Nanyang Junior College

JC1 H2 Mathematics 2019

Lecture Notes

Chapter	Topic
5	Inequalities and Equations
6	Sequences and Series

Name:____________________

CT: 19__

Chapter 5

Inequalities & Equations

At the end of this chapter, students should be able to:
(a) find the numerical solution of an equation using a graphing calculator;
(b) solve inequalities of the form $\frac{f(x)}{g(x)} > 0$ (including cases involving $<$, $\geq$ and $\leq$) where $f(x)$ and $g(x)$ are linear expressions or quadratic expressions (either factorisable or always positive);
(c) solve inequalities by graphical methods with/without the use of a graphing calculator;
(d) solve a system of linear equations using a graphing calculator;
(e) use an equation or system of linear equations to model and solve practical problems, and interpret the solution in the context of a problem.

5.1 Solving Equations Using GC (Independent Learning)

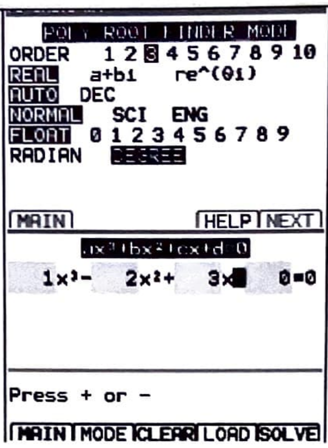
5.1.1 Roots of Polynomials

When solving equations, one should take note of the type of equations given and the type of solutions required. For polynomials that do not require exact solutions, one may use the pre-installed app "PlySmlt2" to acquire the roots.

Keep in mind that a polynomial of degree n will have a maximum of n real roots.

Example 1: Find the real root(s) of the equation $x^3 - 2x^2 + 3x = 1$.

Solution:	**GC Screenshots**	**ThinkZone**
Rewrite equation in the form $f(x) = 0$ where $f(x)$ is a polynomial, i.e. $x^3 - 2x^2 + 3x - 1 = 0$.		
Under , select option **PlySmlt2** and followed by option **1:POLYNOMIAL ROOT FINDER**.	 	What are some methods used in 'O' level to find roots of an equation?

<table>
<tr>
<td>Select order 3 for degree 3 polynomial and real roots.</td>
<td>


</td>
<td>Without a GC, would you be able to find the roots to 3 decimal places of accuracy?

There are numerical methods that can be used to estimate the roots.
You can read up on

- Bisection Method ,
- Newton's Raphson Method and
- Linear Interpolation Method .
</td>
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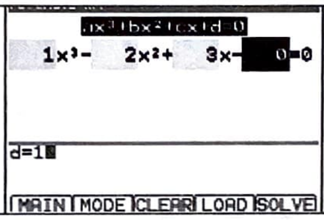
<tr>
<td>Key in the coefficients for the polynomial and press table f5 graph to solve.</td>
<td>


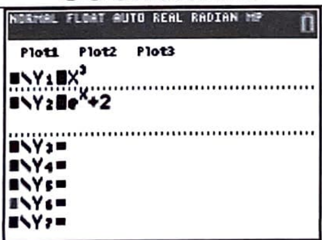
</td>
<td></td>
</tr>
<tr>
<td>∴ The root of the equation is ≈ 0.430.</td>
<td></td>
<td></td>
</tr>
</table>

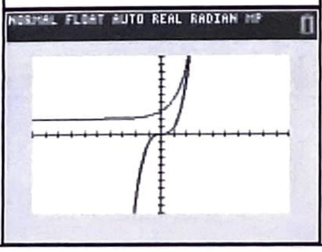
5.1.2 Graphical Method

For non-polynomials, equations can be solved by regarding the expressions on both sides of the equality as two separate functions. Then the solutions to the equations are the x-intercepts of the points of intersections between both graphs.
(You may refer to "My First Step in using TI-84Plus CE for H1 and H2 Math" Annex A, Page 72).

Example 2: Find the real root(s) of the equation $x^3 - e^x = 2$.

<table>
<tr>
<td>Solution:</td>
<td>GC Screenshots</td>
<td>ThinkZone</td>
</tr>
<tr>
<td>You can rewrite equation in the form $x^3 = e^x + 2$ and consider the intersection of the graphs of $y = x^3$ and $y = e^x + 2$. Press stat plot f1 y= and key in the equations for the 2 graphs.

Press table f5 graph .</td>
<td>


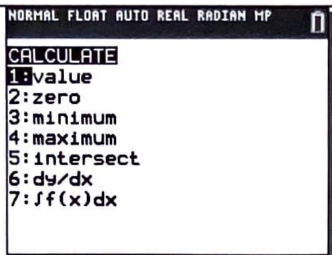


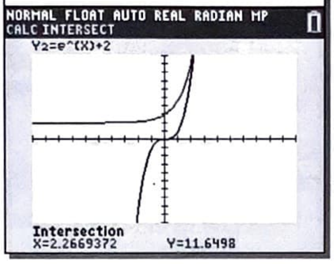
</td>
<td></td>
</tr>
</table>

Press calc f4 **2nd** **trace** to select **CALC** and select option **5:intersect.** Shift the cursor closer to the root you want to estimate for both graphs and press enter. Just press enter for estimate.

A root is $x = 2.27$ (3sf).

Another root is $x = 4.47$ (3sf).





Note, GC cannot be used if the question include words such as solving it “exactly” or “without the use of GC” or “algebraically”.

The use of GC must be coupled with

- Care on the GC keystrokes,
- Window Setting and
- Zoom factor.

Note, sometimes, what you see is not what you should get!

Note: For this question you could sketch the graphs of $y = x^3 - 2$ and $y = e^x$ or the graphs of $y = x^3 - e^x$ and $y = 2$ or the graph of $y = x^3 - e^x - 2$ (then in this case you try to obtain the ‘zero’, i.e. x-intercept, instead).

For the cases where the intersection points appear to be outside of the standard window screen you can always adjust the tblset f2 **window** setting.

ThinkZone: Can you pinpoint what’s wrong with the following working to find the roots to the equation $x = \sqrt{x+2}$?

$x = \sqrt{x+2}$

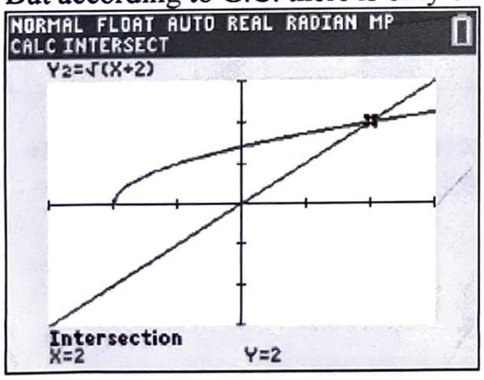
$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

So, $x = -1$ and 2.

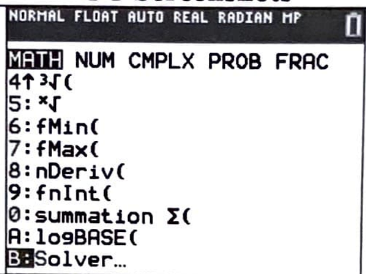
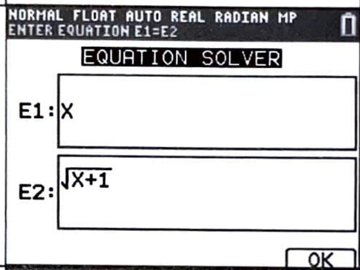
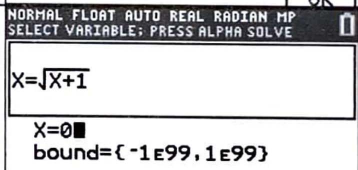
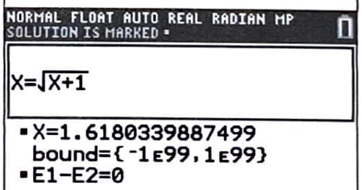
But according to G.C. there is only one root $x = 2$.



5.1.3 Equation Solver

An alternative to solving equations that do not require exact answers is to use the in-built GC programme "Equation Solver". The GC will provide numerical approximations to the solutions of the equation.

Example 3: Solve the equation $x = \sqrt{x+1}$.

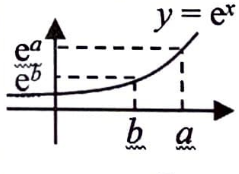
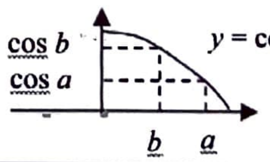
Solution:	**GC Screenshots**	**ThinkZone**
test A Press math and select `C:Numeric Solver`. (Scroll up to reach the bottom of the list)		
Key in the respective expressions that you want to equate. table f5 Press graph to select OK (You may also scroll down.)		You may also try $x - \sqrt{x+1} = 0$
Type in an approximated guess, and press table f5 graph to select Solve. For older GCs, press ALPHA ENTER.	 	

Note: The Numeric Solver requires an initial approximation in order for the calculator to obtain an answer. It may not be helpful in questions where you do not know the total number of real roots. Also, the Numeric Solver can fail, depending on the nature of the equation.

5.2 Introduction to Inequalities (Independent Reading)

Basic Concepts and Rules for Manipulating Inequalities

Let $a, b, c, d \in \mathbb{R}$.

Property	**Example**
(i) If $a > b$ and $b > c$, then $a > c$. (Transitive Property)	If $5 > 3$ and $3 > 2$, then $5 > 2$.
(ii) If $a > b$ and $c > 0$, then $ac > bc$. If $a > b$ and $c > 0$, then $\left(\frac{a}{c} > \frac{b}{c}\right)$. Multiplying/dividing both sides of an inequality by a **positive** real number **preserves** the inequality sign.	If $6 > 3$, then $6\times 2 > 3\times 2$ and $\frac{6}{2} > \frac{3}{2}$.
(iii) If $a > b$ and $c < 0$, then $ac < bc$. If $a > b$ and $c < 0$, then $\left(\frac{a}{c} < \frac{b}{c}\right)$. Multiplying/dividing both sides of an inequality by a **negative** real number **changes** the inequality sign.	If $6 > 3$, then $6\times(-2) < 3\times(-2)$ and $\frac{6}{-2} < \frac{3}{-2}$.
(iv) If $a > b$ and $c > d$, then $a + c > b + d$. **Warning:** But $a - c > b - d$ is not necessary true. e.g. $3 > 2$ and $7 > 1$ but $3 - 7 \ngtr 2 - 1$	If $6 > 3$ and $5 > 4$, then $6 + 5 > 3 + 4$.
(v) If $a > b > 0$ and $n > 0$, $a^2 > b^2,\ a^3 > b^3, \ldots,\ a^n > b^n$ and $\frac{1}{a} < \frac{1}{b},\ \frac{1}{a^2} < \frac{1}{b^2}, \ldots,\ \frac{1}{a^n} < \frac{1}{b^n}$ **Warning:** In general, $a > b \Rightarrow a^2 > b^2$ is not true if a, b are not both positive. e.g. $-3 > -4$ but $(-3)^2 < (-4)^2$; $2 > -3$ but $(2)^2 < (-3)^2$	If $6 > 3$, then $6^2 > 3^2,\ 6^3 > 3^3, \ldots$ and $\frac{1}{6} < \frac{1}{3},\ \frac{1}{6^2} < \frac{1}{3^2},\ \frac{1}{6^3} < \frac{1}{3^3}, \ldots$
(vi) If $a > b$, $f(a) > f(b)$ if f is a strictly increasing function and $f(a) < f(b)$ if f is a strictly decreasing function.	$a > b$ and $e^a > e^b$  $0 < b < a < \frac{\pi}{2}$ and $\cos a < \cos b$ 

5.3 Solving Polynomial Inequalities in One Variable

To solve any inequality, we need to manipulate it to any of the form $f(x) > 0$, $f(x) < 0$, $f(x) \leq 0$, or $f(x) \geq 0$ where $f(x)$ is a polynomial.

General steps in solving polynomial inequalities

1) **Express** the inequality in the form $f(x) > 0$, $f(x) < 0$, $f(x) \leq 0$ or $f(x) \geq 0$. To facilitate the ease of solving, one should manipulate the inequality such that the coefficient of the term with the highest power is positive.
2) **Factorise** $f(x)$ completely or by completing the square.
3) **Find the roots** of the equation $f(x) = 0$.
4) **Identify the required interval** by using the **test point method**.

5.3.1 Solving quadratic inequalities

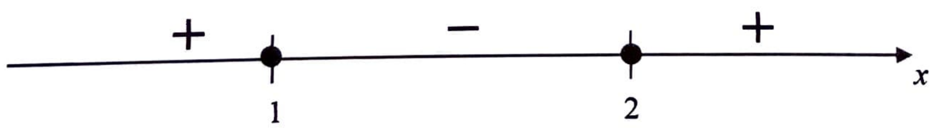
Example 4: Solve the inequality $x^2 - 3x \geq -2$.

Solution:

1) Rearrange into $x^2 - 3x + 2 \geq 0$. Note that the coefficient of x^2 is 1 which is positive.
2) Factorise f(x) completely: $(x-1)(x-2) \geq 0$
3) The roots are $x = 1$ or 2.
4) Identifying the required interval:
   a) Mark the roots on the number line. The use of solid dots or circles on the number line is used to include or exclude the end points of an interval respectively.
   b)

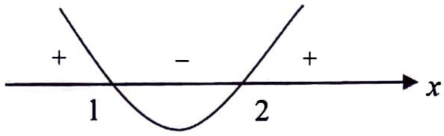
   c) Text the signs in each interval (hence the name test point method) by suitable values in each region. For example:
      - Substitute $x = 0$ into the expression $(x-1)(x-2)$ to determine the sign for the leftmost interval. We will get a positive sign when it is substituted in.
      - Substitute $x = 1.5$ into the expression $(x-1)(x-2)$ to determine the sign for the middle interval. We will get a negative sign when it is substituted in.
      - Substitute $x = 3$ into the expression $(x-1)(x-2)$ to determine the sign for the rightmost interval. We will get a positive sign when it is substituted in. Note that by ensuring that the coefficient of x^2 to be positive, the rightmost region in the number line will be positive.

Thus we have the following diagram.



For $x^2 - 3x \geq -2$, we need $x \leq 1$ or $x \geq 2$.

This is similar and consistent when we draw the graph of $y = x^2 - 3x + 2$.



That is positive polarity when $x > 2$ or $x < 1$ and negative polarity for $1 < x < 2$.

Example 5: Find the exact solution to the inequality $x^2 + 3x - 1 > 0$.

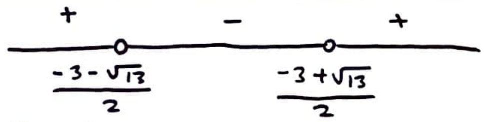
Solution:

Factorise the polynomial $x^2 + 3x - 1$ by completing the square.

$$\left(x+\frac{3}{2}\right)^2 - \frac{13}{4} > 0$$

$$\left(x+\frac{3}{2}+\sqrt{\frac{13}{4}}\right)\left(x+\frac{3}{2}-\sqrt{\frac{13}{4}}\right) > 0$$

$$\left(x-\left(\frac{-3-\sqrt{13}}{2}\right)\right)\left(x-\left(\frac{-3+\sqrt{13}}{2}\right)\right) > 0$$



For $x^2+3x-1>0$, we need

$x < \frac{-3-\sqrt{13}}{2}$ or $x > \frac{-3+\sqrt{13}}{2}$

ThinkZone

What is a keyword in this question?

When you have a quadratic factor you cannot factorise nicely, complete the square to factorise it into factors with surds.

What are some values to substitute in to test? $\sqrt{13} < \sqrt{16} = 4$

Example 6: Find the range of values of x such that $3x < x^2 + 5$.

Solution:

$3x < x^2 + 5$

$x^2 - 3x + 5 > 0$

$(x-\frac{3}{2})^2 - \frac{9}{4} + 5 > 0$

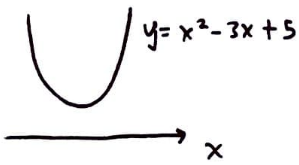
$(x-\frac{3}{2})^2 + \frac{11}{4} > 0$

since $(x-\frac{3}{2})^2 \geq 0$ for all $x \in \mathbb{R}$,

therefore $(x-\frac{3}{2})^2 + \frac{11}{4} > 0$ for $x \in \mathbb{R}$

$\Rightarrow x \in \mathbb{R}$

ThinkZone



Can you interpret the geometrical meaning to this result?

What would the solution be if it is of the form $(x+a)^2 + b^2 < 0$? $x \in \varnothing$

solution set = $\varnothing$

Self-Review 1: Solve algebraically the inequality $x^2 - 4x + 5 < 0$. [no solution]

$x^2 - 4x + 5 < 0$

$(x-2)^2 + 1 < 0$

Since $(x-2)^2 \geq 0$ for $x \in \mathbb{R}$, $(x-2)^2 + 1 > 0$ for $x \in \mathbb{R}$

~~∴ solution set = ∅~~

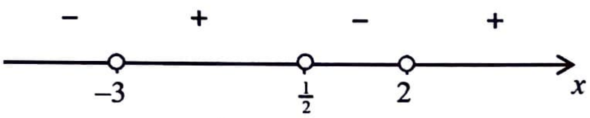
Thus, $(x-2)^2 + 1 \not< 0$ for all $x \in \mathbb{R}$

5.3.2 Solving inequalities involving higher power polynomials

Example 7: Without the use of GC, solve the following inequalities:
(i) $(x+3)(2x-1)(2-x)<0$ (ii) $(x+3)^2(2x-1)(2-x)<0$ (iii) $(x+3)^3(2x-1)(2-x)<0$

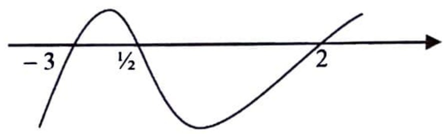
Solution:

(i) $(x+3)(2x-1)(2-x)<0$
$(x+3)(2x-1)(x-2)>0$

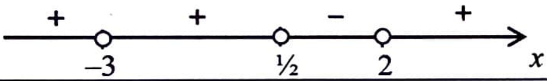


$\therefore -3<x<\frac{1}{2}$ or $x>2$

ThinkZone
Notice that if we draw the curve $y=(x+3)(2x-1)(x-2)$, the polarity is positive for $-3<x<\frac{1}{2}$ or $x>2$



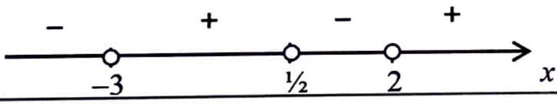
(ii) $(x+3)^2(2x-1)(2-x)<0$
$(x+3)^2(2x-1)(x-2)>0$



$\therefore x<\frac{1}{2}$, $x\neq -3$ or $x>2$

(ii) Does the sign always change?

(iii) $(x+3)^3(2x-1)(2-x)<0$
$(x+3)^3(2x-1)(x-2)>0$

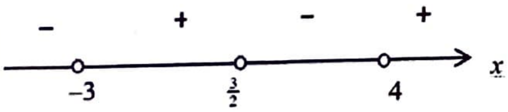


$-3<x<\frac{1}{2}$ or $x>2$

Could you generalise the results for $(x+3)^{2n}(2x-1)(2-x)<0$ and $(x+3)^{2n+1}(2x-1)(2-x)<0$?

Example 8: (Independent learning) Solve the inequality $-2x^3+5x^2+21x-36<0$.

Solution:
$-2x^3+5x^2+21x-36<0$
$2x^3-5x^2-21x+36>0$



$-3<x<\frac{3}{2}$ or $x>4$

ThinkZone
GC can be used directly to solve the inequality as the question did not specify the method to be used.
Use ***poly root finder*** to get the roots.

If asked to solve algebraically:
1. Factorise the expression $y=2x^3-5x^2-21x+36$ completely.
2. Find the roots of $y=0$ and determine the range of values of x that make $y>0$ using the test point method.

Example 9: By completing the square, show that $4x^2-4x+3$ is always positive for all real values of x.

Hence, solve the inequality $(4x^2-4x+3)(2x-3)\geq 0$.

Solution:

$$4x^2-4x+3=4\left(x^2-x\right)+3$$
$$=4\left[\left(x-\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right]+3$$
$$=4\left(x-\frac{1}{2}\right)^2+2>0 \text{ for all } x\in\mathbb{R}$$

since $\left(x-\frac{1}{2}\right)^2\geq 0$ for all $x\in\mathbb{R}$.

ThinkZone

$(4x^2-4x+3)(2x-3)\geq 0$

since $4x^2-4x+3\geq 0$ for all $x\in\mathbb{R}$,

$2x-3\geq 0$

$x\geq\frac{3}{2}$

What is a keyword in this part of the question?

From earlier, $4x^2-4x+3>0$ for all real values of x. Thus we can divide both sides of the inequality with $4x^2-4x+3$ without changing the inequality sign. (Recall Section 5.2(iii))

Self-Review 2: Without the use of GC, solve the inequality $(x^2-3x+4)(x+1)(x-2)\geq 0$.

Example 10: Without the use of GC, solve the inequality $0<x^2-x\leq 12$.

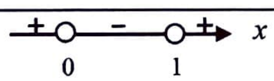
Solution:

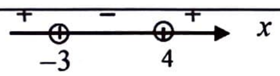
We can rewrite the inequality as follows:

$0<x^2-x$ and $x^2-x\leq 12$

$x(x-1)>0$ $\quad$ $x^2-x-12\leq 0$

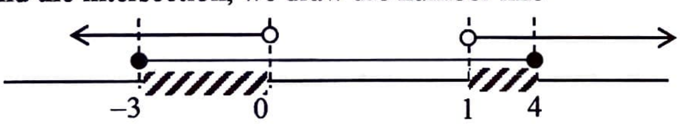
$(x+3)(x-4)\leq 0$





$\therefore x<0$ or $x>1$ **and** $\therefore -3\leq x\leq 4$

To find the intersection, we draw the number line



$\therefore -3\leq x<0$ **or** $1<x\leq 4$.

ThinkZone

You have to solve each inequality separately. Both inequalities must be satisfied.

Do note the connecting words used.

Select the region(s) that satisfy both inequalities.

Do note the connecting words used.

5.4 Inequalities Involving Rational Functions

A rational function is a fraction of the form $\frac{\mathrm{g}(x)}{\mathrm{h}(x)}$ where $\mathrm{g}(x)$ and $\mathrm{h}(x)$ are polynomial and $\mathrm{h}(x)\neq 0$.

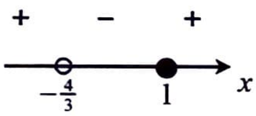
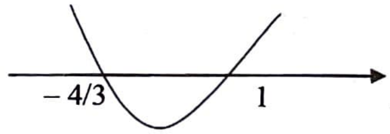
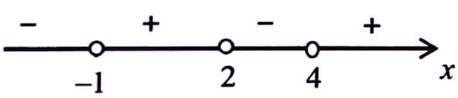
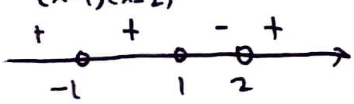
General steps in solving inequalities involving rational functions

1)	**Express** the inequality in the form $\frac{g(x)}{h(x)}>0$, $\frac{g(x)}{h(x)}<0$, $\frac{g(x)}{h(x)}\le 0$ or $\frac{g(x)}{h(x)}\ge 0$ where $\frac{g(x)}{h(x)}$ is a rational function. To facilitate the ease of solving, manipulate the inequality such that both $g(x)$ and $h(x)$ are polynomials where the coefficient of the terms with the highest power is positive.
2)	**Factorise** $g(x)$ and $h(x)$ completely.
3)	Use the **test point method** to solve the resulting inequality.
4)	**Check** that the solution set does <u>**not**</u> include those values of x for which the denominator $h(x)=0$.

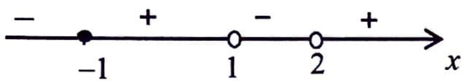
Note: **Do not** 'cross-multiply' any factor(s) unless you are sure that the factor is always positive or always negative. If the factor involved is positive, then the inequality sign remains unchanged. However, if the factor involved is negative, the inequality sign will be reversed.

Example 11: Without the use of GC, find the solution sets of the following inequalities:

(a) $\frac{1-x}{3x+4}\le 0$ (b) $\frac{(x+1)(x-2)}{x-4}>0$ (c) $\frac{(1+x)^2}{(x-1)(x-2)}>0$ (d) $\frac{(1+x)^3}{(x-1)(x-2)}\ge 0$

Solution	**ThinkZone**
(a) $\frac{1-x}{3x+4}\le 0$, $x\ne -\frac{4}{3}$ $\frac{x-1}{3x+4}\ge 0$  The solution set is $\{x\in\mathbb{R} : x<-\frac{4}{3} \text{ or } x\ge 1\}$.	Remember to exclude from your answer all those values of x for which the denominator = 0. We can also represent the solution set as $\left(-\infty, -\frac{4}{3}\right)\cup[1, \infty)$. Observe that the expression $\frac{x-1}{3x+4}$ has the same polarity as $y=(x-1)(3x+4)$.  Care must be exercised to exclude the value x that leads to denominator being zero.
(b) $\frac{(x+1)(x-2)}{x-4}>0$, $x\ne 4$  The solution set is $\{x\in\mathbb{R} : -1<x<2 \text{ or } x>4\}$.	If the inequality was $\frac{(x+1)^2(x-2)^2}{x-4}>0$, how would you solve it?
(c) $\frac{(1+x)^2}{(x-1)(x-2)}>0$ $x\ne 1,2$  $\{x\in\mathbb{R}: x<1, x\ne -1 \text{ or } x>2\}$	

(d) $\dfrac{(1+x)^3}{(x-1)(x-2)} \geq 0, \qquad x \neq 1, x \neq 2$



$\{x \in \mathbb{R} : -1 \leq x < 1 \text{ or } x > 2\}$

Is the result consistent with your hypothesis in ThinkZone of Example 7?

Example 12: Without the use of GC, find the solution set of the inequality $\dfrac{x+1}{x-1} \geq \dfrac{6}{x}$.

Solution:

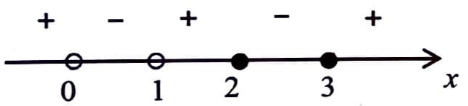
$\dfrac{x+1}{x-1} \geq \dfrac{6}{x} \qquad x \neq 0,\ 1$

$\dfrac{x+1}{x-1} - \dfrac{6}{x} \geq 0$

$\dfrac{x(x+1)-6(x-1)}{x(x-1)} \geq 0$

$\dfrac{x^2-5x+6}{x(x-1)} \geq 0$

$\dfrac{(x-2)(x-3)}{x(x-1)} \geq 0$



The solution set is $\{x \in \mathbb{R} : x < 0 \text{ or } 1 < x \leq 2 \text{ or } x \geq 3\}$.

ThinkZone

Step 1: Express the inequality in the form $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \geq 0$.

Step 2: Factorise $\mathrm{g}(x)$ and $\mathrm{h}(x)$ completely.

Step 3: Use the Test Point method.

Step 4: Exclude those values of x for which the denominator $\mathrm{h}(x) = 0$.

Example 13: Without the use of GC, solve the inequality $\dfrac{1}{(1+x)^2} \geq 1$.

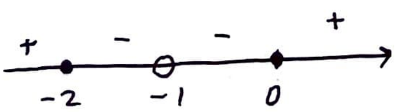
Solution:

$\dfrac{1}{(1+x)^2} \geq 1, \quad x \neq -1$

$1 - \dfrac{1}{(1+x)^2} \leq 0$

$\dfrac{(1+x)^2-1}{(1+x)^2} \leq 0$

$\dfrac{x^2+2x}{(1+x)^2} \leq 0$



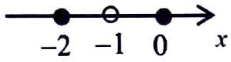
$-2 \leq x < -1$ or $-1 < x \leq 0$

ThinkZone

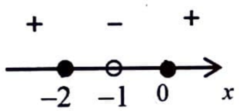
Step 1: Express the inequality in the form $\dfrac{\mathrm{g}(x)}{\mathrm{h}(x)} \leq 0$.

Step 2: Factorise and $\mathrm{h}(x)$ completely.

Step 3: Use the Test Point method.

 $-2 \le x < -1$ or $-1 < x \le 0$	Step 4: Exclude those values of x for which the denominator $\mathrm{h}(x) = 0$.

Alternatively, we can solve in the following manner too.

Solution:	**ThinkZone**
$\frac{1}{(1+x)^2} \ge 1, \quad x \ne -1$ $(1+x)^2 \le 1 \Rightarrow x^2 + 2x \le 0 \Rightarrow x(x+2) \le 0$	Since $x \ne -1$, $(1+x)^2 > 0$, we can multiply both sides of the inequality by $(1+x)^2$
OR	
$\frac{1}{(1+x)^2} \ge 1$ $1 \ge (1+x)^2$ $(1+x)^2 - 1 \le 0$ $[(1+x)-1][(1+x)+1] \le 0$ $x(x+2) \le 0$	Note it is of the form $a^2 - b^2 = (a-b)(a+b)$
 Hence $-2 \le x \le 0$ and $x \ne -1$, i.e. $-2 \le x < -1$ or $-1 < x \le 0$	

Self-Review 3:

1. (i) Prove that $4x^2 + 4x + 3$ is positive for all real values of x.

$4x^2 + 4x + 3 = 4(x^2 + x + \frac{3}{4})$

$= 4(x + \frac{1}{2})^2 + 2$

Since $4(x+\frac{1}{2})^2 \ge 0$ for all $x \in \mathbb{R}$,

$4(x+\frac{1}{2})^2 + 2 > 0$, for all $x \in \mathbb{R}$

(ii) Without the use of GC, solve $\frac{4x^3 + 4x^2 + 3x}{x^2 + x - 2} \le 0$.

[$x < -2$ or $0 \le x < 1$]

~~since $4(x+\frac{1}{2})^2 + 2 > 0$~~

~~$x^2 + x - 2 < 0$~~

~~$(x-1)(x+2) < 0$~~

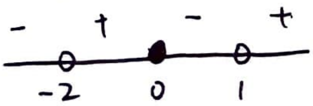
~~$x^2 + x - 2 < 0$~~

$\frac{x(4x^2+4x+3)}{x^2+x-2} \le 0$

Since $4x^2+4x+3$ is positive for all real values of x

$\frac{x}{x^2+x-2} \le 0$

$\frac{x}{(x-1)(x+2)} \le 0, \quad x \ne -2, 1$



$x < -2$ or $0 \le x < 1$

2. Solve the inequality $\frac{3-x}{x-2} \le 4$.

[$x < 2$ or $x \ge \frac{11}{5}$]

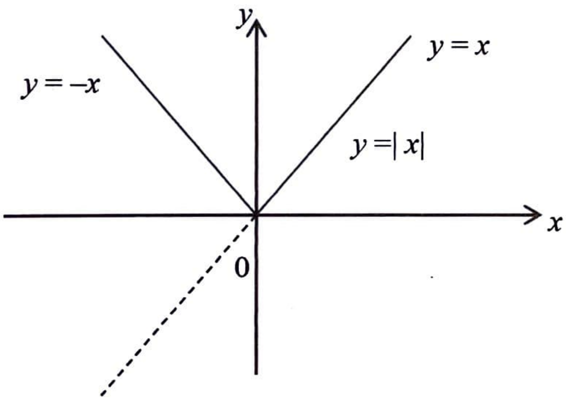
TEST POINT METHOD (REVISITED)

- Make one side of the inequality 0
- Factorise the non-zero side fully.
- For any quadratic factor which cannot be factorised fully, either
  Case 1: complete the square and factorise into factors involving surds, or
  Case 2: complete the square to get a quantity that is always positive or negative and hence simplify the inequality
- Draw the number line based on the roots of the factors.
- If all factors were ensured to have a positive coefficient for the term with the highest power, then the rightmost region will always be positive.
- As you move from right to left of a root:
  - use the opposite sign (positive or negative) if the root comes from a factor with odd power
  - use the same sign if the root comes from a factor with even power
- If there are factors in the denominator, take note on the roots to be excluded in the solution set. (Place empty circles on the number line at the roots to ensure these values will not be included.)
- Based on the sign of the inequality, decide which regions or values are required.

5.5 Inequalities involving the Modulus Function

5.5.1 Basic Concept on Modulus Function

For $x \in \mathbb{R}$, the modulus of x is defined as $|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$



E.g.
$|3| = 3$,
$|-2| = -(-2) = 2$

ThinkZone
Can you identify any line of symmetry?

Are the modulus graphs always V-shape?

Example 14: Sketch the following graphs, without the use of GC:
(i) $y = |x+3|$ (ii) $y = -1-|x-2|$

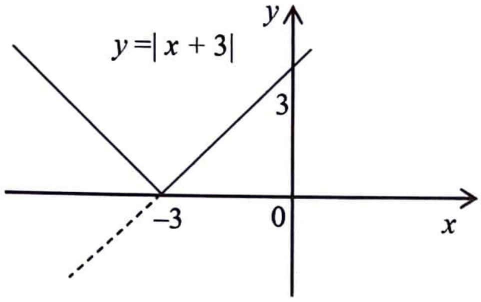
Solution:

(i) $y = |x+3|$

Method 1:

Sketch the graph of $y = x+3$.

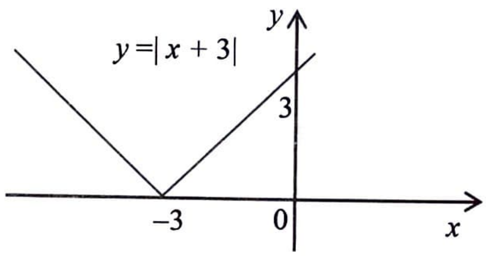
Reflect the portion of the graph of $y = x+3$ that is below the x-axis about the x-axis.



Method 2:

By definition, $y = |x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \\ -(x+3) & \text{if } x+3 < 0 \end{cases}$

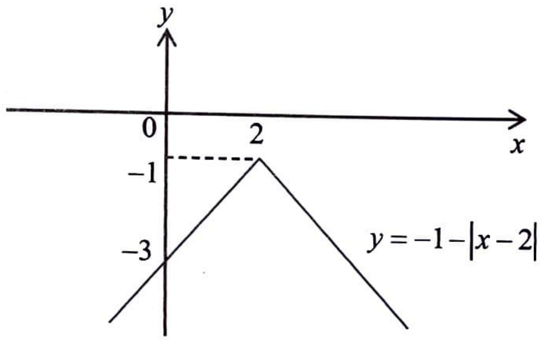
$= \begin{cases} x+3 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$



(ii)

$y = -1-|x-2| = \begin{cases} -1-(x-2) & \text{if } x \geq 2 \\ -1-(-x+2) & \text{if } x < 2 \end{cases}$

$= \begin{cases} 1-x & \text{if } x \geq 2 \\ x-3 & \text{if } x < 2 \end{cases}$



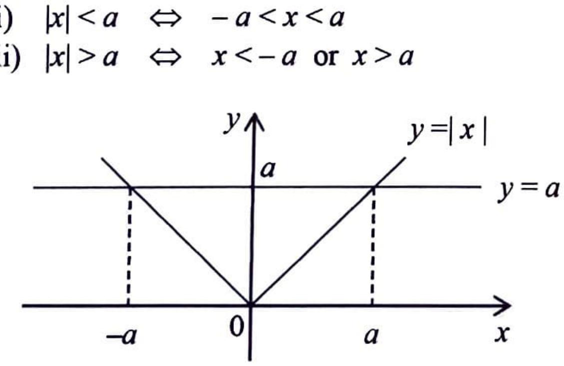
Self-Review 4: Sketch $y = |x+3| - 2$

5.5.2 Inequalities and Modulus Function

For $x,\ y \in \mathbb{R}$ and $a > 0$,

(i) $|x| < a \iff -a < x < a$

(ii) $|x| > a \iff x < -a$ or $x > a$



THINKZONE:
What if a is negative?

If a is negative,

$\left|\frac{x}{a}\right| > a$, solution set : $\cancel{x}\in\mathbb{R}$

$|x| < a$, solution set $\cancel{x} = \emptyset$ or $\{\}$

once you square both sides you can remove modulus

(iii) $|x|>|y| \Leftrightarrow x^2>y^2$ (Recall from Table 5.1(v) that if $a>b$ and $a>0,\ b>0$ then $a^2>b^2$)

These results still hold true if we replace "<" by "$\le$" and ">" by "$\ge$".

Sometimes the modulus function is also referred to as absolute function. Mathematicians tend to use $|x|$ notation. Computing scientists tend to use abs(x) especially for programming.

5.5.3 Properties of Modulus Function

For $x, y \in \mathbb{R}$,

(i) $|x^2| = |x|^2 = x^2$

(ii) $|xy| = |x|\,|y|$

(iii) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, provided $y \neq 0$

Example 15: Find algebraically the solution sets of the following inequalities:
(i) $|2x-1|>7$ and
(ii) $|4-x|<3$.

Solution:	**ThinkZone**
(i) $\lvert 2x-1\rvert>7$ By definition, $2x-1<-7$ or $2x-1>7$ $2x<-6$ or $2x>8$ $x<-3$ or $x>4$ The solution set is $\{x\in\mathbb{R} : x<-3 \text{ or } x>4\}$.	Recall: If $a>0$, $\lvert x\rvert>a \Leftrightarrow x<-a$ or $x>a$ What do you think the solution will be if it was $\lvert 2x-1\rvert>-7$?
(ii) $\lvert 4-x\rvert<3$ $\lvert -(x-4)\rvert<3$ $\lvert x-4\rvert<3$ By definition, $-3<x-4<3$ $1<x<7$ The solution set is $\{x\in\mathbb{R} : 1<x<7\}$.	Recall: $\lvert x\rvert<a,\ a>0 \Leftrightarrow -a<x<a$ Would the solution change if we do not take the step $\lvert -(x-4)\rvert<3$?

Note: This question can also be solved by squaring both sides (since both sides are positive). If the question did not specify the need to use an algebraic method, we can use graphical method (see **Section 5.6**) to solve the question.

even it cannot use gc

Example 16: Using an algebraic method, find the solution set of the inequality $\left|\dfrac{2x+1}{x-3}\right|>1$, $x\neq 3$.

Solution:

$\left|\dfrac{2x+1}{x-3}\right| > 1$ $\quad \boxed{x \neq 3}$

$|2x+1| > |x-3|$

$(2x+1)^2 > (x-3)^2$

$(2x+1)^2 - (x-3)^2 > 0$

$(2x+1+x-3)(2x+1-x+3) > 0$

$(3x-2)(x+4) > 0$

$+ \quad - \quad + \quad +$

$-4 \quad 2/3 \quad 3 \quad x$

The solution set is $\{x \in \mathbb{R} : x < -4 \text{ or } x > \frac{2}{3},\ x \neq 3\}$.

ThinkZone

When both sides of the inequality contain **only modulus functions**, we always square both sides! (Why?)

Self-Review 4: Using an algebraic method, find the solution set of the inequality $\left|\dfrac{x+1}{x-3}\right| > 1$, $x \neq 3$.

$\left|\dfrac{x+1}{x-3}\right| > 1 \quad (x+1)^2 - (x-3)^2 > 0 \quad 8x - 8 > 0$

$(x+1-x+3)(2x-2) > 0 \quad \{x \in \mathbb{R} : x > 1, x \neq 3\}$

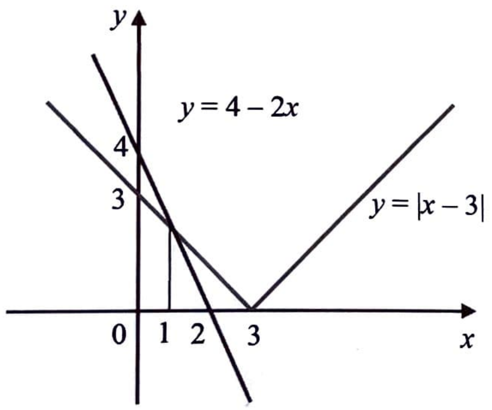
$[\{x \in \mathbb{R} : x > 1, x \neq 3\}]$

Solving inequalities using Graphical Method

Example 17: Solve $|x-3| < 4 - 2x$, giving your answer in exact form.

Solution:

Let $y = |x-3|$ and $y = 4 - 2x$



To find the x-coordinate of the intersection point

$$-x + 3 = 4 - 2x$$

$$x = 1$$

From graph, $|x-3| < 4 - 2x \quad \Rightarrow \quad x < 1$.

ThinkZone

Why don't we use the property $|x| < a \Leftrightarrow -a < x < a$?

By definition,

$$y = |x-3| = \begin{cases} x-3 & \text{if } x \geq 3 \\ -x+3 & \text{if } x < 3 \end{cases}$$

Look out for where the blue graph lies below the black graph.

Can we square both sides for this question?

Self-Review 5:

1) Sketch the graphs of $y=|x-2|$ and $y=\sqrt{x}$. Hence solve the inequality $|x-2|>\sqrt{x}$.

[$0\le x<1$ or $x>4$]

2) Using graphical method, solve the inequality $|3x-12|<\frac{5}{2}x$, giving your answers in exact form.

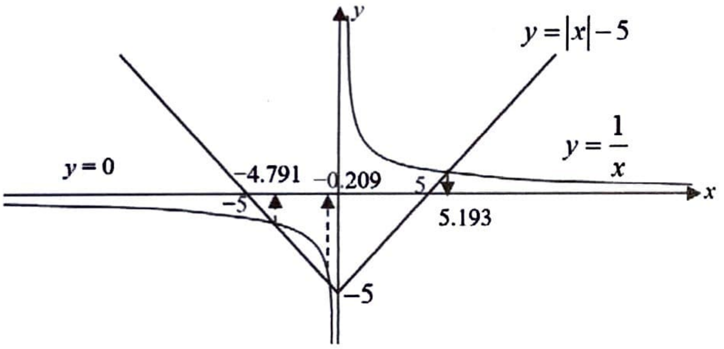
[$\frac{24}{11}<x<24$]

3) By considering the graphs of $y=|2x-1|$ and $y=2+|x+1|$. Hence or otherwise, solve the inequality $|2x-1|-|x+1|>2$.

[$x<-\frac{2}{3}$ or $x>4$]

Example 18: Sketch the graphs of $y=|x|-5$ and $y=\frac{1}{x}$ on a single diagram, showing clearly the important features of each graph and the x-coordinates of the points of intersection, giving your answers to 3 decimal places. Hence solve the inequality $|x|<\frac{5x+1}{x}$.

Solution:

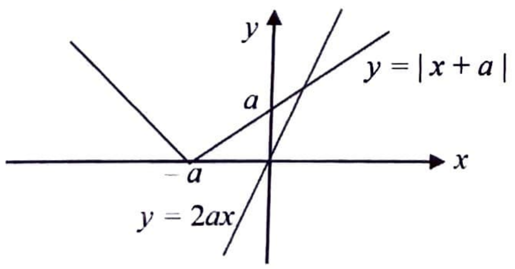


$$|x|<\frac{5x+1}{x} \Rightarrow |x|<5+\frac{1}{x}$$

$$\Rightarrow |x|-5<\frac{1}{x}$$

Thus, $-4.791<x<-0.209$ or $0<x<5.193$

Example 19: On the same axes, sketch the graphs of $y=|x+a|$ and $y=2ax$, where a is a constant and $a>½$. Hence, solve $|x+a|>2ax$.

Solution:	**ThinkZone**
	How do we handle the unknown constant a? What are some details we need to take note when sketching the two graphs? How would the answer

At point of intersection,

$2ax = x + a$

$x = \dfrac{a}{2a-1}$ note that $a > \dfrac{1}{2}$

$\therefore |x+a| \geq 2ax$ when $x < \dfrac{a}{2a-1}$

change if $0 < a < ½$?

5.6 Solving Inequalities by Substitution

Example 20: Solve the inequality $\dfrac{4-x}{x-2} > 3$ and hence solve (i) $\dfrac{4-e^x}{e^x-2} > 3$, (ii) $\dfrac{4-|x|}{|x|-2} > 3$, and (iii) $\dfrac{4x-1}{1-2x} > 3$.

Solution:

$\dfrac{4-x}{x-2} > 3,\ x \neq 2$

$\dfrac{4-x}{x-2} - 3 > 0$

$\dfrac{10-4x}{x-2} > 0$

$\dfrac{2x-5}{x-2} < 0$

$2 < x < \dfrac{5}{2}$

+ − +

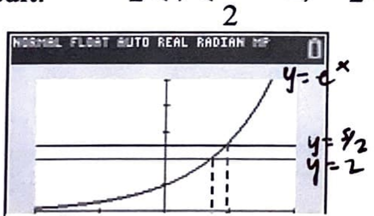
2 $\frac{5}{2}$ x

ThinkZone

(i) $\dfrac{4-e^x}{e^x-2} > 3$

Let $v = e^x$, thus $\dfrac{4-v}{v-2} > 3$.

By earlier result: $2 < v < \dfrac{5}{2} \Rightarrow 2 < e^x < \dfrac{5}{2}$.



ln2 ln(5/2)

$\ln 2 < x < \ln(\frac{5}{2})$

Ensure that the expression containing the variable is of similar form.

Compare $\dfrac{4-e^x}{e^x-2} > 3$ with $\dfrac{4-x}{x-2} > 3$.

*Because $y = \ln x$ is strictly increasing.

(ii) $\dfrac{4-|x|}{|x|-2} > 3$

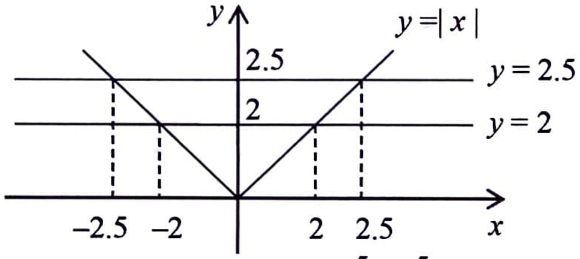
Let $y = |x|$, thus $\dfrac{4-y}{y-2} > 3$

Ensure that the expression containing the variable is of similar form.

By earlier result: $2 < y < \frac{5}{2}$

$$\boxed{2 < |x| < \frac{5}{2}}$$

By the graphical approach,

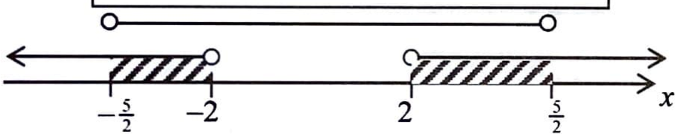


From the graph, for $2 < |x| < \frac{5}{2}$, $-\frac{5}{2} < x < -2$ or $2 < x < \frac{5}{2}$.

Alternatively,

$|x| > 2$ **and** $|x| < \frac{5}{2}$

$$\boxed{\begin{array}{c} x < -2 \text{ or } x > 2 \text{ and} \\ -\frac{5}{2} < x < \frac{5}{2} \end{array}}$$



$\therefore -\frac{5}{2} < x < -2$ **or** $2 < x < \frac{5}{2}$

Compare $\frac{4-|x|}{|x|-2} > 3$ with $\frac{4-x}{x-2} > 3$

How would you solve the problem if we have $-2 < |x| < \frac{5}{2}$?

(iii) $\frac{4x-1}{1-2x} > 3$

$\frac{x(4-\frac{1}{x})}{x(\frac{1}{x}-2)} > 3 \quad , x \neq 0$

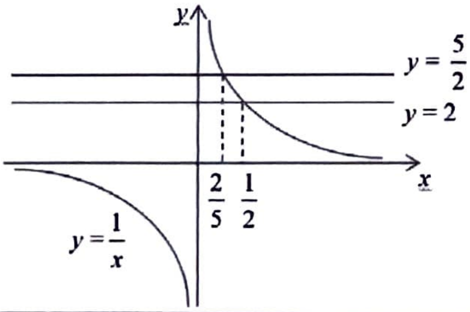
$\frac{4-\frac{1}{x}}{\frac{1}{x}-2} > 3$

Let $u = \frac{1}{x}$, thus $\frac{4-u}{u-2} > 3$.

By earlier result: $2 < u < \frac{5}{2}$

$\Rightarrow \quad 2 < \frac{1}{x} < \frac{5}{2}$

By the graphical approach,



Rearrange $\frac{4x-1}{1-2x} > 3$ to get $\frac{4-\frac{1}{x}}{\frac{1}{x}-2} > 3$.

Compare $\frac{4-\frac{1}{x}}{\frac{1}{x}-2} > 3$ with $\frac{4-x}{x-2} > 3$

Can you solve the inequality $2 < \frac{1}{x} < \frac{5}{2}$ algebraically?

Thus $\frac{2}{5}<x<\frac{1}{2}$.	

Example 21: 2018/Promo/RIJC/Q3

(i) Without the using a calculator, solve $\frac{x-6}{2x^2+3x-2}\le 1$.

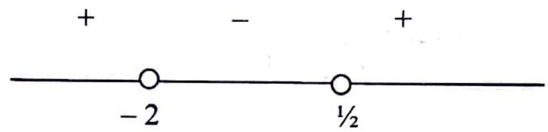
(ii) Hence, solve the inequality $\frac{\sin 2x+6}{2\cos^2 2x+3\sin 2x}\le 1$ for $0\le x\le \pi$.

Solution:

(i) $\frac{x-6}{2x^2+3x-2}\le 1$

$\frac{x-6}{(2x-1)(x+2)}-1\le 0 \qquad x\ne -2,\frac{1}{2}$

$\frac{x-6-2x^2-3x+2}{(2x-1)(x+2)}\le 0$



$\frac{-2x^2-2x-4}{(2x-1)(x+2)}\le 0$

$\frac{x^2+x+2}{(2x-1)(x+2)}\ge 0$

Since $x^2+x+2=\left(x+\frac{1}{2}\right)^2+1\frac{3}{4}>0$, $\frac{x^2+x+2}{(2x-1)(x+2)}\ge 0$ when $(2x-1)(x+2)>0$

$x<-2$ or $x>\frac{1}{2}$

(ii) $\frac{\sin 2x+6}{2\cos^2 2x+3\sin 2x}\le 1$

$\frac{\sin 2x+6}{2\left(1-\sin^2 2x\right)+3\sin 2x}\le 1$

$\frac{\sin 2x+6}{-2\sin^2 2x+3\sin 2x+2}\le 1$

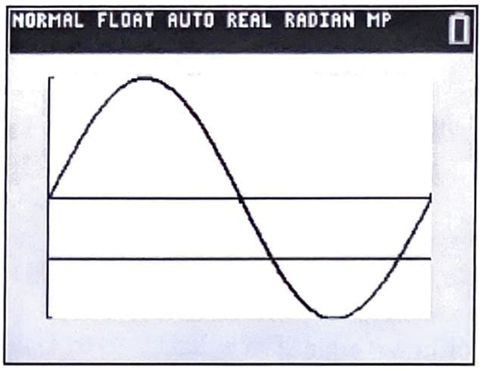
$\frac{-\sin 2x-6}{2\sin^2 2x-3\sin 2x-2}\le 1$

$\frac{(-\sin 2x)-6}{2(-\sin 2x)^2+3(-\sin 2x)-2}\le 1$

Using (i), we have $-\sin 2x<-2$ or $-\sin 2x>\frac{1}{2}$

i.e. $\sin 2x>1$ which is impossible or $\sin 2x<-\frac{1}{2}$

Sketching the graphs:



$\sin 2x < -\frac{1}{2}$ when $\frac{7\pi}{6} < 2x < \frac{11\pi}{6}$

i.e. $\frac{7\pi}{12} < x < \frac{11\pi}{12}$

5.7 Solving Systems of Linear Equations

The equation $a_1x + a_2y = b$ is a **linear** equation with 2 unknowns, x and y. A linear **equation in n unknowns $x_1, x_2, \ldots. x_n$ is an equation of the form**

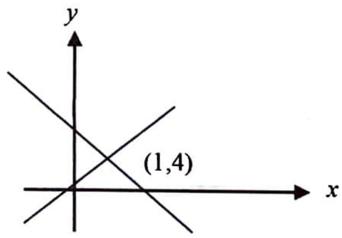
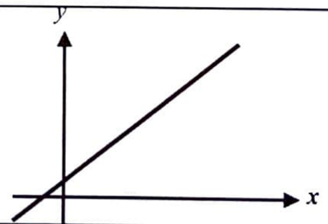
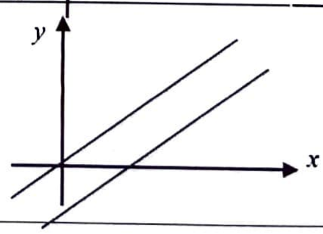
$$a_1x_1 + a_2x_2 + \ldots\ldots + a_nx_n = b \qquad \text{where } a_1, a_2 \ldots\ldots, a_n, b \text{ are real constants.}$$

Examples of **linear** equations	Examples of **non-linear** equations
• $x - 2y + 3z = 120$, • $2x_1 + 3x_2 - 44x_3 = 1$	• $xy = 1$ • $x^2z - 3y = 10$

Considering a system of two linear equations in two variables, x and y, which has the general form:

$$ax + by = r$$
$$cx + dy = s$$

Graphing the equations results in one of the three possible solutions:

1.	The two lines in the linear system intersect at a single point i.e. the system of linear equations has exactly one solution (or a unique solution).	
2.	The two lines coincide i.e. the system of linear equations has infinitely many solutions.	
3.	The lines are parallel i.e. the system of linear equations has no solution.	

Every system of linear equations (regardless of the number of equations and number of variables) will satisfy exactly one of the following:
(i) exactly one solution (a unique solution) ;
(ii) infinitely many solutions or;
(iii) no solution.

A system of equations that has no solution is said to be **inconsistent**.
If there is at least one solution, the system of equations is said to be **consistent**.

We will make use of the graphing calculator to solve systems of linear equations.

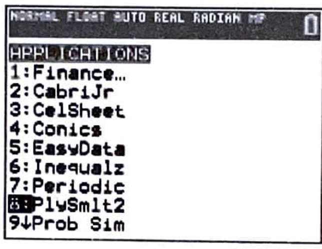
Example 22: (Independent Learning)
Solve the simultaneous equations:

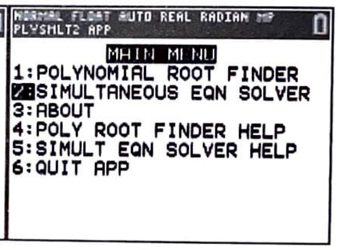
$$x - y = 1$$
$$3x + y = 2$$

Solution:

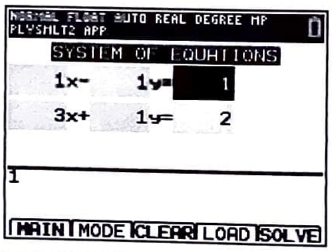
1. Under 

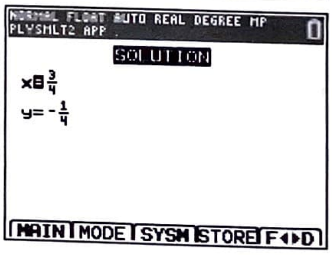
, select option **PlySmlt2**, followed by option **2:SIMULTANEOUS EQN SOLVER**.





2. Press **[NEXT]** and key in the SYSTEM OF EQUATIONS and press **[SOLVE]**.





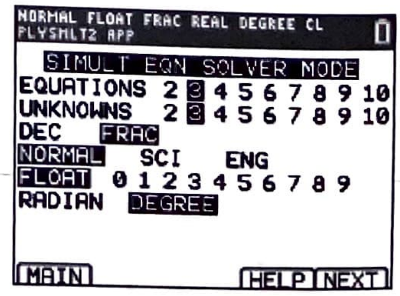
Thus, we get the solution $x = \frac{3}{4}$ and $y = -\frac{1}{4}$.

Example 23: (Independent Learning)
Solve the system of equations:

$$x - 2 = -2y - 3z$$
$$2x + 3z = 3 - 5y$$
$$x + 8z = 4$$

Solution:
Re-arranging the equations we have:

$$x + 2y + 3z = 2$$
$$2x + 5y + 3z = 3$$

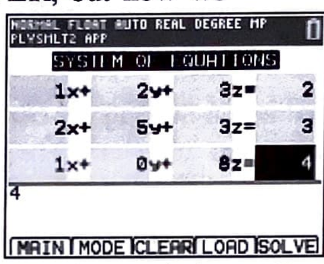


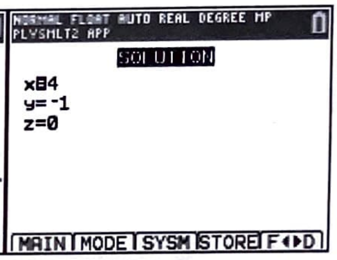
$$x \qquad + 8z = 4$$

Just as in the previous example, under  we select option **PlySmlt2** and **2:SIMULTANEOUS EQN SOLVER**, but now we have 3 EQUATIONS and 3 UNKNOWNS.

Key in the SYSTEM MATRIX accordingly and press **[SOLVE]**.

Thus, $x = 4, y = -1$ and $z = 0$.





Not all systems of linear equations in 3 variables have unique solutions. They may have infinite solutions or no solution.

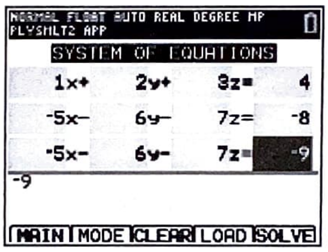
Example 24: (Independent Learning)
Solve the system of equations:

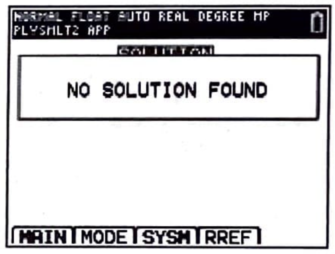
(i)
$$x + 2y + 3z = 4$$
$$-5x - 6y - 7z = -8$$
$$-5x - 6y - 7z = -9$$

(ii)
$$x + 2y + 3z = 4$$
$$-5x - 6y - 7z = -8$$
$$2x + 4y + 6z = 8$$

Solution:

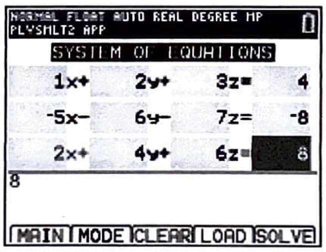
(i) If we key into the GC as in the previous examples, we have the following screen shots:

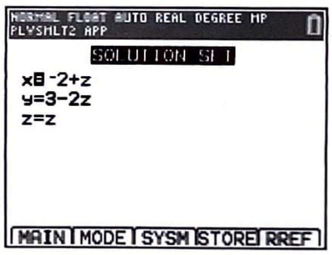




Thus, we say the system is inconsistent and has no solution.

(ii) If we key into the GC as in the previous examples, we have the following screen shots:





This is equivalent to:
$$x = -2 + z$$
$$y = 3 - 2z$$
$$z = z$$

This means that the variable 'z' is a 'free' variable which we can assign any real value.

By letting $z = k$, $k \in \mathbb{R}$, we have
$$x = k - 2$$
$$y = -2k + 3$$
$$z = k$$

The system of equations has infinitely many solutions.

5.9 Solving Practical Problems Using Systems of Linear Equations

In life, some problems can be modelled by mathematics. By converting the problem into a mathematical one, we may get linear equations that can be solved easily. After which, we then contextualise the solution back into the problems we were concerned with.

Example 25: A company specializes in making toy cars. It makes three types of toy cars, Toy Car A, Toy Car B and Toy Car C. The amount of materials required (in units) to make a car of each type is given below:

	Plastic	Rubber	Metal
Toy Car A	2	1	1
Toy Car B	3	4	5
Toy Car C	1	3	2
Amount of material available	100	150	130

If all the materials available are used, how many cars of each type does the company make?

Solution:

Let x be the number of toy cars of type A.
Let y be the number of toy cars of type B.
Let z be the number of toy cars of type C.

$2x+3y+z=100$
$x+4y+3z=150$
$x+5y+2z=130$
using GC, $x=20, y=10, z=30$
The company makes 20, 10 and 30 cars of type A, Type B and Type C respectively.

ThinkZone

1) You must define the variables to the context of the questions. Use the phrase "*how many cars of each type*" to define the variables.

2) Formulate the system of linear equations.

3) Key the 3 equations into the system matrix in GC and solve.

Example 26: A company that rents small moving trucks wants to purchase 25 trucks with a combined capacity of exactly 28000 cubic feet. Three different types of trucks are available for purchase: a 10-feet truck with a capacity of 350 cubic feet, a 14-feet truck with a capacity of 700 cubic feet, and a 24-feet truck with a capacity of 1400 cubic feet. What are the possible combinations trucks the company could buy?
Suppose a client wants to rent all 25 trucks from the rental company.
If the rental company charges \$115 per day for a 10-feet truck, \$150 per day for a 14-feet truck and \$200 per day for a 24-feet truck, how many of each type of trucks should the company now purchase in order to charge the highest price?

Solution:
Let x_1, x_2, x_3 represent the number of 10-feet, 14-feet and 24-feet truck purchased respectively.

We can set up a pair of equations,

$$x_1 + x_2 + x_3 = 25$$
$$350x_1 + 700x_2 + 1400x_3 = 28000 \Rightarrow x_1 + 2x_2 + 4x_3 = 80$$

Using G.C to solve in terms of x_3, we get

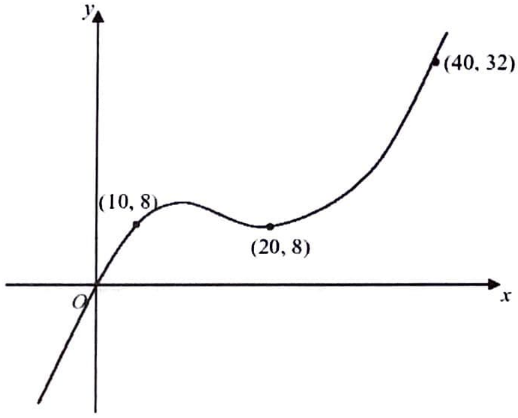
$x_1 = -30 + 2x_3$ $\Rightarrow -30 + 2x_3 \geq 0$

$x_2 = 55 - 3x_3$ $x_3 \geq 15$

$x_3 = x_3$ $\Rightarrow x_3 \leq 18\frac{1}{3}$ $x_3 \leq 18$

Since x_1, x_2, x_3 are all non-negative whole numbers, we can deduce that $15 \leq x_3 \leq 18$. So, the possibilities are (0,10,15), (2,7,16), (4,4,17) and (6,1,18).
To charge the highest price, the company should purchase 10 and 15 of the 14-feet and 24-feet trucks respectively.

Example 27: A cubic curve $y = \mathrm{f}(x)$ passes through the points with coordinates (0, 0), (10, 8), (20, 8) and (40, 32) as shown in the diagram on the right.



(a) Write and solve a system of simultaneous linear equations to find f(x).

(b) Find the range of values of k for the line $y = k$ to intersect graph of $y = \mathrm{f}(x)$ at three distinct points, giving your answers correct to 3 decimal places.

Solution

(a) Let f(x) = ax^3+bx^2+cx+d.
Since the curve passes through the origin, this means that $d = 0$.
Hence with only 3 unknowns left, we have three independent equations.

At (10, 8),
$1000a + 100b + 10c = 8$ — (1)
At (20, 8),
$8000a + 400b + 20c = 8$ → (2)
At (40, 32)
$64000a + 1600b + 40c = 32$ — (3)

Using GC, $a = \frac{1}{500}$, $b = -\frac{1}{10}$, $c = \frac{8}{5}$.

Hence,

$$\mathrm{f}(x) = \frac{1}{500}x^3 - \frac{1}{10}x^2 + \frac{8}{5}x$$

where $a = \frac{1}{500}$, $b = -\frac{1}{10}$, $c = \frac{8}{5}$, $d = 0$.

(b) Using GC to find the x-coordinates of the minimum and maximum points, $x = 20$ or 13.3333.

ThinkZone

To find the maximum and minimum points of the graph, we can either

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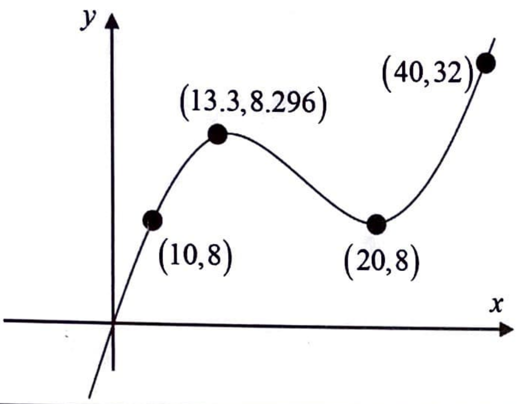
At $x = 13.3333$, $y = 8.296$ (to 3 d.p.) (maximum y-value)

At $x = 20$, $y = 8.000$ (minimum y-value)

Hence, the range of values of k for the line $y = k$ to intersect the graph of $y = \mathrm{f}(x)$ at three distinct points is $8.000 < k < 8.296$.



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(i) differentiate f(x), or
(ii) use GC to find the maximum and minimum points.

Use Zoom or adjust the window settings for each turning point in your GC.

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Chapter 6

Sequences and Series

In this unit, student will

- understand the concepts of sequence and series for finite and infinite case;
- relate sequence as a function $y = f(n)$ where n is a positive integer;
- recognise arithmetic progression (AP) and geometric progression (GP);
- use the formulae for n^{th} term and for the sum of the first n terms to solve problems involving AP or GP;
- understand the relationship between u_n (the nth term) and S_n (the sum to n terms);
- understand the condition for convergence of a geometric series, and use the formula for the sum to infinity of a convergent geometric series;
- solve practical problems involving arithmetic and geometric series;
- use Σ notation;
- obtain the value of a series using the method of difference.

1.1 Sequences

Definition: A **sequence** is defined as an ordered set of objects. In most cases, these objects are numbers. Sequences occur in nature, such as the patterns on snail shells and seed heads in flowers and in man-made application such as the world of finance.

Consider these sequences:

1. J, A, S, O, N, D, J, …
2. M, W, F, S, T, T, S, …
3. 2, 4, 6, 8, 10, …
4. 1, 3, 7, 15, 31, …

How can these sequences be described?

1. Initial letter of each month (in English) beginning with July.
2. Initial letter of days, starting with Monday, going forward each time by two days.
3. Beginning with 2, each subsequent term is 2 more than the previous term.
4. Beginning with 1, each subsequent term is double the previous term plus 1.

In our syllabus, we are only studying the sequences of numbers.
In order to describe sequences mathematically, some notation is required.

Notation	Description
u_n	n^{th} term of a sequence (the subscript refers to the position of the term in the entire sequence).
S_n	the sum of the first n terms $= u_1 + u_2 + u_3 + \dots + u_n$ Note: If the question is asking for the sum of first n even (odd) terms, we cannot use the notation S_n.
a	Initial or first term of a sequence.

For example, study the sequence

$$\begin{array}{ccccccccc} 1, & 3, & 7, & 15, & 31, \ldots & & & \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & \\ u_1, & u_2, & u_3, & u_4, & u_5, \ldots & u_k, & u_{k+1}, \ldots & u_n, \ldots \end{array}$$

Observation 1: In the example above, there is an explicit formula for u_n. Observe that $u_n = 2^n - 1$

$$u_3 = 2^3 - 1 = 7$$
$$u_4 = 2^4 - 1 = 15$$

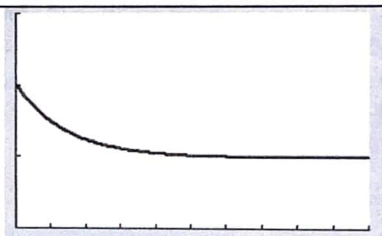
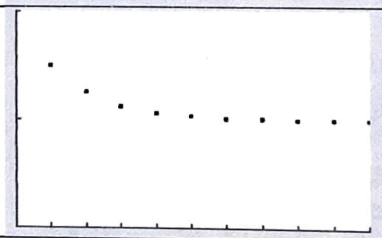
Observation 2: To obtain S3, which is the sum of the first 3 terms , $S_3 = u_1 + u_2 + u_3 = 11$

Similarly, $S_5 = u_1 + u_2 + u_3 + u_4 + u_5 = 57$

Observation 3: The sequence has an infinite number of terms (represented by …).

If a sequence has finite number of terms, we call it a **finite** sequence.
If a sequence has infinite number of terms, we call it an **infinite** sequence.

Now consider the following functions and their respective graphs:

$f(x) = 1 + \frac{1}{2^x}$ for $x \in \mathbb{R}$, $x > 0$,	
$g(x) = 1 + \frac{1}{2^x}$ for $x \in \mathbb{Z}^+$	

Observe that both the graphs have the same trend, except that $g(x)$ are discrete points.

In fact, the range of g is $\left\{\frac{3}{2}, \frac{5}{4}, \frac{9}{8}, \ldots\right\}$ with a domain {1, 2, 3…}.

You can relate $g(x)$ to the sequence defined by $u_n = 1 + \frac{1}{2^n}$ for $n \in \mathbb{Z}^+$.

1.2 Generating Sequences

1.2.1 Generating sequences from the given formula

A sequence is defined by $u_n = 1 + \frac{1}{2^n}$ for $n \in \mathbb{Z}^+$. We can obtain the terms by substituting suitable values of n.

For example, $u_3 = 1 + \frac{1}{2^3} = 1.125$, $u_5 = 1 + \frac{1}{2^5} = 1.03125$

1.2.2 Use of the graphing calculator to generate sequences

Please refer to page 26-27 of "My First Step in Using TI-84 Plus CE for H1 and H2 Math" for the keystrokes to generate a sequence.

Remark: We will use u_n to denote the general term of a sequence in this chapter. Another notation u(n) is used mainly for GC calculation.

1.3 Series

A **series** is the sum of a finite or infinite sequence of terms.

Examples of series

(a) $1 + 2 + 3 + \cdots + 100$ (finite series)

(b) $1 + 4 + 9 + 16 + \cdots$ (infinite series)

1.4 Convergence of Sequences & Series

A sequence of real numbers u_1, u_2, u_3, $\ldots$ is said to **converge** to a real number a if $u_n \to a$ as $n \to \infty$.

For example, the sequence $u_n = 1 + \frac{1}{2^n}$ for $n \in \mathbb{Z}^+$ converges to 1 as $n \to \infty$. Thus, we say that 1 is the limit of the sequence $u_n = 1 + \frac{1}{2^n}$ for $n \in \mathbb{Z}^+$.

Mathematically, we can write $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\left(1 + \frac{1}{2^n}\right) = 1$.

However, the sequence $u_n = 2^n$ for $n \in \mathbb{Z}^+$ does not have a limit, i.e. the sequence $u_n = 2^n$ for $n \in \mathbb{Z}^+$ **diverges**.

Exercise
Do the following sequences converge?

(a) $u_n = \frac{1}{2^{n-1}}$ yes (b) $u_n = 1 + (-1)^n$ no (c) $u_n = n^2$ no

where $n \in \mathbb{Z}^+$.

A finite series always has a finite value. For example, $1 + 2 + 3 + \ldots + 100 = 5050$.
An infinite series, however, may or may not have a finite value. For example, the infinite series $1 + 2 + 3 + \ldots$ does not have a finite value. The infinite series $-1 + 1 - 1 + 1 - 1 + \cdots$ whose terms alternate between -1 and 1 does not have a value. However, the infinite series $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ converges to 1.

SECTION A: Arithmetic and Geometric Progression

1. Arithmetic Progression (AP)

An **arithmetic progression** is a sequence in which each term (other than the first term) differs from the previous one by the same fixed number (positive or negative or zero) known as **common difference**, d.

Examples of AP:

(a) $2, 5, 8, 11, \ldots$ is an AP with first term, $a = 2$ and common difference, $d = 3$

(b) $6, -4, -14, -24, \ldots$ is an AP with first term, $a = 6$ and common difference, $d = -10$

(c) $-1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \ldots$ is an AP with first term, $a = -1$ and common difference, $d = \frac{1}{3}$.

Question: Is the sequence $1, 1, 1, 1, \ldots$ an AP?

Note: An AP is completely defined by the first term a and the common difference d.
In general, if an AP has first term a and common difference d, then the AP can be written as

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad a+4d, \ldots$$

$u_1 = a$

$u_2 = a + d$

$u_3 = a + 2d$

$u_4 = a + 3d$

$\vdots$

In general, the n^{th} term of an AP can be written as

$$\boxed{u_n = a + (n-1)d, \ n \in \mathbb{Z}^+}$$

Example 1
Consider the arithmetic sequence $4, 11, 18, 25, 32, \ldots$

(i) Find an expression for u_n, the n^{th} term of the sequence.

(ii) Find u_{12}, the 12^{th} term of the sequence.

(iii) Is (a) 602 (b) 711 a member of this sequence?

Solution:

	Think Zone:
	Why must n be an integer? Does $n = 86.4$ make sense?

Example 2

The 3rd term of an arithmetic progression is 10 and the 7th term is 34. Find the 1st term and the common difference.

Solution:

Given: $u_3 = 10 \Rightarrow a + (3-1)d = 10$ $a + 2d = 10$ .... (1) $u_7 = 34 \Rightarrow a + (7-1)d = 34$ $a + 6d = 34$ .... (2) Solving (1) & (2): $a = -2,\ d = 6.$ The first term is -2 and the common difference is 6.	**Think Zone:** How do you formulate equations using the given information? Recall how to use the GC to solve simultaneous equations.

Example 3

(a) If m, $(2m-1)$ and $(4m-7)$ are three consecutive terms of an AP, find the value of m.

(b) If m, $(2m-1)$ and $(4m-7)$ are the first, second and fifth term of an AP. Find the value of m.

Solution:

(a) Since the three consecutive terms form an AP, $(2m-1)-m=(4m-7)-(2m-1)$ $m-1=2m-6$ $m=5$ (b) Let d be the common difference. $u_2-u_1=d \Rightarrow (2m-1)-m=d$ $m-d=1$ ---(1) $u_5-u_1=4d \Rightarrow (4m-7)-m=4d$ $3m-4d=7$ --- (2) Solving: $m=-3,\ d=-4.$	**Think Zone:** Why do we start the first line of working with $(2m-1)-m=(4m-7)-(2m-1)$?

Self-Review 1

(a) If three consecutive terms of an AP has a sum of 33 and a product of 1056, find the three numbers of this AP. [6, 11, 16]

(b) In a certain AP, the sum of the first and fifth term is 4 and the sixth term is 4 times of the third term. Find the first term and the common difference. [$a=-2$, $d=2$]

$\cancel{a+d+a+2d+a+3d = 3a+6d = 33}$ $\cancel{a+2d}$

$\cancel{a+3d}$

$\cancel{(a+d)(a+2d)(a+3d)=1056}$

$\cancel{(a^2+3ad+2d^2)(a+33)=1056}$

$\cancel{a^3+3a^2d}+2ad^2+33a^2+99ad+66d^2=1056$

$a^3+(3d+33)a^2+$

An **arithmetic series** is the sum of the terms of an arithmetic progression.
Recall that S_n is the sum of the first n terms in a sequence.
For an AP, the formula of S_n is given by

$$\boxed{\begin{aligned} S_n &= \frac{n}{2}\left[2a + (n-1)d\right] \\ &= \frac{n}{2}(a+l), \text{ where } l \text{ is the last term} \end{aligned}}$$

Proof (Independent Reading):

$S_n = a + (a + d) + (a + 2d) + (a + 3d) + (a + 4d) + \cdots + [a + (n-2)d] + [a + (n-1)d]$

We can write the series in two different manner:

$$S_n = a + (a+d) + \cdots + [a+(n-2)d] + [a+(n-1)d]$$
$$S_n = [a+(n-1)d] + [a+(n-2)d] + \cdots + a+d + a$$

Adding these two equations, we have

$$2S_n = \underbrace{[2a+(n-1)d]+[2a+(n-1)d]+\cdots+[2a+(n-1)d+[2a+(n-1)d]}_{n \text{ of such terms}}$$

Thus, $2S_n = n[2a+(n-1)d]$

$$\Rightarrow \quad S_n = \frac{n}{2}[2a+(n-1)d]$$
$$= \frac{n}{2}[a+a+(n-1)d]$$
$$= \frac{n}{2}[a+l] \quad \text{where } l = a+(n-1)d \text{ is the } n^{\text{th}} \text{ (or last) term of the sequence}$$

This method of finding S_n of an AP is known as the Gaussian method.
Interesting history behind this method: https://nrich.maths.org/2478

Example 4

The 8^{th} term of an arithmetic progression is 11 and the 15^{th} term is 21. Find the sum of the first 20 terms. Find the least number of terms in the progression that must be taken for the sum to exceed 2020.

Solution:

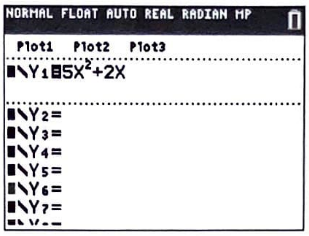
	Think Zone:
$u_8 = a + (8-1)d$ $11 = a + 7d$ .... (1) $u_{15} = a + (15-1)d$ $21 = a + 14d$ .... (2) Solving: $a = 1,\ d = \frac{10}{7}$.	
$S_{20} = \frac{20}{2}\left[2(1)+(20-1)\frac{10}{7}\right] = 291\frac{3}{7}$	Recall the formula for the sum of an AP. What information do you need to find S_{20}?

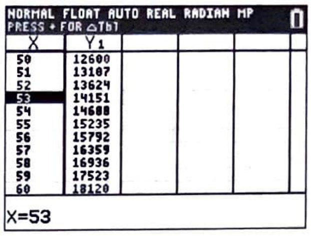
Press 

to key the expression into Y1.

Press 

to look up the TABLE.





Algebraic method of solving:

$5n^2 + 2n > 14140$

$5n^2 + 2n - 14140 > 0$

$n > 52.979$ or $n < -53.379$ (rej.)

$\therefore$ least $n = 53$

Example 5

Find the number of positive integers, less than 1000, that are divisible by 7.
Hence find the sum of the positive integers, less than 1000, that are divisible by 7.
Find also the sum of the positive integers, less than 1000, that are **not** divisible by 7.

Solution:

The largest integer <1000 and is divisible by 7 is 994.

$u_n = a + (n-1)d$

$994 = 7 + (n-1)7$

$987 = (n-1)7$

$n - 1 = 141 \Rightarrow n = 142$

$S_n = \frac{n}{2}(a + l)$

$S_{142} = \frac{142}{2}(7 + 994) = 71071$

Sum of positive integers, less than 1000, that are **not** divisible by 7

$= \frac{999}{2}(1 + 999) - 71071$

$= 428429$

Think Zone:

Alternatively, $\frac{1000}{7} = 142\frac{6}{7}$. Hence number of positive integers <1000 and are divisible by 7 is 142.

Try listing out the first few positive integers that are divisible by 7. Can you recognise the sequence? Is this an AP?

Recall the formula to find the sum of an AP. What do you need?

Extension: Can you find the sum of positive integers, less than 1000, that are divisible by 2 **and** 7?

Interesting read: How to tell if a number is divisible by 7 (or other prime numbers less than 50)? http://www.savory.de/maths1.htm

Self-Review 2

(a) The 7th term of an AP is 13 and the 12th term is 25. Find the least number of terms in the progression that must be taken for the sum to exceed 1000. [30]

(b) The sum of the first n terms of an AP is given by $S_n = pn + qn^2$. Given that $S_3 = 54$ and $S_5 = 145$, find the values of p and q. $\left[p = \frac{3}{2} \text{ and } q = \frac{11}{2}\right]$

2. Geometric Progression (GP)

A **geometric progression** (GP) is a sequence in which each term (other than the first term) is obtained from the previous one by multiplying by the same **non-zero** constant (positive or negative) known as **common ratio, *r***.

Examples of GP:

(a) 1, 2, 4, 8, 16, ... is a GP with first term, $a = 1$ and common ratio, $r = 2$.

(b) 64, –16, 4, –1, ... is a GP with first term, $a = 64$ and common ratio, $r = -\frac{1}{4}$.

Question: Is 1, 1, 1, 1, ... a GP? Can a sequence be both an AP or GP?

In general, if a GP has **first term *a*** and **common ratio *r***, then the GP can be written as

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \ldots$$

$u_1 = a$

$u_2 = ar$

$u_3 = ar^2$

$u_4 = ar^3$

$\vdots$

In general, the n^{th} term of a GP can be written as

$$\boxed{u_n = ar^{n-1}, \ n \in \mathbb{Z}^+}$$

A **geometric series** is the sum of the terms of a geometric progression.
Recall that S_n is the sum of the first n terms in a sequence.
For a GP, the formula of S_n is given by

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} = \frac{a(r^n-1)}{r-1}, \quad r \neq 1}$$

Remark: We will use $S_n = \dfrac{a(r^n-1)}{r-1}$ if $r > 1$ and $S_n = \dfrac{a(1-r^n)}{1-r}$ if $r < 1$.

Question: How do we find S_n when $r = 1$?

Proof (Independent Reading):

$S_n = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}$ .... (1)

$r S_n = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n$ .... (2)

(1) – (2), $S_n - rS_n = a - ar^n$

$(1-r)\, S_n = a\,(1-r^n)$

$\therefore S_n = \dfrac{a(1-r^n)}{1-r}$

Example 6

The 3rd term of a geometric progression is 36 and the 6th term is $\frac{243}{2}$.
Find the first term, the common ratio and the sum of the first eight terms.

Solution:

Let a be the first term and r the common ratio of the GP. given: $u_n = ar^{n-1}$ $u_3 = 36 \Rightarrow ar^2 = 36 - (1)$ $u_6 = \frac{243}{2} \Rightarrow ar^5 = \frac{243}{2} - (2)$ $\frac{u_6}{u_3} = \frac{ar^5}{ar^2} = \frac{243/2}{36} \Rightarrow r^3 = \frac{27}{8}$ $r = \frac{3}{2}$	**Think Zone** How do you formulate equations using the given information? Recall the formula for S_n. How do you relate the terms to the formula?

$S_n = \frac{a(r^n-1)}{r-1} = \frac{16((\frac{3}{2})^8-1)}{3/2-1} = 788\frac{1}{8}$

Example 7
If $(m-4)$, $(2m+1)$ and $(10m+5)$ are three consecutive terms of a GP with non-zero terms, find the possible value for m.

Solution:

Since the three terms form a GP, $\frac{2m+1}{m-4} = \frac{10m+5}{2m+1}$ $(2m+1)^2 = (10m+5)(m-4)$ $6m^2 - 39m - 21 = 0$ $m = 7$ or $m = -\frac{1}{2}$ (rejected as $m = -\frac{1}{2} \Rightarrow 2m+1 = 0$) Hence $m = 7$.	**Think Zone:** How do you link up the three terms by an equation given that they are three consecutive terms of a GP? Check if all values are acceptable.

Example 8
A geometric progression has positive terms. The sum of the first six terms is nine times the sum of the first three terms. The seventh term is 320. Find the common ratio and the first term. Find the smallest value of n such that the sum to first n terms of the progression exceeds 10^6.

Solution:

$S_6 = 9S_3$ $\quad u_7 = ar^6 = 320$ $\frac{a(r^6-1)}{r-1} = \frac{9a(r^3-1)}{r-1}$ ~~$r^6 - 1 = 9r^3 - 9$~~ ~~$r^6 - 9r^3 + 8 = 0$~~ ~~$r^3(r^3 - 9) = 0$~~ ~~$r = 0$ or $r = \sqrt[3]{9}$~~ ~~$a(\sqrt[3]{9})^6 = 320$~~ ~~$a = \frac{320}{81}$~~ $(r^3-1)(r^3+1) = 9(r^3-1)$ $(r^3+1-9)(r^3-1) = 0$ $(r^3-8)(1-r^3) = 0$ $(r^3-8)(1-r^3) = 0$ $r^3 = 8$ or $r^3 = 1$	**Think Zone:** Think of the formula you should use and form equations with a and r. Why is $r = 1$ rejected?

$r = 2$ or $r = 1$ (rejected) If $r=1$, $9S_3 \neq S_6$ $(2S_3 = S_6)$

$(2^6)a = 320$
$a = 5$

$\frac{5(1-2^n)}{1-2} > 10^6$

From GC,

$u_7 = ar^6 = 320$

$a(2^6) = 320$

$a = 5$

$S_n > 10^6$

$\frac{5(1-2^n)}{1-2} > 10^6$

From GC,

n	S_n
17	$655355 < 10^6$
18	$1.31\times10^6 > 10^6$

$\therefore$ smallest $n = 18$.

Algebraic way of solving

$$S_n > 10^6$$

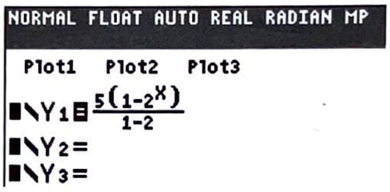
$$\frac{5\left(1-2^n\right)}{1-2} > 10^6$$

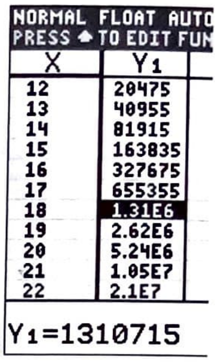
$$2^n > 200001$$

$$n\ln 2 > \ln 200001$$

$$n > 17.6$$

$\therefore$ smallest $n = 18$





Self-Review 3

A GP has positive terms. The sum of the first three terms of the GP is 52. The fourth term is 9 times of the second term. Find the common ratio and the first term. Find the smallest value of n such that the sum to first n terms of the progression exceeds 1000. [$a = 4$, $r = 3$, smallest $n = 6$]

Example 9

An arithmetic progression has first term a and non-zero common difference d. The n^{th} term, denoted by u_n, is such that u_1, u_4 and u_8 are consecutive terms in a geometric progression. Show that $d = \frac{a}{9}$.

Solution:

$$\frac{u_4}{u_1} = \frac{u_8}{u_4}$$

$$\frac{a+3d}{a} = \frac{a+7d}{a+3d}$$

$$(a+3d)^2 = a(a+7d)$$

$$a^2 + 6ad + 9d^2 = a^2 + 7ad$$

$$ad = 9d^2$$

Since $d \neq 0$, $a = 9d \Rightarrow d = \frac{a}{9}$ (shown)

2.1 Sum of an Infinite Geometric Series

Consider the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. The common ratio is $\frac{1}{2}$ and this infinite series converges.

This means that $S_n = \frac{a\left(1-r^n\right)}{1-r}$ will approach the value of $\frac{a}{1-r}$ when $n \to \infty$ if $|r| < 1$ (since $r^n \to 0$ when $n \to \infty$). Thus, the infinite series converges to $S_\infty = \frac{a}{1-r}$ which is known as **the sum to infinity.**

$$\boxed{S_\infty = \frac{a}{1-r} \text{ for } -1 < r < 1}$$

If $|r| > 1$, the infinite series is said to be **divergent** and the sum becomes infinitely large (since $r^n \to \infty$ when $n \to \infty$).

Note:

$$\begin{tikzcd}
 & \boxed{S_\infty \text{ exists}} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\boxed{-1 < r < 1} \arrow[rr, leftrightarrow] & & \boxed{\text{Infinite series converges}}
\end{tikzcd}$$

Remark: S_∞ can be more than or less than S_n depending on the value of a and r. Can you think of a case when $S_n > S_\infty$? occurs when a or r is -ve.

Example 10

Find the sum to infinity of the geometric series $7+\frac{2}{7}+\frac{4}{343}+\frac{8}{16087}+\cdots$. Find also the least number of terms of the series that must be taken to give a sum which exceeds 99.99% of the sum to infinity.

Solution:

Since this is a geometric series,

$a=7, \quad r=\frac{2}{7}\div 7=\frac{2}{49}$

$$S_\infty=\frac{7}{1-\frac{2}{49}}=\frac{343}{47}$$

$$S_n > 0.9999S_\infty$$

$$\frac{a(1-r^n)}{1-r} > 0.9999\left(\frac{a}{1-r}\right)$$

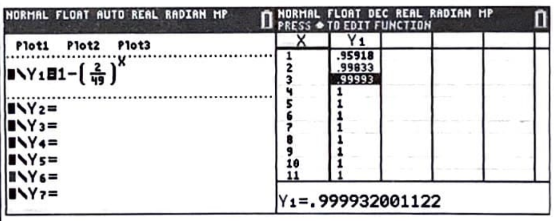
$$1-\left(\frac{2}{49}\right)^n > 0.9999$$

From GC

n	$1-\left(\frac{2}{49}\right)^n$
2	$0.99833 < 0.9999$
3	$0.99993 > 0.9999$

$\therefore$ least $n=3$

Thinkzone:



Algebraic way of solving

$$S_n > 0.9999S_\infty$$

$$\frac{a(1-r^n)}{1-r} > 0.9999\left(\frac{a}{1-r}\right)$$

$$1-\left(\tfrac{2}{49}\right)^n > 0.9999$$

$$\left(\tfrac{2}{49}\right)^n < 0.0001$$

$$n\ln\left(\tfrac{2}{49}\right) < \ln 0.0001$$

Note the change in inequality sign!

$$n > \frac{\ln 0.0001}{\ln\left(\frac{2}{49}\right)} = 2.88$$

$\therefore$ least $n=3$

Example 11

The sum of the first n terms of a GP is given by $S_n = 1-\left(\frac{3}{4}\right)^n$. Find the first term and the common ratio. Hence, find an expression for the sum to infinity of the **odd numbered** terms i.e. $u_1 + u_3 + u_5 + \cdots$.

Solution:

Given $S_n = 1-\left(\frac{3}{4}\right)^n$, $a = S_1 = \frac{1}{4}$

$S_2 = a + ar$

$1-\left(\frac{3}{4}\right)^2 = \frac{1}{4}+\frac{1}{4}r$

$r = \frac{3}{4}$

$u_1 + u_3 + u_5 + u_7 + \ldots = a + ar^2 + ar^4 + ar^6 + \ldots$
(This is a new GP with first term a and common ratio r^2.)

$$S_\infty = \frac{a}{1-r^2} = \frac{\frac{1}{4}}{1-\left(\frac{3}{4}\right)^2} = \frac{4}{7}$$

Think Zone:
Instead of using $S_2 = a + ar$, you can also use $S_2 = \frac{a\left(1-r^2\right)}{1-r} = 1-\left(\frac{3}{4}\right)^2$. However, this approach involves solving a quadratic equation which may make the working slightly more complicated.

List out the first few terms of the GP and identify the odd numbered terms:
$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5, \ldots$

Note: Odd-numbered terms $\neq$ Odd numbers

Exercise: Find the sum to infinity of $u_3 + u_6 + u_9 + u_{12} + \ldots$

Example 12 (Independent learning)

Express the recurring decimal 20.03030....... as a fraction.

Solution:

$$\begin{aligned}
20.030303\ldots &= 20 + 0.03 + 0.0003 + 0.000003 + \ldots \\
&= 20 + \left[\frac{3}{100} + \frac{3}{100}\left(\frac{1}{100}\right) + \frac{3}{100}\left(\frac{1}{100}\right)^2 + \ldots\right] \\
&= 20 + \frac{\frac{3}{100}}{1-\frac{1}{100}} = 20 + \frac{1}{33} \\
&= 20\frac{1}{33}
\end{aligned}$$

Self-Review 4
The sum to infinity of a geometric series is 162. The sum of the first three terms is 114. Find the least value of n for which the sum to n terms differs from the sum to infinity by less than 0.2. [17]

3. Test for Sequences

3.1 Test for an Arithmetic Progression

The sequence $u_1, u_2, u_3, \ldots u_n, u_{n+1}, \ldots$ is an AP **if and only if**

$\boldsymbol{u_{n+1} - u_n}$ **= constant** (i.e. independent of n) for **all** positive integers n.

This constant is the **common difference** $\boldsymbol{d}$.

Note:
In general, the n^{th} term of <u>**ANY**</u> sequence can be written as $\boldsymbol{u_n = S_n - S_{n-1},\ n \in \mathbb{Z}^+,\ n \geq 2}$

Example 13
The sum of the first n terms of a progression is $n^2 + 3n$. Find an expression for the n^{th} term and show that the progression is arithmetic.

Solution:

$S_n = n^2 + 3n$ $S_{n-1} = (n-1)^2 + 3(n-1)$ $= n^2 + n - 2$ $\therefore\ u_n = S_n - S_{n-1}$ $= n^2 + 3n - (n^2 + n - 2)$ $= 2n + 2$ $u_{n+1} - u_n = [2(n+1)+2] - [2n+2]$ $= 2$ Since $u_{n+1} - u_n$ is a constant, the progression is arithmetic.	**Think Zone:** It is insufficient to conclude that the progression is arithmetic by only showing $u_3 - u_2 = u_2 - u_1 = \text{constant}$. You **MUST** work with the general term to show. Is the progression also arithmetic when the sum of the first n terms is equal to any quadratic expression?

3.2 Test for a Geometric Progression

The sequence $u_1, u_2, u_3, \ldots u_n, \ldots$ is a GP **if and only if**

$\dfrac{u_{n+1}}{u_n}$ = constant (independent of n) for **all** positive integers n.

This constant is the **common ratio,** $\boldsymbol{r}$.

Note:
In general, the n^{th} term of <u>**ANY**</u> sequence can be written as $\boldsymbol{u_n = S_n - S_{n-1},\ n \in \mathbb{Z}^+,\ n \geq 2}$

Example 14

The sum of the first n terms of a progression is $3^n - 1$. Show that it is geometric. State the common ratio and the first term. Find the value of n for which the difference between the sum of the first $2n$ terms and the sum of the first n terms is 6480.

Solution:

Given $S_n = 3^n - 1$

$U_n = S_n - S_{n-1}$

$= (3^n - 1) - (3^{n-1} - 1)$

$= 3^n(1 - 3^{-1})$

$= \frac{2}{3}(3^n)$

$\frac{U_{n+1}}{U_n} = \frac{\cancel{\frac{2}{3}}(3^{n+1})}{\cancel{\frac{2}{3}}(3^n)}$

$= 3$ = constant

Since $\frac{U_{n+1}}{U_n}$ is a constant the progression is geometric.

$u_1 = S_1 = 3^1 - 1 = 2$

Common ratio, $r = 3$

$|S_{2n} - S_n| = 6480$

$(3^{2n} - 1) - (3^n - 1) = 6480$

$3^{2n} - 3^n = 6480$

From GC

n	$\lvert S_{2n} - S_n \rvert$
3	702
4	6480

$\therefore\ n = 4$

Think Zone

It is insufficient to conclude that the progression is geometric by only showing $\frac{u_3}{u_2} = \frac{u_2}{u_1} =$ constant.

You **MUST** work with the general term to show.

Why is there a modulus sign? difference

Why is the modulus sign removed? $S_{2n} > S_n$ as $a > 0$ and $r > 0$

Algebraic way of solving:

$(3^n)^2 - 3^n - 6480 = 0$

Let $y = 3^n$, then $y^2 - y - 6480 = 0$

Using GC (use APPS -> PlySmlt2),

$y = 3^n = -80$ (rejected) or $y = 3^n = 81 = 3^4$

$\therefore\ n = 4$

Self-Review 5

(a) The sum of the first n terms of a progression is $21n - 3n^2$. Show that it is arithmetic.

(b) The sum of the first n terms of a progression is $\frac{3^{n+1} - 3}{2}$. Show that it is geometric.

4. Practical Problems Involving Arithmetic or Geometric Progression

Example 15

A 28.5 m length of rope is cut into pieces whose lengths are in arithmetic progression with a common difference of d m. Given that the length of the shortest and longest pieces are 1 m and 3.75 m respectively, find the number of pieces and the value of d.

Solution: We first use a diagram to help us visualise the information

$S_n = 28.5$

1 m $1+d$ $1+2d$ ……   3.75 m $1+(n-1)d$

$28.5 = \frac{n}{2}(1+3.75)$

$n = 12$

$3.75 = 1+11d$

$d = 0.25$

Annotate the important information in the question and use the relevant formula.

Example 16

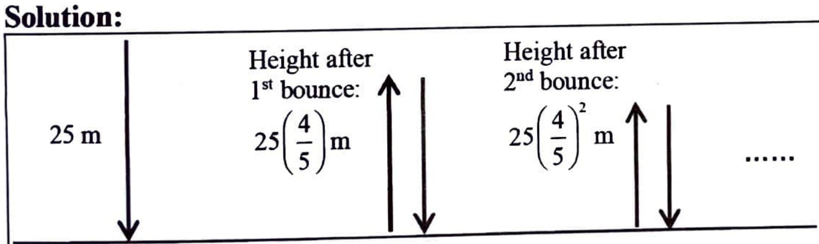
A rubber ball is dropped vertically from a height of 25 metres and rises to $\frac{4}{5}$ of its previous height on bouncing. If this ratio of height remains constant,

(i) what is the height it rises after the first bounce? How high does it rise immediately after 7 bounces? Give your answer to nearest cm.

(ii) after how many bounces does the ball rise to a height of 12.8m?

(iii) through what distance will it have moved before it comes to rest?

Solution:



(i) $a = 25$, $r = \frac{4}{5}$

Height after first bounce = $25\left(\frac{4}{5}\right) = 20$ m

Height after 7 bounces = $25\left(\frac{4}{5}\right)^7 = 5.24$ m

$= 524$ cm

(ii) $12.8 = 25\left(\frac{4}{5}\right)^n$

$n = 3$

Think Zone:

We first use a diagram to help us visualise the information

total distance moved $= 25+2[25(\frac{4}{5})+25(\frac{4}{5})^2+25(\frac{4}{5})^3+\ldots]$ $=25+2(25)(\frac{4}{5})[1+(\frac{4}{5})+(\frac{4}{5})^2+\ldots]$ $=25+2(25)(\frac{4}{5})(\frac{1}{1-\frac{4}{5}})$ $=225\text{m}$	(iii) How many times do you think the ball would have hit the floor before it comes to a rest?

Example 17 (N97/I/15)
A bank has an account for investors. Interest is added to the account at the end of each year at a fixed rate of 5% of the amount in the account at the beginning of that year. A man and woman both invest money.

(a) The man decides to invest $\$x$ at the beginning of one year and then a further $\$x$ at the beginning of the second and subsequent year. He also decides that he will not draw any money out of the account but just leave it, and any interest, to build up.

(i) How much will there be in the account at the end of one year, including the interest?

(ii) Show that, at the end of n years, when the interest for the last year has been added, he will have a total of $\$21(1.05^n-1)x$ in his account.

(iii) After how many complete years will he have, for the first time, **at least** $\$12x$ at his account?

(b) The woman decides that, to assist her in her everyday expenses, she will withdraw the interest as soon as it has been added. She invests $\$y$ at the beginning of each year, Show that, at the end of n years, she will have received a total of $\$\frac{1}{40}n(n+1)y$ in interest.

Solution:

(a)(i) Amount in account at the end of one year $= x(0.05)+x$
$=1.05x$

(a)(ii)

At the end of n^{th} year	Amount received
1^{st}	$x(0.05)+x=1.05x$
2^{nd}	$(1.05x+x)1.05=(1.05)^2x+1.05x$
3^{rd}	$[(1.05)^2x+(1.05)x+x]1.05=(1.05)^3x+(1.05)^2x+(1.05x)$
$\vdots$	
nth	$(1.05)^nx+(1.05)^{n-1}x+\cdots(1.05)^2x+(1.05)x$ $=\frac{1.05x(1-(1.05)^n)}{1-1.05}=\frac{1.05x(1.05^n-1)}{1.05-1}$ $=21x(1.05^n-1)$

(a)(iii) $21x[1.05^n - 1] \geq 12x$

$21[1.05^n - 1] \geq 12$

From GC,

n	$21[1.05^n - 1]$
9	$11.578 < 12$
10	$13.207 > 12$

He needs at least 10 complete years.

(b)

At the beginning of n^{th} years	Amount in the account	Interest
1^{st}	y	$y(0.05) = 0.05y$
2^{nd}	$2y$	$2y\,(0.05)$
3^{rd}	$3y$	$3y\,(0.05)$
$\vdots$	$\vdots$	$\vdots$
n^{th}	ny	$ny\,(0.05)$

At the end of n years,

Interest received $= 0.05y + (0.05)2y + (0.05)3y + \cdots + (0.05)ny$

$= 0.05y(1 + 2 + 3 + \cdots + n)$

$= 0.05y\left[\dfrac{n(1+n)}{2}\right]$

$= \dfrac{1}{40}n(n+1)y$ (shown)

Summary

	AP	GP
u_n	$a + (n-1)d$	ar^{n-1}
S_n	$\frac{n}{2}[2a + (n-1)d]$ or $\frac{n}{2}[a + l]$	$\dfrac{a(1-r^n)}{1-r}$, $r \neq 1$
S_∞	N.A.	$\dfrac{a}{1-r}$, $\lvert r \rvert < 1$
To prove	$u_{n+1} - u_n =$ constant	$\dfrac{u_{n+1}}{u_n} =$ constant
Given S_n to find u_n	$u_n = S_n - S_{n-1}$	
		S_∞ exists ↔ $-1 < r < 1$ ↔ Infinite series converges

SECTION B: Σ – The Summation Notation

1. Introduction and Terminologies

The sum of a series can be written using the 'Σ' (read as 'Sigma') notation. The notation 'Σ' is the capital letter of the Greek alphabet σ which corresponds to the word 'sum'. We will study some standard results associated with the sigma notation and use them in summing some standard series.

Suppose we want to express the sum of all positive **integers** from 1 to 100 (inclusive). We may write it as '$1+2+3+\ldots+100$' without listing all the 100 integers. Using the sigma notation, this sum can be written even more concisely as $\sum_{r=1}^{100} r$, i.e. $\sum_{r=1}^{100} r = 1+2+3+\ldots+100$.

We read this as "the sum of terms of the form r, where r takes all integer values from 1 to 100".

Consider another example: $\sum_{r=4}^{10} \frac{1}{2^r} = \frac{1}{2^4}+\frac{1}{2^5}+\cdots+\frac{1}{2^{10}}$. From this example, it is worth to note the following:

1. The first integer value that r takes need not necessarily be 1.

2. The series can be written in another way using the sigma notation. For example, we may write the series as $\sum_{r=3}^{9} \frac{1}{2^{r+1}} = \frac{1}{2^4}+\frac{1}{2^5}+\cdots+\frac{1}{2^{10}}$. Check to see for yourself that the new representation give the same series. In fact, there are many different ways of writing the same series using the sigma notation. In other words, there is no unique way of expressing a series using the sigma notation.

In general, let $u_m, u_{m+1}, u_{m+2}, \ldots, u_n$ be a sequence of numbers where $m \le n$. Then the sum $u_m + u_{m+1} + u_{m+2} + \cdots + u_n$ can be expressed as $\sum_{r=m}^{n} u_r$.

Note:

1. 'u_r' is called the **general term** of the summation. It is used to denote an expression written in terms of r only. In other words, we say that u_r is a **function** of r. However, it is possible for u_r to be a **constant**.

2. The letter 'r' is called the **index** of the summation which takes **only integer** values. It is a **dummy variable** used to denote the integer values that must be 'fed into' u_r to generate the entire series. The variable 'r' can thus be replaced by another variable say 'i'. For example, $\sum_{r=m}^{n} u_r = \sum_{i=m}^{n} u_i$ as both represent the same series.

3. The first integer value 'm' and the last integer value 'n' that r takes are called the **lower limit** and the **upper limit** of the summation respectively. The lower limit is never larger than the upper limit. That is, $m \le n$.

4. The number of terms of the series is given by $\boldsymbol{n - m + 1}$.

Note: Recall in the chapter of AP/GP, you have learnt that S_n represents the sum of the first n terms of a series, so in summation notation, we will write $S_n = \sum_{r=1}^{n} u_r$ where u_r represents the r^{th} term of the AP/GP sequence.

2. Working with the Sigma Notation and the Use of GC

In this section, we introduce the basic rules in working with the Σ notation, and how to use the GC to evaluate a given sum.

Example 1
Write the following series using the sigma notation and determine the number of terms in the series whenever appropriate.

(a) $\sqrt{10}+\sqrt{11}+\sqrt{12}+\cdots+\sqrt{75}$ $= \sum_{r=10}^{75}\sqrt{r}$ number of terms $= 75-10+1 = 66$

(b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2000}$ $= \sum_{r=1}^{1000}\frac{1}{2r}$

(c) $1^3+3^3+5^3+\cdots+2013^3$ $\sum_{r=1}^{1007}(-1+2r)^3$

(d) $1-2+3-4+\cdots$ to 2011 terms. Write down also the last term of this series.

(e) $3+7+11+\cdots+8051$

(f) $1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots$. Write down also the 20th term of this series.

(g) $1+2x+3x^2+4x^3+\cdots$ where $|x|<1$.

Solution	**Notes**
(a) $\sqrt{10}+\sqrt{11}+\sqrt{12}+\cdots+\sqrt{75} =$ Number of terms =	
(b) $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots+\frac{1}{2000} =$ Number of terms =	An even number can be written in the form '$2r$' where r is an integer.
(c) $1^3+3^3+5^3+\cdots+2013^3 =$ Number of terms =	An odd number can be written in the form '$2r-1$' where r is an integer. **Exercise:** Try expressing the same series using the general term '$2r+1$' instead. How would the lower and upper limits be different now?
(d) $1-2+3-4+\cdots$ to 2011 terms $= \sum_{r=1}^{2011}(-1)^{r+1}r$ Last term of the series is =	Check that the representation is correct by writing out the a few terms. **Note:** The inclusion of '$(-1)^{r+1}$' creates the alternating signs in the series.
(e) The pattern is an AP with the first term $a=3$ and common difference $d=4$. Therefore the general term $u_r = a+(r-1)d = 3+(r-1)4 = 4r-1$. When $u_r = 8051 = 4r-1$, $r = 2013$ $3+7+11+\cdots+8051 = \sum_{r=1}^{2013}(4r-1)$	

(f)	$1+\frac{1}{4}+\frac{1}{4^2}+\frac{1}{4^3}+\cdots =$ $\sum_{r=0}^{\infty}\frac{1}{4^r}$ 20^{th} term of the series = $\frac{1}{4^{19}}$	The number of terms is infinite since the series does not terminate. Hence, (f) is an **infinite series.** Does this infinite series converge? If it does, what is its value?
(g)	$1+2x+3x^2+4x^3+\cdots =$ $\sum_{r=1}^{\infty} rx^{r-1}$	In each term, the coefficient is always one more than the power of x. So we may write the general r^{th} term in the form rx^{r-1}. Check that the representation is correct by writing out the first 3 terms.

Self-Review 1

(a) Write $\sum_{r=2}^{20} r^4$ as a series of terms, giving the first 3 terms and the last term. How many terms are there in the series? What is the 10^{th} term of the series? $[2^4+3^4+4^4+\cdots+20^4\,;\,19;\,11^4]$

(b) Write, but do not evaluate, the following series using the sigma notation. State also the number of terms of the series where appropriate.

(i) $1+\frac{1}{2}+\frac{1}{4}+\ldots+\frac{1}{128}$ $[\sum_{r=1}^{8}\frac{1}{2^{r-1}}\,;\,8]$

(ii) $2-4+6-8+\cdots$ to 15 terms $[\sum_{r=1}^{15}(-1)^{r+1}(2r)]$

(iii) $2+9+16+23+\cdots+14072$ $\binom{n}{r}a^n b^{n-r}$ $[\sum_{r=1}^{2011}(7r-5)\,,\,2011]$

$(a+b)^2 = a^2+2ab+b^2 = \Sigma$

$(a+b)^n = a^n + na^{n-1}b + \cdots nab^{n-1} + b^n = \sum_{r=0}^{n}$

Use of the GC in finding the sum of a series

Use the GC to find the value of $\sum_{r=5}^{100}(r^2+1)$. $= 1a^{2-(+1)}b^{1-1} +$

Press [A-lock alpha] [tblset f2 window] [L2 Z 2] to get the summation template.	NORMAL FLOAT AUTO REAL RADIAN MP Σ ( )
Use the arrow keys to move to different blanks on the template. Key in the values accordingly. You can choose any letter to denote the variable, in this case, X from [link X,T,θ,n] is used.	NORMAL FLOAT AUTO REAL RADIAN MP $\sum_{X=5}^{100}(X^2+1)$ 338416

Note: If the lower limit and upper limit of the summation are 'known' numbers, then you can or should use the calculator to compute the sum. However, there are instances when the limits are given as unknown constants such as m or n. In such instances, we will need to apply the properties and standard results of summation.

3. Properties and Standard Results of Summation

3.1 Properties of Summation

For any two integers m and n such that $m \le n$, and for any constant a,

1. $\sum_{r=m}^{n}[\mathrm{f}(r) \pm \mathrm{g}(r)] = \sum_{r=m}^{n}\mathrm{f}(r) \pm \sum_{r=m}^{n}\mathrm{g}(r)$

2. $\sum_{r=m}^{n}[a\mathrm{f}(r)] = a\left[\sum_{r=m}^{n}\mathrm{f}(r)\right]$

3. $\sum_{r=m}^{n}\mathrm{f}(r) = \sum_{r=1}^{n}\mathrm{f}(r) - \sum_{r=1}^{m-1}\mathrm{f}(r)$ (for $m > 1$)

where $\mathrm{f}(r)$ and $\mathrm{g}(r)$ are denoted as some functions of r.

Proof:

Proof of Property 1 [Independent Reading]:

$$\sum_{r=m}^{n}[\mathrm{f}(r) \pm \mathrm{g}(r)] = [\mathrm{f}(m) \pm \mathrm{g}(m)] + [\mathrm{f}(m+1) \pm \mathrm{g}(m+1)] + \cdots + [\mathrm{f}(n) \pm \mathrm{g}(n)]$$

$$= [\mathrm{f}(m) + \mathrm{f}(m+1) + \cdots + \mathrm{f}(n)] \pm [\mathrm{g}(m) + \mathrm{g}(m+1) + \cdots + \mathrm{g}(n)] \text{ (regrouping)}$$

$$= \sum_{r=m}^{n}\mathrm{f}(r) \pm \sum_{r=m}^{n}\mathrm{g}(r).$$

Proof of Property 2 [Independent Reading]:

$$\sum_{r=m}^{n}[a\mathrm{f}(r)] = a\mathrm{f}(m) + a\mathrm{f}(m+1) + \cdots + a\mathrm{f}(n)$$

$$= a[\mathrm{f}(m) + \mathrm{f}(m+1) + \cdots + \mathrm{f}(n)] = a\left[\sum_{r=m}^{n}\mathrm{f}(r)\right]$$

Proof of Property 3:

$$\sum_{r=m}^{n}\mathrm{f}(r) = \mathrm{f}(m) + \mathrm{f}(m+1) + \cdots + \mathrm{f}(n)$$

$$= [\mathrm{f}(1) + \mathrm{f}(2) + \cdots + \mathrm{f}(m-1) + \mathrm{f}(m) + \mathrm{f}(m+1) + \cdots + \mathrm{f}(n)]$$
$$- [\mathrm{f}(1) + \mathrm{f}(2) + \cdots + \mathrm{f}(m-1)]$$

$$= \sum_{r=1}^{n}\mathrm{f}(r) - \sum_{r=1}^{m-1}\mathrm{f}(r).$$

3.2 Standard Results of Summation

1. $\sum_{r=m}^{n}a = (n-m+1)a$ where a is a constant, $n \ge 1$ and $m \le n$.

In particular, when $m = 1$, $\sum_{r=1}^{n}a = na$.

E.g. $\sum_{r=3}^{10}7 = 7+7+7+7+7+7+7+7 = (10-3+1)(7) = 8(7) = 56$.

2. $\sum_{r=1}^{n} r = 1+2+3+...+n = \frac{n(1+n)}{2}$ (sum of arithmetic series)

E.g. $\sum_{r=1}^{10} r = 1+2+3+...+10 = \frac{10}{2}(1+10) = 55$.

3. $\sum_{r=1}^{n} a^r = a+a^2+a^3+...+a^n = \frac{a(1-a^n)}{1-a}$ (sum of geometric series)

Note: To apply formula (2) or (3) directly, the lower limit must be 1.

It is important to distinguish clearly the meanings of the letters r, a and n in the summation, that is, which is a *dummy variable* and which is a *constant.*

E.g. $\sum_{r=5}^{10} a3^r = a\left(3^5+3^6+...+3^{10}\right) = \frac{a3^5(1-3^6)}{1-3} = 88452a$. **$r$ is the variable here!**

vs $\sum_{a=5}^{10} a3^r = 3^r\left(5+6+...+10\right) = 3^r\left(\frac{10-5+1}{2}(5+10)\right) = 45\left(3^r\right)$ **a is the variable here!**

Example 2

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, simplify $\sum_{r=1}^{n}\left(2r^3+r^2-r\right)$ in terms of n.

Solution

$$\sum_{r=1}^{n}\left(2r^3+r^2-r\right)$$

$$= \sum_{r=1}^{n} 2r^3 + \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \quad \text{(by property 1)}$$

$$= 2\sum_{r=1}^{n} r^3 + \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} r \quad \text{(by property 2)}$$

$$= 2\left[\frac{n^2(n+1)^2}{4}\right] + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n^2(n+1)^2}{2} + \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6}\left[3n(n+1)+(2n+1)-3\right] \quad \text{(factorising)}$$

$$= \frac{n(n+1)\left(3n^2+5n-2\right)}{6}$$

$$= \frac{n(n+1)(n+2)(3n-1)}{6}$$

$\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ are standard summation results, but you are not required to memorise them. The question would have to give you the result if you are expected to use them.

You should simplify your expression by factorisation. After applying the results, it is unwise to expand the terms as this will inevitably lead to complicated working and algebraic errors.

ct dr → sum of AP of n+1 terms

Example 3

Find, in terms of n, (a) $\sum_{r=0}^{n}(2n+1-2r)$ (b) $\sum_{r=0}^{n}\left(2^{r-1}\right)$.

Solution	
(a) $\sum_{r=0}^{n}(2n+1-2r)=\sum_{r=0}^{n}(2n+1)-2\sum_{r=0}^{n}r$ $=(n-0+1)(2n+1)-2(0+1+2+...+n)$ $=(n+1)(2n+1)-2\left[\frac{n}{2}(1+n)\right]$ $=(n+1)[2n+1-n]$ $=(n+1)^2$	Note that r is the ***dummy variable*** which is changing and n is a ***constant*** which remains unchanged as r changes.
(b) $\sum_{r=0}^{n}(2^{r-1}) = 2^{-1}+2^{0}+2^{1}+\dots+2^{n-1}$ $=\frac{2^{-1}(1-2^{n+1})}{1-2}$	Write out to see the pattern.

$=\frac{1}{2}(2^{n+1}-1)$

Example 4

Given that $\sum_{r=1}^{n}r^2=\frac{n}{6}(n+1)(2n+1)$, show that $\sum_{r=n+1}^{2n}(2r-1)^2=\frac{n}{3}(28n^2-1)$.

Solution	
(a) $\sum_{r=n+1}^{2n}(2r-1)^2$ $=\sum_{r=n+1}^{2n}(4r^2-4r+1)$ $=4\sum_{r=n+1}^{2n}r^2-4\sum_{r=n+1}^{2n}r+\sum_{r=n+1}^{2n}1$ $=4\left(\sum_{r=1}^{2n}r^2-\sum_{r=1}^{n}r^2\right)-4\left[\frac{n}{2}(n+1+2n)\right]+(2n-n-1+1)(1)$ $=4\left(\frac{2n}{6}(2n+1)(4n+1)-\frac{n}{6}(n+1)(2n+1)\right)-2n(3n+1)+n$ $=\frac{2n}{3}(2n+1)[2(4n+1)-(n+1)]-n(6n+1)$ **(Factorise!)** $=\frac{2n}{3}(2n+1)(7n+1)-n(6n+1)$ $=\frac{n}{3}[28n^2+18n+2-18n-3]$ $=\frac{n}{3}(28n^2-1)$ (shown)	To simplify $\sum_{r=n+1}^{2n}r^2$, we use the property $\sum_{r=m}^{n}f(r)=\sum_{r=1}^{n}f(r)-\sum_{r=1}^{m-1}f(r)$ to rewrite the sum as a difference of two sums both of which start from $r=1$. This would then allow us to use the result $\sum_{r=1}^{n}r^2=\frac{n}{6}(n+1)(2n+1)$. However, this procedure is not used for $\sum_{r=n+1}^{2n}r$ as it is an arithmetic series. The sum can be obtained by applying the result: $\frac{\text{no. of terms}}{2}(\text{first term}+\text{last term})$

Example 5

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$.

(a) Find $\sum_{r=1}^{n} r(r+1)(r+2)$ in terms of n, fully factorising your answer. [Independent reading]

(b) Find, in terms of n, the sum $4\times3 + 7\times5 + 10\times7 + \cdots +$ to n terms.

Solution

(a) $\sum_{r=1}^{n} r(r+1)(r+2)$

$$= \sum_{r=1}^{n} (r^3 + 3r^2 + 2r)$$

$$= \frac{n^2(n+1)^2}{4} + 3\left[\frac{n(n+1)(2n+1)}{6}\right] + 2\left[\frac{n(n+1)}{2}\right]$$

$$= \frac{n(n+1)}{4}\left[n(n+1) + 2(2n+1) + 4\right]$$

Factorise!

$$= \frac{n(n+1)(n^2+5n+6)}{4}$$

$$= \frac{1}{4}n(n+1)(n+2)(n+3)$$

(b) 4, 7, 10,… to n terms is an AP with $a = 4$, $d = 3$.

Find a relationship between the numbers.

So the r^{th} term is $u_r = a+(r-1)d = 4+(r-1)(3) = 3r+1$

3, 5, 7, … to n terms is an AP with $a = 3$, $d = 2$.

So the r^{th} term is $u_r = a+(r-1)d = 3+(r-1)(2) = 2r+1$

The general r^{th} term of the series is therefore given by

$u_r = (3r+1)(2r+1)$

Hence $4\times3 + 7\times5 + 10\times7 + \cdots +$ to n terms

$$= \sum_{r=1}^{n} [(3r+1)(2r+1)]$$

$$= \sum_{r=1}^{n} (6r^2+5r+1)$$

$$= \sum_{r=1}^{n} 6r^2 + \sum_{r=1}^{n} 5r + \sum_{r=1}^{n} 1$$

$$= 6\left[\frac{n}{6}(n+1)(2n+1)\right] + 5\left[\frac{n}{2}(n+1)\right] + n$$

$$= \frac{n}{2}\left[2(n+1)(2n+1) + 5(n+1) + 2\right]$$

$$= \frac{n}{2}(4n^2 + 11n + 9)$$

Self-Review 2

Given that $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$ and $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$

(a) Find, in terms of n, $\sum_{r=1}^{n}(r+1)^2$. $\left[\frac{n}{6}(2n^2+9n+13)\right]$

Find the value of n such that $\sum_{r=1}^{n}(r+1)^2 = 4n^2-5$. $[n=6]$

(b) Find the series $2\times1^2+3\times2^2+4\times3^2+\cdots$ to n terms, in terms of n. $\left[\frac{1}{12}n(n+1)(n+2)(3n+1)\right]$

3.3 Change of Index in Summation

When solving summation questions, it is sometimes useful to **change the index** of the original summation notation so as to deduce the formula of another summation notation. The following example will illustrate the steps in doing so.

Example 6 (Revisit)

Given that $\sum_{r=1}^{n} r(r+1)(r+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$, deduce the sum $\sum_{r=1}^{n} r(r^2-1)$.

Solution	
Observe that $\sum_{r=1}^{n} r(r^2-1) = \sum_{r=1}^{n} r(r-1)(r+1)$	1. Arrange the factors in ascending order for $\sum_{r=1}^{n}(r-1)r(r+1)$ to follow the ascending pattern
$= \sum_{r=1}^{n}(r-1)r(r+1)$	in $\sum_{r=1}^{n} r(r+1)(r+2)$.
$= \sum_{k=0}^{n-1} k(k+1)(k+2)$	2. If we let '$r-1=k$', then $(r-1)r(r+1) = k(k+1)(k+2)$. We must then change the index of the summation from r to k, and as such, the limits must change accordingly:
$= \sum_{k=1}^{n-1} k(k+1)(k+2)$	Lower limit: When $r=1$, $k=r-1=1-1=0$; Upper limit: When $r=n$, $k=r-1=n-1$.
$= \frac{1}{4}(n-1)n(n+1)(n+2)$	3. Check that the expressions are now the 'same' (same function **and lower** limit). When the expressions are the 'same', we can use the given result by modifying the end value.
$= \frac{1}{4}n(n^2-1)(n+2)$.	In this case, we replace n by $n-1$ in the previous result.

Self-review 3

Given that $\sum_{r=1}^{N}\frac{(r+2)^2-5}{(r+1)(r+3)} = N-\frac{5}{3}+\frac{2}{N+2}+\frac{2}{N+3}$, deduce the sum $\sum_{r=3}^{N}\frac{r^2-5}{r^2-1}$ in terms of N.

$\sum_{r=3}^{N}\frac{r^2-5}{r^2-1}$ $= \sum_{r=3}^{N}\frac{r^2-5}{(r+1)(r-1)}$ $= \sum_{k=1}^{N-2}\frac{(r+2)^2-5}{(r+1)(r+3)}$ $= (N-2)-\frac{5}{3}$

$\left[N-\frac{11}{3}+\frac{2}{N}+\frac{2}{N+1}\right]$

Example 7 Find the least value of n for which $\sum_{r=1}^{n}\left(\frac{1}{2r}\right) > 2$.

Solution:

$\sum_{r=1}^{n}\left(\frac{1}{2r}\right) > 2$.

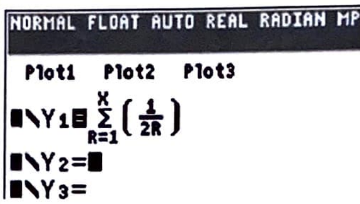
Using GC,

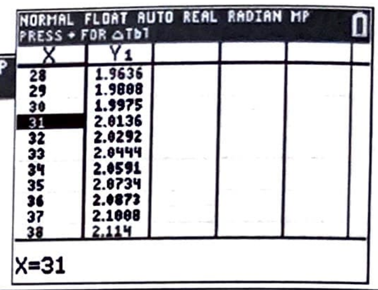
n	$\sum_{r=1}^{n}\left(\frac{1}{2r}\right)$
30	$1.9975 < 2$
31	$2.0136 > 2$

Hence, the least value for n is 31.

Since question did not specify the method to use, we can the GC to solve. As we are solving for the value of n, let 'X' be n in the GC.

You may use any other letters from the GC to denote the index r. For example, Press ALPHA X to input 'R' as the index.





Self-Review 4 Find the least value of n for which $\sum_{r=1}^{n} r^2 > 2013$. [18]

4. S_n, S_∞ and Convergence of Series

Recall that S_n represents the sum of the first n terms of a sequence, i.e. $S_n = \sum_{r=1}^{n} u_n$.

Since S_∞ represents the sum to infinity, we can write $S_\infty = \lim_{n\to\infty} S_n = \sum_{r=1}^{\infty} u_r$.

Recall that for a GP, if $|r| < 1$, then the series is convergent and the limiting value of the series is $S_\infty = \frac{a}{1-r}$ which is also the sum to infinity.

Is the arithmetic series a convergent or divergent series?

Example 8 Determine if the following series S_n is convergent and if so, find S_∞.

(a) $\frac{n+1}{n^2}$ (b) $\frac{n^2-2}{2n^2+1}$ (c) $\frac{n^2+2}{n-10}$

Solution

(a) As $n \to \infty$

$S_n = \frac{n+1}{n^2} = \frac{1}{n} + \frac{1}{n^2} \to 0$

therefore the series is convergent and $S_\infty = 0$

Another way of writing is

$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$

(b) $S_n = \frac{n^2-2}{2n^2+1} = \frac{1-\frac{2}{n^2}}{2+\frac{1}{n^2}} \to \frac{1}{2}$, since $\frac{1}{n^2} \to 0$ as $n \to \infty$.

Therefore the series is convergent and $S_\infty = \frac{1}{2}$.

(c) As $n \to \infty$, $S_n = \frac{n^2+2}{n-10} \to \infty$, therefore the series is divergent.

Example 9

(a) An arithmetic series has first term 3 and common difference 2, and the sum of the first n terms is denoted by A_n. Write down an expression for A_n in terms of n, and hence find $\sum_{n=1}^{N} A_n$.

(b) A geometric series has first term 3 and common ratio $\frac{1}{2}$ and the sum of the first n terms is denoted by G_n. Write down an expression for G_n in terms of n. Find $\sum_{n=1}^{N} G_n$ and deduce the value of $\sum_{n=1}^{N} G_n - 6N$ as $N \to \infty$.

[You may use the results $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$ as appropriate.]

Solution	
(a) $A_n = \frac{n}{2}[2(3)+(n-1)2] = \frac{n}{2}(4+2n) = n(n+2)$ $\sum_{n=1}^{N} A_n = \sum_{n=1}^{N} n(n+2)$ $= \sum_{n=1}^{N} n^2 + 2\sum_{n=1}^{N} n$ $= \frac{N(N+1)(2N+1)}{6} + 2\left[\frac{N(N+1)}{2}\right]$ $= \frac{N(N+1)}{6}[(2N+1)+6]$ $= \frac{N(N+1)(2N+7)}{6}$	Note that n is denoted as the *dummy variable* and N is a constant in this question. **Factorise!**
(b) $G_n = \frac{3\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} = 6\left[1-\left(\frac{1}{2}\right)^n\right]$ $\sum_{n=1}^{N} G_n = \sum_{n=1}^{N} 6\left[1-\left(\frac{1}{2}\right)^n\right]$ $= 6\sum_{n=1}^{N} 1 - 6\sum_{n=1}^{N}\left(\frac{1}{2}\right)^n$ $= 6(N-1+1) - 6\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^N}\right)$ $= 6N - 6\left(\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^N\right)}{1-\frac{1}{2}}\right)$ $= 6N - 6\left(1-\left(\frac{1}{2}\right)^N\right)$ $\lim_{N\to\infty}\left(\sum_{n=1}^{N} G_n - 6N\right) = \lim_{N\to\infty}\left[-6\left(1-\left(\frac{1}{2}\right)^N\right)\right] = -6$ since $\lim_{N\to\infty}\left(\frac{1}{2}\right)^N = 0$.	

Example 10 (Independent Learning)

The r^{th} term, u_r, of a series is given by $u_r = \left(\frac{1}{3}\right)^{3r-2} + \left(\frac{1}{3}\right)^{3r-1}$.

Express $\sum_{r=1}^{n} u_r$ in the form $A\left(1-\frac{B}{27^n}\right)$, where A and B are constants. Hence deduce the sum to infinity of the series.

Solution:

$$\begin{aligned}\sum_{r=1}^{n} u_r = \sum_{r=1}^{n}\left[\left(\frac{1}{3}\right)^{3r-2}+\left(\frac{1}{3}\right)^{3r-1}\right] &= \sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r-2}+\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r-1}\\ &= 9\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r}+3\sum_{r=1}^{n}\left(\frac{1}{3}\right)^{3r}\\ &= 12\sum_{r=1}^{n}\left(\frac{1}{27}\right)^{r}\\ &= 12\left(\frac{\frac{1}{27}\left[1-\left(\frac{1}{27}\right)^n\right]}{1-\left(\frac{1}{27}\right)}\right)\\ &= \frac{6}{13}\left(1-\frac{1}{27^n}\right)\end{aligned}$$

Hence $\lim_{n\to\infty}\sum_{r=1}^{n} u_r = \lim_{n\to\infty}\frac{6}{13}\left(1-\frac{1}{27^n}\right) = \frac{6}{13}$ since $\lim_{n\to\infty}\left(\frac{1}{27^n}\right)=0$.

SECTION C: Method of Difference

In this section, we will learn how to find the sum of a special series where the general term u_r can be expressed in the form $\mathrm{f}(r)-\mathrm{f}(r-1)$, for some function $\mathrm{f}(r)$.

As an illustration, suppose we want to find the sum to n terms of a series where the general term is given by $u_r = \mathrm{f}(r)-\mathrm{f}(r-1)$.

Then we have

$$\begin{aligned}\sum_{r=1}^{n} u_r &= \sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]\\ &= \quad \cancel{\mathrm{f}(1)}-\mathrm{f}(0)\\ &\quad + \cancel{\mathrm{f}(2)}-\cancel{\mathrm{f}(1)}\\ &\quad + \cancel{\mathrm{f}(3)}-\cancel{\mathrm{f}(2)}\\ &\quad + \vdots\\ &\quad + \cancel{\mathrm{f}(n-2)}-\cancel{\mathrm{f}(n-3)}\\ &\quad + \cancel{\mathrm{f}(n-1)}-\cancel{\mathrm{f}(n-2)}\\ &\quad + \mathrm{f}(n)-\cancel{\mathrm{f}(n-1)}\\ &= \mathrm{f}(n)-\mathrm{f}(0)\end{aligned}$$

Since each term of the series can be written in the form $\mathrm{f}(r)-\mathrm{f}(r-1)$ for $r=1, 2, \ldots, n$, we see that terms of the series cancel out, leaving only ‘f(n) – f(0)’. This method of summation is known as the **Method of Difference**.

This type of series is also known as a telescoping series.

When dealing with questions involving the method of difference, it is often necessary to rewrite the general term in the desired form '$\mathrm{f}(r)-\mathrm{f}(r-1)$' or its equivalent (e.g. $\mathrm{f}(r+1)-\mathrm{f}(r)$) before obtaining the sum to n terms of the series.

Example 1

Show that $r^3-(r-1)^3=3r^2-3r+1$. Hence find the sum $\sum_{r=1}^{n}(3r^2-3r+1)$ in terms of n.

Deduce an expression, in terms of n, for the sum $\sum_{r=1}^{n} r^2$.

Solution:

$$r^3-(r-1)^3=r^3-(r^3-3r^2+3r-1)$$
$$=3r^2-3r+1$$

$$\sum_{r=1}^{n}(3r^2-3r+1)=\sum_{r=1}^{n}\left[r^3-(r-1)^3\right]$$

Show the cancellation.

$$\begin{aligned}
&= \cancel{1^3}-0^3 && \leftarrow r=1\\
&+ \cancel{2^3}-\cancel{1^3} && \leftarrow r=2\\
&+ \cancel{3^3}-\cancel{2^3} &&\\
&\ \ \vdots &&\\
&+ \cancel{(n-1)^3}-\cancel{(n-2)^3} && \leftarrow r=n-1\\
&+ n^3-\cancel{(n-1)^3} && \leftarrow r=n\\
&= n^3-0^3 &&\\
&= n^3 &&
\end{aligned}$$

$$\sum_{r=1}^{n}(3r^2-3r+1)=n^3$$

$$3\sum_{r=1}^{n}r^2-3\sum_{r=1}^{n}r+\sum_{r=1}^{n}1=n^3$$

$$3\sum_{r=1}^{n}r^2=n^3+3\sum_{r=1}^{n}r-\sum_{r=1}^{n}1$$

$$=n^3+3\left[\frac{n}{2}(1+n)\right]-n$$

$$=\frac{n}{2}\left[2n^2+3(1+n)-2\right]$$

$$=\frac{n}{2}\left[2n^2+3n+1\right]$$

Factorise!

$$=\frac{n}{2}(n+1)(2n+1)$$

Hence $\sum_{r=1}^{n}r^2=\frac{n}{6}(n+1)(2n+1)$.

Note: This example demonstrates how the standard result for $\sum_{r=1}^{n}r^2$ can be derived.

Example 2

It is given that $f(r)=r(r+1)!$. By considering $f(r)-f(r-1)$, find in terms of n, the sum of the series $5(2!)+10(3!)+17(4!)+\cdots+(n^2+1)(n!)$.

Solution:

$f(r)-f(r-1)=r(r+1)!-(r-1)r!$
$=r![r(r+1)-(r-1)]$
$=r!(r^2+1)$
$=(r^2+1)r!$

$(r+1)!=r!\times(r+1)$

$5(2!)+10(3!)+17(4!)+\cdots+(n^2+1)(n!)$

$=\sum_{r=2}^{n}(r^2+1)r!$

$=\sum_{r=2}^{n}[f(r)-f(r-1)]$

Note the limits of the summation. Check that the representation is correct by writing out the first 3 terms.

$$
\begin{aligned}
= & \ \cancel{f(2)}-f(1)\\
& +\cancel{f(3)}-\cancel{f(2)}\\
& + \quad \vdots\\
& +f(n)-\cancel{f(n-1)}
\end{aligned}
$$

Show the cancellation.

$=f(n)-f(1)$
$=n(n+1)!-1(2!)$
$=n(n+1)!-2$

Example 3

Express $\dfrac{2}{(r-1)(r+1)}$ in partial fractions and hence find $\sum_{r=3}^{n}\dfrac{2}{(r-1)(r+1)}$ in terms of n.

Explain why $\sum_{r=3}^{n}\dfrac{2}{(r-1)(r+1)}$ is always less than $\dfrac{5}{6}$ for all positive integers n

Obtain the value of $\sum_{r=3}^{\infty}\dfrac{1}{(r-1)(r+1)}$ and hence find $\sum_{r=2}^{\infty}\dfrac{1}{(r-1)(r+1)}$.

Solution:

$$\frac{2}{(r-1)(r+1)}=\frac{A}{r-1}+\frac{B}{r+1}$$

$\Rightarrow\ 2=A(r+1)+B(r-1)$

Put $r=1$: $\quad A=1$
Put $r=-1$: $\quad B=-1$

$$\therefore\ \frac{2}{(r-1)(r+1)}=\frac{1}{r-1}-\frac{1}{r+1}.$$

Refer to MF26 for the partial fraction formula.

You may also use the cover up method to find A and B quickly.

$\sum_{r=3}^{n} \frac{2}{(r-1)(r+1)}$

$= \frac{1}{2} - \frac{1}{4}$ $r=3$

$+ \frac{1}{3} - \frac{1}{5}$ $r=4$

$+ \frac{1}{4} - \frac{1}{6}$ $r=5$

$\vdots$

$+ \frac{1}{n-3} - \frac{1}{n-1}$ $r=n-2$

$+ \frac{1}{n-2} - \frac{1}{n}$ $r=n-1$

$+ \frac{1}{n-1} - \frac{1}{n+1}$ $r=n$

$= \frac{1}{2} + \frac{1}{3} - \frac{1}{n} - \frac{1}{n+1}$

$= \frac{5}{6} - \frac{2n+1}{n(n+1)}$

Since $\frac{2n+1}{n(n+1)} > 0$ for all $n \in \mathbb{Z}^+$,

$$\sum_{r=3}^{n} \frac{2}{(r-1)(r+1)} = \frac{5}{6} - \frac{2n+1}{n(n+1)} < \frac{5}{6} \text{ for all } n \in \mathbb{Z}^+$$

$$\sum_{r=3}^{\infty} \frac{1}{(r-1)(r+1)} = \frac{1}{2}\sum_{r=3}^{\infty} \frac{2}{(r-1)(r+1)}$$

$$= \frac{1}{2}\left[\lim_{n\to\infty} \sum_{r=3}^{n} \frac{2}{(r-1)(r+1)}\right]$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{5}{6} - \frac{2n+1}{n(n+1)}\right) = \frac{5}{12}$$

since $\frac{2n+1}{n(n+1)} \to 0$ as $n \to \infty$.

$$\sum_{r=2}^{\infty} \frac{1}{(r-1)(r+1)} = \frac{1}{(2-1)(2+1)} + \sum_{r=3}^{\infty} \frac{1}{(r-1)(r+1)}$$

$$= \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$$

Note:

- The cancellation is 2 rows down, and not the immediate rows. Use the denominator to help you see the cancellation.
- If there are k terms that are not cancelled at the beginning (in this case 2 terms), then there are also k terms that are not cancelled at the end (in this case 2 terms).

Example 4 (FM J89/I/2 modified)

Prove that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}+\sqrt{r-1}} = \sqrt{n}$. Hence,

(i) show that $\sum_{r=1}^{n} \frac{1}{2\sqrt{r}} < \sqrt{n}$,

(ii) find the least value of n such that $\sum_{r=1}^{n} \frac{1}{\sqrt{r}+\sqrt{r-1}} > 10$.

Solution:

$\frac{1}{\sqrt{r}+\sqrt{r-1}} = \frac{\sqrt{r}-\sqrt{r-1}}{\left(\sqrt{r}+\sqrt{r-1}\right)\left(\sqrt{r}-\sqrt{r-1}\right)}$ $= \frac{\sqrt{r}-\sqrt{r-1}}{\left(\sqrt{r}\right)^2-\left(\sqrt{r-1}\right)^2}$ $= \sqrt{r}-\sqrt{r-1}$	Rationalise the denominator
$\sum_{r=1}^{n} \frac{1}{\sqrt{r}+\sqrt{r-1}} = \sum_{r=1}^{n}\left(\sqrt{r}-\sqrt{r-1}\right)$ $= \sqrt{1}-\sqrt{0}$ $+\sqrt{2}-\sqrt{1}$ $+\sqrt{3}-\sqrt{2}$ $\vdots$ $+\sqrt{n-1}-\sqrt{n-2}$ $+\sqrt{n}-\sqrt{n-1}$ $= \sqrt{n}$.	
Since $\sqrt{r}$ + [redacted] $\sqrt{r-1} < \sqrt{r}+\sqrt{r}$ for all $r = 1, 2, 3 \ldots$ $\sqrt{r}+\sqrt{r-1} < 2\sqrt{r}$ $\frac{1}{2\sqrt{r}} < \frac{1}{\sqrt{r}+\sqrt{r-1}}$ summing both sides from $r=1$ to $r=n$ gives $\sum_{r=1}^{n}\frac{1}{2\sqrt{r}} < \sum_{r=1}^{n}\frac{1}{\sqrt{r}+\sqrt{r-1}} = \sqrt{n}$	
(ii) $\sum_{r=1}^{n} \frac{1}{\sqrt{r}+\sqrt{r-1}} > 10$ $\Rightarrow \sqrt{n} > 10 \Rightarrow n > 100$. So the least value is $n = 101$.	

Self-Review 1

Express $\frac{1}{r^2+3r+2}$ in partial fractions and hence show that $\sum_{r=1}^{n} \frac{1}{r^2+3r+2} = \frac{1}{2} - \frac{1}{n+2}$.

Deduce the exact value of $\sum_{r=1}^{\infty} \frac{1}{r^2+3r+2}$. [$\frac{1}{2}$]

Example 5

Given that $\dfrac{7}{r}-\dfrac{5}{r+1}-\dfrac{2}{r+2}=\dfrac{9r+14}{r(r+1)(r+2)}$, use the method of difference to find $S_n=\displaystyle\sum_{r=1}^{n}\frac{9r+14}{r(r+1)(r+2)}$, giving your answer in the form $k-\mathrm{f}(n)$, where k is a constant.

(i) Deduce that $S_n<8$ and find the value of S_∞.

(ii) Obtain the least value of n such that S_∞ is larger than S_n by less than 0.2.

Solution:

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$$S_n=\sum_{r=1}^{n}\frac{9r+14}{r(r+1)(r+2)}=\sum_{r=1}^{n}\left(\frac{7}{r}-\frac{5}{r+1}-\frac{2}{r+2}\right)$$

$$\begin{aligned}
&=\left(\frac{7}{1}-\frac{5}{2}-\frac{2}{3}\right)\\
&+\left(\frac{7}{2}-\frac{5}{3}-\frac{2}{4}\right)\\
&+\left(\frac{7}{3}-\frac{5}{4}-\frac{2}{5}\right)\\
&+\quad\vdots\\
&+\left(\frac{7}{n-2}-\frac{5}{n-1}-\frac{2}{n}\right)\\
&+\left(\frac{7}{n-1}-\frac{5}{n}-\frac{2}{n+1}\right)\\
&+\left(\frac{7}{n}-\frac{5}{n+1}-\frac{2}{n+2}\right)
\end{aligned}$$

$$=\frac{7}{1}-\frac{5}{2}+\frac{7}{2}-\frac{2}{2}-\frac{2}{n+1}-\frac{5}{n+1}-\frac{2}{n+2}$$

$$=8-\frac{7}{n+1}-\frac{2}{n+2}$$

$$=8-\frac{7(n+2)+2(n+1)}{(n+1)(n+2)}$$

$$=8-\frac{9n+16}{(n+1)(n+2)}$$

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Write the terms in their original form (e.g. do not simplify $\frac{2}{4}$ to $\frac{1}{2}$).

Use the denominator as a guide to find terms to cancel.

Since there are 3 terms that are not cancelled at the beginning, then there are also 3 terms that are not cancelled at the end.

Alternative method:

Rewrite the sum as

$$\sum_{r=1}^{n}\left(\frac{7}{r}-\frac{5}{r+1}-\frac{2}{r+2}\right)$$

$$=\sum_{r=1}^{n}\left[\left(\frac{7}{r}-\frac{7}{r+1}\right)+\left(\frac{2}{r+1}-\frac{2}{r+2}\right)\right]$$

$\vdots$

to create two telescoping series

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(i) Since $\dfrac{9n+16}{(n+1)(n+2)}>0$ for all $n\in\mathbb{Z}^+$,

$$S_n=8-\frac{9n+16}{(n+1)(n+2)}<8.$$

$$S_\infty=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[8-\frac{9n+16}{(n+1)(n+2)}\right]$$

$$=8 \qquad \text{since } \frac{9n+16}{n^2+3n+2}\to 0 \text{ as } n\to\infty$$

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(ii)

$$S_\infty - S_n < 0.2$$

$$8-\left[8-\frac{9n+16}{(n+1)(n+2)}\right]<0.2$$

$$\frac{9n+16}{(n+1)(n+2)}<0.2$$

From GC:

n	$\frac{9n+16}{(n+1)(n+2)}$
43	$0.20354 > 0.2$
44	$0.19903 < 0.2$

From the table, the least value of n is 44.

Example 6

It is given that $U_1 = \mathrm{e}^2$ and $U_{r+1} - U_r = (\mathrm{e}-1)\mathrm{e}^{r+1}$ where $r \in \mathbb{Z}^+$. By considering $\sum_{r=1}^{n-1}(U_{r+1}-U_r)$, show that $U_n = \mathrm{e}^{n+1}$. Hence find the product of the terms $U_2, U_3, U_4, \ldots, U_n$ in terms of n.

Solution:

$$\begin{aligned}\sum_{r=1}^{n-1}(U_{r+1}-U_r) &= \cancel{U_2} - U_1 \\ &\quad + U_3 - \cancel{U_2} \\ &\quad \vdots \\ &\quad + U_n - \cancel{U_{n-1}} \\ &= U_n - U_1 \\ &= U_n - \mathrm{e}^2\end{aligned}$$

Note that we have used $n-1$ for the upper index. This is done to ensure that we have U_n in the simplified expression.

On the other hand,

$$\begin{aligned}\sum_{r=1}^{n-1}(U_{r+1}-U_r) &= \sum_{r=1}^{n-1}(\mathrm{e}-1)\mathrm{e}^{r+1} \\ &= (\mathrm{e}-1)\sum_{r=1}^{n-1}\mathrm{e}^{r+1} \\ &= (\mathrm{e}-1)\cdot\frac{\mathrm{e}^2(\mathrm{e}^{n-1}-1)}{\mathrm{e}-1} \\ &= \mathrm{e}^{n+1}-\mathrm{e}^2\end{aligned}$$

Note that this is a geometric sum with first term e^2, common ratio e and $n-1$ terms in the sum.

Thus $U_n - \mathrm{e}^2 = \mathrm{e}^{n+1} - \mathrm{e}^2 \Rightarrow U_n = \mathrm{e}^{n+1}$

$$\begin{aligned}U_2 \cdot U_3 \cdot U_4 \cdots U_n &= \mathrm{e}^3 \cdot \mathrm{e}^4 \cdots \mathrm{e}^{n+1} \\ &= \mathrm{e}^{3+4+\cdots+(n+1)} \\ &= \mathrm{e}^{\frac{1}{2}(n-1)(3+n+1)} \\ &= \mathrm{e}^{\frac{1}{2}(n-1)(n+4)}\end{aligned}$$

We have made use of indices law. The sum is arithmetic with $n-1$ terms.