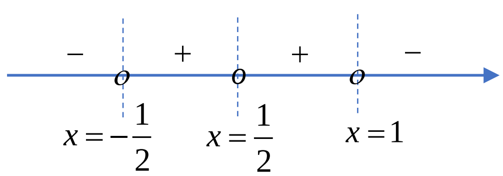
2023 H2MA Prelim Paper 1 Markers Comments

1	Solution [6] Inequality P1 Q1	
(i)	$\frac{4x^2-4x+1}{1+x-2x^2}<0$ $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$	Many students did not factorize the denominator correctly, missing out the negative sign to $(x-1)(2x+1)$, which resulted in errors to solutions
	Method 1: $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$ Since $(2x-1)^2 \geq 0$ for all real values of x, $(1-x)(2x+1)<0$ $\therefore x<-\frac{1}{2}$ or $x>1$	Some students still incorrectly wrote $(2x-1)^2>0$ or stated always positive.
	Method 2: $\frac{(2x-1)^2}{(1-x)(2x+1)}<0$  $\therefore x<-\frac{1}{2}$ or $x>1$	
(ii)	$\frac{(2^{x+1}-1)^2}{1+2^x-2^{2x+1}}\leq 0$ $\frac{(2(2^x)-1)^2}{1+(2^x)-2(2^x)^2}\leq 0$ Replace x with 2^x,	Most students could identify the correct replacement to use. However, most students missed out $\leq$ in the question and thus missed out the solution $x=-1$. Many students also did not give the reason for rejection correctly i.e. $2^x>0$ for all real x

	$2^x < -\frac{1}{2}$ OR $2^x > 1$ OR $2^x = \frac{1}{2}$ (rejected, since 2^x is positive for all real values of x.) $x > 0$ $x = -1$		

2	Solution [6] Complex Numbers P1 Q2	
(i)	$\left\|\frac{z-1}{z^*(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\left\|\frac{(a+\mathrm{i})-1}{(a+\mathrm{i})^*(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\left\|\frac{(a-1)+\mathrm{i}}{(a-\mathrm{i})(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\frac{\|(a-1)+\mathrm{i}\|}{\|(a-\mathrm{i})\|\|(1+\mathrm{i})\|}=\frac{1}{2}$ $\frac{\sqrt{(a-1)^2+1^2}}{\sqrt{a^2+1^2}\sqrt{1^2+1^2}}=\frac{1}{2}$ $2\sqrt{a^2-2a+1+1}=\sqrt{2}\sqrt{a^2+1}$ $4\left(a^2-2a+2\right)=2\left(a^2+1\right)$ $2a^2-4a+4=a^2+1$ $a^2-4a+3=0$ $a=1$ or 3 (rejected since $a<2$)	Some students incorrectly applied properties of modulus. Common errors include: • $\|(a+\mathrm{i})-1\|=(a-1)^2-1$ • $\|z-1\|=\|z\|-1$ • $\left\|\frac{z-1}{z^*(1+\mathrm{i})}\right\|=\frac{1}{2}$ $\Rightarrow \frac{z-1}{z^*(1+\mathrm{i})}=\pm\frac{1}{2}$ It is necessary to state both roots for a before rejecting the one which doesn't satisfy the criteria.
(ii)	Consider $\arg\left(\frac{z-1}{z^*w}\right)^n$ $=n\arg\left(\frac{z-1}{z^*w}\right)$ $=n\left[\arg(z-1)-\arg z^*-\arg w\right]$ $=n\left[\arg(\mathrm{i})-\arg(1-\mathrm{i})-\frac{\pi}{4}\right]$ $=n\left[\frac{\pi}{2}-\left(-\frac{\pi}{4}\right)-\frac{\pi}{4}\right]$ $=\frac{n\pi}{2}$ If $\left(\frac{z-1}{z^*w}\right)^n$ is purely imaginary and negative, then:	Most students attempted to find $\arg\left(\frac{z-1}{z^*w}\right)^n$ using properties of argument, however only some were successful. Note: It is incorrect to assume w = 1 + i just because arg(w) = $\frac{\pi}{4}$ Many students were not able to identify correctly the possible values of the argument for purely imaginary and negative.

	$\arg\left(\dfrac{z-1}{z^*w}\right)^n = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \dfrac{11\pi}{2}, \ldots$ or $\dfrac{3\pi}{2} + 2k\pi,\ k \in \mathbb{Z}^+$ Thus, $\dfrac{n\pi}{2} = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}, \dfrac{11\pi}{2}, \ldots$ $n = 3,\ 7,\ 11,\ \ldots$ 3 smallest positive values of $n = 3, 7, 11$	

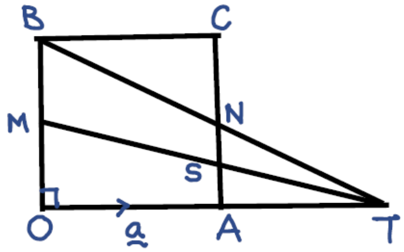
3	Solution [7] Summation P1 Q3	
(i)	$u_{n-1}-2u_n+u_{n+1}$ where $n\geq 2$ $=\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}$ $=\frac{n(n+1)-2(n-1)(n+1)+(n-1)n}{(n-1)n(n+1)}$ $=\frac{n^2+n-2n^2+2+n^2-n}{n[(n-1)(n+1)]}$ $=\frac{2}{n(n^2-1)}$ $=\frac{2}{n^3-n}$ $A=2$	Most students were able to get the result except for those who made careless mistakes in algebraic. There were a few students who adopted an incorrect approach to substitute in n = 1 to deduce A. That is not correct as that only gives a particular case and does not show that the result is applicable for all n at least 2
(ii)	$\sum_{n=2}^{N}\frac{1}{n^3-n}$ $=\frac{1}{2}\sum_{n=2}^{N}\frac{2}{n^3-n}$ $=\frac{1}{2}\sum_{n=2}^{N}(u_{n-1}-2u_n+u_{n+1})$ $=\frac{1}{2}\left[\begin{array}{ccc} u_1 & -2u_2 & \cancel{+u_3} \\ +u_2 & \cancel{-2u_3} & +u_4 \\ \cancel{+u_3} & -2u_4 & +u_5 \\ \dots & \dots & \dots \\ +u_{N-3} & -2u_{N-2} & \cancel{+u_{N-1}} \\ +u_{N-2} & \cancel{-2u_{N-1}} & +u_N \\ \cancel{+u_{N-1}} & -2u_N & +u_{N+1} \end{array}\right]$ $=\frac{1}{2}(u_1-u_2-u_N+u_{N+1})$ $=\frac{1}{2}\left(1-\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$	Most students were able to carry out method of differences. A few missed out the 1/2 in their working. Some students committed the error of writing the last few rows in terms of n instead of N.

(iii)	$\sum_{n=2}^{\infty}\frac{1}{n^3-n}=\lim_{N\to\infty}\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ Since $\frac{1}{N}\to 0$ as $N\to\infty$, $\frac{1}{N+1}\to 0$ as $N\to\infty$ $\therefore$ It converges.	Students should note that to explain convergence, there is a need to **explicitly explain** $\frac{1}{N}\to 0$, $\frac{1}{N+1}\to 0$ as $N\to\infty$ It is not sufficient to just state the limit of the sum is ¼ . Also some students wrote as $n\to\infty$ but we are supposed to consider N instead. Lastly, a number of students incorrectly stated $\frac{1}{n^3-n}\to 0$ as $n\to\infty$, this does not explain that the sum converges.
(iv)	$\sum_{n=3}^{N}\frac{1}{(n-2)(n-1)(n)}$ $=\sum_{n+1=3}^{n+1=N}\frac{1}{(n+1-2)(n+1-1)(n+1)}$ (Replace n with $n+1$.) $=\sum_{n=2}^{N-1}\frac{1}{(n-1)(n)(n+1)}$ $=\sum_{n=2}^{N-1}\frac{1}{n^3-n}$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N}\right)$	Students who succeeded in identifying the correct replacement tended to make mistake in not updating the upper limit, resulting in error in solutions. There were some students who replaced n by $n-1$ by going from (ii) to (iv) but lacked the understanding in knowing what that implied and how to get the correct solution. Have included that as an alternative method for students to see the difference. But this method is more cumbersome and not recommended!
	(Alternative, but not recommended)	

	$\sum_{n=2}^{N}\frac{1}{n^3-n}=\sum_{n=2}^{N}\frac{1}{n(n-1)(n+1)}$ $=\sum_{n-1=2}^{n-1=N}\frac{1}{(n-1)(n-2)(n)}$ (n replaced by $n-1$) $=\sum_{n=3}^{N+1}\frac{1}{(n-1)(n-2)(n)}$ From (ii), $\sum_{n=2}^{N}\frac{1}{n^3-n}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ Thus, $\sum_{n=3}^{N+1}\frac{1}{(n-1)(n-2)(n)}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N}+\frac{1}{N+1}\right)$ $\therefore\sum_{n=3}^{N}\frac{1}{(n-1)(n-2)(n)}$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N-1+1}\right)$ $=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{N-1}+\frac{1}{N}\right)$	

4	Solution [13] Complex Numbers P1 Q4	
(i)	From: $x^4-4x^3+6x^2-ax+b=0$ Given that $x=x_0$ is a root, then: $x_0^{\ 4}-4x_0^{\ 3}+6x_0^{\ 2}-ax_0+b=0$ ---eqn (1) Consider applying conjugate on both sides: $\left(x_0^{\ 4}-4x_0^{\ 3}+6x_0^{\ 2}-ax_0+b\right)*=\left(0\right)*$ $\left(x_0^{\ 4}\right)*-\left(4x_0^{\ 3}\right)*+\left(6x_0^{\ 2}\right)*-\left(ax_0\right)*+\left(b\right)*=0$ Since coefficients are all real, then $\left(a\right)*=a$ and $\left(b\right)*=b$ $\left(x_0*\right)^4-4\left(x_0*\right)^3+6\left(x_0*\right)^2-a\left(x_0*\right)+b=0$ Therefore, x_0* is a root as well. Alternatively, Substitute $x=x_0^*$ into LHS of eqn (1): $\left(x_0*\right)^4-4\left(x_0*\right)^3+6\left(x_0*\right)^2-a\left(x_0*\right)+b$ $=\left(x_0^{\ 4}\right)*-4\left(x_0^{\ 3}\right)*+6\left(x_0^{\ 2}\right)*-a\left(x_0\right)*+b$ $=\left(x_0^{\ 4}-4x_0^{\ 3}+6x_0^{\ 2}-ax_0+b\right)*$ since *a*, *b* are real $=\left(0\right)*=0$ Thus x_0* is also a root.	This question was not well attempted and there is one misconception that $x_0*=-x_0$ Some of the students verify that x_0* is the root without much explanation. There is a difference between "**Show**" and "**Verify**"
(ii)	Using Remainder Theorem: Since $x=2-\mathrm{i}$ is a root of the equation, $\left(2-\mathrm{i}\right)^4-4\left(2-\mathrm{i}\right)^3+6\left(2-\mathrm{i}\right)^2-a\left(2-\mathrm{i}\right)+b=0$ $-7-24\mathrm{i}-4\left(2-11\mathrm{i}\right)+6\left(3-4\mathrm{i}\right)-2a+a\mathrm{i}+b=0$ $3-4\mathrm{i}-2a+a\mathrm{i}+b=0$ Comparing the real and imaginary parts, $3-2a+b=0$ and $-4+a=0$ $a=4$ and $b=5$ $x^4-4x^3+6x^2-4x+5=0$	The responses from the students were mixed. The students could have used their GC to evaluate some of the terms (or to check their answers) to prevent the careless mistakes in evaluating the terms. Some students used part (i) answer to obtain $x=2+\mathrm{i}$ to obtain the other roots but there are slips in their calculation which led to incorrect answers. Students should learn to present their workings clearly and

		coherently and not all over the place as it is not easy to follow the flow of working.
	Using Factor Theorem: $x^4-4x^3+6x^2-4x+5$ $\equiv\left[x-(2-\mathrm{i})\right]\left[x-(2+\mathrm{i})\right]\left(x^2+Ax+B\right)$ Comparing constant term: $5=B(2+\mathrm{i})(2-\mathrm{i})\Rightarrow B=1$ Substitute $x=2$: $16-32+24-8+5$ $\equiv\left[2-(2-\mathrm{i})\right]\left[2-(2+\mathrm{i})\right]\left(4+2A+B\right)$ $5=4+2A+1\Rightarrow A=0$ For $x^2+Ax+B=0\Rightarrow x^2+1=0$ $x=\mathrm{i}$ or $-\mathrm{i}$ The roots are: $x=2-\mathrm{i},\ 2+\mathrm{i},\ -\mathrm{i},\ \mathrm{i}$	As "*a*" and "*b*" were used in the previous part, students should avoid using "*a*" and "*b*" again when factorizing the equation. A much more efficient method of solving this question is • Sub $x=2-\mathrm{i}$ into the given equation to solve for *a* and *b* • Use the GC to simplify the terms • With the values of *a* and *b*, use the GC to directly solve for the roots
(iii)	$by^4-ay^3+6y^2-4y+1=0$ $1-4y+6y^2-ay^3+by^4=0$ Divide throughout by y^4, $\left(\frac{1}{y}\right)^4-4\left(\frac{1}{y}\right)^3+6\left(\frac{1}{y}\right)^2-a\left(\frac{1}{y}\right)+b=0$ Replace x with $\frac{1}{y}$, Then $\frac{1}{y}=2-\mathrm{i},\ 2+\mathrm{i},\ -\mathrm{i},\ \mathrm{i}$ $y=\frac{2}{5}+\mathrm{i}\frac{1}{5},\ \frac{2}{5}-\mathrm{i}\frac{1}{5},\ \mathrm{i},\ -\mathrm{i}$	There are some students who simply ignored the questions and proceed **without** the use of the "hence". Some of the students managed to identify the substitution but they **did not simply the terms and this resulted in a loss of mark**. These simplification can be done with the use of a calculator, for example $\frac{1}{i}=-i$.

5	Solution [7] Abstract Vectors P1 Q5	
(i)	 By Ratio Theorem, $\overrightarrow{OS}=\dfrac{\overrightarrow{OC}+3\overrightarrow{OA}}{1+3}$ $\overrightarrow{OS}=\dfrac{1}{4}\overrightarrow{OC}+\dfrac{3}{4}\overrightarrow{OA}$ $\overrightarrow{OS}=\dfrac{1}{4}(\mathbf{a}+\mathbf{b})+\dfrac{3}{4}\mathbf{a}$ Since $\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AC}$ $\overrightarrow{AC}=\overrightarrow{OB}=\mathbf{b}$ $\overrightarrow{OS}=\mathbf{a}+\dfrac{1}{4}\mathbf{b}$	There is a need for students to understand and correctly write the vector notation. This part is generally quite well attempted. One common mistake is that the students have $\overrightarrow{OS}=\mathbf{a}+\dfrac{1}{5}\mathbf{b}$ instead
(ii)	$\overrightarrow{MS}=\overrightarrow{OS}-\overrightarrow{OM}$ $=\left(\mathbf{a}+\dfrac{1}{4}\mathbf{b}\right)-\dfrac{1}{2}\mathbf{b}=\mathbf{a}-\dfrac{1}{4}\mathbf{b}$ $l_{MS}:\mathbf{r}=\overrightarrow{OM}+\lambda\overrightarrow{MS}$ $\mathbf{r}=\dfrac{1}{2}\mathbf{b}+\lambda\left(\mathbf{a}-\dfrac{1}{4}\mathbf{b}\right),\ \lambda\in\mathbb{R}$ $\overrightarrow{BN}=\overrightarrow{ON}-\overrightarrow{OB}$ $=\dfrac{1}{2}\left[\mathbf{a}+(\mathbf{a}+\mathbf{b})\right]-\mathbf{b}=\mathbf{a}-\dfrac{1}{2}\mathbf{b}$ $l_{BN}:\mathbf{r}=\overrightarrow{OB}+\mu\overrightarrow{BN}$ $\mathbf{r}=\mathbf{b}+\mu\left(\mathbf{a}-\dfrac{1}{2}\mathbf{b}\right),\ \mu\in\mathbb{R}$	There are numerous slips in the simplification of the vectors. Some students used the same variable for both vector equation of line MS and BN and this led to confusion for part (iii). Some students only find the direction of the lines MS and BN and not finding the vector equation. Another common mistake is that students see $\overrightarrow{ON}=\dfrac{1}{2}\mathbf{b}$ where it should have been $\overrightarrow{AN}=\dfrac{1}{2}\mathbf{b}$
(iii)	At T, $\dfrac{1}{2}\mathbf{b}+\lambda\left(\mathbf{a}-\dfrac{1}{4}\mathbf{b}\right)=\mathbf{b}+\mu\left(\mathbf{a}-\dfrac{1}{2}\mathbf{b}\right)$	2 parallel lines **MIGHT NOT** result in them being collinear unless there is a common point on the 2

	Since **a** and **b** are non parallel and non zero, comparing coefficients of **a,** $\lambda = \mu$ Comparing coefficients of **b,** $\frac{1}{2} - \frac{\lambda}{4} = 1 - \frac{\mu}{2}$ Solving, $\mu = 2 = \lambda$ $\therefore \overrightarrow{OT} = \frac{1}{2}\mathbf{b} + 2\left(\mathbf{a} - \frac{1}{4}\mathbf{b}\right) = 2\mathbf{a}$ Since $\overrightarrow{OT} = 2\overrightarrow{OA}$, then *O*, *T* and *A* are colliear points with *O* (or *A* or *T*) as the common point.	lines There is **a need to state that there is a common point** and not just simply state that $\overrightarrow{OT} = 2\overrightarrow{OA}$.

6	Solution [7]	
	Let $y=\dfrac{ax+b}{x-a}$. $xy-ay=ax+b$ $x(y-a)=ay+b$ $x=\dfrac{ay+b}{y-a}$ $\therefore\ f^{-1}(x)=\dfrac{ax+b}{x-a}$ f is self-inverse.	This part is generally well attempted by most except for some slips in simplifying the algebra to find $\mathrm{f}^{-1}(x)$.
	Symmetrical about $y = x$.	
	$R_f=\mathbb{R}\setminus\{a\}$ Since $D_g=\mathbb{R}, R_f\subseteq D_g$. $\therefore$ gf exists. $D_{gf}=D_f=\mathbb{R}\setminus\{a\}$ $\mathbb{R}\setminus\{a\}\xrightarrow{f}\mathbb{R}\setminus\{a\}\xrightarrow{g}(0,c)\cup(c,e)$ or $(0,e)\setminus\{c\}$	Marks will be given to what the assessors can read and see from your writing. You need to understand the meaning of the symbols used in set notation as "$\cup$" and "$\cap$" mean differently and hence this resulted in quite a lot of students getting the wrong answer. Some students skipped the steps for finding the R_{gf} and hence when the provided answer is incorrect, there **is no marks** awarded.

<table>
<tr><td>7</td><td>Solution [8] AP GP</td><td></td></tr>
<tr><td>(a)</td><td>Let first term be $u_1 = a$ and common difference d.
Middle term $u_{\frac{n+1}{2}} = a + \left(\frac{n+1}{2} - 1\right)d$
Last term $u_n = a + (n-1)d$

Given $a + a + \left(\frac{n+1}{2} - 1\right)d = a + (n-1)d$
$a + \frac{nd}{2} + \frac{d}{2} - d = nd - d$
$\frac{nd}{2} = a + \frac{d}{2}$
$n = \frac{2a+d}{d}$

$n = \frac{2a}{d} + 1$ OR $d = \frac{2a}{n-1}$

Sub $n = \frac{2a}{d} + 1$ into the middle term
$u_{\frac{n+1}{2}} = a + \left(\frac{n+1}{2} - 1\right)d$:

$u_{\frac{n+1}{2}}$
$= a + \left(\frac{\frac{2a}{d} + 1 + 1}{2} - 1\right)d$
$= a + \left(\frac{2 + \frac{2a}{d}}{2} - 1\right)d$
$= a + \left(1 + \frac{a}{d} - 1\right)d$
$= a + \left(\frac{a}{d}\right)d$
$= 2a$</td><td>Generally may students are able to identify the correct form for the middle term. Some common mistake includes simply dividing n by 2 which will give a product of a fraction to d.
Alternatively, some students correctly identified that the relationship between the 3 terms differs by the same amount of common difference.
Solutions which only shows for certain values of n, eg. n = 5 needs to extend their working to show for the all odd numbers for n in order to get the full credit.
Many unsuccessful attempts have misread the question interpreting the sum of the first and middle terms as the sum of all terms from the first term to the middle term.</td></tr>
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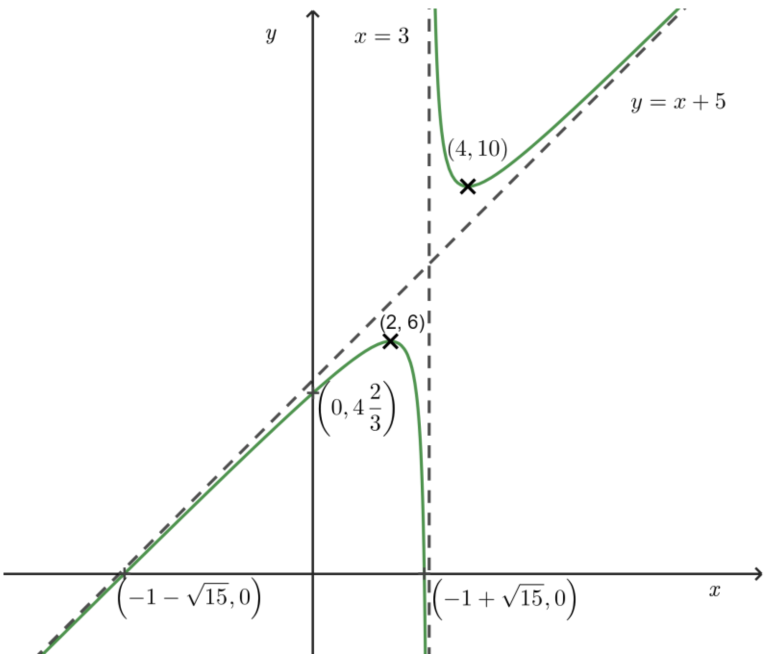
	OR Alternatively Sub $d=\dfrac{2a}{n-1}$ into the middle term $u_{\frac{n+1}{2}}=a+\left(\dfrac{n+1}{2}-1\right)d$: $u_{\frac{n+1}{2}}$ $=a+\left(\dfrac{n+1}{2}-1\right)\left(\dfrac{2a}{n-1}\right)$ $=a+\left(\dfrac{n-1}{2}\right)\left(\dfrac{2a}{n-1}\right)$ $=2a$	
(b) (i)	Given $T_1=3p^3,\ r=\dfrac{p}{q^2}$ And $m=3p^3\left(\dfrac{p}{q^2}\right)^{n-1}$ $m=3p^3\left(\dfrac{p^{n-1}}{q^{2(n-1)}}\right)$ $m=3\left(\dfrac{p^{n+2}}{q^{2n-2}}\right)$ $\ln m=\ln 3+\ln p^{n+2}-\ln q^{2n-2}$ $\ln m=\ln 3+(n+2)\ln p-(2n-2)\ln q$ $A=2,\ B=2$	Generally done well. Unsuccessful attempts include defining $m=3p^3\left(\dfrac{p}{q^2}\right)^n$ and applying laws of logarithm. Several solutions show carelessness in their calculations.
(ii)	Since it is a convergent GP, then: $\lvert r\rvert<1$. $-1<\dfrac{p}{q^2}<1$ $q=2p$	Many did not realise the "convergent GP" information is to be used here to get the inequality for p. Most which presented $\lvert r\rvert<1$ correctly proceeded to show their lack of skills in manipulating inequalities including multiplying both sides with p without any justification that p > 0, or even flipping the fractions on both sides of

	$-1<\frac{p}{(2p)^2}<1$ $-1<\frac{1}{4p}<1$ $-4<\frac{1}{p}<4$ $p<-\frac{1}{4}$ OR $p>\frac{1}{4}$ (Rejected, since all terms are positive.)	the inequality without checking if the inequality sign should change. In many cases, since p is positive, many such solutions do end up at the correct answer but will not be given credit. Many solutions did not realise also the "positive terms" give the restriction for p to be positive, hence not rejecting the negative range of value for p.

8	Solution [8] Macluarin's Series	
(i)	$\left(1-x^2\right)f''(x)-xf'(x)=0$ (Given) Differentiating, $\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$ At $x=0$, substitute into: $\left(1-x^2\right)f''(x)-xf'(x)=0$ $f''(0)-0=0$ $f''(0)=0$ At $x=0$, substitute into: $\left(1-x^2\right)f'''(x)-2xf''(x)-xf''(x)-f'(x)=0$ $f'''(0)-0-0-f'(0)=0$ $f'''(0)=-1$ Thus, $f(x)$ $=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)+\ldots$ $=1+x(-1)+\frac{x^2}{2!}(0)+\frac{x^3}{3!}(-1)+\ldots$ $=1-x-\frac{x^3}{6}+\ldots$	Some students were quite loss in this question and could not begin, leaving the whole question unanswered. There were some who did not proceed to find the 3rd derivative. If that is the case, the answer will end up with 2 non-zero terms instead of 3.
(ii)	$f(x)=1-x-\frac{x^3}{6}+\ldots$ $f'(x)=-1-\frac{x^2}{2}+\ldots$ $\frac{f'(x)}{f(x)}=\frac{-1-\frac{x^2}{2}+\ldots}{1-x-\frac{x^3}{6}+\ldots}$ $=\left(-1-\frac{x^2}{2}+\ldots\right)\left(1-x-\frac{x^3}{6}+\ldots\right)^{-1}$	Many students could see the need to use the binomial series for $(1-x)^{-1}$. However, they had used with an extra x^3 term $\left(1-x-\frac{x^3}{6}+\ldots\right)^{-1}$. That had created a more complicated set of workings since we only need to expand up to and including x^2.

	$=\left(-1-\frac{x^2}{2}+\ldots\right)(1-x+\ldots)^{-1}$ (Need not consider x^3 – term anymore.) $=\left(-1-\frac{x^2}{2}+\ldots\right)\left(1+(-1)(-x)+\frac{(-1)(-2)}{2!}(-x)^2\ldots\right)$ $=\left(-1-\frac{x^2}{2}+\ldots\right)(1+x+x^2\ldots)$ $=\left(-1-x-x^2-\frac{x^2}{2}+\ldots\right)$ $=-1-x-\frac{3}{2}x^2+\ldots$	A handful of students derived the series using the first principle. This was not accepted because it is a "hence" question.
(iii)	$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}=-1-x-\frac{3}{2}x^2+\ldots$ Integrating both sides with respect to x, $\int\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x=\int-1-x-\frac{3}{2}x^2+\ldots\ldots\,\mathrm{d}x$ $\ln\lvert\mathrm{f}(x)\rvert=\left[-x-\frac{1}{2}x^2+\ldots\right]+C$ When $x=0$, $\mathrm{f}(0)=1$: $\ln\lvert 1\rvert=C\Rightarrow C=0$ $\ln\lvert\mathrm{f}(x)\rvert=-x-\frac{1}{2}x^2+\ldots$	Majority had used the wrong method in this question. The question required us to "deduce" so we need to use results in (ii) and link it to $\ln\lvert\mathrm{f}(x)\rvert$. Many students simply used (i) and expand $\ln\lvert\mathrm{f}(x)\rvert$ using the ln (1 + x) formula in MF26, which does not contain modulus.

9	Solution [10] Curve Sketching	
(i)	Since $x=3$, $c=-3$ $y=x+5+\dfrac{d}{x-3}=\dfrac{x^2+2x-15+d}{x-3}=\dfrac{ax^2+bx-14}{x-3}$ Therefore $a=1$ and $b=2$.	Almost all could do this, except some with careless mistakes.
(ii)	Graph of $y=\dfrac{x^2+2x-14}{x-3}$  When rounded to 3 s.f, the axial intercepts are (0, 4.67), (−4.87, 0) and (2.87, 0).	Graphs were well drawn and points were labelled clearly. A small number of students had only the bottom portion of the graph, probably due to the settings in the GC. Students ought to be aware that such rational function graphs always come in 2 branches.

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<td>(iii)</td>
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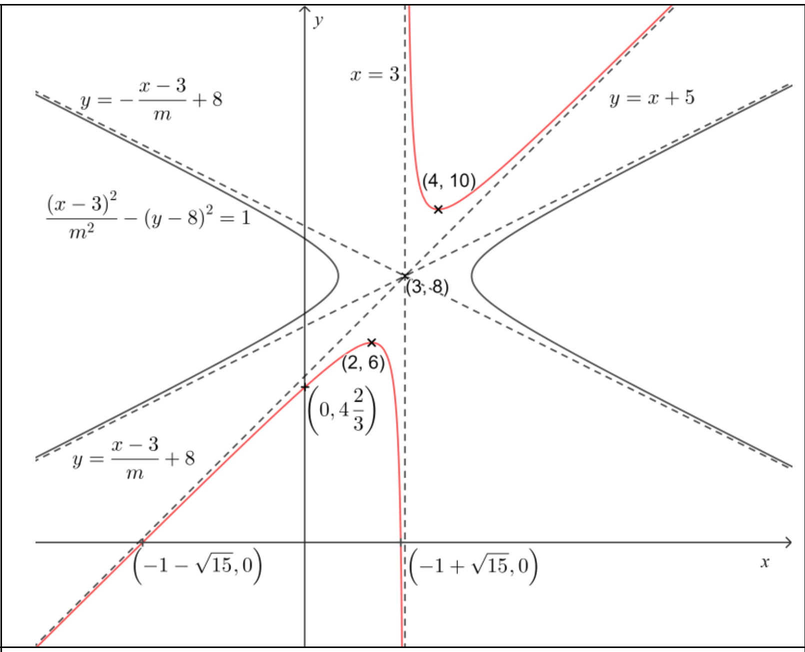

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<td>Majority could identify the hyperbola and write down the equations of the asymptotes, though some did not get the centre right.</td>
</tr>
<tr>
<td>(iv)</td>
<td>
The asymptotes of $\frac{(x-3)^2}{m^2}-(y-8)^2=1$ ----(1) are
$y=\frac{x-3}{m}+8$ and $y=-\frac{x-3}{m}+8$

$y=\frac{x^2+2x-14}{x-3}$ ----(2) has asymptotes $x=3$ and
$y=x+5$

All the asymptotes meet at the point $(3,8)$.
Observing the graphs, (1) and (2) intersect when the asymptote of (1) with positive gradient, $y=\frac{x-3}{m}+8$ must be between the asymptotes of (2) i.e. $y=x+5$ and $x=3$.

Thus, gradient of $y=\frac{x-3}{m}+8$ need to be between the gradient of $y=x+5$ and gradient of the vertical line, $x=3$.

$1<\frac{1}{m}$
Therefore, $0<m<1$
</td>
<td>Majority did not get this right or had left it unanswered.

The crucial point here is to observe that 1/m is the gradient of the asymptote of the ellipse and compare this with the gradient of the asymptote of the original curve which is 1, to establish the number of intersections of the 2 graphs.</td>
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10	Solution [10] Applications of Differentiation	
(i)	$x=\sqrt{4+t^2}$ $\quad y=t^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{t}{\sqrt{4+t^2}}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}t}=2t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}t}\div\frac{\mathrm{d}x}{\mathrm{d}t}$ $=2\sqrt{4+t^2}$ When $t=p$, $x=\sqrt{4+p^2}$, $y=p^2$ and $\frac{\mathrm{d}y}{\mathrm{d}x}=2\sqrt{4+p^2}$. Equation of tangent $y-p^2=2\sqrt{4+p^2}\left(x-\sqrt{4+p^2}\right)$ $y=2\sqrt{4+p^2}\,x-8-p^2$	Most students were able to solve this part, except for those who made careless mistakes in finding dy/dx and equation of tangent
(ii)	When $y=0$, $x=\frac{8+p^2}{2\sqrt{4+p^2}}$ x-coordinate of M $=\frac{1}{2}\left(\sqrt{4+p^2}+\frac{8+p^2}{2\sqrt{4+p^2}}\right)$ $=\frac{16+3p^2}{4\sqrt{4+p^2}}$ Coordinates of M are $\left(\frac{16+3p^2}{4\sqrt{4+p^2}},\frac{p^2}{2}\right)$ Parametric equation of curve traced out by M $x=\frac{16+3p^2}{4\sqrt{4+p^2}}$ $y=\frac{p^2}{2}$ $\quad (1)$ From (1), $p^2=2y$ Cartesian equation of line $x=\frac{16+3(2y)}{4\sqrt{4+2y}}$ $2x\sqrt{4+2y}=8+3y$	Many students did not know how to do this part. For those who attempted, most stopped at finding the coordinates of M but did not know how to proceed.

(iii)	Required area $=\int_2^3 y\ \mathrm{d}x$ $=\int_0^{\sqrt{5}} y\frac{\mathrm{d}x}{\mathrm{d}t}\ \mathrm{d}t$ $=\int_0^{\sqrt{5}} \left(t^2\right)\left(\frac{t}{\sqrt{4+t^2}}\right)\mathrm{d}t$ $=\int_0^{\sqrt{5}} \frac{t^3}{\sqrt{4+t^2}}\ \mathrm{d}t\ \ (shown)$ Let $u=t^2$ then $\frac{\mathrm{d}u}{\mathrm{d}t}=2t$	Many students stopped at showing the integral and did not know how to proceed. Some gave the incorrect limits.
	Let $\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{t}{\sqrt{4+t^2}}$ then $v=\frac{1}{2}\int\ 2t\left(4+t^2\right)^{-\frac{1}{2}}\ \mathrm{d}t=\left(4+t^2\right)^{\frac{1}{2}}$ $\int_0^{\sqrt{5}} \frac{t^3}{\sqrt{4+t^2}}\ \mathrm{d}t$ $=\left[t^2\sqrt{4+t^2}\right]_0^{\sqrt{5}}-\int_0^{\sqrt{5}} 2t\sqrt{4+t^2}\ \mathrm{d}t$ $=\left[5\sqrt{9}-0\right]-\left[\frac{2}{3}\left(4+t^2\right)^{\frac{3}{2}}\right]_0^{\sqrt{5}}$ $=15-\frac{2}{3}\left[(9)^{\frac{3}{2}}-(4)^{\frac{3}{2}}\right]$ $=\frac{7}{3}$ units^2	For students who recognized that integration by parts was necessary, most chose the incorrect u to use. It should be noted that the question specifically asked students to **<u>evaluate the integral</u>** to find the **<u>exact area</u>**. These key phrases suggest that using GC to evaluate the integral will not be acceptable. Also using the cartesian form of the integral directly instead of parametric form to evaluate is also not acceptable.

11	Solution [12] Differentiation	
(i)	A is at $(1,0)$. B is at (x,y) where $y^2=1-\frac{x^2}{4}$ $r^2=(x-1)^2+(y-0)^2$ $=(x-1)^2+1-\frac{x^2}{4}$ $=\frac{1}{4}(3x^2-8x+8)$ $F=\frac{4k}{3x^2-8x+8}$	Many candidates are not able to reason that the distance between $(1,0)$ and (x,y) is $(x-1)^2+(y-0)^2$
(ii)	$F=\frac{4k}{3x^2-8x+8}$ $\frac{\mathrm{d}F}{\mathrm{d}x}=-4k(3x^2-8x+8)^{-2}(6x-8)$ $=\frac{-8k(3x-4)}{(3x^2-8x+8)^2}$ At stationary point, $\frac{\mathrm{d}F}{\mathrm{d}x}=0$. $6x-8=0$ $x=\frac{4}{3}$ **Method 1: Using 2nd Derivative to verify nature** $\frac{\mathrm{d}^2F}{\mathrm{d}x^2}=\frac{(3x^2-8x+8)^2(-24k)+8k(3x-4)(2)(3x^2-8x+8)(6x-8)}{(3x^2-8x+8)^4}$ When $6x-8=0,$ i.e. $x=\frac{4}{3}$: $\frac{\mathrm{d}^2F}{\mathrm{d}x^2}=-3.375k<0$, therefore maximum. Note: $\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right\|_{x=\frac{4}{3}}$ can be found easily using GC.	Most candidates are aware that they need to find $\frac{\mathrm{d}F}{\mathrm{d}x}$ and set $\frac{\mathrm{d}F}{\mathrm{d}x}=0$, to locate the x value of the stationary point. When implementing the second derivative test, candidates are required to give the value of $\left.\frac{\mathrm{d}^2F}{\mathrm{d}x^2}\right\|_{x=\frac{4}{3}}$. This value can be easily found using GC. They are not allowed to

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Method 2: Using 1st Derivative to verify nature

<table>
<tr><td>x</td><td>1.33</td><td>$\frac{4}{3}$</td><td>1.34</td></tr>
<tr><td>$\frac{dF}{dx}$</td><td>0.00112k</td><td>0</td><td>-0.0225k</td></tr>
<tr><td></td><td>/</td><td>-</td><td>\</td></tr>
</table>

$$F_{\max} = \frac{4k}{3\left(\frac{4}{3}\right)^2 - 8\left(\frac{4}{3}\right) + 8}$$

$$= \frac{3k}{2}$$

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simply state $\left.\frac{d^2F}{dx^2}\right|_{x=\frac{4}{3}} < 0$

Similarly for the first derivative test, candidates are required to provide the values of x and $\frac{dF}{dx}$, to establish that there is a maximum point at $x = \frac{4}{3}$

Candidates who failed to provide numerical values when carrying out the 1st / 2nd derivative test will be penalized.

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<td>(iii)</td>
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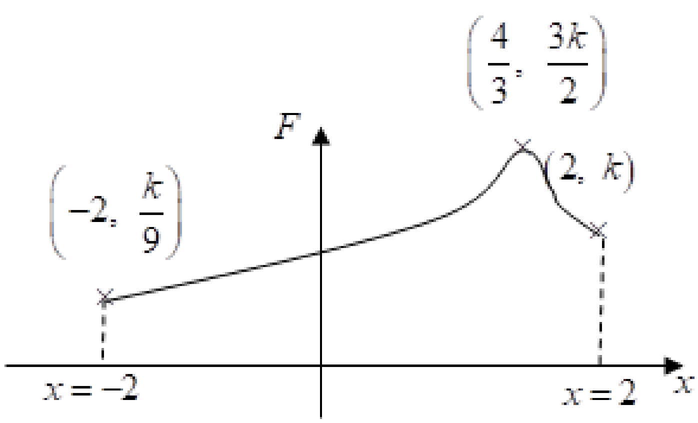

Note that $y^2 = 1 - \frac{x^2}{4} \Rightarrow x = \pm 2\sqrt{1 - y^2}$

Therefore $-2 \le x \le 2$

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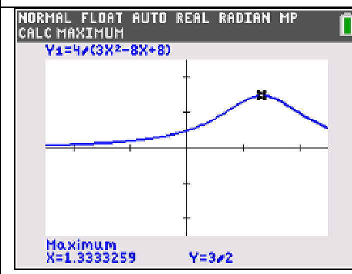

Candidates need to take into account the **<u>context</u>** of the question. Since B is in motion along the curve defined by $y^2 = 1 - \frac{x^2}{4}$.

Therefore $-2 \le x \le 2$.

In addition, $F = \frac{k}{r^2}$, implying that F is always positive.

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<tr>
<td>(iv)</td>
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Minimum F occurs when B is farthest from A.
i.e. when B is $(-2, 0)$, and thus $r = 3$.

$$F_{\min} = \frac{k}{3^2}$$

$$F_{\min} = \frac{k}{9}$$

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(v)	By symmetry, *B* must be at $(0,1)$ or $(0,-1)$ with x-coordinate $=0$ With $x=0$, $F=\dfrac{4k}{0+8}=\dfrac{k}{2}$	Candidates **<u>need to answer in context.</u>** In this case, since B is in motion on the curve $\dfrac{x^2}{4}+y^2=1$, B is equidistant from the foci at $(0,1)$ or $(0,-1)$

• Answers given as (0,b) and (0,−b) did not apply the knowledge in context, and are not acceptable.

<table>
<tr><td>12</td><td>Solution [13] Differential Equations</td><td></td></tr>
<tr><td>(i)</td><td>
$$\frac{\mathrm{d}Q_{in}}{\mathrm{d}t}=k$$
$$\frac{\mathrm{d}Q_{out}}{\mathrm{d}t}\propto\sqrt{Q}$$
$$\frac{\mathrm{d}Q_{out}}{\mathrm{d}t}=c\sqrt{Q},\ \ c\in\mathbb{R}$$
$$\frac{\mathrm{d}Q}{\mathrm{d}t}=\frac{\mathrm{d}Q_{in}}{\mathrm{d}t}-\frac{\mathrm{d}Q_{out}}{\mathrm{d}t}$$
$$=k-c\sqrt{Q}$$

Starting with a new clean tank:
When t = 0, Q = 0, $\frac{\mathrm{d}Q}{\mathrm{d}t}=5$ ---(*)
$$\frac{\mathrm{d}Q}{\mathrm{d}t}=k-c\sqrt{Q}$$
$$5=k-c\sqrt{0}$$
$$k=5$$

With filter in a new clean tank, level of pollution stabilizes at 75 units:
As $t\to\infty,\ \ Q\to 75,\ \ \frac{\mathrm{d}Q}{\mathrm{d}t}\to 0$, ----(**)
$$\frac{\mathrm{d}Q}{\mathrm{d}t}=5-c\sqrt{Q}$$
$$0=5-c\sqrt{75}$$
$$c=\frac{1}{\sqrt{3}}$$
$$\therefore\ \frac{\mathrm{d}Q}{\mathrm{d}t}=5-\sqrt{\frac{Q}{3}}\ \text{(shown)}$$
</td><td>While the majority of solutions are able to show the result, their presentations often lack sufficient workings to warrant full credit. In many cases, they omit the necessary steps for integrating the information to substitute for the variable. In fact, most of the solutions left out the first 4 lines of the suggested solution which presents how the DE is formed in the first place which is in fact critical for this show problem. As it is a show question, the fact that information from the context is extracted to substitute into the variables to find unknowns should be made clear as well.</td></tr>
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(ii)	$\frac{\mathrm{d}Q}{\mathrm{d}t}=5-\sqrt{\frac{Q}{3}}$ ----(1) $x=\sqrt{\frac{Q}{3}}$ $Q=3x^2$ ----(2) Differentiate with respect to *t*, $\frac{\mathrm{d}Q}{\mathrm{d}t}=6x\frac{\mathrm{d}x}{\mathrm{d}t}$ ----(3) Sub (2) and (3) into (1): $6x\frac{\mathrm{d}x}{\mathrm{d}t}=5-x$	Most solutions are able to show the result here as well but again lacks most of the critical explanations necessary for full credit. Most found $\frac{\mathrm{d}Q}{\mathrm{d}x}$ or $\frac{\mathrm{d}x}{\mathrm{d}Q}$ instead but failed to show how $\frac{\mathrm{d}Q}{\mathrm{d}t}$ is related to them jumping into the result instead. Such lack of explanation invalidates the whole proof. One critical misconception found in some solution is that $\frac{\mathrm{d}x}{\mathrm{d}Q}=\frac{1}{2\sqrt{3}\sqrt{Q}}$ implies $2\sqrt{3}\sqrt{\mathrm{Q}}\mathrm{d}x=\mathrm{d}Q$. To clarify, $\frac{\mathrm{d}x}{\mathrm{d}Q}$ is <u>NOT</u> a fraction and should never be treated as one.

(iii)	$6x\frac{\mathrm{d}x}{\mathrm{d}t}=5-x$ $\int \frac{6x}{5-x}\,\mathrm{d}x=\int \mathrm{d}t$ $6\int -1+\frac{5}{5-x}\,\mathrm{d}x=\int \mathrm{d}t$ $-x-5\ln\lvert 5-x\rvert=\frac{1}{6}t+c$ $x+\ln\lvert 5-x\rvert^5=\frac{-t}{6}-c$ $\ln\lvert 5-x\rvert^5=-x-\frac{t}{6}-c$ $\lvert 5-x\rvert^5=\mathrm{e}^{-x-\frac{t}{6}-c}$ $(5-x)^5=\pm\mathrm{e}^{-c}\,\mathrm{e}^{-x}\mathrm{e}^{-\frac{t}{6}}$ $(5-x)^5\,\mathrm{e}^x=A\mathrm{e}^{\frac{-t}{6}}$ where $A=\pm\mathrm{e}^{-c}$ -----(*) $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\mathrm{e}^{\frac{-t}{6}}$ When t = 0, Q = 0: $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=A\mathrm{e}^{\frac{-t}{6}}$ $\left(5-\sqrt{\frac{0}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{0}{3}}}=A\mathrm{e}^{0}$ $A=5^5=3125$ $\left(5-\sqrt{\frac{Q}{3}}\right)^5\mathrm{e}^{\sqrt{\frac{Q}{3}}}=3125\mathrm{e}^{\frac{-t}{6}}$ Therefore, a = 5, b = 5, m = 3125, and $p=\frac{-1}{6}$.	Most solutions presented are able to correctly separate the variables and do the integration reasonable well, but are unable to manipulate into a credible general solution at (*). Many did not manage to handle the removal of the modulus well, simply dropping it without justification. Some have attempted to convert the arbitrary constants but did not manage to present the correct steps. Many have issues with working with the powers with some applying the laws of indices incorrectly. Many solutions also did not manage to identify the correct initial values for substitution to find the arbitrary constants while some did it prematurely.

(iv)	When $Q = 48$, $\left(5-\sqrt{\frac{48}{3}}\right)^5 e^{\sqrt{\frac{48}{3}}} = 3125e^{\frac{-t}{6}}$ $t = -6\ln\left(\frac{e^4}{3125}\right) = 24.3$ days (3 s.f.) Without a filter, the pollutant level would reach 48 units in $t = 48 / 5 = 9.6$ days. Therefore the filter is effective.	Most did not attempt this part due to lack of time or simply did not get an answer from part (iii).