National Junior College
2016 – 2017 H2 Mathematics

Topic 5: Differentiation and its Applications

Key Questions to Answer:

1. What does the derivative tell you about the behaviour of the graph ?
   - *What does differentiation by first principles mean?*
   - *How do you differentiate polynomials by first principles?*
   - *What is the geometrical interpretation of* $\mathrm{f}'(x)$ *?*
2. What are the meanings of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{f}(x)$?
   - *Is* $\frac{\mathrm{d}y}{\mathrm{d}x}$ *a fraction?*
   - *Does* $\frac{\mathrm{d}x}{\mathrm{d}y}$ *exist? If it does, what does* $\frac{\mathrm{d}x}{\mathrm{d}y}$ *mean?*
3. How do you differentiate the following functions
   - *polynomial functions,*
   - *trigonometric functions,*
   - *exponential functions,*
   - *logarithmic functions,*
   - *as well as constant multiples, sums and differences of any combination of the above functions?*
4. What are the rules that are useful in differentiation and how do you use them?
5. Under what circumstances would you need to use implicit differentiation, parametric differentiation or logarithmic differentiation?
6. What does $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ mean?
   - *Is this the same as* $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)$ *or* $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ *?*
7. How do you relate the concavity of the graph with the first and second derivative?
   - *Is the graph of* $y=\frac{1}{x}$ *an example of a graph that is concave upwards? Why or why not?*
8. How do you identify a local maximum or minimum on the graph?
   - *What is the difference between a local and a global maximum/minimum?*
   - *What is the first derivative test?*
   - *What is the second derivative test? Why does it work? Does it work all the time?*
   - *When should you use the first derivative test instead of the second derivative test?*
9. How do you find the equation of the tangent?
   - *What is the relationship between the gradient of the tangent and that of the normal?*
   - *How do you find the equation of the normal?*
10. What do you see in common between chain rule, implicit differentiation, parametric differentiation and connected rates of change?

Note: This topic is a build-up from the basic calculus knowledge acquired under the O-level Additional Mathematics syllabus. Knowledge of O-level calculus is assumed.

§1 Differentiation Techniques

1.1 Differentiation by First Principles

Recall that the gradient of a straight line is the *rate of change of y with respect to x*, i.e.

$$\frac{\text{change in } y}{\text{change in } x}.$$

For example, the line $y = 2x$ has gradient 2, which means that every unit change in x results in 2 units change in y or equivalently, the rate of change of y with respect to x is 2.

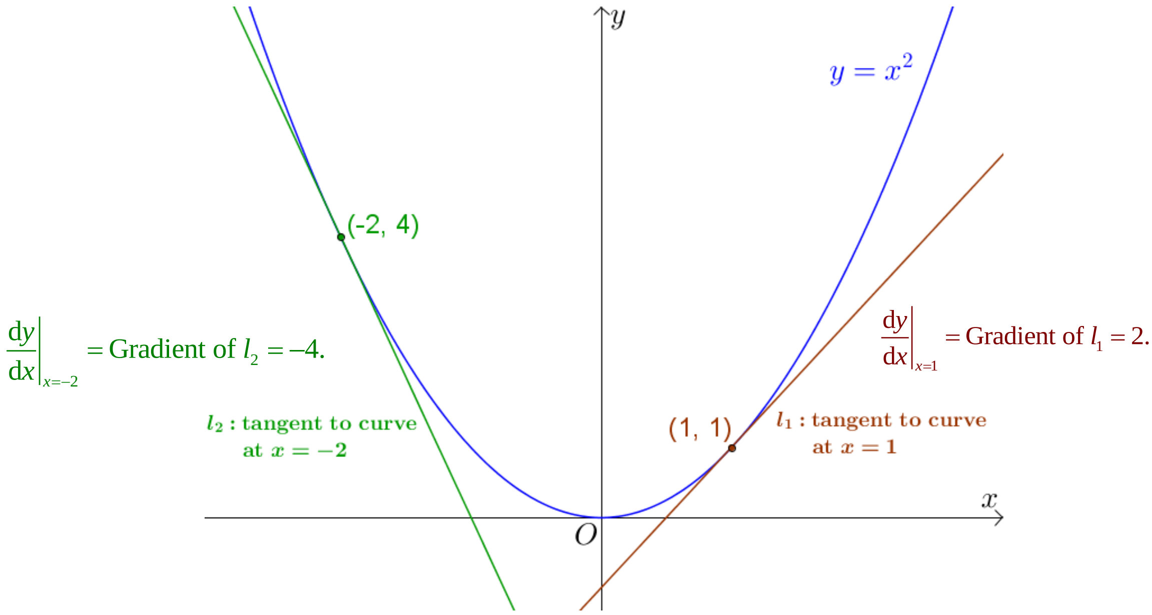
For a curve, recall that the gradient is found by differentiation. For example, for the curve $y = x^2$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x.$$

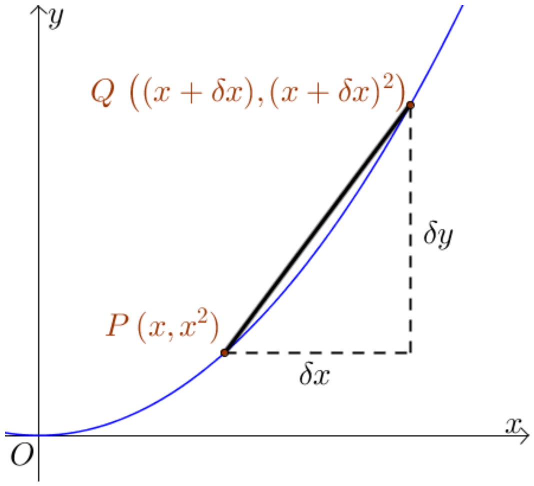
Hence the gradients of the curve $y = x^2$ at $x = -2$ and $x = 1$ are -4 and 2 respectively. We can see that the gradient of a curve **differs** for different points on the curve.

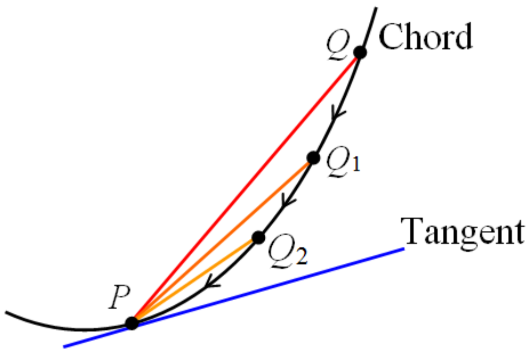
How do we find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the first place?

Recall that the gradient of a curve at a given point is equal to the gradient of the **tangent** to the curve at the same point, as seen from the graph below:



To find the gradient of the tangent to the curve at a general point $P(x, \mathrm{f}(x))$ on the curve $y = x^2$, consider the following diagram (left):





Gradient of the **chord** $PQ = \dfrac{\delta y}{\delta x} = \dfrac{(x+\delta x)^2 - x^2}{(x+\delta x)-x}$.

As $Q \;\to\; P$ along the curve, i.e. $\delta x \to 0$

the chord $PQ \;\to\;$ tangent at P (see above diagram (right))

$\Rightarrow$ gradient of the chord $PQ \;\to\;$ gradient of tangent at P.

Therefore the gradient of tangent to the curve at P is given by

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{(x+\delta x)^2 - x^2}{(x+\delta x)-x} \\
&= \lim_{\delta x \to 0} \frac{x^2 + 2x \cdot \delta x + (\delta x)^2 - x^2}{\delta x} \\
&= \lim_{\delta x \to 0} (2x + \delta x) \\
&= 2x.
\end{aligned}$$

In general, we have the following definition.

Definition 1.1.1 (Derivative of a Function)

If f is a function, the gradient of the curve $y = \mathrm{f}(x)$ is defined to be

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) = \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$$

if this limit exists. We call this limit the *derivative* or the *gradient function* of $\mathrm{f}(x)$.

The process of obtaining the derivative by finding $\lim_{\delta x \to 0} \dfrac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x}$ is known as **differentiation by first principles**.

Example 1.1.2

Find the gradient function of the following curves from first principles.

(i) $y = \dfrac{1}{x}$, (ii) $y = \sin x$, (iii) $y = \mathrm{e}^x$.

Solution:

(i) Using first principles,

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\left(\dfrac{1}{x+\delta x}\right) - \left(\dfrac{1}{x}\right)}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{x - (x+\delta x)}{(x+\delta x)(x)\delta x} \\
&= \lim_{\delta x \to 0} \frac{-\delta x}{(x+\delta x)(x)\delta x} \\
&= \lim_{\delta x \to 0} \frac{-1}{(x+\delta x)x} \\
&= -\frac{1}{x^2}.
\end{aligned}$$

(ii) Using first principles,

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\sin(x+\delta x) - \sin x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{2\cos\left(\dfrac{2x+\delta x}{2}\right)\sin\left(\dfrac{\delta x}{2}\right)}{\delta x} \\
&= \left[\lim_{\delta x \to 0} \cos\left(x + \frac{\delta x}{2}\right)\right] \times \left[\lim_{\delta x \to 0} \frac{\sin\left(\dfrac{\delta x}{2}\right)}{\left(\dfrac{\delta x}{2}\right)}\right] \\
&= \cos x.
\end{aligned}$$

Note: $\lim_{\delta x \to 0} \dfrac{\sin \delta x}{\delta x} = 1.$

(iii) Using first principles,

$$\begin{aligned}
\mathrm{f}'(x) &= \lim_{\delta x \to 0} \frac{\mathrm{f}(x+\delta x) - \mathrm{f}(x)}{\delta x} = \lim_{\delta x \to 0} \frac{\mathrm{e}^{(x+\delta x)} - \mathrm{e}^x}{\delta x} \\
&= \lim_{\delta x \to 0} \frac{\mathrm{e}^x\left[\mathrm{e}^{\delta x} - 1\right]}{\delta x} \\
&= \mathrm{e}^x \lim_{\delta x \to 0} \frac{\left(\mathrm{e}^{\delta x} - 1\right)}{\delta x} \\
&= \mathrm{e}^x.
\end{aligned}$$

Note: $\lim_{\delta x \to 0} \dfrac{\mathrm{e}^{\delta x} - 1}{\delta x} = 1.$

1.2 Differentiation of Functions using Standard Results

The following basic derivatives and differentiation techniques are assumed knowledge from the 'O'-level syllabus:

Basic Functions	y	$\frac{\mathrm{d}y}{\mathrm{d}x}$
Polynomials ($n \in \mathbb{Z}^+$)	x^n	nx^{n-1}
	$(ax+b)^n$	$an(ax+b)^{n-1}$
Trigonometric Functions	$\sin(\mathrm{f}(x))$	$\mathrm{f}'(x)\cdot\cos(\mathrm{f}(x))$
	$\cos(\mathrm{f}(x))$	$-\mathrm{f}'(x)\cdot\sin(\mathrm{f}(x))$
	$\tan(\mathrm{f}(x))$	$\mathrm{f}'(x)\cdot\sec^2(\mathrm{f}(x))$
	$\mathrm{cosec}(\mathrm{f}(x))$	$-\mathrm{f}'(x)\cdot\mathrm{cosec}(\mathrm{f}(x))\cdot\cot(\mathrm{f}(x))$
	$\sec(\mathrm{f}(x))$	$\mathrm{f}'(x)\cdot\sec(\mathrm{f}(x))\cdot\tan(\mathrm{f}(x))$
	$\cot(\mathrm{f}(x))$	$-\mathrm{f}'(x)\cdot\mathrm{cosec}^2(\mathrm{f}(x))$
Logarithmic and Exponential Functions	$\ln(\mathrm{f}(x))$	$\frac{\mathrm{f}'(x)}{(\mathrm{f}(x))}$
	$\mathrm{e}^{\mathrm{f}(x)}$	$\mathrm{f}'(x)\cdot\mathrm{e}^{\mathrm{f}(x)}$

Chain Rule	If $y=\mathrm{f}(u)$ and $u=\mathrm{f}(x)$, then $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}u}\times\frac{\mathrm{d}u}{\mathrm{d}x}.$$
Product Rule	If $y = uv$ where u, v are functions of x, then $$\frac{\mathrm{d}y}{\mathrm{d}x}=v\frac{\mathrm{d}u}{\mathrm{d}x}+u\frac{\mathrm{d}v}{\mathrm{d}x}.$$
Quotient Rule	If $y=\frac{u}{v}$ where u, v are functions of x and v is non-zero, then $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{v\frac{\mathrm{d}u}{\mathrm{d}x}-u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}.$$

Refer to Appendix B for ways to use your graphing calculator to

- evaluate the derivative at a point, and
- plot the graph of the derivative function without doing the actual differentiation.

The following are additional results for the A-levels. For proofs, refer to Appendix A:

	y	$\dfrac{\mathrm{d}y}{\mathrm{d}x}$
Inverse Trigonometric Functions	$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert<1$
	$\sin^{-1}\left(\mathrm{f}(x)\right)$	$\dfrac{\mathrm{f}'(x)}{\sqrt{1-\left[\mathrm{f}(x)\right]^2}}$, $\lvert\mathrm{f}(x)\rvert<1$
	$\cos^{-1} x$	$\dfrac{-1}{\sqrt{1-x^2}}$, $\lvert x\rvert<1$
	$\cos^{-1}\left(\mathrm{f}(x)\right)$	$\dfrac{-\mathrm{f}'(x)}{\sqrt{1-\left[\mathrm{f}(x)\right]^2}}$, $\lvert\mathrm{f}(x)\rvert<1$
	$\tan^{-1} x$	$\dfrac{1}{1+x^2}$
	$\tan^{-1}\left(\mathrm{f}(x)\right)$	$\dfrac{\mathrm{f}'(x)}{1+\left[\mathrm{f}(x)\right]^2}$
Logarithmic and Exponential Functions	a^x	$a^x \cdot \ln a$
	$a^{\mathrm{f}(x)}$	$\mathrm{f}'(x)\cdot a^{\mathrm{f}(x)}\cdot \ln a$
	$\log_a\left(\mathrm{f}(x)\right)$	$\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}\cdot \log_a \mathrm{e}$

Relationship between $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}x}{\mathrm{d}y}$	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$. However, $\dfrac{\mathrm{d}^n y}{\mathrm{d}x^n}\neq\dfrac{1}{\frac{\mathrm{d}^n x}{\mathrm{d}y^n}}$ in general.

Example 1.2.1

Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ if (a) $y=\sin^{-1}(1-2x)$, (b) $y=\tan^{-1}\left(\dfrac{x}{2}+1\right)$, (c) $y=5^{4x}$,

(d) $y=\cos^{-1}\left(x^2-1\right)$, (e) $y=\log_4(x+2)$.

Solution:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{-2}{\sqrt{1-(1-2x)^2}}=\dfrac{-2}{\sqrt{4x-4x^2}}$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{1}{2}}{1+\left(\frac{x}{2}+1\right)^2}=\dfrac{2}{x^2+4x+8}$.

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=4\left(5^{4x}\right)\ln 5$.

(d) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-\dfrac{2x}{\sqrt{1-(x^2-1)^2}}=-\dfrac{2x}{\sqrt{2x^2-4x^4}}$.

(e) $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{x+2}\log_4 \mathrm{e}$.

Alternatively, convert to natural logarithm before differentiating: $\log_4(x+2)=\dfrac{\ln(x+2)}{\ln 4}$.

1.3 Implicit Differentiation

So far, we have dealt with functions that are mostly explicit (e.g. $y = x^3 + 3$ and $y = \tan x$). However, the dependent variable y may not always be expressed in terms of the independent variable x explicitly (e.g. $5xy^2 - x^3y + 3 = 0$).

In such cases, we will use the **chain rule** to find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. Such a process is known as **implicit differentiation**.

Example 1.3.1

Differentiate the following with respect to x.

(i) y, (ii) y^2, (iii) xy, (iv) $\dfrac{\mathrm{d}y}{\mathrm{d}x}$, (v) $\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$.

Solution:

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}(y) = \dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) $\dfrac{\mathrm{d}}{\mathrm{d}x}(y^2) = \dfrac{\mathrm{d}}{\mathrm{d}y}(y^2) \cdot \dfrac{\mathrm{d}y}{\mathrm{d}x} = 2y\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

(iii) $\dfrac{\mathrm{d}}{\mathrm{d}x}(xy) = x\dfrac{\mathrm{d}y}{\mathrm{d}x} + y$.

(iv) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right) = \dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(v) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right] = 2\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

In general,

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(y^n) = \dfrac{\mathrm{d}}{\mathrm{d}y}(y^n)\dfrac{\mathrm{d}y}{\mathrm{d}x} = ny^{n-1}\dfrac{\mathrm{d}y}{\mathrm{d}x}$, where n is a rational number.

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(y)) = \dfrac{\mathrm{d}}{\mathrm{d}y}(\mathrm{f}(y))\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(y)\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

Example 1.3.2

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y if (a) $x^2+y^3=xy$, (b) $3x^2-7y^2+4xy-8x=0$,

(c) $\cos^{-1}y=x+y$, (d) $y=\tan^{-1}(y-x)$.

Solution:

(a) $x^2+y^3=xy$.

Differentiate implicitly w.r.t. x, we get

$$2x+3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=x\frac{\mathrm{d}y}{\mathrm{d}x}+y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2x-y}{x-3y^2}.$$

Note: $\frac{\mathrm{d}}{\mathrm{d}x}(y^3)=\frac{\mathrm{d}}{\mathrm{d}y}(y^3)\times\frac{\mathrm{d}y}{\mathrm{d}x}=3y^2\frac{\mathrm{d}y}{\mathrm{d}x}$.

(b) $3x^2-7y^2+4xy-8x=0$.

Differentiate implicitly w.r.t. x, we get

$$6x-14y\frac{\mathrm{d}y}{\mathrm{d}x}+\left(4x\frac{\mathrm{d}y}{\mathrm{d}x}+4y\right)-8=0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{6x+4y-8}{14y-4x}=\frac{3x+2y-4}{7y-2x}.$$

(c) $\cos^{-1}y=x+y$.

Differentiate implicitly w.r.t. x, we get

$$-\frac{1}{\sqrt{1-y^2}}\cdot\frac{\mathrm{d}y}{\mathrm{d}x}=1+\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(1+\frac{1}{\sqrt{1-y^2}}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{1+\frac{1}{\sqrt{1-y^2}}}$$

$$=-\frac{\sqrt{1-y^2}}{\sqrt{1-y^2}+1}.$$

Alternatively,

$\cos^{-1}y=x+y$

$\Rightarrow y=\cos(x+y)$.

Differentiate implicitly w.r.t. x, we get

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\sin(x+y)\cdot\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)$$

$$\Rightarrow\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\sin(x+y)}{1+\sin(x+y)}.$$

(d) $y=\tan^{-1}(y-x)\Rightarrow\tan y=y-x$.

Differentiate implicitly w.r.t. x,

$$\left(\sec^2 y\right)\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\sec^2 y-1\right)=-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}\left(\tan^2 y\right)=-1$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=-\cot^2 y.$$

Example 1.3.3

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x and y if (a) $y = x^x,\ x > 0$, (b) $y = (\ln x)^x$.

Solution:

(a) $y = x^x$

Taking "ln" on both sides, we get $\ln y = x \ln x$

Differentiate implicitly with respect to x, we get

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{1}{x}\right) + (1)\ln x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1 + \ln x) = x^x(1 + \ln x).$$

(b) $y = (\ln x)^x$

$\ln y = x \ln(\ln x)$

Differentiate implicitly w.r.t. x,

$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \cdot \ln(\ln x) + x\left(\frac{1}{\ln x} \cdot \frac{1}{x}\right)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y\left[\ln(\ln x) + \frac{1}{\ln x}\right]$$

$$= (\ln x)^x\left[\ln(\ln x) + \frac{1}{\ln x}\right].$$

Note: If the expression to be differentiated is defined explicitly, the derivative should be in **explicit form** as well.

Note: The process of taking "ln" on both sides before carrying out implicit differentiation is called **logarithmic differentiation.**

In general, logarithmic differentiation is useful for expressions of the form u^v, where both u and v are **non-constant** expressions of x and/or y.

1.4 Parametric Differentiation

We have seen curves defined by a pair of parametric equations. We shall look at how to evaluate the gradient function in such cases.

Given a set of parametric equations: $\begin{cases} x = \mathrm{f}(t) \\ y = \mathrm{g}(t) \end{cases}$, we may evaluate $\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{f}'(t) \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = \mathrm{g}'(t). \end{cases}$

The gradient function can be found as: $\boxed{\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}t} . \dfrac{\mathrm{d}t}{\mathrm{d}x} = \dfrac{\left[\dfrac{\mathrm{d}y}{\mathrm{d}t}\right]}{\left[\dfrac{\mathrm{d}x}{\mathrm{d}t}\right]} = \dfrac{\mathrm{g}'(t)}{\mathrm{f}'(t)}}$.

$\boxed{\text{Note: } \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{\frac{\mathrm{d}t}{\mathrm{d}x}}}$

Note that the gradient function will also generally be in terms of the parameter t.

Example 1.4.1

Find the gradient function for the following curves defined parametrically.

(a) $x = t^3,\ y = t^2 - t,$ (b) $x = \sqrt{t}\ ,\ y = \cos t,$ (c) $x = t^2,\ y = t + \dfrac{1}{t}.$

Solution:

(a) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 2t - 1,\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = 3t^2$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]}{\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]} = \frac{2t-1}{3t^2} = \frac{2}{3t} - \frac{1}{3t^2}.$$

(b) $\dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{1}{2}t^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{t}},\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = -\sin t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]}{\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]} = \frac{-\sin t}{\frac{1}{2\sqrt{t}}} = -2\sqrt{t} \cdot \sin t.$$

(c) $\dfrac{\mathrm{d}y}{\mathrm{d}t} = 1 - \dfrac{1}{t^2},\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = 2t$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\left[\frac{\mathrm{d}y}{\mathrm{d}t}\right]}{\left[\frac{\mathrm{d}x}{\mathrm{d}t}\right]} = \frac{1 - \frac{1}{t^2}}{2t} = \frac{t^2 - 1}{2t^3}.$$

Example 1.4.2

It is given that $x=\sec\theta+\tan\theta$ and $y=\operatorname{cosec}\theta+\cot\theta$. Show that

$$x+\frac{1}{x}=2\sec\theta \text{ and } y+\frac{1}{y}=2\operatorname{cosec}\theta.$$

Hence show that $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1+y^2}{1+x^2}$.

Solution:

$$\begin{aligned}
x+\frac{1}{x}&=\sec\theta+\tan\theta+\frac{1}{\sec\theta+\tan\theta}\\
&=\frac{(\sec\theta+\tan\theta)^2+1}{\sec\theta+\tan\theta}\\
&=\frac{\sec^2\theta+\tan^2\theta+2\sec\theta\tan\theta+1}{\sec\theta+\tan\theta}\\
&=\frac{2\sec^2\theta+2\sec\theta\tan\theta}{\sec\theta+\tan\theta}\\
&=\frac{2\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}\\
&=2\sec\theta \quad \text{(shown).}
\end{aligned}$$

$$\begin{aligned}
y+\frac{1}{y}&=\operatorname{cosec}\theta+\cot\theta+\frac{1}{\operatorname{cosec}\theta+\cot\theta}\\
&=\frac{(\operatorname{cosec}\theta+\cot\theta)^2+1}{\operatorname{cosec}\theta+\cot\theta}\\
&=\frac{\operatorname{cosec}^2\theta+\cot^2\theta+2\operatorname{cosec}\theta\cot\theta+1}{\operatorname{cosec}\theta+\cot\theta}\\
&=\frac{2\operatorname{cosec}^2\theta+2\operatorname{cosec}\theta\cot\theta}{\operatorname{cosec}\theta+\cot\theta}\\
&=\frac{2\operatorname{cosec}\theta(\operatorname{cosec}\theta+\cot\theta)}{\operatorname{cosec}\theta+\cot\theta}\\
&=2\operatorname{cosec}\theta \quad \text{(shown).}
\end{aligned}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\theta}=-\operatorname{cosec}\theta\cot\theta-\operatorname{cosec}^2\theta=-\operatorname{cosec}\theta(\cot\theta+\operatorname{cosec}\theta).$$
$$\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec\theta\tan\theta+\sec^2\theta=\sec\theta(\tan\theta+\sec\theta).$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-\operatorname{cosec}\theta(\cot\theta+\operatorname{cosec}\theta)}{\sec\theta(\tan\theta+\sec\theta)}=\frac{-\frac{1}{2}\left(y+\frac{1}{y}\right)y}{\frac{1}{2}\left(x+\frac{1}{x}\right)x}=-\frac{1+y^2}{1+x^2} \quad \text{(shown).}$$

§2 Geometrical Results of the Gradient Function

2.1 Increasing and Decreasing Functions

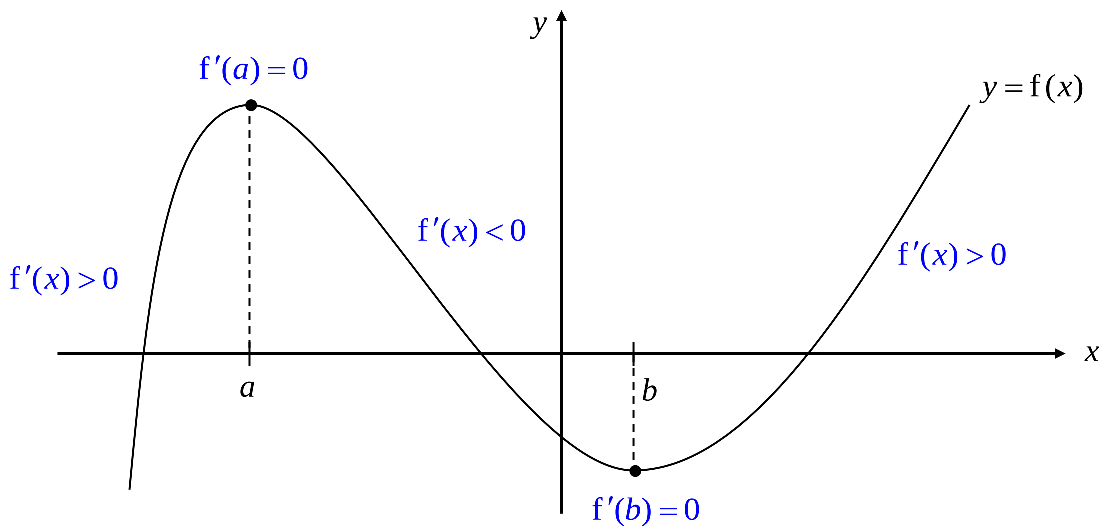
A function is said to be strictly increasing (resp. strictly decreasing) when the value of $\mathrm{f}(x)$ increases (resp. decreases) as the value of x increases. Mathematically, we have:

Definition 2.1.1 (Increasing and Decreasing Functions)

A function f is said to be *strictly increasing* (resp. *strictly decreasing*) if $a < b$ implies that $\mathrm{f}(a) < \mathrm{f}(b)$ $\left(\text{resp. } \mathrm{f}(a) > \mathrm{f}(b)\right)$.

Examples: (Strictly increasing functions) $y = x,\ y = x^3 + x.$
(Strictly decreasing function) $y = -2x + 1.$

For differentiable functions that are neither strictly increasing nor strictly decreasing, we can still determine the interval (say I) such that the curve is **upward** or **downward sloping** on I. For example, consider the following graph of $y = \mathrm{f}(x)$.



Range of values of x, I	Value of $\mathrm{f}'(x)$ or $\dfrac{\mathrm{d}y}{\mathrm{d}x}$	Gradient of curve $y = \mathrm{f}(x)$ is	On I, curve is…	On I, function f is…
$x < a$ or $x > b$	> 0	positive	upward sloping	strictly increasing
$a < x < b$	< 0	negative	downward sloping	strictly decreasing

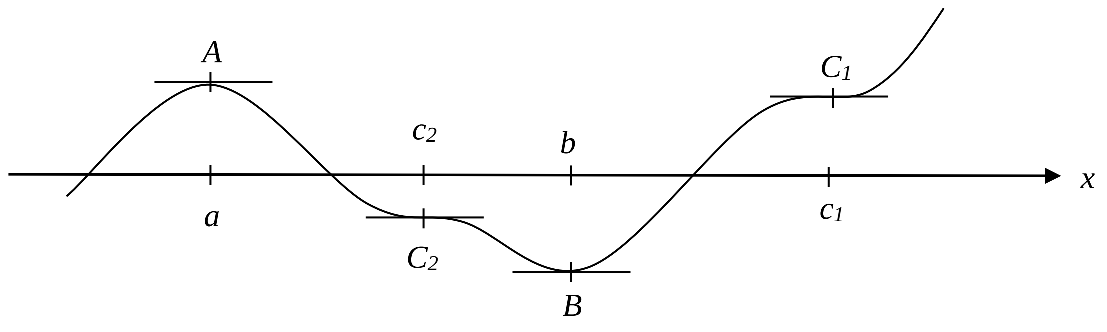
Hence we can conclude that

(a) If $\boxed{\mathrm{f}'(x) > 0 \quad \text{or} \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} > 0}$ for all x in an interval I, then f is strictly increasing on I.

(b) If $\boxed{\mathrm{f}'(x) < 0 \quad \text{or} \quad \dfrac{\mathrm{d}y}{\mathrm{d}x} < 0}$ for all x in an interval I, then f is strictly decreasing on I.

2.2 Stationary Points

Stationary points are points on the curve where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x) = 0$.



There are 3 types of stationary points – local minimum, local maximum and stationary point of inflexion.

From the above diagram, A is a **local maximum point**.
B is a **local minimum point**.
C_1 and C_2 are **stationary points of inflexion**.

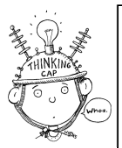
A and B are called **turning points**, because the sign of the gradient to the curve changes at these points (either from negative to positive or vice versa), and hence the curve "turns" at these points.

In general, we have the following definition:

Definition 2.2.1 (Local maximum and minimum points)

A point $\left(k, \mathrm{f}(k)\right)$ lying on the curve $y = \mathrm{f}(x)$ is said to be a *local maximum* (resp. *minimum*) *point* if $\mathrm{f}(k)$ is greatest (resp. least) value of $\mathrm{f}(x)$ in an **immediate neighbourhood** of k, i.e. for a small interval of values of x containing k.

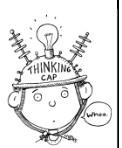
In the above definition, $\mathrm{f}(k)$ need not be the greatest (least) value of $\mathrm{f}(x)$ for all x in D_f and hence the word "*local*". On the other hand, we look at all the values of $\mathrm{f}(x)$ in D_f to identify the global minimum/maximum point.



WONDER

Must the global minimum/maximum be a stationary point?

Answer: No.



WONDER

Can a graph have no global minimum/maximum point?

Answer: Yes. For instance, consider $\mathrm{f}(x) = 2x,\ 0 < x < 1$.
Note that 0 and 2 does not lie in the range of f.

2.3 Determining the Nature of Stationary Points

If a point is known to be stationary, we can use the following methods to determine if it is a local maximum, a local minimum or a stationary point of inflexion.

First Derivative Test (Sign Test)

Check the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ just before and just after a stationary point. The following table shows all possible outcomes.

x	a^-	a	a^+	b^-	b	b^+	c_1^-	c_1	c_1^+	c_2^-	c_2	c_2^+
Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$	+	0	−	−	0	+	+	0	+	−	0	−
Shape of curve	/	—	\	\	—	/	/	—	/	\	—	\
Nature	MAXIMUM			MINIMUM			INFLEXION			INFLEXION		

UNDERSTAND

What values should you choose to test for the nature of stationary point at $x = 2$?

Answer: To be safe, choose $x = 1.99$ and $x = 2.01$ (or values that are even closer to $x = 2$).

Second Derivative Test

Check the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the stationary point. The following shows all possible outcomes.

Sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at *stationary point*	Negative	Positive	Zero
Nature	MAXIMUM	MINIMUM	**Inconclusive!**

Notes:

1. The choice of first / second derivative test depends on the equation of the given curve and the relative ease of doing each test.
2. Note that some points of inflexion are non-stationary; hence please do not simply write "point of inflexion" but rather "stationary point of inflexion".

UNDERSTAND

Why does the 2nd derivative test work?

Answer: Refer to 1st derivative test and consider the gradient of $y = \mathrm{f}'(x)$.

QUESTION

Why does the 2nd derivative test fail when $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at a stationary point?

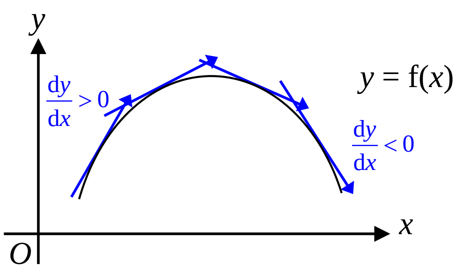
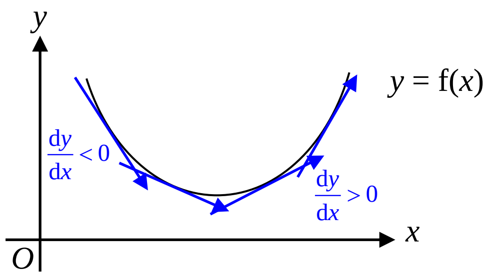
Answer: Consider and sketch the graphs of $y = x^4$ and $y = -x^4$, which has a stationary point at $x = 0$.

2.4 Concavity / Curvature

Recall that

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \text{rate of change of } \frac{dy}{dx} \text{ with respect to } x,$$

i.e. $\frac{d^2y}{dx^2}$ is the gradient function for $\frac{dy}{dx}$. Hence we have the following results:

<table>
<tr><th>Concave downwards</th><th>Concave upwards</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>As x increases, the gradient of the curve decreases, i.e. $f'(x)$ is decreasing.

$f'(x)$ is decreasing $\Leftrightarrow$ $f''(x) < 0$.</td><td>As x increases, the gradient of the curve increases, i.e. $f'(x)$ is increasing.

$f'(x)$ is increasing $\Leftrightarrow$ $f''(x) > 0$.</td></tr>
</table>

Therefore, if $f''(x) < 0$ (resp. $f''(x) > 0$) for all x in an interval I, then curve of $y = f(x)$ **concave downwards** (resp. **concave upwards**) on I.

Example 2.4.1

Determine the concavity of $y = f(x) = x^3 - 3x^2 + 1$ for (a) $x > 1$, (b) $x < 1$.

Solution:

$f(x) = x^3 - 3x^2 + 1 \Rightarrow f'(x) = 3x^2 - 6x$ and $f''(x) = 6x - 6$.

(a) $x > 1$: $f''(x) = 6x - 6 > 0$ $\therefore$ curve is concave upwards on $(1, \infty)$.

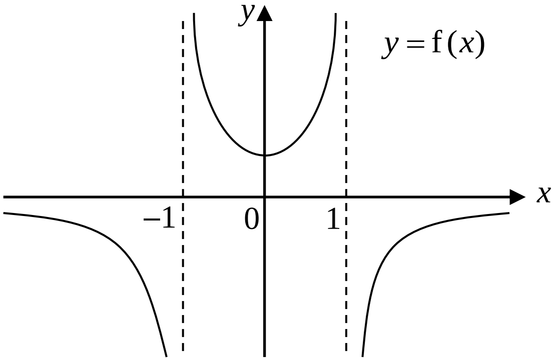
(b) $x < 1$: $f''(x) = 6x - 6 < 0$ $\therefore$ curve is concave downwards on $(-\infty, 1)$.

Example 2.4.2

Determine the largest set of values of x for which the graph given below is:

(i) strictly increasing, (ii) strictly decreasing, (iii) concave upwards,

(iv) concave downwards.

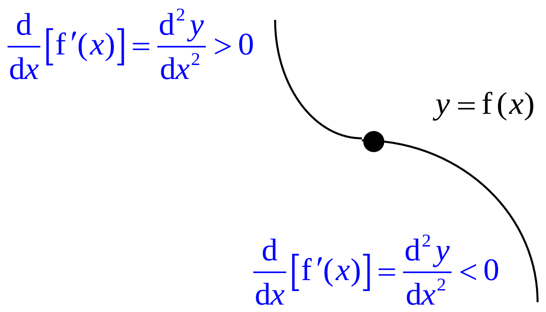


Solution:

(i) $(0,\,1)\cup(1,\infty)$, (ii) $(-\infty,-1)\cup(-1,0)$,

(iii) $(-1,\,1)$, (iv) $(-\infty,-1)\cup(1,\infty)$.

2.5 Graph of the Derivative Function, $y = \mathrm{f}'(x)$

Graph of $y=\mathrm{f}(x)$	**Graph of** $y=\mathrm{f}'(x)$	**Mathematical Verification**
Stationary points	x-intercepts	The x-intercepts of $y=\mathrm{f}'(x)$ occurs when $\mathrm{f}'(x)=0$, i.e. at the same x-values for the stationary points on the curve $y=\mathrm{f}(x)$.
Points of inflexion	Turning points	 At a point of inflexion, the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes from positive to negative (or vice versa), hence $\mathrm{f}'(x)$ changes from strictly increasing to strictly decreasing (or vice versa). This implies that $y=\mathrm{f}'(x)$ has a turning point.
• Gradient of $y=\mathrm{f}(x)$ is positive. • Gradient of $y=\mathrm{f}(x)$ is negative.	• $y=\mathrm{f}'(x)$ is above the x-axis. $[\,y=\mathrm{f}'(x)>0\,]$ • $y=\mathrm{f}'(x)$ is below the x-axis. $[\,y=\mathrm{f}'(x)<0\,]$	• $y=\mathrm{f}'(x)>0$ same as $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]>0$. • $y=\mathrm{f}'(x)<0$ same as $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)]<0$.
• $y=\mathrm{f}(x)$ has vertical asymptote $x=a$. • $y=\mathrm{f}(x)$ has horizontal asymptote $y=b$. • $y=\mathrm{f}(x)$ has oblique asymptote $y=mx+c$.	• $y=\mathrm{f}'(x)$ has vertical asymptote $x=a$. • $y=\mathrm{f}'(x)$ has horizontal asymptote $y=0$. • $y=\mathrm{f}'(x)$ tends to *m*.	• Vertical asymptote(s) implies that the limiting value of $\mathrm{f}'(x)$ is infinite and undefined. • As the curve approaches a horizontal asymptote, the gradient of the curve gradually reduces to zero. • As the curve approaches an oblique asymptote, the gradient of the curve gradually tends to *m*.

CHECK

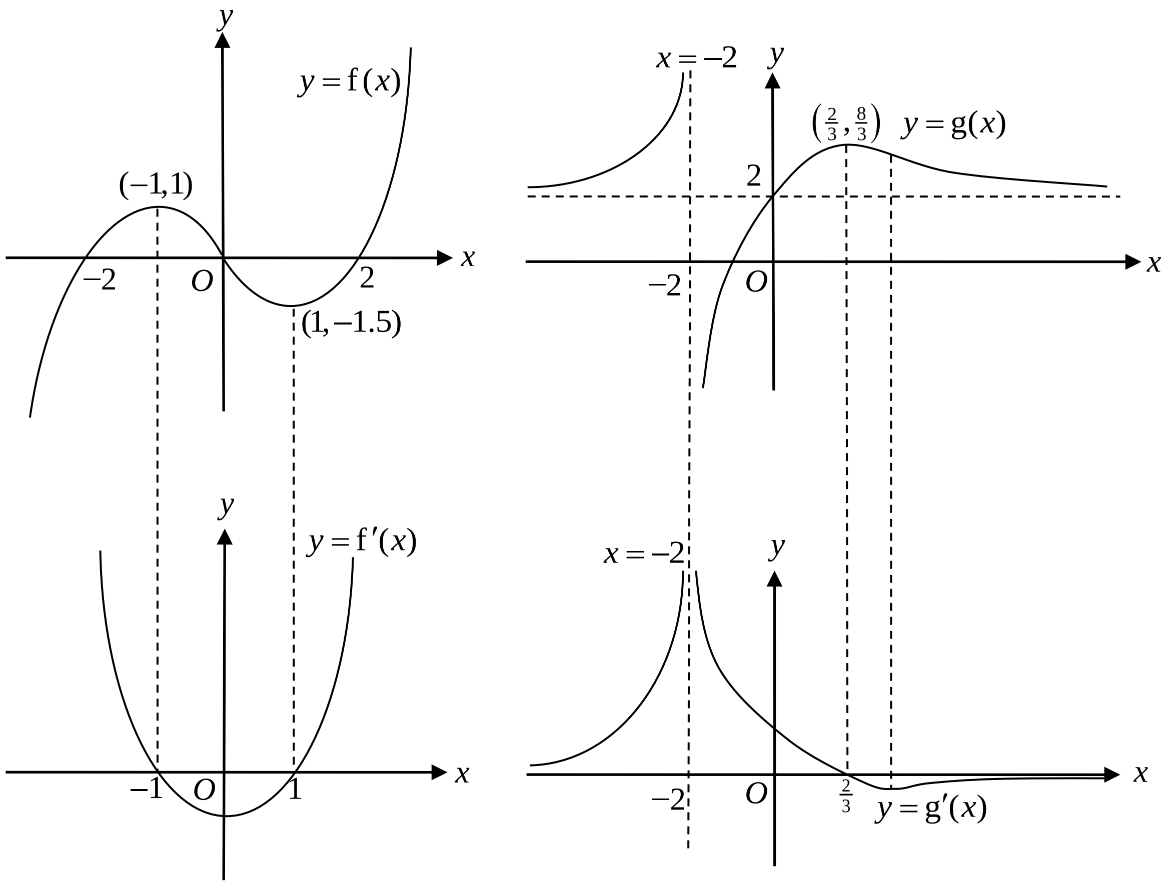
Will the axial intercepts in the graph of $y = \mathrm{f}(x)$ remain in the graph of $y = \mathrm{f}'(x)$?

Answer: Generally not. (One exception is when a stationary point is also an x-intercept). Can you think of any other exception(s)?

Example 2.5.1

The following is a sketch of the curves $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$. Sketch, on separate diagrams, the graphs of $y = \mathrm{f}'(x)$ and $y = \mathrm{g}'(x)$.

Solution:



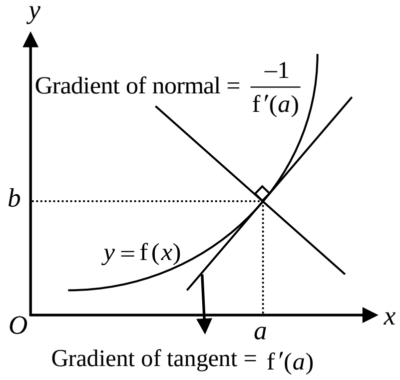
§3 Tangents and Normals

3.1 Equations of the Tangent and the Normal of a Curve at a Point (*a*, *b*)

From the 'O'-level syllabus, we have the following:

For a given point (a, b) on a curve $y = \mathrm{f}(x)$,

Equation of tangent	$y - b = \mathrm{f}'(a)(x - a)$
Equation of normal	$y - b = \dfrac{-1}{\mathrm{f}'(a)}(x - a)$



Example 3.1.1

The normal to the curve $y = (x-1)^2$ at the point $P(2,1)$ cuts the curve at another point Q.

(i) Find the equation of the normal.
(ii) Obtain the coordinates of the point Q.

Solution:

(i) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2(x-1).$

At $P(2,1)$, gradient of curve = 2(2 – 1) = 2. Thus gradient of normal at P = $-\dfrac{1}{2}$.

Equation of normal: $y - 1 = -\dfrac{1}{2}(x-2) \Rightarrow 2y = 4 - x$

(ii) When the normal cuts the curve again, $\dfrac{(4-x)}{2} = (x-1)^2 \Rightarrow x^2 - \dfrac{3}{2}x - 1 = 0$

$$\Rightarrow (x-2)\left(x + \frac{1}{2}\right) = 0.$$

Thus $x = 2$ or $x = -\dfrac{1}{2}$.

Therefore x-coordinate of the point Q is $-\dfrac{1}{2}$. ($\because x = 2$ corresponds to P)

Substituting $x = -\dfrac{1}{2}$ into the equation of the curve, we get $y = \left(-\dfrac{1}{2} - 1\right)^2 = \dfrac{9}{4}$.

Thus, coordinates of Q is $\left(-\dfrac{1}{2}, \dfrac{9}{4}\right)$.

Example 3.1.2

Find the euatqions of the tangents to the curve $y=\left(\frac{1}{2}x-1\right)^3$ which are parallel to the line $y=\frac{3}{2}x-5$.

Solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x}=3\left(\frac{1}{2}x-1\right)^2\cdot\frac{1}{2}.$$

Gradient of tangents = $\frac{3}{2}$ $\Rightarrow \frac{3}{2}\left(\frac{1}{2}x-1\right)^2=\frac{3}{2}$

$$\Rightarrow \left(\frac{1}{2}x-1\right)^2=1$$

$$\Rightarrow \frac{1}{2}x-1=\pm1$$

$$\Rightarrow x=0 \text{ or } 4.$$

When $x = 0$, $y = -1$.

Equation of tangent at $(0, -1)$ is

$$y-(-1)=\frac{3}{2}(x-0)$$

$$y=\frac{3}{2}x-1.$$

When $x = 4$, $y = 1$.

Equation of tangent at $(4, 1)$ is

$$y-1=\frac{3}{2}(x-4)$$

$$y=\frac{3}{2}x-5.$$

Note that the results hold even if the curve $y=\mathrm{f}(x)$ is defined parametrically or implicitly, which we will observe in the following examples.

Example 3.1.3 (Involves Implicit Differentiation)

Find the equation of the tangent to the curve $y^3-2xy^2+3x^2-3=0$ at the point (2 ,3).

Solution:

Differentiating implicitly w.r.t. *x*, we get

$$3y^2\frac{\mathrm{d}y}{\mathrm{d}x}-4xy\frac{\mathrm{d}y}{\mathrm{d}x}-2y^2+6x=0 \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2y^2-6x}{3y^2-4xy}.$$

When $x=2$, $y=3$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2(3)^2-6(2)}{3(3)^2-4(2)(3)}=2.$

Equation of tangent is $y-3=2(x-2) \;\Rightarrow\; y=2x-1.$

Example 3.1.4 (Involves Implicit Differentiation)

Given that $x^2-2xy+2y^2=4$, find an expression for $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of *x* and *y*. Hence find the coordinates of each point on the curve at which the tangent is parallel to the *x*-axis.

Solution:

$x^2-2xy+2y^2=4$ --- (1)

Differentiating implicitly w.r.t. *x*, we get

$$2x-2x\frac{\mathrm{d}y}{\mathrm{d}x}-2y+4y\frac{\mathrm{d}y}{\mathrm{d}x}=0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2y-2x}{4y-2x}=\frac{y-x}{2y-x}.$$

For tangent to the parallel to the *x*-axis, $\frac{\mathrm{d}y}{\mathrm{d}x}=0 \;\Rightarrow\; y=x$ --- (2)

Substitute (2) into (1), we get

$$\begin{aligned} y^2-2y^2+2y^2&=4\\ y^2&=4\\ y&=2 \quad \text{or} \quad y=-2\\ x&=2 \quad \text{or} \quad x=-2 \end{aligned}$$

Required coordinates are (2, 2) and (−2, −2).

Example 3.1.5 (Involves Parametric Differentiation)

Find the equations of the tangents and normals to the curve defined parametrically as

$$y = t^3 + 1, \;\; x = t^2$$

at the points where the curve cuts the line $x = 4$.

Solution:

$$\left.\begin{aligned} y = t^3 + 1 \quad &\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 \\ x = t^2 \quad &\Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 2t \end{aligned}\right\} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{3t^2}{2t} = \frac{3}{2}t\,.$$

When $x = 4$,	$t = -2$	or	$t = 2$
Thus,	$y = -7$	or	$y = 9$
And	$\frac{\mathrm{d}y}{\mathrm{d}x} = -3$	or	$\frac{\mathrm{d}y}{\mathrm{d}x} = 3$

Equation of tangents:

$$y + 7 = -3(x - 4) \quad \text{or} \quad y - 9 = 3(x - 4)$$
$$y = -3x + 5 \quad \text{or} \quad y = 3x - 3$$

Equation of normals:

$$y + 7 = \frac{1}{3}(x - 4) \quad \text{or} \quad y - 9 = -\frac{1}{3}(x - 4)$$
$$y = \frac{1}{3}x - \frac{25}{3} \quad \text{or} \quad y = -\frac{1}{3}x + \frac{31}{3}\,.$$

Example 3.1.6 (Involves Parametric Differentiation)

A curve is defined parametrically as

$$x = 2 + t, \;\; y = 1 - t^2 .$$

Show that the normal to the curve at the point with parameter *t* has equation

$$x - 2ty = 2t^3 - t + 2 .$$

The normal at the point *T*, where $t = 2$ cuts the curve again at the point *P*, where $t = p$. Show that $4p^2 + p - 18 = 0$ and hence deduce the coordinates of *P*.

Solution:

$$\left.\begin{aligned} y = 1 - t^2 \quad &\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2t \\ x = 2 + t \quad &\Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \end{aligned}\right\} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}.\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)}{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)} = \frac{-2t}{1} = -2t.$$

Equation of normal at point with parameter *t*:

$$\begin{aligned} y - (1 - t^2) &= \frac{1}{2t}(x - (2 + t)) \\ 2ty - 2t + 2t^3 &= x - 2 - t \\ 2t^3 - t + 2 &= x - 2ty \quad \text{(shown).} \end{aligned}$$

Equation of normal when *t* = 2,

$$\begin{aligned} 2(2)^3 - (2) + 2 &= x - 2(2)y \\ 16 &= x - 4y \quad \text{---} \quad (1) \end{aligned}$$

When *t* = *p*, $y = 1 - p^2$ and $x = 2 + p$.

Substitute into (1), we get

$$\begin{aligned} 16 &= 2 + p - 4(1 - p^2) \\ 4p^2 + p - 18 &= 0 \quad \text{(shown).} \end{aligned}$$

Solving, we obtain $\quad (4p + 9)(p - 2) = 0$.

Thus, $\quad p = 2 \text{ (point } T\text{)} \;\text{ or }\; p = -\frac{9}{4} \text{(point } P\text{)}$.

Coordinates of *P* is $\left(-\frac{1}{4}, -\frac{65}{16}\right)$.

§4 Practical Problems involving Differentiation

4.1 Maxima & Minima

We can apply differentiation to maximise or minimise a quantity given some restrictions / limitations.

For example, a farmer wishes to find out how to enclose a rectangular piece of land with his fixed amount of fencing, say, 1000 m on all sides of the rectangle (restriction). How would you advise him to obtain the largest area (quantity) for his herd to graze the land?

Such problems exist in the real world especially in the fields of economics, sciences and manufacturing sectors as we always try to minimise cost and/or maximise profit.

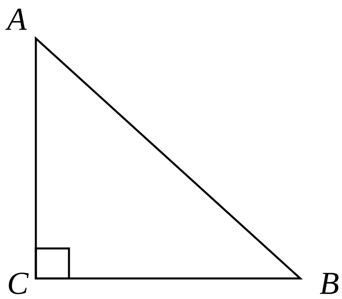
Example 4.1.1 (Function of a single variable)

Triangle *ABC* has a right angle at *C*. The shape of the triangle can vary but the sides *BC* and *CA* have a fixed total length of 10 cm. Find the maximum area of the triangle.

Solution:

Step 1: Express the quantity to be maximised as a function of one variable.

Let side *BC* be x, hence side $CA = 10 - x$.



Therefore area (P) of right-angle triangle is

$$P = \frac{1}{2}(10 - x)x = 5x - \frac{x^2}{2}.$$

Step 2: Differentiate w.r.t. the variable.

$$\frac{\mathrm{d}P}{\mathrm{d}x} = 5 - x.$$

Step 3: Find stationary value(s) of the variable.

For stationary values, $\frac{\mathrm{d}P}{\mathrm{d}x} = 0 \Rightarrow 5 - x = 0 \Rightarrow x = 5.$

When $x = 5\text{cm}$, $P = 12.5\text{cm}^2$.

Step 4: Check whether stationary value gives a maximum.

Using the 2nd derivative test, we get $\frac{\mathrm{d}^2P}{\mathrm{d}x^2} = -1 < 0$.

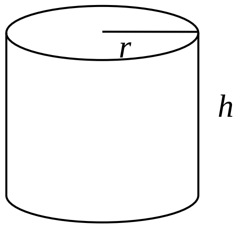
Hence the triangle has a maximum area of 12.5cm^2 when x is 5 cm.

Example 4.1.2 (Function of 2 variables)

A closed cylindrical can has height h and base radius r and its volume is 0.01 cubic units. Show that the surface area S is given by $S = 2\pi r^2 + \dfrac{1}{50r}$. Hence find the value of r for which S is a minimum.

Solution:

$S = 2\pi r^2 + 2\pi rh$ --- (1)



We can see that S is a function of 2 variables, r and h.
We now use $V = 0.01$ units^3 to remove the variable h.

$$V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2} = \frac{1}{100\pi r^2}.$$

Substituting $h = \dfrac{1}{100\pi r^2}$ in (1), we get

$$S = 2\pi r^2 + 2\pi r \frac{1}{100\pi r^2}$$

$$S = 2\pi r^2 + \frac{1}{50r} \text{ (shown).}$$

To minimise S, we consider

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \frac{1}{50r^2}.$$

For stationary values of S,

$$\frac{\mathrm{d}S}{\mathrm{d}r} = 4\pi r - \frac{1}{50r^2} = 0 \Rightarrow r = \frac{1}{\sqrt[3]{200\pi}}.$$

Using the second derivative test,

$$\frac{\mathrm{d}^2S}{\mathrm{d}r^2} = 4\pi + \frac{1}{25r^3} > 0 \text{ (since } r > 0\text{).}$$

Thus, S is minimum when $r = \dfrac{1}{\sqrt[3]{200\pi}}$.

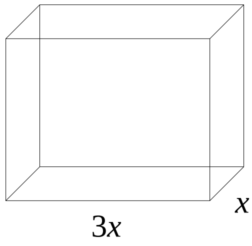
Example 4.1.3

An opened rectangular tank of capacity 1203 cm^3 is to be constructed using materials of negligible thickness. The length of the tank is to be three times its breadth (which is denoted as x cm). If the material needed for making the tank is denoted as A cm^2 , show that

$$A = \frac{3208}{x} + 3x^2.$$

Find the value of x for which A will be a minimum.

Solution:



Let h cm be height of the tank.

$h = \dfrac{1203}{3x^2} = \dfrac{401}{x^2}$.

$A = 3x(x) + 3x\left(\dfrac{401}{x^2}\right)(2) + x\left(\dfrac{401}{x^2}\right)(2) = 3x^2 + \dfrac{3208}{x}$ (shown).

$$\begin{aligned} \frac{\mathrm{d}A}{\mathrm{d}x} &= 6x - \frac{3208}{x^2} \\ \Rightarrow\quad 6x - \frac{3208}{x^2} &= 0 \\ \Rightarrow\quad x &\approx 8.12. \end{aligned}$$

Using the second derivative test,

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 6 + \frac{6416}{x^3} > 0 \text{ since } x > 0 .$$

Therefore $\min A \approx 592.9$ cm^2.

4.2 Connected Rates of Change

In this section, we consider problems involving the rates of change for two related variables.

For example, if the area of a metal frame, say A, is related to the temperature of the metal frame, x, by $A = \mathrm{f}(x)$, how can we find the rate of change of A at a certain time t, given the value of x and the rate of change of x at time t?

To solve such problems,

(i) Determine the rate of change to be found: $\dfrac{\mathrm{d}A}{\mathrm{d}t}$.

(ii) Identify what you are given: $\dfrac{\mathrm{d}x}{\mathrm{d}t}$.

(iii) By the Chain rule, we know that the given rate of change and the rate of change to be found are related by the identity

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Hence we need to find $\dfrac{\mathrm{d}A}{\mathrm{d}x} = \mathrm{f}'(x)$ at the value of x at time t and find $\dfrac{\mathrm{d}A}{\mathrm{d}t}$ using the above relation.

Example 4.2.1

[The volume and the surface area of a sphere with radius r are $V = \dfrac{4}{3}\pi r^3$ and $A = 4\pi r^2$ respectively.]

A spherical balloon is being inflated, and at the instant when its radius is 10 cm, its surface area is increasing at a rate of 6.4 cm^2 s^{-1}.
Find the rate of increase, at the same instant, of **(i)** the radius, **(ii)** the volume**.**

Solution:

(i) $A = 4\pi r^2$, $\quad \dfrac{\mathrm{d}A}{\mathrm{d}t} = 6.4 \text{ cm}^2\text{s}^{-1}$

Since $\dfrac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$ and $\dfrac{\mathrm{d}A}{\mathrm{d}t} = \dfrac{\mathrm{d}A}{\mathrm{d}r} \cdot \dfrac{\mathrm{d}r}{\mathrm{d}t}$,

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}A}{\mathrm{d}t}}{\frac{\mathrm{d}A}{\mathrm{d}r}} = \frac{6.4}{8\pi(10)} \quad (\text{when } r = 10)$$

$$= 0.0255 \text{ cm s}^{-1} (\text{to 3 s.f.})$$

(ii) $V = \dfrac{4}{3}\pi r^3$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2.$$

When $r = 10$,

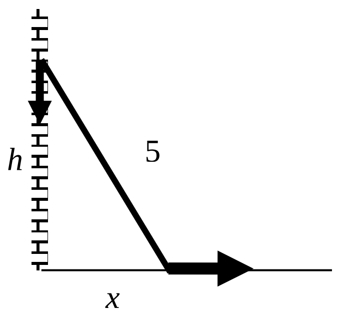
$$\frac{\mathrm{d}V}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}r} \cdot \frac{\mathrm{d}r}{\mathrm{d}t} = 4\pi(10)^2 \cdot (0.025465)$$

$$= 32 \text{ cm}^3 \text{ s}^{-1}.$$

Example 4.2.2

A ladder 5 m long is leaning against a vertical wall. The bottom of the ladder is pulled away along the ground from the wall at a constant rate of 0.4 ms^{-1}. How quickly will the top of the ladder be falling when the bottom of the ladder is 3 m from the wall?

Solution:



$h^2 + x^2 = 25 \quad \Rightarrow \quad h = \sqrt{25 - x^2}$.

$$\begin{aligned}\frac{\mathrm{d}h}{\mathrm{d}t} &= \frac{\mathrm{d}h}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{25 - x^2}\right] \times (0.4) \\ &= \frac{-x}{\sqrt{25 - x^2}} \times 0.4.\end{aligned}$$

When x = 3 m, $\dfrac{\mathrm{d}h}{\mathrm{d}t} = \dfrac{(-3) \times 0.4}{\sqrt{25 - 3^2}} = -\dfrac{1.2}{\sqrt{16}} = -0.3\text{ ms}^{-1}$.

The top of the ladder will be falling at a rate of 0.3 ms^{-1}.

Appendix A: Proofs of Results on Differentiation in Section 1.2 (Page 6)

1. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} x\right)=\dfrac{1}{\sqrt{1-x^2}}$, $|x|<1$.

Proof: Let $y=\sin^{-1} x$. Note that for $\sin^{-1} x$ to be defined, $|x|<1$. Then $\sin y = x$.

Differentiating implicitly w.r.t. $x, \cos y \dfrac{\mathrm{d}y}{\mathrm{d}x}=1 \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\cos y}$. $-$ (1)

Since $\sin^2 y+\cos^2 y=1$, $\cos^2 y = 1-\sin^2 y \Rightarrow$ $\cos y=\sqrt{1-\sin^2 y}=\sqrt{1-x^2}$. $-$ (2)

Substituting (2) into (1), $\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{1}{\cos y}=\dfrac{1}{\sqrt{1-x^2}}$, $|x|<1$ (shown).

QUESTION

In (2), why do we consider only the positive square root $\sqrt{1-x^2}$ and not the negative square root $-\sqrt{1-x^2}$?

Answer: The range of principal values of $y=\sin^{-1} x$ is $-\dfrac{\pi}{2}\le y\le\dfrac{\pi}{2}$.
This means that $\cos y$ is non-negative.

The proof for $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\cos^{-1} x\right)=-\dfrac{1}{\sqrt{1-x^2}}$, $|x|<1$ is similar, and is left as an exercise.

2. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1} x\right)=\dfrac{1}{1+x^2}$.

Proof: Let $y=\tan^{-1} x$. Then $\tan y = x$.

Differentiating implicitly w.r.t. $x, \sec^2 y\dfrac{\mathrm{d}y}{\mathrm{d}x}=1$.

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{\sec^2 y}=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2} \text{ (shown).}$$

3. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right)=a^x \ln a$

Proof: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a^x\right)=\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x\ln a}\right)=\mathrm{e}^{x\ln a}\cdot \ln a = a^x \ln a$ (shown).

Note that $a^x=\mathrm{e}^{x\ln a}$ for any positive rational number a.

4. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\log_a x\right] = \dfrac{1}{x}\cdot\log_a \mathrm{e}$.

Proof: $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\log_a x\right) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{\ln a}\right)$ (by Change of Base Law for logarithms)

$$= \frac{1}{\ln a}\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x\right)$$

$$= \frac{\log_a \mathrm{e}}{\log_a a}\cdot\frac{1}{x} \text{ (by Change of Base Law for logarithms)}$$

$$= \frac{1}{x}\cdot\log_a \mathrm{e}.$$

For results 1 to 4 above, the variants of the results (by replacing x with $\mathrm{f}(x)$) follow from the Chain Rule. For example, if we let $y = \mathrm{f}(x)$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}\left[\mathrm{f}(x)\right] &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1} y\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}y}\left(\sin^{-1} y\right)\cdot\frac{\mathrm{d}y}{\mathrm{d}x}\\ &= \frac{1}{\sqrt{1-y^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]\\ &= \frac{\mathrm{f}'(x)}{\sqrt{1-\left[\mathrm{f}(x)\right]^2}}.\end{aligned}$$

5. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}}$.

Proof: Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x) = 1$.

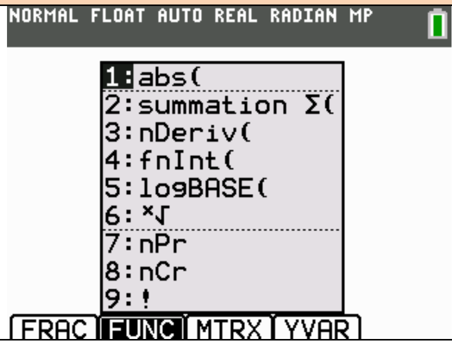
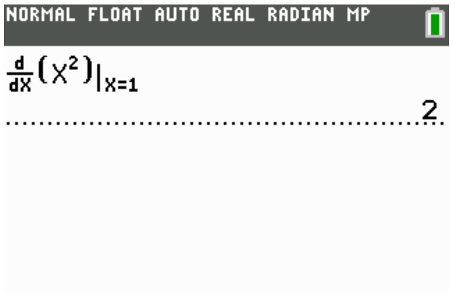
By treating x as a function of y and differentiating implicitly w.r.t. x, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}(x)\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 1 &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 1\\ &\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} = 1\\ &\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} \text{ (shown).}\end{aligned}$$

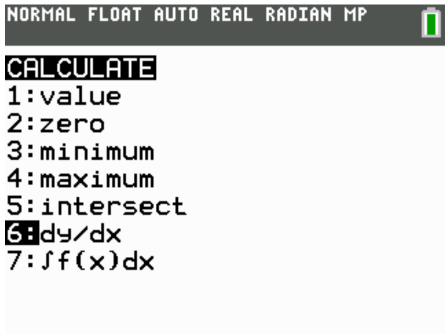
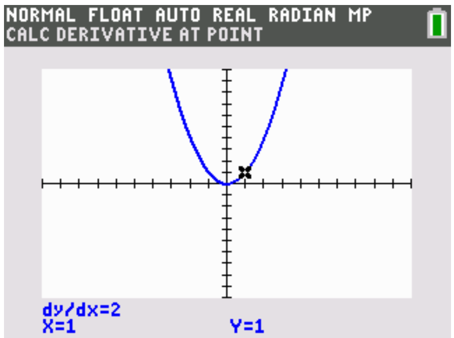
Appendix B: Use of GC involving derivatives

The graphing calculator can be used to determine the numerical derivative at a point on the curve. The tables below show two methods to obtain the gradient of $y = x^2$ at $x = 1$.

Method 1

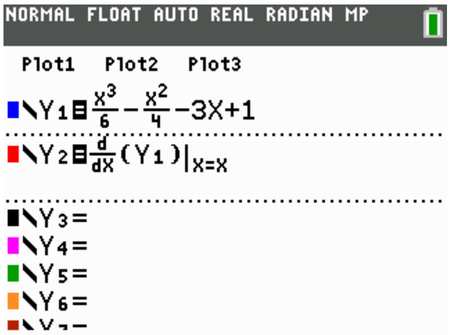
Steps	Screenshot
Make sure you are at the home screen. Press ALPHA WINDOW 3 .	
Press X,T,θ,n X,T,θ,n x^2) 1 . Then press ENTER . The numerical value of the derivative of the function will appear on the next line.	

Method 2

Steps	Screenshot
Sketch the graph of the function you are interested in. For example, $Y_1 = X^2$. Press 2nd TRACE . Look for [6: dy/dx] Press 6	
If you are interested in the value of $\left.\frac{dy}{dx}\right\|_{x=1}$, press 1 in the next screen. Then press ENTER . The value of $\left.\frac{dy}{dx}\right\|_{x=1}$ is 2.	

Suppose that we are given the equation $y = \mathrm{f}(x)$. The graphing calculator can be used to show the graph of the derivative function $y = \mathrm{f}'(x)$ without actually doing differentiation to find $\mathrm{f}'(x)$. As an illustration, consider

$$\mathrm{f}(x) = \frac{1}{6}x^3 - \frac{1}{4}x^2 - 3x + 1, \quad x \in \mathbb{R}.$$

Steps	**Screenshot**
Enter the function $\mathrm{Y}_1 = \frac{1}{6}x^3 - \frac{1}{4}x^2 - 3x + 1$ in your GC.	
At Y_2 enter the derivative function by pressing  You may want to distinguish the two graphs by adjusting the thickness of the curves. (Scroll to the left until the cursor reaches the “slanted line” besides the function and hit ENTER repeatedly.)	
Graph the two functions.