

Solution:

$y + y^2 = 2xy + 3x^2$

Differentiate with respect to *x*,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(y + y^2\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(2xy + 3x^2\right) \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x(1)\frac{\mathrm{d}y}{\mathrm{d}x} + 2(1)y + 6x \\ \frac{\mathrm{d}y}{\mathrm{d}x} + 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 2x\frac{\mathrm{d}y}{\mathrm{d}x} + 2y + 6x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{2y + 6x}{2y - 2x + 1}\end{aligned}$$

Example 5 Differentiate x^x, $x > 0$ with respect to x.

Solution:

Let $y = x^x$

$\ln y = x \ln x$

Differentiate with respect to x,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln y) = \frac{\mathrm{d}}{\mathrm{d}x}(x \ln x)$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x \cdot \frac{1}{x} + 1 \cdot \ln x$$

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \ln x)$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = x^x(1 + \ln x)$$

§6 **Higher Derivatives**

Let $y = \mathrm{f}(x)$, we define

1st derivative	$\mathrm{f}'(x) = \mathrm{f}^{(1)}(x) = \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}}{\mathrm{d}x}(y)$
2nd derivative	$\mathrm{f}''(x) = \mathrm{f}^{(2)}(x) = \dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$
3rd derivative	$\mathrm{f}'''(x) = \mathrm{f}^{(3)}(x) = \dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$
4th derivative	$\mathrm{f}''''(x) = \dfrac{\mathrm{d}^4 y}{\mathrm{d}x^4} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^3 y}{\mathrm{d}x^3}\right)$
nth derivative	$\mathrm{f}^{(n)}(x) = \dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}}\right)$ $= \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\ldots\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\right)\right)\right)$ $(n \text{ times})$

Note:

(1) In general, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n} \neq \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^n$. E.g. $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} \neq \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$

(2) $\dfrac{1}{\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)} = \dfrac{\mathrm{d}x}{\mathrm{d}y}$ but this is **NOT TRUE** for higher derivatives. E.g. $\dfrac{1}{\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)} \neq \dfrac{\mathrm{d}^2 x}{\mathrm{d}y^2}$

(3) In general, $\mathrm{f}^n(x) \neq \mathrm{f}^{(n)}(x)$. E.g. $\mathrm{f}^2(x) = \mathrm{ff}(x)$ and $\mathrm{f}^{(2)}(x) = \dfrac{\mathrm{d}^2}{\mathrm{d}x^2}\left[\mathrm{f}(x)\right]$.

Example 6 If $y^2 = \sin 2x + \cos 2x$, show that $y\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y^2 = 0$.

Solution:

$y^2 = \sin 2x + \cos 2x$

Differentiate with respect to *x*,

$2y\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x - 2\sin 2x$

Differentiate with respect to *x*,

$$2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -4\sin 2x - 4\cos 2x$$

$$2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -4(\sin 2x + \cos 2x)$$

$$y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -2y^2$$

$$\therefore y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y^2 = 0 \text{ (Shown)}$$

§7 Differentiation of Parametric Equations

A relationship between x and y may also be expressed in terms of a third variable, called a **parameter**. e.g. $x=1-t$ ----- (1)

$y=t^2-4$ ----- (2)

In the above example, t is the parameter.

Equations (1) and (2) are called **Parametric Equations** of the curve.

The derivatives of functions given parametrically are obtained using the **Chain Rule.**

If $x=\mathrm{f}(t)$ and $y=\mathrm{g}(t)$, then $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\mathrm{d}y}{\mathrm{d}t}\times\dfrac{\mathrm{d}t}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}}$

Example 7

A curve has parametric equations $x=1+t^2$ and $y=\tan^{-1}t$. Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *t*.

Solution:

$$\frac{\mathrm{d}x}{\mathrm{d}t}=2t,\quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{1}{1+t^2}$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x}&=\frac{\mathrm{d}y}{\mathrm{d}t}\times\frac{\mathrm{d}t}{\mathrm{d}x}\\&=\frac{1}{1+t^2}\times\frac{1}{2t}\\&=\frac{1}{2t\left(1+t^2\right)}\end{aligned}$$

§8 Numerical Value of the Derivative at a Given Point

Example 8

Find the derivative of $\sqrt{\left(x^2-x+2\right)}$ at $x=2$.

Solution:

Method 1: Find Algebraically

$$\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{\left(x^2-x+2\right)}=\frac{\mathrm{d}}{\mathrm{d}x}(x^2-x+2)^{\frac{1}{2}}=\frac{1}{2}(x^2-x+2)^{-\frac{1}{2}}(2x-1)=\frac{2x-1}{2\sqrt{\left(x^2-x+2\right)}}$$

When $x=2$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{2(2)-1}{2\sqrt{\left(2^2-2+2\right)}}=\dfrac{3}{4}$

Method 2: Using GC [CALC] Command

Press !

Enter $Y_1=\sqrt{\left(x^2-x+2\right)}$

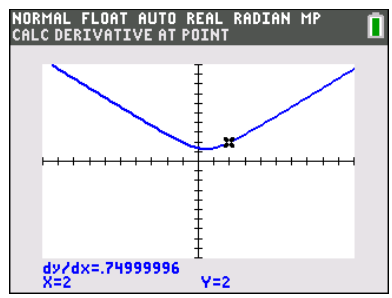
Press %

Press ` $ è

Choose option 6: $\mathrm{d}y/\mathrm{d}x$

Enter '2' for $x=2$, press e

The derivative of $\sqrt{\left(x^2-x+2\right)}$ at $x=2$ is 0.75



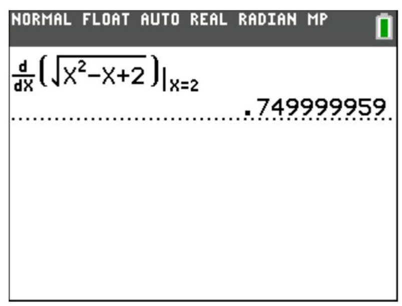
Method 3: Using GC [nDeriv] Command

Either press m , choose 8: nDeriv **(** and press e or

Press a @ 3 for the nDeriv function.

Complete the command $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{\left(x^2-x+2\right)}\right)\Big|_{x=2}$ and press e .

The derivative of $\sqrt{\left(x^2-x+2\right)}$ at $x=2$ is 0.75



Example 9

Evaluate the derivative of each of the following curves at the specified points.

(i) $y=x^2$ at $x=1$

(ii) $y=\ln\left(2x^3\right)$ at $x=5$

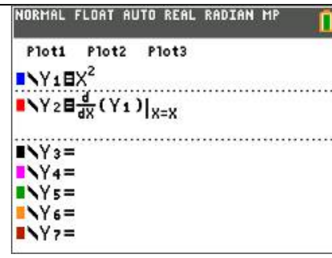
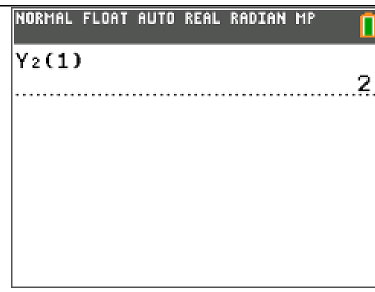
(iii) $x=t^2+1,\ y=t^3-t$ at $t=1.5$

(iv) $x=\sin 3t,\ y=2+\cos 3t$ at $t=-0.2$

Solution:

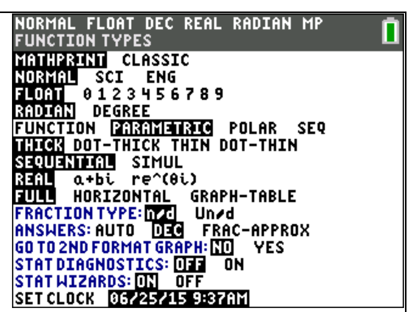
(i) Using GC, when $x=1$, $\frac{dy}{dx}=2$

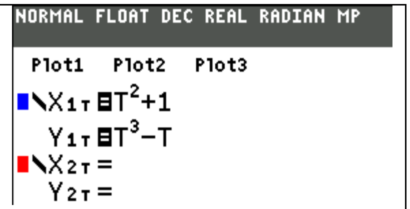
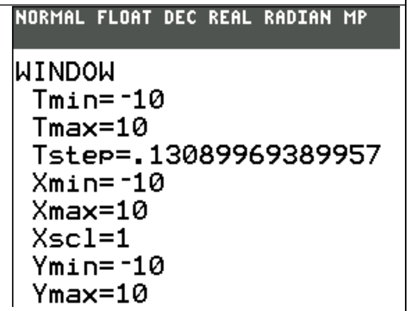
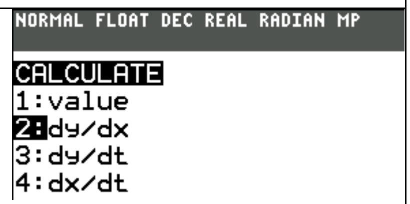
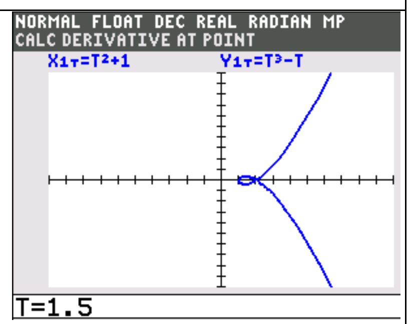
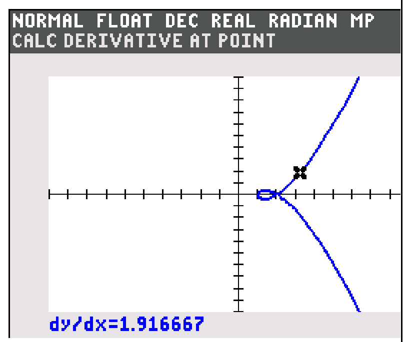
<u>Method 4: Using graph function (</u><u>useful for finding</u> **<u>Higher Order Derivative)</u>**

Press ! . Enter the equation of the graph $y_1=x^2$ Enter the derivative function keystrokes for $y_2=\frac{d}{dx}(y_1)$ i.e. Press a @ 3 for the nDeriv function. Press x . (Differentiate w.r.t x) To get function $\boxed{Y_1}$, press a $ e .	
Quit to main screen. Retrieve the differentiated function y_2 by pressing a $ e To evaluate the differentiated function at $x=1$ Note: • Y_2 gives the 1st order derivative function ie **differentiated** function • **<u>Extension:</u>** Hence, to find 2nd order derivative function, key in under $y_3=\frac{d}{dx}(y_2)$	

(ii) Using GC, when $x=5$, $\frac{dy}{dx}=0.6$

(iii) Using GC, when $t=1.5$, $\frac{dy}{dx}=1.92$ (3 s.f)

Press M Press ; 3 times and > to select parametric mode. Press ` ‖ M to go back to main screen.	

Press ! Key in $x = t^2 + 1, \ y = t^3 - t$.	
Adjust your window settings accordingly. Set $\mathrm{T_{min}} = -10$.	
Press ` $ and select Option: 2 to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.	
Press $\boxed{1.5}$ to find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at t = 1.5 and press e to get the next screen shown. Hence the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at t = 1.5 is 1.92 (3s.f).	 

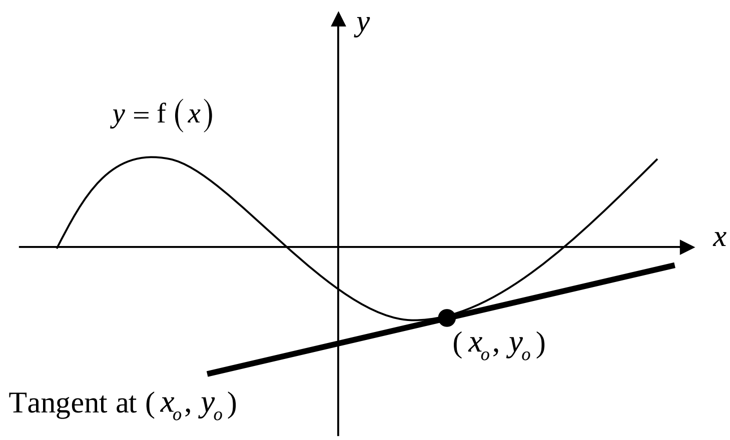
(iv) Using GC, when $t = -0.2$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.684$ (3 s.f.)

§9 <u>Gradient of a Point on a Curve</u>

Given a curve $y = \mathrm{f}(x)$, the gradient of a point (x_o, y_o) on the curve can be found by evaluating $\frac{\mathrm{d}y}{\mathrm{d}x}$ at that point i.e.

Gradient at point $(x_o, y_o) = \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\|_{x=x_o}$
$=$ Gradient of the tangent at (x_o, y_o)

Note: Tangent to the curve at (x_o, y_o) is a line touching the curve at (x_o, y_o)

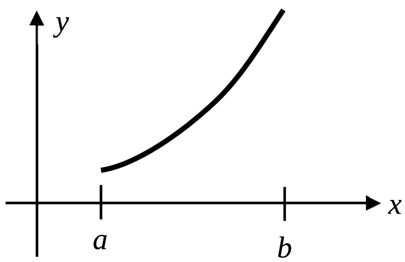
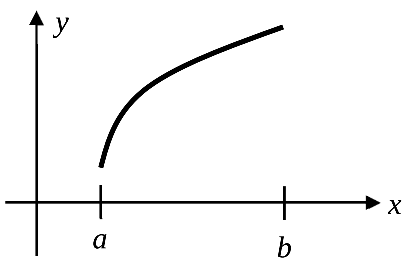
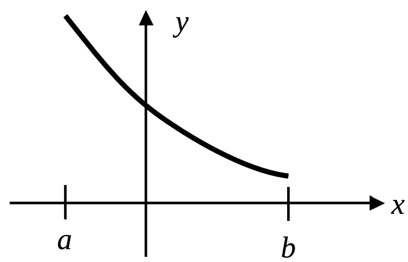
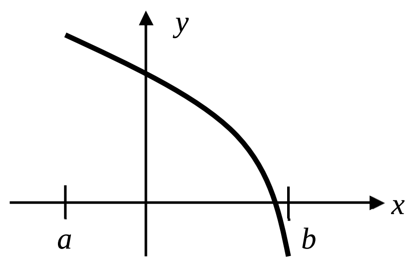


§9.1 **Increasing and decreasing function**	$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$	As x increases, y increases.
	$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$	As x increases, y decreases.
§9.2 **Stationary points**	$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	As x increases, there is no change in y.

9.1 <u>Increasing and Decreasing Functions</u>

- A function f is ***strictly increasing*** on the interval *I* if, for each $x_1 < x_2$ in *I*, $\mathrm{f}(x_1) < \mathrm{f}(x_2)$.
- A function f is ***strictly decreasing*** on the interval *I* if, for each $x_1 < x_2$ in *I*, $\mathrm{f}(x_1) > \mathrm{f}(x_2)$.

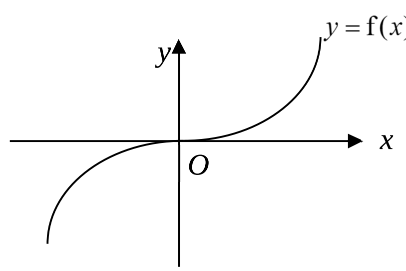
Let $y = \mathrm{f}(x)$

Strictly Increasing Functions	**Strictly Decreasing Functions**
f is a strictly increasing function means: • As x increases, $\mathrm{f}(x)$ increases	f is a strictly decreasing function means • As x increases, $\mathrm{f}(x)$ decreases.
 	 
$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ on $(a,b) \Rightarrow$ f is strictly increasing on (a,b) **[Proof not required in syllabus]**	$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ on $(a,b) \Rightarrow$ f is strictly decreasing on (a,b) **[Proof not required in syllabus]**

Think about this! Is it true that f is strictly increasing on $(a,b) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} > 0$ on (a,b).

Ans: No.

Consider $\mathrm{f}(x) = x^3$, from the graph of $y = \mathrm{f}(x)$, f is strictly increasing for $x \in \mathbb{R}$.



<u>**But**</u> $\mathrm{f}'(x) = 3x^2 \not> 0$ for all $x \in \mathbb{R}$, e.g. $\mathrm{f}'(0) = 0$.

Note: f is a strictly increasing function on $(a,b) \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} \geq 0$ on (a,b).

Example 10

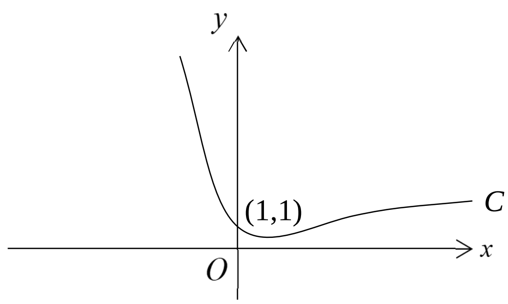
A curve C is defined by the parametric equations

$$x = t + \mathrm{e}^{t}, \quad y = t + \mathrm{e}^{-t}.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t, and hence find the coordinates of the stationary point of C.

(ii) Sketch the curve C.

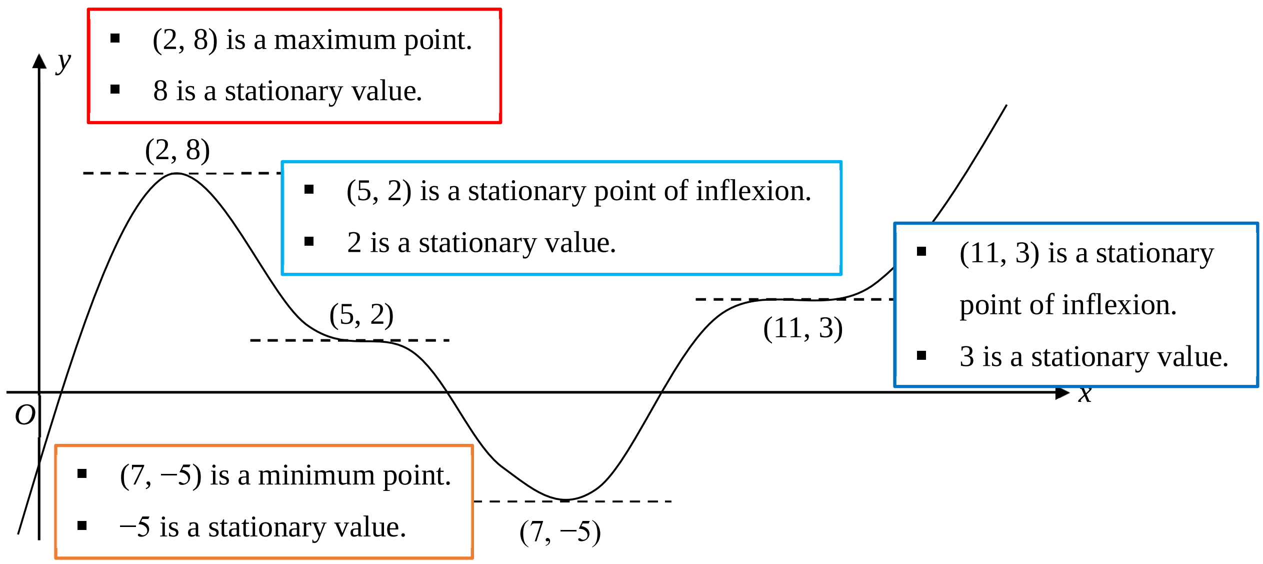
(iii) Hence, find the range of values of x for which the curve is strictly increasing.

(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \mathrm{e}^{t}$, $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \mathrm{e}^{-t}$ $\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^{t}}$ For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\frac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^{t}} = 0 \Rightarrow t = 0$ When $t = 0$, $x = 0 + \mathrm{e}^{0} = 1$ $y = 0 + \mathrm{e}^{-0} = 1$ The coordinates of the stationary point is (1, 1).
(ii)	
(iii)	For the graph to be strictly increasing, $x \geq 1$

9.2 Stationary Point

A point on a curve $y = \mathrm{f}(x)$ is called a **stationary point** if the gradient $\frac{\mathrm{d}y}{\mathrm{d}x}$ at that point is zero. There are 3 types of stationary points, namely

(a) maximum points,
(b) minimum points and
Turning points
(c) points of inflexion.



Note: At the stationary point of $y = \mathrm{f}(x)$,

- the gradient of the curve ($\frac{\mathrm{d}y}{\mathrm{d}x}$) is zero
- the tangent to the curve is parallel to the *x*-axis

9.3 To Determine the Nature of Stationary Point

Let the x-coordinate of the stationary point be a.

<u>Method 1</u> (First Derivative Test)

Find the signs of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for $x = a^-$ and $x = a^+$.

Note: a^- is a value just a little less than a.

a^+ is a value just a little greater than a.

x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	/	—	\

There is a **<u>maximum point</u>** at $x = a$.

x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	\	—	/

There is a **<u>minimum point</u>** at $x = a$.

x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	/	—	/

There is a **<u>stationary point of inflexion</u>** at $x = a$.

x	a^-	a	a^+
$\frac{\mathrm{d}y}{\mathrm{d}x}$	\	—	\

There is a **<u>stationary point of inflexion</u>** at $x = a$.

<u>Method 2</u> (Second Derivative Test)

Find the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ for $x = a$.

- If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, then there is a **<u>minimum point</u>** at $x = a$.
- If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, then there is a **<u>maximum point</u>** at $x = a$.
- If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, then there is **<u>no conclusion</u>**. Use method 1(First Derivative Test) to determine the nature of the stationary point.

Note: We usually use method 2 (Second Derivative Test) when $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ can be found easily.

Refer to Annex A on the different types of stationary points that can arise when $\left(\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0\right)$.

Example 11

Find the exact coordinates of the stationary points of the following curves and determine the nature of these stationary points.

(a) $y=\dfrac{x^2-x+3}{x+2}$ **(b)** $y=x^3$

Solution:

(a) $y=\dfrac{x^2-x+3}{x+2}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-1)(x+2)-(x^2-x+3)(1)}{(x+2)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{x^2+4x-5}{(x+2)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(x-1)(x+5)}{(x+2)^2}$$

At stationary points, $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0\Rightarrow(x-1)(x+5)=0$

$\therefore\ x=-5$ or $x=1$.

When $x=-5,\ y=-11$

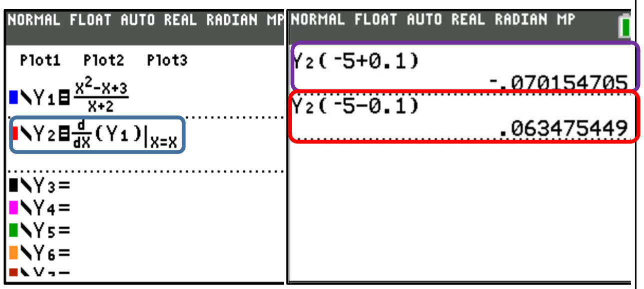
When $x=1,\ y=1$

The stationary points are $(-5,-11)$ and $(1,1)$.

<u>Method 1</u> (First Derivative Test)

x	-5^-	-5	-5^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	/	—	\

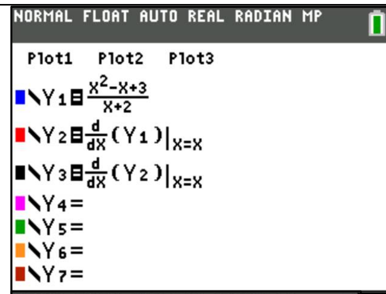
$\therefore(-5,-11)$ is a maximum point.

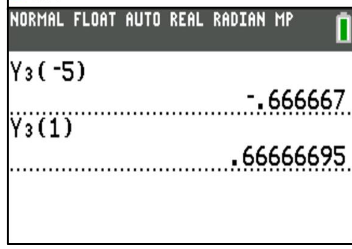


x	1^-	1	1^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	\	—	/

$\therefore(1,1)$ is a minimum point.

Method 2 (Second Derivative Test)





When $x=-5,\ y=-11,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=-0.666667<0$

$\therefore (-5,-11)$ is a maximum point.

When $x=1,\ y=1,\ \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}=0.667>0$

$\therefore (1,1)$ is a minimum point.

(b) $y=x^3$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3x^2=0$$

$\therefore x=0$

When $x=0$, $y=0$

x	0^-	0	0^+
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	/	—	/

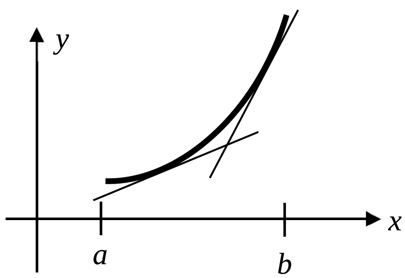
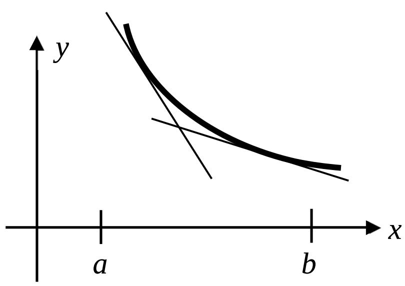
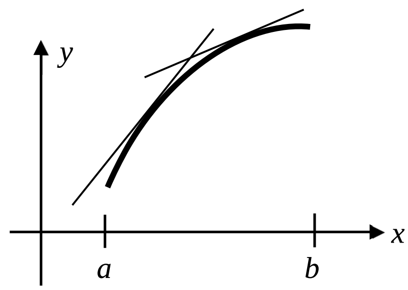
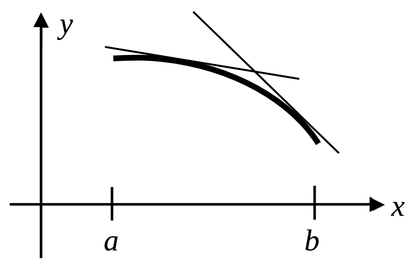
$\therefore (0,0)$ is a stationary point of inflexion.

9.4 Concavity of Curves

- A curve is concave upwards over an interval (a, b) if the curve is **above the tangent** in that interval.

- A curve is concave downwards over an interval (a, b) if the curve is **below the tangent** in that interval.

Let $y = \mathrm{f}\left(x\right)$.

Concave Upwards	**Concave Downwards**
f concave upwards means: • As x increases, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ increases	f concave downwards means: • As x increases, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ decreases
 	 
$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ on $\left(a, b\right) \Rightarrow$ f concave upwards on $\left(a, b\right)$. **[Proof not required in syllabus]**	$\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ on $\left(a, b\right) \Rightarrow$ f concave downwards on $\left(a, b\right)$. **[Proof not required in syllabus]**

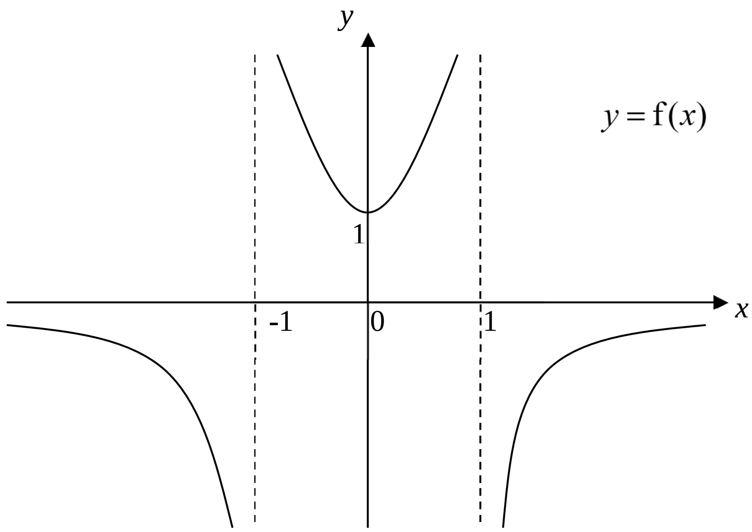
Example 12 Determine the set of values of x for which the graph given below is

(i) strictly increasing,

(ii) strictly decreasing,

(iii) concave upwards,

(iv) and, concave downwards.



Solution:

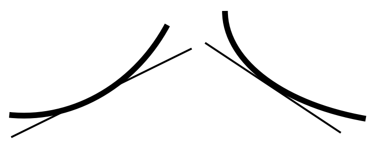
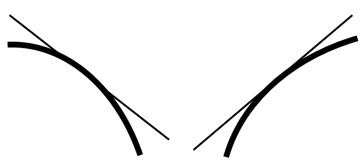
(i) $\{x \in \mathbb{R} : 0 \le x < 1\}$ or $\{x \in \mathbb{R} : x > 1\}$

(ii) $\{x \in \mathbb{R} : -1 < x \le 0\}$ or $\{x \in \mathbb{R} : x < -1\}$

(iii) $\{x \in \mathbb{R} : -1 < x < 1\}$

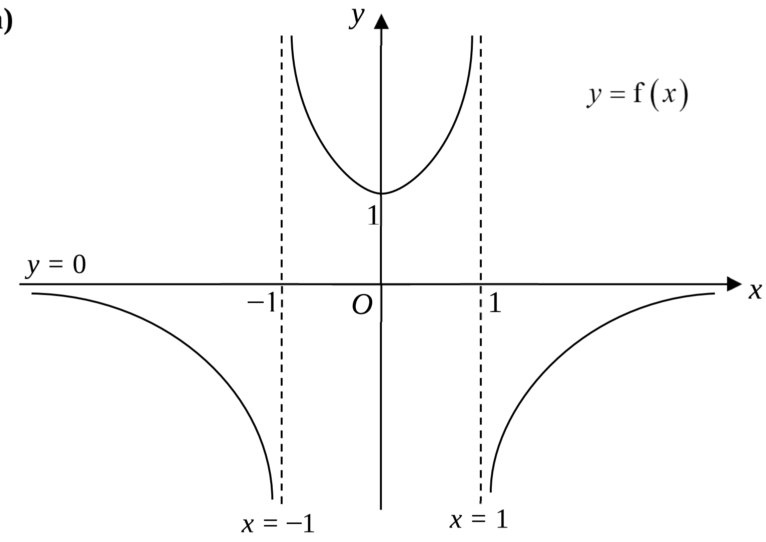
(iv) $\{x \in \mathbb{R} : x < -1\}$ or $\{x \in \mathbb{R} : x > 1\}$

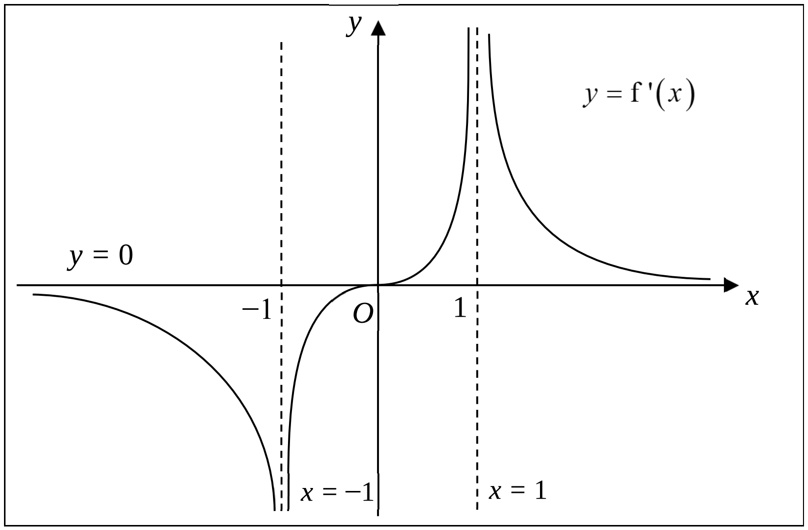
9.5 Relationship between the Graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}'(x)$

	Graph of $y=\mathrm{f}(x)$	**Values of** $\mathrm{f}'(x)$	**Graph of** $y=\mathrm{f}'(x)$
Stationary Points	Stationary Point: (x_1, y_1)	$\mathrm{f}'(x)=0$ at $x=x_1$	x-intercept: $(x_1, 0)$
Asymptotes	Vertical asymptote: $x = a$	$\mathrm{f}'(x)\to\pm\infty$, as $x\to a$	Vertical asymptote: $x = a$
	Horizontal asymptote: $y = b$ *i.e.* $\mathrm{f}(x)\to b$, as $x\to\pm\infty$	$\mathrm{f}'(x)\to 0$, as $x\to\pm\infty$	Horizontal asymptote: $y = 0$
	Oblique asymptote: $y=ax+b$ *i.e.* $\mathrm{f}(x)\to ax+b$, as $x\to\pm\infty$	$\mathrm{f}'(x)\to a$, as $x\to\pm\infty$	Horizontal asymptote: $y = a$
Gradient	Positive gradient.	$\mathrm{f}'(x)>0$	$y=\mathrm{f}'(x)$ is above x–axis.
	Negative gradient.	$\mathrm{f}'(x)<0$	$y=\mathrm{f}'(x)$ is below x–axis.
Concavity	Concave upwards on (a,b) 	$\mathrm{f}''(x)>0 \Rightarrow \mathrm{f}'(x)$ increases as x increases on (a,b)	Increasing on (a,b)
	Concave downwards on (a,b) 	$\mathrm{f}''(x)<0 \Rightarrow \mathrm{f}'(x)$ decreases as x increases on (a,b)	Decreasing on (a,b)

Example 13 Given the graphs of $y=\mathrm{f}(x)$ below, sketch the graph of $y=\mathrm{f}'(x)$.

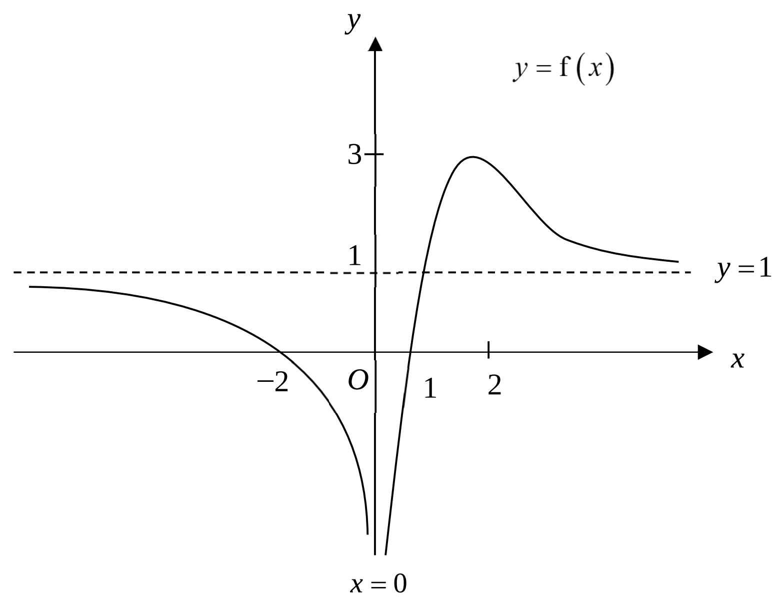
(a)

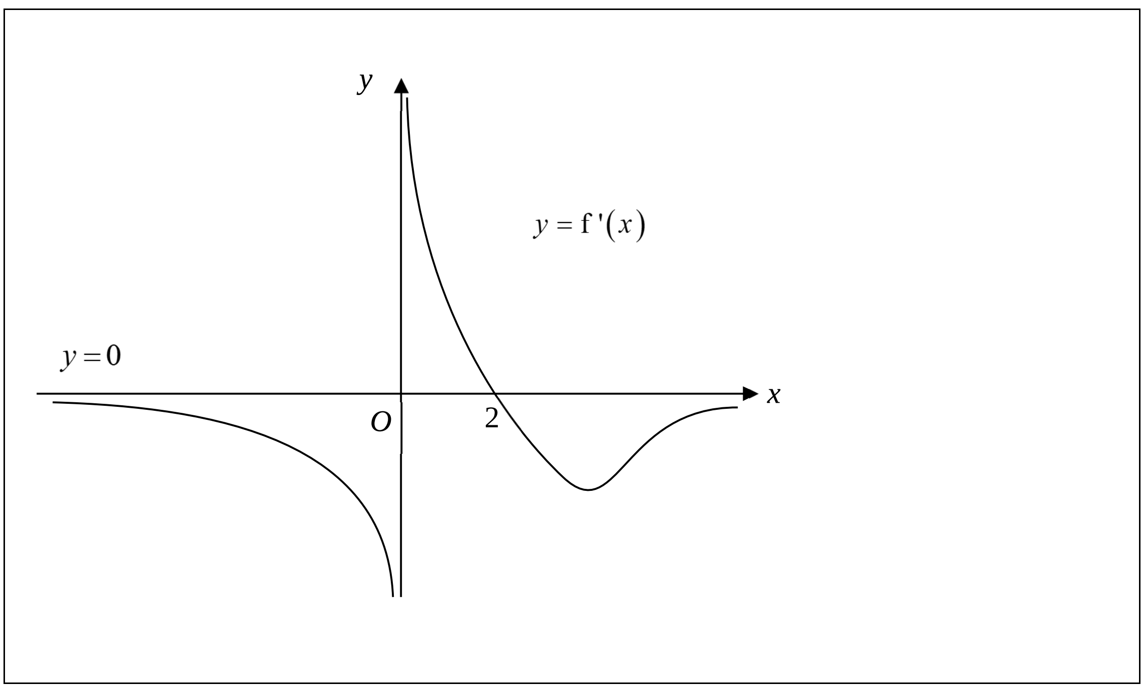




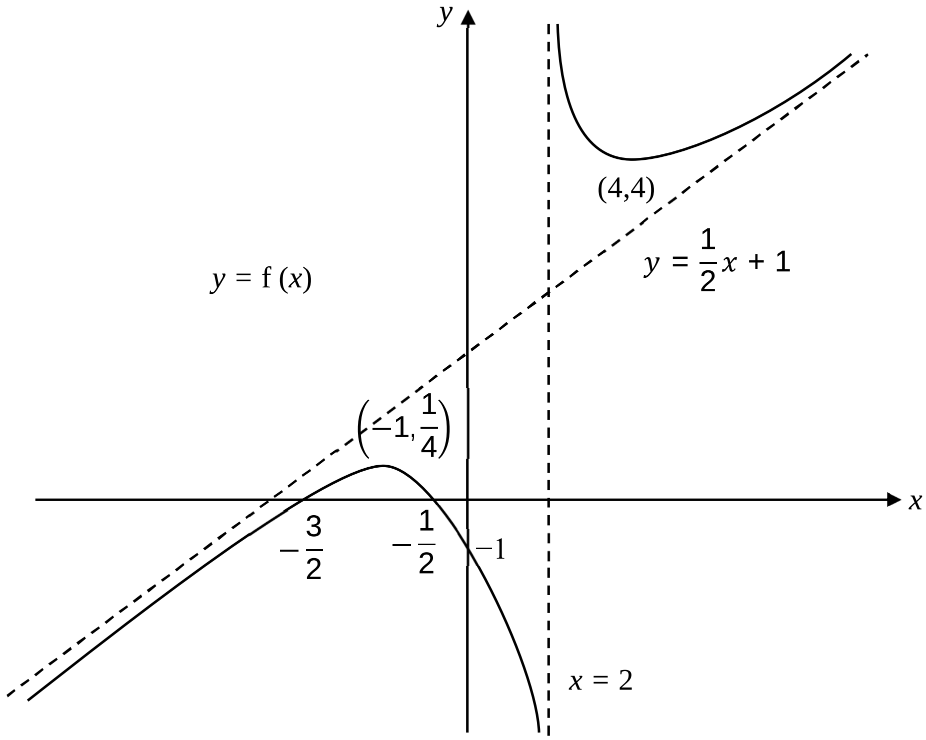
		Graph of $y=\mathrm{f}(x)$	**Graph of** $y=\mathrm{f}'(x)$
Stationary Points		Stationary point: $(0,1)$	x-intercept: $(0,0)$
Asymptotes		Vertical asymptotes: $x=-1$ and $x=1$	Vertical asymptotes: $x=-1$ and $x=1$
		Horizontal asymptote: $y=0$	Horizontal asymptote: y = 0
Gradient and Concavity	For $x<-1$	Negative gradient and concave downwards.	$y=\mathrm{f}'(x)$ is below x–axis and is decreasing.
	For $-1<x<0$	Negative gradient and concave upwards.	$y=\mathrm{f}'(x)$ is below x–axis and is increasing.
	For $0<x<1$	Positive gradient and concave upwards.	$y=\mathrm{f}'(x)$ is above x–axis and is increasing.
	For $x>1$	Positive gradient and concave downwards.	$y=\mathrm{f}'(x)$ is above x–axis and is decreasing.

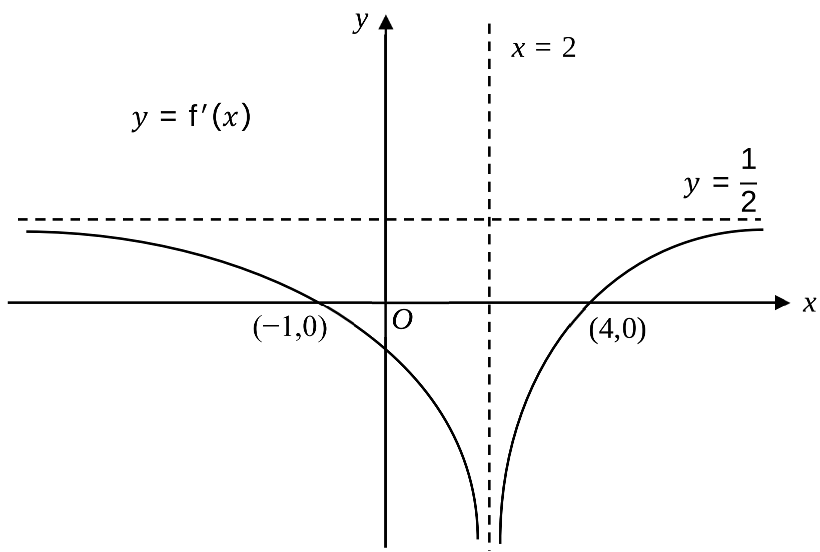
(b)





(c)



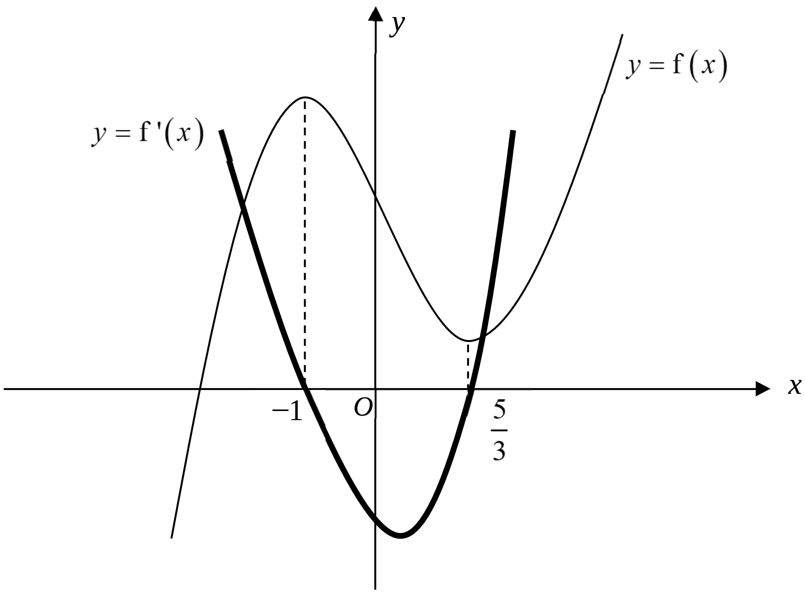


9.6 Drawing Derivative Graphs using a Graphing Calculator

Example 14

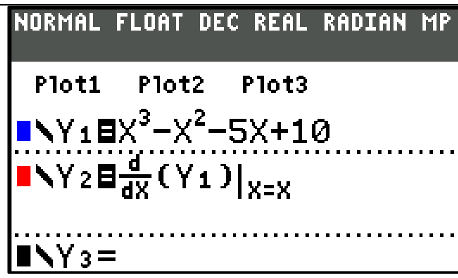
Sketch the graph of $y = f(x) = x^3 - x^2 - 5x + 10$ and that of its derivative $\frac{dy}{dx}\left(= f'(x)\right)$.

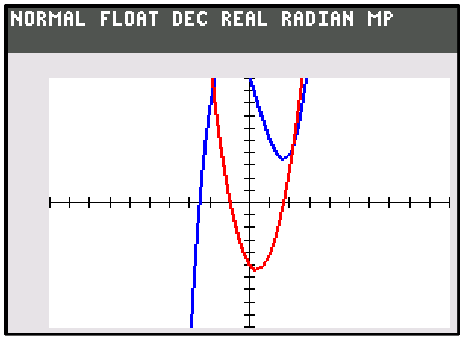
Solution:



Enter the equation of the original function under $\boxed{Y_1}$

Enter the derivative function under $\boxed{Y_2}$





Graphs of $y = x^3 - x^2 - 5x + 10$ (in blue) and its derivative(in red) $\frac{dy}{dx}\left(= f'(x)\right)$.

Annex A: Proof of Differentiation & Second Derivative Test

1 Differentiation of Trigonometric Functions

Proof: $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec x) = \sec x \tan x$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(\sec x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{\cos x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\cos x\right)^{-1}\\ &= -\left(\cos x\right)^{-2}\left(-\sin x\right)\\ &= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x}\\ &= \sec x \tan x\end{aligned}$$

2 Differentiation of Inverse Trigonometric Functions

Proof:

(1) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1} x) = \dfrac{1}{\sqrt{1-x^2}}$

Let $y = \sin^{-1} x$ where $-\dfrac{\pi}{2} \le y \le \dfrac{\pi}{2}$ and $-1 \le x \le 1$.

$\Rightarrow \sin y = x$

Differentiating with respect to x, $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin y) = \dfrac{\mathrm{d}}{\mathrm{d}x}(x)$

$$\Rightarrow \cos y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y} \quad \left(\because -\frac{\pi}{2} < y < \frac{\pi}{2}\right)$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sqrt{1-x^2}}$$

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos^{-1} x) = -\dfrac{1}{\sqrt{1-x^2}}$

Let $y = \cos^{-1} x$ where $0 \le y \le \pi$ and $-1 \le x \le 1$.

Differentiating with respect to *x*,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\cos y) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

$$-\sin y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sin y} \quad (\because 0 < y < \pi)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-\cos^2 y}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}}$$

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} x) = \dfrac{1}{1+x^2}$

(Proof will be discussed in tutorial)

Let $y = \tan^{-1} x$ where $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Differentiating with respect to *x*,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\tan y) = \frac{\mathrm{d}}{\mathrm{d}x}(x)$$

$$\sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\sec^2 y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+\tan^2 y}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$

3 To Determine the Nature of Stationary Point

Examples of $\left(\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0\right)$**:**

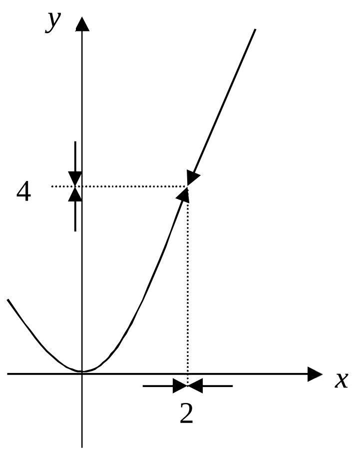
	$y = x^3$	$y = x^4$	$y = -x^4$
$\dfrac{\mathrm{d}y}{\mathrm{d}x}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3$	$\dfrac{\mathrm{d}y}{\mathrm{d}x} = -4x^3$
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$	$x = 0, y = 0$	$x = 0, y = 0$	$x = 0, y = 0$
$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$ at stationary point	$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 6x = 0$	$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 12x^2 = 0$	$\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -12x^2 = 0$
Nature of stationary point	**Stationary point of Inflexion**	**Minimum Point**	**Maximum Point**

Using 2nd derivative test, when the second derivative equals to zero, the result is inconclusive.

Annex B

Limit of a function

Consider the behaviour of a function $f(x) = x^2$ in the neighbourhood of $x = 2$. The following table gives the corresponding values of $f(x)$ for values of x close to 2, but not equal to 2.



x	1.9	1.99	1.999
$f(x)$	3.6100	3.9601	3.9960

x	2.1	2.01	2.001
$f(x)$	4.4100	4.0401	4.0040

It appears that as x approaches 2, $f(x)$ gets closer and closer to the value of 4. This is called the limiting value or the limit of $f(x)$ as x tends to 2.

Therefore, given a function $f(x)$, as x gets closer and closer to a, if $f(x)$ gets closer and closer to a fixed number L, we say that the limit of $f(x)$ exists and L is the limit of $f(x)$ as x approaches a. We write

$$\boxed{\lim_{x \to a} f(x) = L}$$

Note: $f(x)$ does not have to be defined at $x = a$. (Refer to the next example.)

Properties of limits

If $\lim_{x\to a} \mathrm{f}(x)$ and $\lim_{x\to a} \mathrm{g}(x)$ both exist, then

Properties	**Examples**
(1) $\lim_{x\to a} c\mathrm{f}(x) = c\lim_{x\to a}\mathrm{f}(x)$ where c is a constant.	$\lim_{x\to 3} 5x = 5\lim_{x\to 3} x = 5(3) = 15$
(2) $\lim_{x\to a}\left[\mathrm{f}(x)\pm\mathrm{g}(x)\right] = \lim_{x\to a}\mathrm{f}(x)\pm\lim_{x\to a}\mathrm{g}(x)$	$\lim_{x\to 2}\left(x^3+x\right) = \lim_{x\to 2}x^3+\lim_{x\to 2}x = 2^3+2 = 10$
(3) $\lim_{x\to a}\left[\mathrm{f}(x)\,\mathrm{g}(x)\right] = \left[\lim_{x\to a}\mathrm{f}(x)\right]\left[\lim_{x\to a}\mathrm{g}(x)\right]$	$\lim_{x\to 2}\left[x\ (x+1)\right] = \left[\lim_{x\to 2}x\right]\left[\lim_{x\to 2}(x+1)\right]$ $= 2(2+1) = 6$
(4) $\lim_{x\to a}\left[\dfrac{\mathrm{f}(x)}{\mathrm{g}(x)}\right] = \dfrac{\lim_{x\to a}\mathrm{f}(x)}{\lim_{x\to a}\mathrm{g}(x)}$ where $\lim_{x\to a}\mathrm{g}(x)\neq 0$.	$\lim_{x\to 3}\left[\dfrac{x+1}{x^2+1}\right] = \dfrac{\lim_{x\to 3}(x+1)}{\lim_{x\to 3}(x^2+1)}$ $= \dfrac{3+1}{3^2+1} = \dfrac{2}{5}$

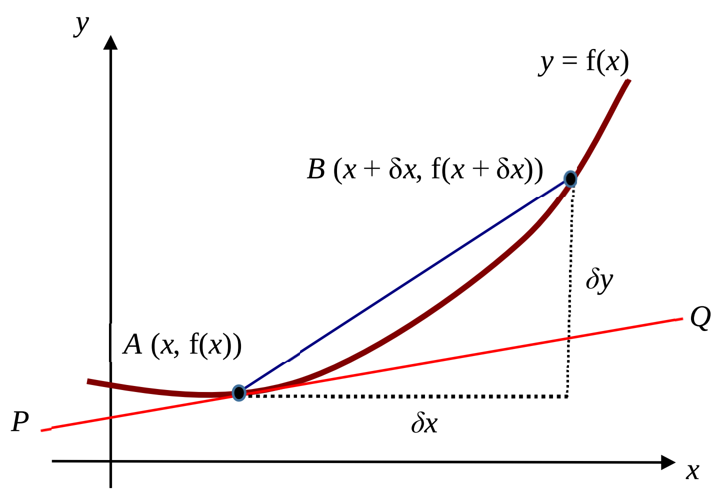
Example Evaluate the following limits:

(a) $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2}$	**(b)** $\lim_{x\to\infty}\dfrac{x}{x^2-2}$
Note that $\dfrac{x^2-x-2}{x-2}$ is undefined at $x=2$. $\lim_{x\to 2}\dfrac{x^2-x-2}{x-2} = \lim_{x\to 2}\dfrac{(x-2)(x+1)}{x-2}$ $= \lim_{x\to 2}(x+1)$ $= 2+1 = 3$	$\lim_{x\to\infty}\dfrac{x}{x^2-2} = \lim_{x\to\infty}\left(\dfrac{\frac{x}{x^2}}{1-\frac{2}{x^2}}\right)$ $= \lim_{x\to\infty}\left(\dfrac{\frac{1}{x}}{1-\frac{2}{x^2}}\right) = 0$

Note: The symbol for infinity (∞) does not represent a real number. The symbol (∞) denotes unbounded limits. $x\to\infty$ means x is increasing without bound and $x\to-\infty$ means x is decreasing without bound.

<u>Differentiation From First Principles [Not examinable]</u>

Consider any two points on the curve $y = \mathrm{f}(x)$ that are close to each other, say $A\left(x, \mathrm{f}(x)\right)$ and $B\left(x+\delta x, \mathrm{f}(x+\delta x)\right)$.



Gradient of chord AB = $\boxed{\dfrac{\delta y}{\delta x} = \dfrac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}}$

As $\delta x \to 0$, $B \to A$, gradient of chord $AB \to$ gradient of tangent PQ.

$$\therefore \text{Gradient of tangent } PQ = \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}$$

For any function $y = \mathrm{f}(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) = \displaystyle\lim_{\delta x \to 0} \frac{\delta y}{\delta x}$ or $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x} = \displaystyle\lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x)-\mathrm{f}(x)}{\delta x}}$

The above process of deriving $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is known as **<u>differentiation from first principles</u>**.

Example: By considering the derivative as a limit, find the first derivative of x^2.

Solution:

Let $y = \mathrm{f}(x) = x^2$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\left(x^2 + 2x \cdot \delta x + (\delta x)^2\right) - x^2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{2x \cdot \delta x + (\delta x)^2}{\delta x} \\ &= \lim_{\delta x \to 0} (2x + \delta x) \\ &= 2x\end{aligned}$$

***Students may also wish to derive the derivative of $\frac{1}{x}$, $\sin x$ and e^x from first principles.

H2 Mathematics (9758)
Chapter 7 Differentiation
Discussion Questions

Level 1

1 Differentiate the following with respect to the variable given:

(a) $\sqrt{x}\cot\left(\sqrt{x}\right)$ **(b)** $z\left(\ln(z)\right)^3$ **(c)** $\log_a(1-3x)^3$, where a is a positive constant and $a \neq 1$

(d) $\ln\left(\sin^2\theta\right)$ **(e)** $3^{x\sin x}$ **(f)** $\dfrac{1}{\cos^{-1}2x}$

(g) $\sin^3 x$ **(h)** $\sqrt{\operatorname{cosec}\left(3x-\dfrac{1}{2}\right)}$ **(i)** $\tan^{-1}\left(1-x^3\right)$

2 **2004/I/14 modified**

Find the x-coordinates of all the stationary points on the curve

$$y=\frac{x^3}{(x+a)^2}$$

where $a>1$ and state, with reasons, the nature of each point. [7]

Level 2

3 Prove that $\dfrac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1}x)=\dfrac{1}{1+x^2}$, where $-\dfrac{\pi}{2}<\tan^{-1}x<\dfrac{\pi}{2}$.

4 **2017(9758)/I/5**

When the polynomial x^3+ax^2+bx+c is divided by $(x-1)$, $(x-2)$ and $(x-3)$, the remainders are 8, 12 and 25 respectively.

(i) Find the values of a, b and c. [4]

A curve has equation $y=\mathrm{f}(x)$, where $\mathrm{f}(x)=x^3+ax^2+bx+c$, with the values of a, b and c found in part **(i)**.

(ii) Show that the gradient of the curve is always positive. Hence explain why the equation $\mathrm{f}(x)=0$ has only one real root and find this root. [3]

(iii) Find the x-coordinates of the points where the tangent to the curve is parallel to the line $y=2x-3$. [3]

5 A curve is defined by the equation $x^3+y^3+3xy=1$. Find the values of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the point (2, –1).

6 **2013/ACJC/II/3 (part)**

A curve C has parametric equations

$$x = 2 + t + \frac{2}{t}, \quad y = 2 - t + \frac{2}{t}$$

where $t < 0$.

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and hence find the exact value of t for which the tangent to the curve at t is parallel to the y-axis. [4]

(b) Find the value of t for which the distance from the point (1, 0) to the curve is the shortest possible. [2]

(c) Find the Cartesian equation of C. [2]

7 **2008(9740)/I/9 Part**

It is given that

$$\mathrm{f}(x) = \frac{ax + b}{cx + d},$$

for non–zero constants a, b, c and d.

(i) Given that $ad - bc \neq 0$, show by differentiation that the graph of $y = \mathrm{f}(x)$ has no turning points. [3]

(ii) What can be said about the graph of $y = \mathrm{f}(x)$ when $ad - bc = 0$? [2]

(iii) Deduce from part **(i)** that the graph of

$$y = \frac{3x - 7}{2x + 1}$$

has a positive gradient at all points of the graph. [1]

8 **2012(9740)/I/8**

The curve C has equation

$$x - y = \left(x + y\right)^2.$$

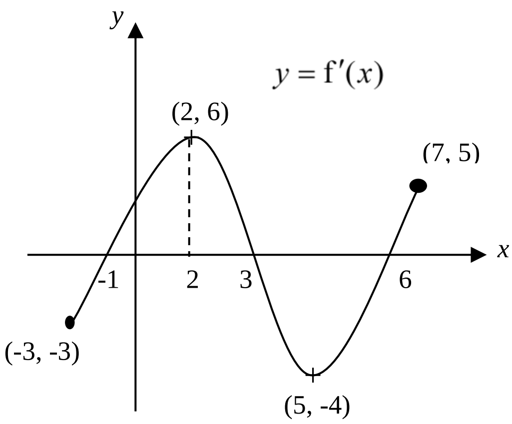
It is given that C has only one turning point.

(i) Show that $1 + \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2}{2x + 2y + 1}$. [4]

(ii) Hence, or otherwise, show that $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\left(1 + \frac{\mathrm{d}y}{\mathrm{d}x}\right)^3$. [3]

(iii) Hence state, with a reason, whether the turning point is a maximum or minimum. [2]

9 The diagram shows the graph of $y = \mathrm{f}'(x)$ on the interval $-3 \le x \le 7$.



State the x-coordinates of all the stationary points on the graph of $y = \mathrm{f}(x)$ for $-3 \le x \le 7$. Determine the nature of the stationary points.

Determine the set of values of x for which the graph of $y = \mathrm{f}(x)$ is

(i) strictly increasing,

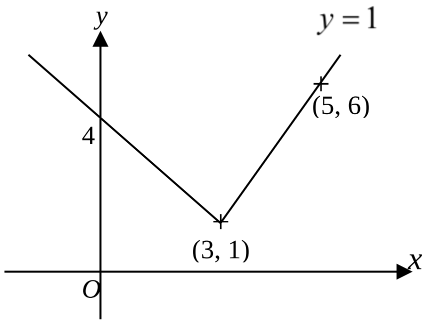
(ii) strictly decreasing,

(iii) concave upwards,

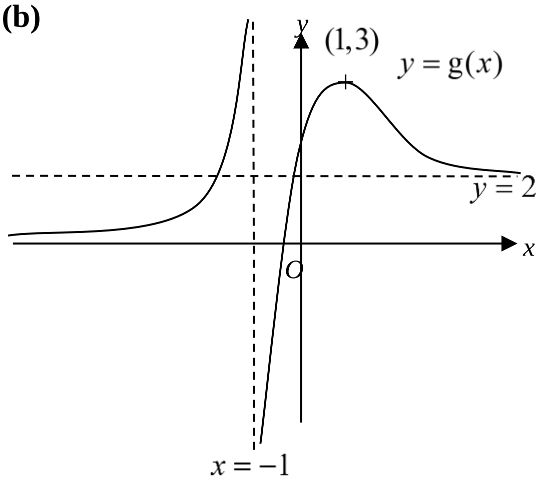
(iv) concave downwards.

10 Sketch the graphs of the derivative functions for each of the functions whose graphs are shown below.

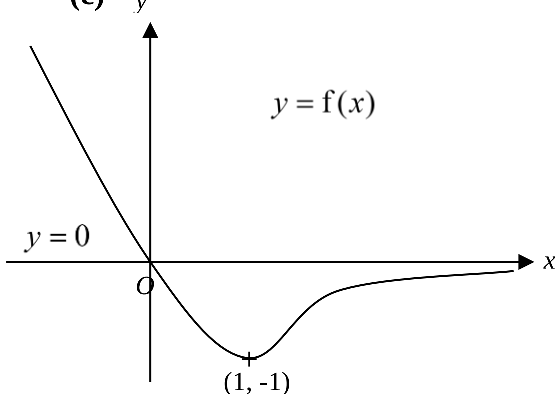
(a)



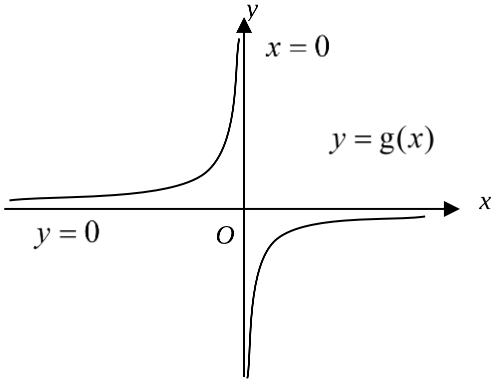
(b)



(c)



(d)



<u>Level 3</u>

11 2015(9740)/I/11 [parts]-modified

A curve C has parametric equations

$$x = \sin^3\theta, \qquad y = 3\sin^2\theta\cos\theta, \qquad \text{for } 0 \le \theta \le \frac{1}{2}\pi.$$

(i) Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\cot\theta - \tan\theta$. [3]

(ii) Show that C has a stationary point when $\tan\theta = \sqrt{k}$, where k is an integer to be determined. Find, in non-trigonometric form, the exact coordinates of the stationary point. [5]

The line with equation $y = ax$, where a is a positive constant, meets C at the origin and at the point P.

(iii) Show that $\tan\theta = \dfrac{3}{a}$ at P. Find the exact value of a such that the line passes through the stationary point of C. [3]

12 2016/MJC/I/9

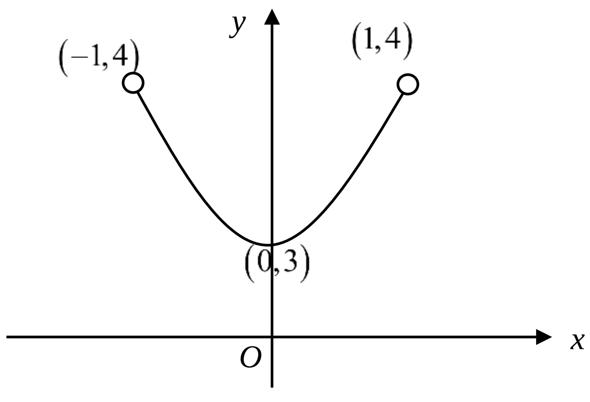
It is given that

$$\mathrm{f}(x) = \frac{x}{\sqrt{(1-x^2)}}, \text{ where } -1 < x < 1.$$

(i) Show by differentiation that f is strictly increasing. [3]

(ii) Sketch the graph of $y = \mathrm{f}(x)$, stating the equations of any asymptotes and the coordinates of any points of intersection with the axes. [3]

The diagram below shows the graph of $y = \mathrm{g}(x)$, which is continuous and differentiable on $(-1, 1)$. It has a minimum turning point at $(0, 3)$.



(iii) It is given that $\mathrm{w}(x) = \mathrm{g}(x)\mathrm{f}(x)$, where $-1 < x < 1$. By finding $\mathrm{w}'(x)$ and using your earlier results in parts **(i)** and **(ii)**, determine the number of stationary points on the graph of w. [4]

Answer Key

No.	Answer
1	**(a)** $\frac{1}{2\sqrt{x}}\cot\left(\sqrt{x}\right)-\frac{1}{2}\operatorname{cosec}^2\left(\sqrt{x}\right)$ **(b)** $3\left(\ln(z)\right)^2+\left(\ln(z)\right)^3$ **(c)** $\frac{9}{\left(\ln a\right)\left(3x-1\right)}$ **(d)** $2\cot\theta$ **(e)** $3^{x\sin x}(\ln 3)\left(x\cos x+\sin x\right)$ **(f)** $\frac{2}{\left(\cos^{-1}2x\right)^2\sqrt{1-4x^2}}$ **(g)** $3\sin^2 x\left(\cos x\right)$ **(h)** $-\frac{3}{2}\cot\left(3x-\frac{1}{2}\right)\sqrt{\operatorname{cosec}\left(3x-\frac{1}{2}\right)}$ **(i)** $\frac{-3x^2}{2-2x^3+x^6}$
2	At $x=0$, it is a stationary point of inflexion. At $x=-3a$, it is a maximum point.
4	**(i)** $a=-\frac{3}{2},\ b=\frac{3}{2}$ and $c=7$. **(ii)** -1.33 **(iii)** -0.145 or 1.15
5	$\frac{dy}{dx}=-1;\ \frac{d^2y}{dx^2}=0$
6	**(a)** $\frac{dy}{dx}=\frac{2+t^2}{2-t^2};\ t=-\sqrt{2}$ **(b)** $t=-0.938$; $\left(x-2\right)^2-\left(y-2\right)^2=8,\quad x\le 2-2\sqrt{2}$
7	The graph is a horizontal line with equation $y=\frac{b}{d},\ x\neq-\frac{d}{c}$.
8	**(iii)** The turning point is a maximum point
9	At $x=-1$, it is a minimum point. At $x=3$, it is a maximum point. At $x=6$, it is a minimum point. **(i)** $\left\{x\in\mathbb{R}:-1\le x\le 3\right\}$ or $\left\{x\in\mathbb{R}:6\le x\le 7\right\}$ **(ii)** $\left\{x\in\mathbb{R}:-3\le x\le -1\right\}$ or $\left\{x\in\mathbb{R}:3\le x\le 6\right\}$ **(iii)** $\left\{x\in\mathbb{R}:-3< x< 2\right\}$ or $\left\{x\in\mathbb{R}:5< x< 7\right\}$ **(iv)** $\left\{x\in\mathbb{R}:2< x< 5\right\}$
11	**(ii)** $\left(\frac{2\sqrt{6}}{9},\frac{2\sqrt{3}}{3}\right)$ **(iii)** $\frac{3\sqrt{2}}{2}$
12	**(iii)** There are no stationary points in the graph of w .

H2 Mathematics (9758)
Chapter 7 Differentiation
Extra Practice Questions

1 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ by implicit differentiation.

(a) $(x+1)y+x^4y^2=1$

(b) $\ln xe^x = y\ln x^2, \quad x>0$

(c) $e^{2y}-e^{x^2+y^2}=\frac{1}{y}$

(d) $\frac{y^2}{x}+(x-y)^2-\cos(xy)=3$

2 **2014(9758)/I/2**

The curve C has equation $x^2y+xy^2+54=0$. Without using a calculator, find the coordinates of the point on C at which the gradient is -1, showing that there is only one such point.

3 Differentiate the following with respect to x, leaving your answer in the simplest form:

(a) $\sin\left(\cos^{-1}(3x)\right)$

(b) $\sin^{-1}(\cos x)$

4 A curve C has parametric equations

$$x=\theta-\sin\theta, \qquad y=1-\cos\theta, \qquad 0\le\theta\le 2\pi.$$

At the point on the curve with parameter α, the gradient is $\frac{1}{2}$.

Show that $2\sin\alpha+\cos\alpha=1$.

5 **2018/AJC/Promo/5 (modified)**

A curve C has equation $x^2-xy+y^2-9=0$.

(i) Show that $(2y-x)\frac{\mathrm{d}y}{\mathrm{d}x}=y-2x$.

(ii) Find the exact coordinates of the stationary points.

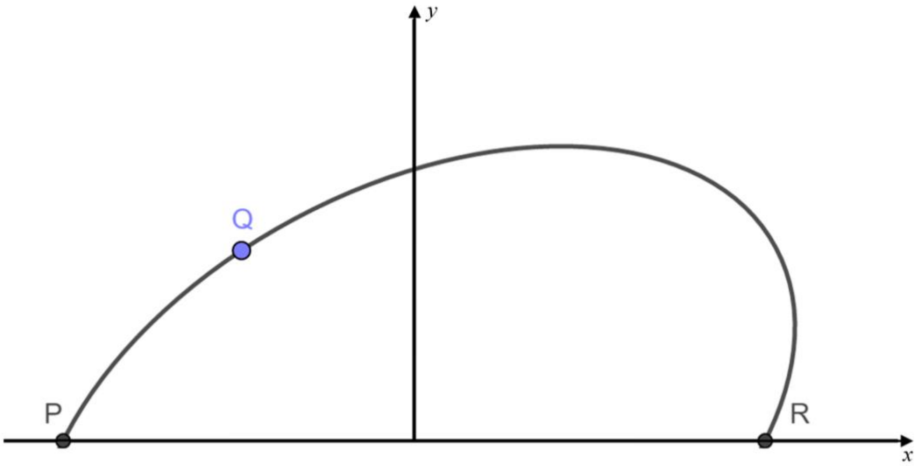
(iii) By further differentiation of the equation in part **(i)**, determine the nature of the stationary points found in part **(ii)**.

6 **2018/MI/Promo/5(a)(i)**

Given a function $\mathrm{f}(x)=x^2e^{x^2}$, for $x\in\mathbb{R}$.

By differentiation, find the range of values of x for which the function is increasing. [4]

7 2018/EJC/Promo/10(i)-(iii)



A curve C has equation $5y^2 - 3xy + 3x^2 - 48 = 0$, with $y \geq 0$. C cuts the x-axis at points P and R.

(i) The point $Q(x, y)$ lies on C as shown on the diagram above. Show that A, the area of the triangle PQR, is given by $A = 4y$. [2]

(ii) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y. [2]

(iii) Hence, find the exact value of x for which A has a maximum value.
[Note: There is no need to show that A is a maximum.] [3]

8 2018/YJC/Promo/7(i)

A curve C has parametric equations

$$x = \mathrm{e}^t \sin t, \; y = \mathrm{e}^{-t} \cos t \qquad \text{for } -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}.$$

Using differentiation, show that C has no stationary points. [3]

9 2017/MJC/Promo/5

Suppose the following facts are known about the function g and its derivatives:

$$\mathrm{g}\left(\frac{\pi}{2}\right) = 1, \;\; \mathrm{g}'\left(\frac{\pi}{2}\right) = 0, \;\; \mathrm{g}''\left(\frac{\pi}{2}\right) = -1$$

Consider the function $\mathrm{f}(x) = \mathrm{e}^{\mathrm{g}(x)}$.

(a) Using differentiation, show that $\mathrm{f}(x)$ has a stationary point at $x = \frac{\pi}{2}$ and determine the nature of this stationary point. [5]

(b) Given that $\mathrm{g}(x)$ is sin x, show that $\mathrm{f}(x)$ is increasing on the interval where $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [2]

Answer Key

No.	Answers
1	**(a)** $-\dfrac{4x^3y^2+y}{x+1+2x^4y}$ **(b)** $\dfrac{x+1-2y}{2x\ln x}$ **(c)** $\dfrac{2xe^{x^2+y^2}}{2e^{2y}-2ye^{x^2+y^2}+y^{-2}}$ **(d)** $\dfrac{\left(\frac{y}{x}\right)^2-2(x-y)-y\sin(xy)}{\frac{2y}{x}-2(x-y)+x\sin(xy)}$
2	$(-3,-3)$
3	**(a)** $\dfrac{-9x}{\sqrt{\left(1-9x^2\right)}}$ **(b)** $\begin{cases}1 & \text{if } \sin x<0\\ -1 & \text{if } \sin x>0\end{cases}$
5	**(ii)** $\left(\sqrt{3},2\sqrt{3}\right)$ and $\left(-\sqrt{3},-2\sqrt{3}\right)$ **(iii)** $\left(\sqrt{3},2\sqrt{3}\right)$ max point and $\left(-\sqrt{3},-2\sqrt{3}\right)$ min point
6	**(a) (i)** $x>0$
7	**(ii)** $\dfrac{dy}{dx}=\dfrac{3y-6x}{10y-3x}$ **(iii)** $x=4\sqrt{\dfrac{3}{17}}$
9	**(a)** Maximum point at $x=\dfrac{\pi}{2}$