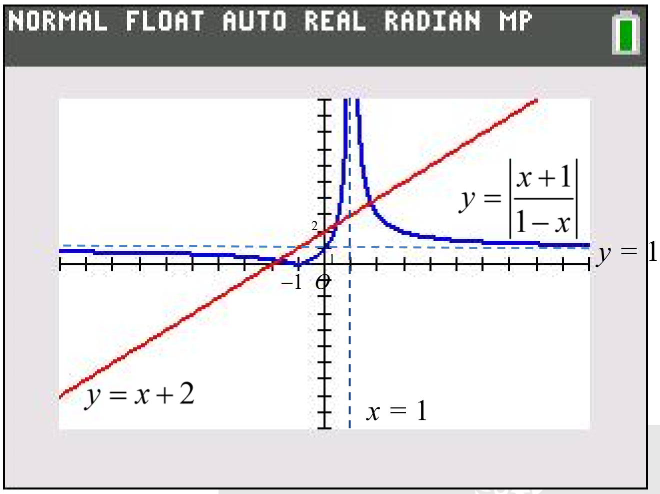
Raffles Institution
H2 Mathematics
Solution for 2015 A-Level Paper 1

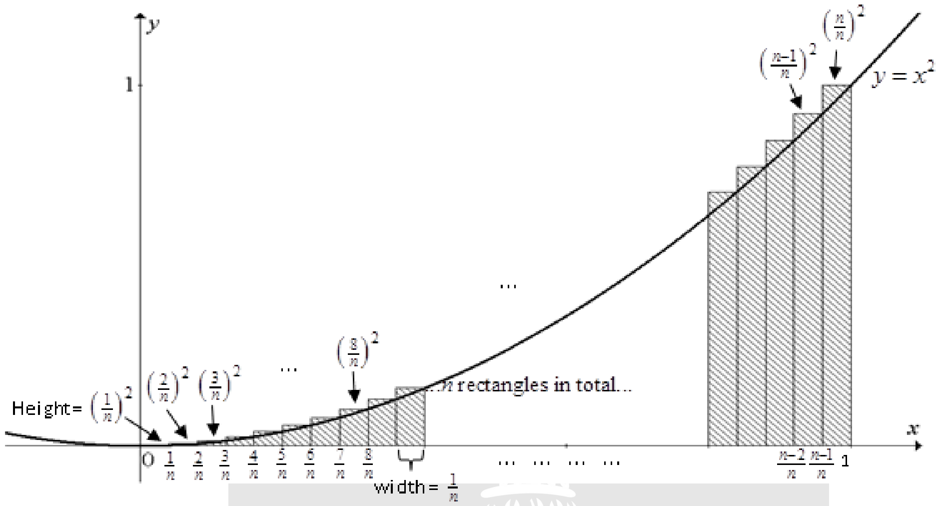
Question 1

No.	Suggested Solution	Remarks for Student
(i)	$y=\frac{a}{x^2}+bx+c$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2a}{x^3}+b$ At $x=1$, $\frac{\mathrm{d}y}{\mathrm{d}x}=2$, we have $-2a+b=2$ ...(1) At $(1.6,-2.4)$, we have $\frac{a}{(1.6)^2}+1.6b+c=-2.4$ ...(2) At $(-0.7,3.6)$, we have $\frac{a}{(-0.7)^2}-0.7b+c=3.6$ ...(3) Using GC to solve (1), (2) and (3): $a=-3.59345\approx-3.593$, $b=-5.18691\approx-5.187$, $c=7.30274\approx7.303$	Can key in $\frac{1}{(1.6)^2}$
(ii)	Using GC, $x=-0.589$ (3 d.p.)	
(iii)	$y=-5.187x+7.303$ (3 d.p.)	

Question 2

No.	Suggested Solution	Remarks for Student
(i)	 Label asymptotes: $y = 1$ and $x = 1$	
(ii)	From graphs and GC, the graphs intersect at $x = -1.73,\ 0.414,\ 1.73$ For the inequality to hold, we have $-1.73 < x < 0.414$ or $x > 1.73$	

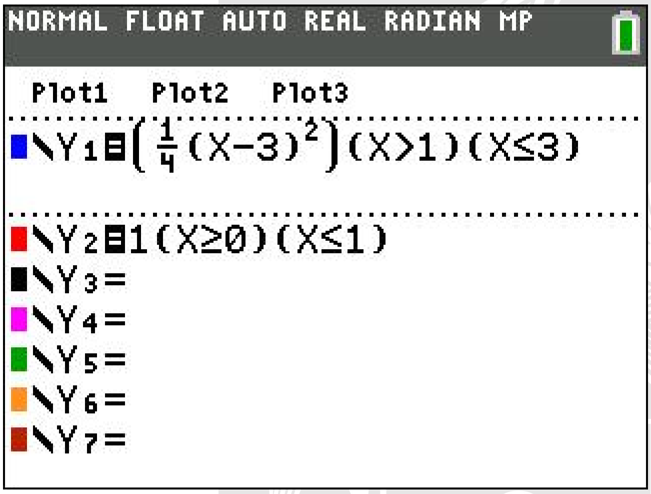
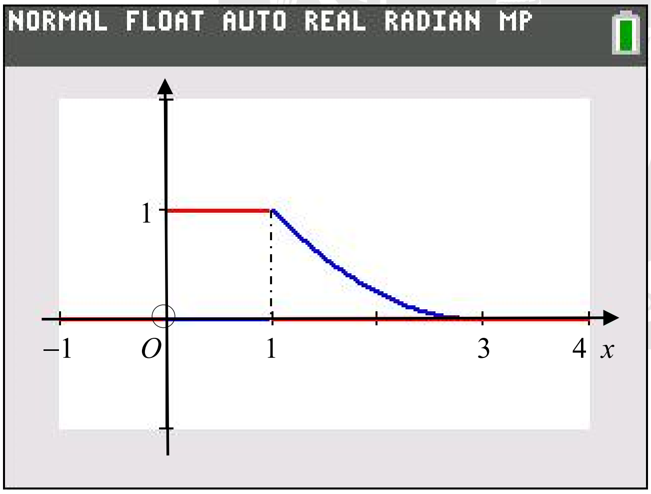
Question 3

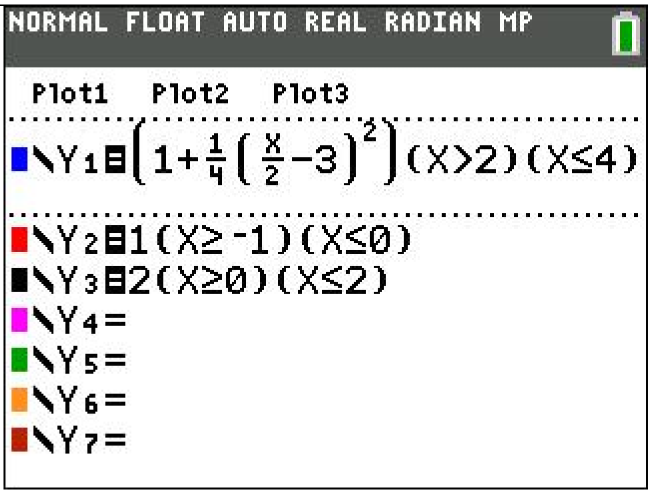
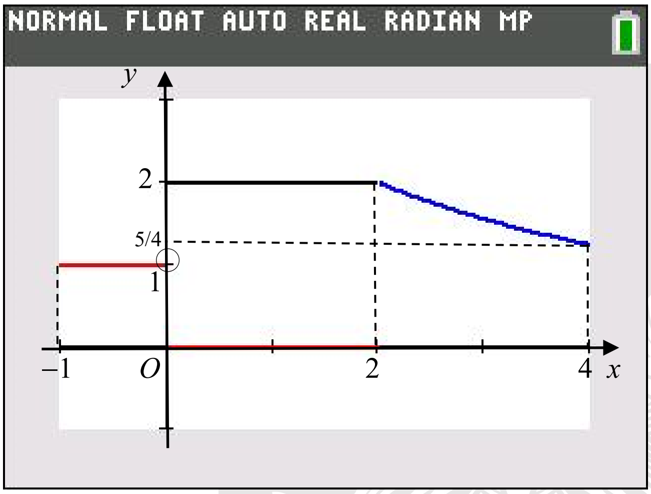
No.	Suggested Solution	Remarks for Student
(i)	 Note that $\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+...+f\left(\frac{n}{n}\right)\right\}$ is the sum of area of the n rectangles in the diagram. As $n\to\infty$, the sum will approach the exact area under the curve. Hence $\lim_{n\to\infty}\frac{1}{n}\left\{f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+...+f\left(\frac{n}{n}\right)\right\}=\int_0^1 f(x)\ dx$	Since f is any continuous function, we can use $y = x^2$ for convenience.
(ii)	$\lim_{n\to\infty}\frac{1}{n}\left\{\frac{\sqrt[3]{1}+\sqrt[3]{2}+...+\sqrt[3]{n}}{\sqrt[3]{n}}\right\}=\int_0^1\sqrt[3]{x}\ dx=\left[\frac{3}{4}x^{\frac{4}{3}}\right]_0^1=\frac{3}{4}$	

Question 4

No.	Suggested Solution	Remarks for Student
	Length of rectangle $= 2x + 2y$ Length of semi-circle $= d - (2x + 2y)$ But length of semi-circle also $= \frac{1}{2}(2\pi x) + 2x = \pi x + 2x = (\pi + 2)x$ Thus, $d - (2x + 2y) = (\pi + 2)x$ $\Rightarrow 2x + 2y = d - 2x - \pi x$ $\Rightarrow y = \frac{1}{2}d - \frac{1}{2}\pi x - 2x$ Total area, $A = xy + \frac{1}{2}\pi x^2$ $= x\left(\frac{1}{2}d - \frac{1}{2}\pi x - 2x\right) + \frac{1}{2}\pi x^2$ $= \frac{1}{2}xd - 2x^2$ $= \frac{1}{2}x(d - 4x)$ This is an quadratic expression with max value attained when $x = \frac{d}{8}$, that is the mid-point of the 2 roots 0 and $\frac{d}{4}$. Thus, max value of $A = \frac{1}{2}\left(\frac{d}{8}\right)\left(d - 4\left(\frac{d}{8}\right)\right) = \frac{1}{32}d^2 \text{ m}^2$ So, $k = \frac{1}{32}$	

Question 5

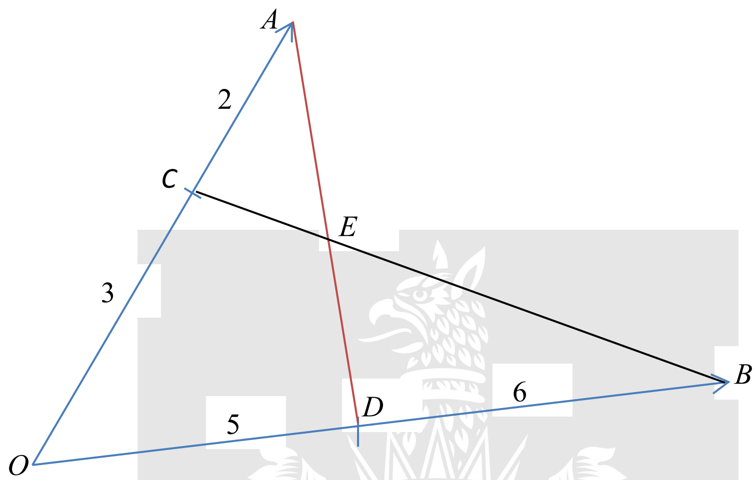
No.	Suggested Solution	Remarks for Student
(i)	$y = x^2$ $\downarrow$ replace x by $(x-3)$ $y = (x-3)^2$ $\downarrow$ replace y by $y/\frac{1}{4}$ $y = \frac{1}{4}(x-3)^2$ Translate 3 units in the positive x directions followed by scaling of factor $\frac{1}{4}$ parallel to y-axis.	
(ii)	  	
(iii)		

	 NORMAL FLOAT AUTO REAL RADIAN MP Plot1 Plot2 Plot3 Y1=(1+1/4(X/2−3)^2)(X>2)(X≤4) Y2=1(X≥−1)(X≤0) Y3=2(X≥0)(X≤2) Y4= Y5= Y6= Y7=  NORMAL FLOAT AUTO REAL RADIAN MP y 2 5/4 1 −1 O 2 4 x		

Question 6

No.	Suggested Solution	Remarks for Student
(i)	$\ln(1+2x)=2x-\frac{(2x)^2}{2}+\frac{(2x)^3}{3}+\ldots\approx 2x-2x^2+\frac{8}{3}x^3$	
(ii)	$ax(1+bx)^c=ax\left[1+bcx+\frac{c(c-1)}{2!}(bx)^2+\frac{c(c-1)(c-2)}{3!}(bx)^3\ldots\right]$ $=ax+abcx^2+\frac{ab^2c(c-1)}{2}x^3+\frac{ab^3c(c-1)(c-2)}{6}x^4\ldots$ Given that $ax+abcx^2+\frac{ab^2c(c-1)}{2}x^3=2x-2x^2+\frac{8}{3}x^3$ Thus, $a=2$ $bc=-1$ $-b(c-1)=\frac{8}{3}\Rightarrow -bc+b=\frac{8}{3}\Rightarrow b=\frac{5}{3}$ $c=-\frac{3}{5}$ Coefficient of x^4 is $\frac{ab^3c(c-1)(c-2)}{6}=\frac{2\left(\frac{5}{3}\right)^3\left(-\frac{3}{5}\right)\left(-\frac{3}{5}-1\right)\left(-\frac{3}{5}-2\right)}{6}=-\frac{104}{27}$	

Question 7

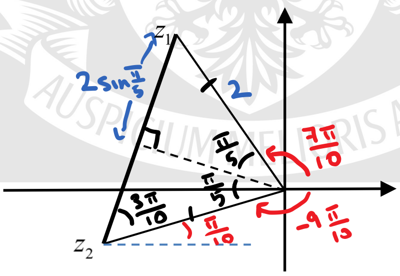
No.	Suggested Solution	Remarks for Student
		
(i)	$\overrightarrow{OC}=\frac{3}{5}\underset{\sim}{a}, \qquad \overrightarrow{OD}=\frac{5}{11}\underset{\sim}{b}$	
(ii)	$l_{BC}:\underset{\sim}{r} =\underset{\sim}{b}+\lambda\overrightarrow{BC},\ \lambda\in\mathbb{R}$ $=\underset{\sim}{b}+\lambda\left(\frac{3}{5}\underset{\sim}{a}-\underset{\sim}{b}\right)$ $=\frac{3}{5}\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}$ $l_{AD}:\underset{\sim}{r} =\frac{5}{11}\mu\underset{\sim}{b}+(1-\mu)\underset{\sim}{a},\ \mu\in\mathbb{R}$	
(iii)	$\frac{3}{5}\lambda\underset{\sim}{a}+(1-\lambda)\underset{\sim}{b}=\frac{5}{11}\mu\underset{\sim}{b}+(1-\mu)\underset{\sim}{a}$ $\frac{3}{5}\lambda=1-\mu\Rightarrow\frac{3}{5}\lambda+\mu=1$ $1-\lambda=\frac{5}{11}\mu\Rightarrow\lambda+\frac{5}{11}\mu=1$ Using GC, $\lambda=\frac{3}{4},\mu=\frac{11}{20}$ Thus, $\overrightarrow{OE}=\frac{3}{5}\left(\frac{3}{4}\right)\underset{\sim}{a}+\left(1-\frac{3}{4}\right)\underset{\sim}{b}=\frac{9}{20}\underset{\sim}{a}+\frac{1}{4}\underset{\sim}{b}$	

	$\overrightarrow{AE}=\frac{9}{20}\underset{\sim}{a}+\frac{1}{4}\underset{\sim}{b}-\underset{\sim}{a}=\frac{1}{4}\underset{\sim}{b}-\frac{11}{20}\underset{\sim}{a}$ $\overrightarrow{ED}=\frac{5}{11}\underset{\sim}{b}-\left(\frac{9}{20}\underset{\sim}{a}+\frac{1}{4}\underset{\sim}{b}\right)=\frac{9}{44}\underset{\sim}{b}-\frac{9}{20}\underset{\sim}{a}=\frac{9}{11}\left(\frac{1}{4}\underset{\sim}{b}-\frac{11}{20}\underset{\sim}{a}\right)=\frac{9}{11}\overrightarrow{AE}$ Thus, $AE:ED=11:9$	

Question 8

No.	Suggested Solution	Remarks for Student
	$1.5\text{h} = 1.5 \times 60 \times 60 = 5400$ s $1.75\text{h} = 1.75 \times 60 \times 60 = 6300$ s	
(i)	$5400 \leq \frac{50}{2}\left[2T + (50-1)(2)\right] \leq 6300$ $5400 \leq 50T + 2450 \leq 6300$ $\therefore 59 \leq T \leq 77$ Set of values of T is $[59, 77]$	
(ii)	$5400 \leq \frac{t\left(1.02^{50}-1\right)}{1.02-1} \leq 6300$ $108 \leq t\left(1.02^{50}-1\right) \leq 126$ $\therefore 63.845 \leq t \leq 74.486$ Set of values of t is $[63.9, 74.4]$ (3 s.f.)	Examiner report does not accept $[63.8, 74.5]$ (3 s.f.)
(iii)	$63.845(1.02)^{50-1} - \left[59 + (50-1)(2)\right] = 11.475 \approx 11$	

Question 9

No.	Suggested Solution	Remarks for Student
(a)	$\dfrac{w^2}{w^*} = \dfrac{(a+\mathrm{i}b)^2}{a-\mathrm{i}b}$ $= \dfrac{(a+\mathrm{i}b)^3}{a^2+b^2}$ $= \dfrac{a^3+3\mathrm{i}a^2b-3ab^2-\mathrm{i}b^3}{a^2+b^2}$ $= \dfrac{(a^3-3ab^2)+\mathrm{i}(3a^2b-b^3)}{a^2+b^2}$ $\dfrac{w^2}{w^*}$ is purely imaginary $\mathrm{Re}\left(\dfrac{w^2}{w^*}\right)=0 \Rightarrow \dfrac{(a^3-3ab^2)}{a^2+b^2}=0$ $\Rightarrow a(a^2-3b^2)=0$ $\Rightarrow a=0$ (rejected, since $a\neq 0$) or $b=\pm\dfrac{a}{\sqrt{3}}$ Thus, $w=a+\mathrm{i}\dfrac{a}{\sqrt{3}}$ or $w=a-\mathrm{i}\dfrac{a}{\sqrt{3}}$.	
b(i)	$z^5=-32\mathrm{i}=2^5\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}\right)}=2^5\mathrm{e}^{\mathrm{i}\left(-\frac{\pi}{2}+2k\pi\right)}=2^5\mathrm{e}^{\mathrm{i}\left(\frac{4k\pi-\pi}{2}\right)}$ $\therefore z=2\mathrm{e}^{\mathrm{i}\frac{(4k-1)\pi}{10}}, \; k=-2,-1,0,1,2$	You may want to list down all values if you have time.
(ii)	 $\arg(z_1)=\dfrac{7\pi}{10}, \; \arg(z_1)=-\dfrac{9\pi}{10}$ $\lvert z_1-z_2\rvert = 2\times 2\sin\dfrac{\pi}{5}$ $\arg(z_1-z_2)=\dfrac{3\pi}{10}+\dfrac{\pi}{10}=\dfrac{2\pi}{5}$	

Alternatively,

$$z_1 = 2e^{i\frac{7\pi}{10}}, z_2 = 2e^{i\frac{-9\pi}{10}}$$

$$\begin{aligned}
z_1 - z_2 &= 2e^{i\frac{7\pi}{10}} - 2e^{i\frac{-9\pi}{10}} \\
&= 2e^{i\left(\frac{-\pi}{10}\right)}\left(e^{i\frac{8\pi}{10}} - e^{i\frac{-8\pi}{10}}\right) \\
&= 2e^{i\left(\frac{-\pi}{10}\right)}\left(2i\sin\frac{8\pi}{10}\right) \\
&= \left(4\sin\frac{4\pi}{5}\right)e^{i\left(\frac{-\pi}{10}\right)}(i) \\
&= \left(4\sin\frac{\pi}{5}\right)e^{i\left(\frac{-\pi}{10}\right)}\left(e^{i\frac{\pi}{2}}\right), \qquad \text{since } \sin\frac{4\pi}{5} = \sin\left(\pi - \frac{\pi}{5}\right) = \sin\frac{\pi}{5} \\
&= \left(4\sin\frac{\pi}{5}\right)e^{i\left(\frac{2\pi}{5}\right)}
\end{aligned}$$

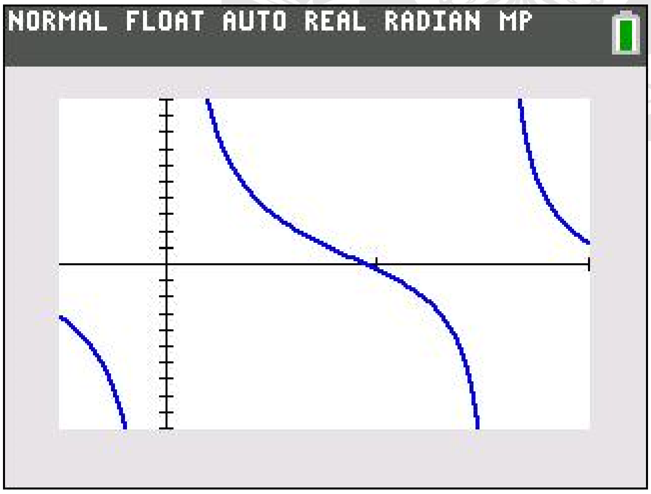
Thus, $\arg(z_1 - z_2) = \frac{2\pi}{5}$ and $|z_1 - z_2| = 4\sin\frac{\pi}{5}$

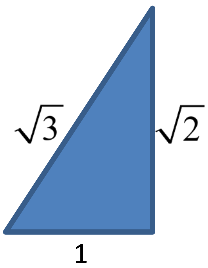
Question 10

No.	Suggested Solution	Remarks for Student
(i)	$A_1 = \int_0^{\frac{\pi}{4}} \sin x \ \mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \ \mathrm{d}x$ $= [-\cos x]_0^{\frac{\pi}{4}} + [\sin x]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ $= \left[\frac{\sqrt{2}}{2} - (-1)\right] + \left(1 - \frac{\sqrt{2}}{2}\right)$ $= 2 - \sqrt{2}$ $A_2 = \int_0^{\frac{\pi}{4}} \cos x \ \mathrm{d}x - \int_0^{\frac{\pi}{4}} \sin x \ \mathrm{d}x$ $= [\sin x]_0^{\frac{\pi}{4}} + [\cos x]_0^{\frac{\pi}{4}}$ $= \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1$ $= \sqrt{2} - 1$ $= \frac{1}{\sqrt{2}}\left(2 - \sqrt{2}\right)$ $= \frac{1}{\sqrt{2}} A_1$ $\therefore \frac{A_1}{A_2} = \sqrt{2}$	
(ii)	$y = \sin x \Rightarrow x = \sin^{-1} y$ Required volume $= \pi \int_0^{\frac{\sqrt{2}}{2}} x^2 \ \mathrm{d}y = \pi \int_0^{\frac{\sqrt{2}}{2}} \left(\sin^{-1} y\right)^2 \ \mathrm{d}y$	
(iii)	$y = \sin u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \cos u$ $y = 0 \Rightarrow u = 0$ $y = \frac{\sqrt{2}}{2} \Rightarrow u = \frac{\pi}{4}$	

	$\pi\int_0^{\frac{\sqrt{2}}{2}} \left(\sin^{-1} y\right)^2 \; \mathrm{d}y = \pi\int_0^{\frac{\pi}{4}} u^2 \cos u \; \mathrm{d}u$ (shown) $=\pi\left\{\left[u^2 \sin u\right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2u\sin u \; \mathrm{d}u\right\}$ $=\pi\left\{\frac{\pi^2\sqrt{2}}{32} - \left(\left[-2u\cos u\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} 2\cos u \; \mathrm{d}u\right)\right\}$ $=\pi\left\{\frac{\pi^2\sqrt{2}}{32} + \frac{\pi\sqrt{2}}{4} - \left[2\sin u\right]_0^{\frac{\pi}{4}}\right\}$ $=\pi\left\{\frac{\pi^2\sqrt{2}}{32} + \frac{\pi\sqrt{2}}{4} - \sqrt{2}\right\}$	

Question 11

No.	Suggested Solution	Remarks for Student
(i)	$x=\sin^3\theta,\qquad y=3\sin^2\theta\cos\theta$ $\dfrac{\mathrm{d}x}{\mathrm{d}\theta}=3\sin^2\theta\cos\theta\qquad \dfrac{\mathrm{d}y}{\mathrm{d}\theta}=6\sin\theta\cos^2\theta-3\sin^3\theta$ $\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}}$ $=\dfrac{6\sin\theta\cos^2\theta-3\sin^3\theta}{3\sin^2\theta\cos\theta}$ $=\dfrac{6\sin\theta\cos^2\theta}{3\sin^2\theta\cos\theta}-\dfrac{3\sin^3\theta}{3\sin^2\theta\cos\theta}$ $=2\cot\theta-\tan\theta$ (shown)	
(ii)	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=0 \Rightarrow 2\cot\theta-\tan\theta=0$ $\Rightarrow 2\left(\dfrac{1}{\tan\theta}\right)-\tan\theta=0$ $\Rightarrow \tan^2\theta=2$ $\therefore \tan\theta=\sqrt{2}$ (reject $\tan\theta=-\sqrt{2}$ as $0\le\theta\le\dfrac{\pi}{2}$) Thus, $k=2$. C has only one stationary point when $\tan\theta=\sqrt{2}$, for $0\le\theta\le\dfrac{\pi}{2}$. Graph of $\dfrac{\mathrm{d}y}{\mathrm{d}\theta}=2\cot\theta-\tan\theta$  Gradient graph indicates there is only one change in sign for $0\le\theta\le\dfrac{\pi}{2}$. Thus, the stationary point is a turning point.	

	Note that the change in sign is from positive to negative. Thus, first derivative test shows that the turning point is a maximum. A right-angled triangle with $\tan\theta = \sqrt{2}$  Thus, $x = \left(\frac{\sqrt{2}}{\sqrt{3}}\right)^3 = \frac{2\sqrt{2}}{3\sqrt{3}}$, $\quad y = 3\left(\frac{\sqrt{2}}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}$ Coordinates = $\left(\frac{2\sqrt{2}}{3\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$	
(iii)	Note that x increases from 0 to 1 for $0 \le \theta \le \frac{\pi}{2}$ Area $= \int_0^1 y \, \mathrm{d}x$ $= \int_0^{\frac{\pi}{2}} \left(3\sin^2\theta\cos\theta\right)\left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right) \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{2}} \left(3\sin^2\theta\cos\theta\right)\left(3\sin^2\theta\cos\theta\right) \mathrm{d}\theta$ $= \int_0^{\frac{\pi}{2}} 9\sin^4\theta\cos^2\theta \, \mathrm{d}\theta$ (shown) $=0.884$ (3 d.p.)	
(iv)	$y = ax \Rightarrow 3\sin^2\theta\cos\theta = a\sin^3\theta$ $\Rightarrow \frac{3}{a} = \tan\theta$ (shown) $\Rightarrow \frac{3}{a} = \sqrt{2}$ $\therefore a = \frac{3}{\sqrt{2}}$	